Electronic Journal of Differential Equations, Vol. 2015 (2015), No. 169, pp. 1–20.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

SOLVABLE PRODUCT-TYPE SYSTEM OF DIFFERENCE EQUATIONS OF SECOND ORDER

STEVO STEVIĆ, MOHAMMED A. ALGHAMDI,
ABDULLAH ALOTAIBI, ELSAYED M. ELSAYED

Abstract. We show that the system of difference equations

$$z_{n+1}=\frac{w_n^a}{z_{n-1}^b},\quad w_{n+1}=\frac{z_n^c}{w_{n-1}^d},\quad n\in\mathbb{N}_0,$$

where $a,b,c,d\in\mathbb{Z}$, and initial values $z_{-1},z_0,w_{-1},w_0\in\mathbb{C}$, is solvable in closed form, and present a method for finding its solutions.

1. Introduction

Difference equations and systems not closely related to differential equations is a topic of considerable recent interest (see, e.g., [1]-[10], [12], [14]-[23], [25]-[56]). Since the appearance of paper [29], in which was explained the formula for solutions to the difference equation in [12], the area of solving difference equations and systems of difference equations reattracted some attention (see, e.g., [1]-[5], [10], [12], [23], [31], [36], [37], [39], [40], [42]-[50], [52]-[56] and the related references therein).

On the other hand, symmetric systems of difference equations and systems of a similar appearance, whose investigation began by Papaschinopoulos, Schinas and their collaborators during the mid of 1990's, is another area which has attracted some recent attention (see, e.g., [10, 18, 19, 20, 21, 25, 26, 27, 36, 39, 40, 42, 44, 45, 46, 47, 48, 49, 50, 51, 53, 54, 56] and the related references therein).

The publication of [28] and [30] initiated a considerable investigation of the boundedness character of some classes of difference equations and systems containing non-integer powers of their variables (see, e.g., [7, 8, 22, 33, 35, 51] and the related references therein). An interesting fact is that these equations and systems are perturbations of some product-type equations and systems of difference equations, usually obtained by using the translation operator

$$\tau_a(s)=a+s,\quad a\in\mathbb{R},\tag{1.1}$$

or the following operator with maximum

$$m_a(s)=\max\{a,s\},\quad a\in\mathbb{R}.\tag{1.2}$$

2010 *Mathematics Subject Classification.* 39A10, 39A20.

Key words and phrases. Solvable system of difference equations; second-order system; product-type system; long-term behavior.

©2015 Texas State University - San Marcos.

Submitted February 7, 2015. Published June 18, 2015.

Note that operator (1.1) can act on the space of complex sequences, unlike operator (1.2), which can act only on the space of real sequences. However, practically there are no results which deal with the equations and systems generated by operator (1.1) on the space of complex sequences.

Properties of solutions to difference equations and systems obtained by using operators (1.1) and (1.2) are frequently closely related to the corresponding product-type ones. For example, in [51], it was studied the following system of difference equations

$$x_{n+1} = \max\Big\{a, \frac{y_n^p}{x_{n-1}^q}\Big\}, \quad y_{n+1} = \max\Big\{a, \frac{x_n^p}{y_{n-1}^q}\Big\}, \quad n \in \mathbb{N}_0, \tag{1.3}$$

with $\min\{a,p,q\} > 0$, where the boundedness character of their positive solutions was completely characterized. Note that system (1.3) can be regarded as a perturbation of the following product-type system of difference equations

$$x_{n+1} = \frac{y_n^p}{x_{n-1}^q}, \quad y_{n+1} = \frac{x_n^p}{y_{n-1}^q}, \quad n \in \mathbb{N}_0. \tag{1.4}$$

If only positive solutions to system (1.4) are considered then it can be solved in closed form. Generally speaking, a great majority of papers on difference equations and systems consider only their positive solutions. One of the reasons is that such equations and systems can be frequently regarded as models of some population or biological models (see, e.g., [11, 31]). For some other applications to difference equations, see, for example [13, 14, 24]. Beside this, their investigation is somewhat simpler than in the general case. Hence, a natural problem, which seems has been neglected so far, is to study behavior of solutions to product-type equations and systems whose initial conditions need not be positive numbers only. This paper is devoted to the problem and can be regarded as a starting point in the investigation.

The following second-order system of difference equations, which is an extension of system (1.4),

$$z_{n+1} = \frac{w_n^a}{z_{n-1}^b}, \quad w_{n+1} = \frac{z_n^c}{w_{n-1}^d}, \quad n \in \mathbb{N}_0, \tag{1.5}$$

where $a,b,c,d \in \mathbb{R}$ and initial values z_{-1}, z_0, w_{-1}, w_0 are positive numbers can be solved in closed form. Namely, by using the method of induction, it can be shown

$$z_n > 0, \quad w_n > 0, \quad \text{for } n \geq -1,$$

which enables us, by taking the logarithm to the both sides of both equations in (1.5), to transform it to a linear second-order system of difference equations with constant coefficients, which is solvable in closed form. If z_{-1}, z_0, w_{-1}, w_0 are complex numbers, then the method cannot be used, since in the case the sequences $(z_n)_{n\geq -1}$ and $(w_n)_{n\geq -1}$ need not be uniquely defined.

Our aim here is to show that in some cases system (1.5) can be solved in closed form also when z_{-1}, z_0, w_{-1}, w_0 are complex numbers. By the obtained formulas we will present some results on the long-term behavior of solutions to system (1.5).

A vector sequence $\vec{z}_n = (z_n^{(1)}, \dots, z_n^{(l)})$, $n \geq -k$, is called *periodic* (or *eventually periodic*) with period $p \in \mathbb{N}$ if there is $n_0 \geq -k$, such that

$$z_{n+p}^{(j)} = z_n^{(j)}, \quad \text{for } n \geq n_0,$$

for every $j \in \{1, \dots, l\}$. Period p is prime if there is no $\hat{p} \in \mathbb{N}$, $\hat{p} < p$ which is a period for the vector sequence. For $p = 1$ the sequences are called eventually constant or trivial (see, e.g., [15]). The periodicity is also one of the areas of considerable interest (see, e.g., [6, 9, 14, 16, 17, 27, 31, 32, 34, 38, 41], and the related references therein). If we say that a solution of system (1.5) is periodic with period p we will tacitly regard that p need not be its prime period.

2. Main result

In this section we prove the main result in this paper, which presents formulas for solutions to system (1.5). Before we formulate and prove it, note that the domain of undefinable solutions ([45]) to system (1.5) is the set

$$\mathcal{U} = \{(z_{-1}, z_0, w_{-1}, w_0) \in \mathbb{C}^4 : z_{-1} = 0 \text{ or } z_0 = 0 \text{ or } w_{-1} = 0 \text{ or } w_0 = 0\}.$$

Hence, such solutions will be excluded from our considerations.

Theorem 2.1. *Assume that $a, b, c, d \in \mathbb{Z}$ and initial values $z_{-1}, z_0, w_{-1}, w_0 \in \mathbb{C} \setminus \{0\}$. Then system (1.5) is solvable in closed form.*

Proof. Let

$$a_1 = a, \quad b_1 = b, \quad c_1 = c, \quad d_1 = d. \tag{2.1}$$

By using the equations in (1.5) we obtain

$$z_{n+1} = \frac{w_n^{a_1}}{z_{n-1}^{b_1}} = \frac{z_{n-1}^{ca_1 - b_1}}{w_{n-2}^{da_1}} = \frac{z_{n-1}^{a_2}}{w_{n-2}^{b_2}}, \tag{2.2}$$

$$w_{n+1} = \frac{z_n^{c_1}}{w_{n-1}^{d_1}} = \frac{w_{n-1}^{ac_1 - d_1}}{z_{n-2}^{bc_1}} = \frac{w_{n-1}^{c_2}}{z_{n-2}^{d_2}}, \tag{2.3}$$

where we define a_2, b_2, c_2 and d_2 as follows

$$a_2 := ca_1 - b_1, \quad b_2 := da_1, \quad c_2 := ac_1 - d_1, \quad d_2 := bc_1.$$

By using (2.2), (2.3) and the equations in (1.5) we further obtain

$$z_{n+1} = \frac{z_{n-1}^{a_2}}{w_{n-2}^{b_2}} = \frac{w_{n-2}^{aa_2 - b_2}}{z_{n-3}^{ba_2}} = \frac{w_{n-2}^{a_3}}{z_{n-3}^{b_3}}, \tag{2.4}$$

$$w_{n+1} = \frac{w_{n-1}^{c_2}}{z_{n-2}^{d_2}} = \frac{z_{n-2}^{cc_2 - d_2}}{w_{n-3}^{dc_2}} = \frac{z_{n-2}^{c_3}}{w_{n-3}^{d_3}}, \tag{2.5}$$

where we define a_3, b_3, c_3 and d_3 as follows

$$a_3 := aa_2 - b_2, \quad b_3 := ba_2, \quad c_3 := cc_2 - d_2, \quad d_3 := dc_2.$$

Assume that

$$z_{n+1} = \frac{w_{n-2k+2}^{a_{2k-1}}}{z_{n-2k+1}^{b_{2k-1}}}, \quad w_{n+1} = \frac{z_{n-2k+2}^{c_{2k-1}}}{w_{n-2k+1}^{d_{2k-1}}}, \tag{2.6}$$

where

$$a_{2k-1} := aa_{2k-2} - b_{2k-2}, \quad b_{2k-1} := ba_{2k-2},$$
$$c_{2k-1} := cc_{2k-2} - d_{2k-2}, \quad d_{2k-1} := dc_{2k-2},$$

and

$$z_{n+1} = \frac{z_{n-2k+1}^{a_{2k}}}{w_{n-2k}^{b_{2k}}}, \quad w_{n+1} = \frac{w_{n-2k+1}^{c_{2k}}}{z_{n-2k}^{d_{2k}}}, \tag{2.7}$$

where

$$a_{2k} := ca_{2k-1} - b_{2k-1}, \quad b_{2k} := da_{2k-1},$$

$$c_{2k} := ac_{2k-1} - d_{2k-1}, \quad d_{2k} := bc_{2k-1},$$

for some $k \in \mathbb{N}$ such that $n \geq 2k-2$.

By using (2.7) and the equations in (1.5) we obtain

$$z_{n+1} = \frac{z_{n-2k+1}^{a_{2k}}}{w_{n-2k}^{b_{2k}}} = \frac{w_{n-2k}^{aa_{2k}-b_{2k}}}{z_{n-2k-1}^{ba_{2k}}} = \frac{w_{n-2k}^{a_{2k+1}}}{z_{n-2k-1}^{b_{2k+1}}}, \tag{2.8}$$

$$w_{n+1} = \frac{w_{n-2k+1}^{c_{2k}}}{z_{n-2k}^{d_{2k}}} = \frac{z_{n-2k}^{cc_{2k}-d_{2k}}}{w_{n-2k-1}^{dc_{2k}}} = \frac{z_{n-2k}^{c_{2k+1}}}{w_{n-2k-1}^{d_{2k+1}}}, \tag{2.9}$$

where we define a_{2k+1}, b_{2k+1}, c_{2k+1} and d_{2k+1} as follows

$$a_{2k+1} := aa_{2k} - b_{2k}, \quad b_{2k+1} := ba_{2k}, \quad c_{2k+1} := cc_{2k} - d_{2k}, \quad d_{2k+1} := dc_{2k}.$$

From (2.8), (2.9) and by using the equations in (1.5) we obtain

$$z_{n+1} = \frac{w_{n-2k}^{a_{2k+1}}}{z_{n-2k-1}^{b_{2k+1}}} = \frac{z_{n-2k-1}^{ca_{2k+1}-b_{2k+1}}}{w_{n-2k-2}^{da_{2k+1}}} = \frac{z_{n-2k-1}^{a_{2k+2}}}{w_{n-2k-2}^{b_{2k+2}}}, \tag{2.10}$$

$$w_{n+1} = \frac{z_{n-2k}^{c_{2k+1}}}{w_{n-2k-1}^{d_{2k+1}}} = \frac{w_{n-2k-1}^{ac_{2k+1}-d_{2k+1}}}{z_{n-2k-2}^{bc_{2k+1}}} = \frac{w_{n-2k-1}^{c_{2k+2}}}{z_{n-2k-2}^{d_{2k+2}}}, \tag{2.11}$$

where we define $a_{2k+2}, b_{2k+2}, c_{2k+2}$ and d_{2k+2} as follows

$$a_{2k+2} := ca_{2k+1} - b_{2k+1}, \quad b_{2k+2} := da_{2k+1},$$
$$c_{2k+2} := ac_{2k+1} - d_{2k+1}, \quad d_{2k+2} := bc_{2k+1}.$$

Hence, this inductive argument shows that relations (2.6) and (2.7) hold for every $k \in \mathbb{N}$ and $n \geq 2k-1$, and that above defined sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ satisfy the following recurrent relations

$$a_{2k} = ca_{2k-1} - b_{2k-1}, \quad b_{2k} = da_{2k-1}, \tag{2.12}$$
$$a_{2k+1} = aa_{2k} - b_{2k}, \quad b_{2k+1} = ba_{2k}, \tag{2.13}$$
$$c_{2k} = ac_{2k-1} - d_{2k-1}, \quad d_{2k} = bc_{2k-1}, \tag{2.14}$$
$$c_{2k+1} = cc_{2k} - d_{2k}, \quad d_{2k+1} = dc_{2k}, \tag{2.15}$$

for $k \in \mathbb{N}$.

From (2.8)-(2.11) we obtain

$$z_{2n+1} = \frac{w_0^{a_{2n+1}}}{z_{-1}^{b_{2n+1}}}, \quad z_{2n+2} = \frac{z_0^{a_{2n+2}}}{w_{-1}^{b_{2n+2}}} \tag{2.16}$$

$$w_{2n+1} = \frac{z_0^{c_{2n+1}}}{w_{-1}^{d_{2n+1}}}, \quad w_{2n+2} = \frac{w_0^{c_{2n+2}}}{z_{-1}^{d_{2n+2}}}, \tag{2.17}$$

for $n \in \mathbb{N}_0$.

Using (2.12) and (2.13) we have

$$a_{2k+1} = aa_{2k} - da_{2k-1}, \quad a_{2k+2} = ca_{2k+1} - ba_{2k}, \quad k \in \mathbb{N},$$

from which it follows that

$$a_{2k+3} - (ac - b - d)a_{2k+1} + bda_{2k-1} = 0, \quad k \in \mathbb{N}, \tag{2.18}$$

$$a_{2k+2} - (ac - b - d)a_{2k} + bda_{2k-2} = 0, \quad k \geq 2. \tag{2.19}$$

From (2.14) and (2.15) we have

$$c_{2k+1} = cc_{2k} - bc_{2k-1}, \quad c_{2k+2} = ac_{2k+1} - dc_{2k}, \quad k \in \mathbb{N},$$

from which it follows that

$$c_{2k+3} - (ac - b - d)c_{2k+1} + bdc_{2k-1} = 0, \quad k \in \mathbb{N}, \tag{2.20}$$

$$c_{2k+2} - (ac - b - d)c_{2k} + bdc_{2k-2} = 0, \quad k \geq 2. \tag{2.21}$$

In what follows we will consider three cases separately; that is, $b = 0$, $d = 0$ and $bd \neq 0$.

Case $b = 0$. In this case equations (2.18)-(2.21) become

$$a_{2k+3} = (ac - d)a_{2k+1}, \quad a_{2k+2} = (ac - d)a_{2k}, \quad k \in \mathbb{N},$$
$$c_{2k+3} = (ac - d)c_{2k+1}, \quad c_{2k+2} = (ac - d)c_{2k}, \quad k \in \mathbb{N},$$

from which it follows that

$$a_{2k+1} = a_1(ac - d)^k = a(ac - d)^k, \tag{2.22}$$

$$a_{2k+2} = a_2(ac - d)^k = ac(ac - d)^k, \tag{2.23}$$

$$c_{2k+1} = c_1(ac - d)^k = c(ac - d)^k, \tag{2.24}$$

$$c_{2k+2} = c_2(ac - d)^k = (ac - d)^{k+1}, \tag{2.25}$$

for $k \in \mathbb{N}_0$.

Using (2.22)-(2.25) in (2.12)-(2.15), as well as the condition $b = 0$, it follows that

$$b_{2k+1} = 0, \quad b_{2k+2} = ad(ac - d)^k, \quad k \in \mathbb{N}_0, \tag{2.26}$$

$$d_{2k+1} = d(ac - d)^k, \quad d_{2k+2} = 0, \quad k \in \mathbb{N}_0. \tag{2.27}$$

Employing (2.22)-(2.27) in (2.16) and (2.17) we obtain that well-defined solutions to system (1.5) in this case are given by the following formulas

$$z_{2n+1} = w_0^{a(ac-d)^n}, \quad z_{2n+2} = \frac{z_0^{ac(ac-d)^n}}{w_{-1}^{ad(ac-d)^n}} \tag{2.28}$$

$$w_{2n+1} = \frac{z_0^{c(ac-d)^n}}{w_{-1}^{d(ac-d)^n}}, \quad w_{2n+2} = w_0^{(ac-d)^{n+1}}, \quad n \in \mathbb{N}_0. \tag{2.29}$$

Case $d = 0$. In this case equations (2.18)-(2.21) become

$$a_{2k+3} = (ac - b)a_{2k+1}, \quad a_{2k+2} = (ac - b)a_{2k}, \quad k \in \mathbb{N},$$
$$c_{2k+3} = (ac - b)c_{2k+1}, \quad c_{2k+2} = (ac - b)c_{2k}, \quad k \in \mathbb{N},$$

from which it follows that

$$a_{2k+1} = a_1(ac - b)^k = a(ac - b)^k, \tag{2.30}$$

$$a_{2k+2} = a_2(ac - b)^k = (ac - b)^{k+1}, \tag{2.31}$$

$$c_{2k+1} = c_1(ac - b)^k = c(ac - b)^k, \tag{2.32}$$

$$c_{2k+2} = c_2(ac - b)^k = ac(ac - b)^k, \tag{2.33}$$

for $k \in \mathbb{N}_0$.

Using (2.30)-(2.33) in (2.12)-(2.15), as well as the condition $d = 0$, it follows that

$$b_{2k+1} = b(ac - b)^k, \quad b_{2k+2} = 0, \quad k \in \mathbb{N}_0, \tag{2.34}$$

$$d_{2k+1} = 0, \quad d_{2k+2} = bc(ac-b)^k, \quad k \in \mathbb{N}_0. \tag{2.35}$$

Employing (2.30)-(2.35) in (2.16) and (2.17) we obtain that well-defined solutions to system (1.5) in this case are given by the following formulas

$$z_{2n+1} = \frac{w_0^{a(ac-b)^n}}{z_{-1}^{b(ac-b)^n}}, \quad z_{2n+2} = z_0^{(ac-b)^{n+1}} \tag{2.36}$$

$$w_{2n+1} = z_0^{c(ac-b)^n}, \quad w_{2n+2} = \frac{w_0^{ac(ac-b)^n}}{z_{-1}^{bc(ac-b)^n}}, \quad n \in \mathbb{N}_0. \tag{2.37}$$

Case $b \neq 0 \neq d$. Let $\lambda_{1,2}$ be the roots of the characteristic polynomial

$$P(\lambda) = \lambda^2 - (ac-b-d)\lambda + bd, \tag{2.38}$$

of the difference equation

$$u_{n+2} - (ac-b-d)u_{n+1} + bdu_n = 0, \quad n \in \mathbb{N}. \tag{2.39}$$

From (2.18)-(2.21) it is clear that the sequences $(a_{2k+1})_{k\in\mathbb{N}_0}$, $(a_{2k})_{k\in\mathbb{N}}$, $(c_{2k+1})_{k\in\mathbb{N}_0}$, $(c_{2k})_{k\in\mathbb{N}}$, are solutions to equation (2.39).

It is known that the general solution of (2.39) has the form

$$u_n = \alpha_1\lambda_1^n + \alpha_2\lambda_2^n, \quad n \in \mathbb{N},$$

if $(ac-b-d)^2 \neq 4bd$, where α_1 and α_2 are arbitrary constants, while in the case $(ac-b-d)^2 = 4bd$, the general solution has the following form

$$u_n = (\beta_1 n + \beta_2)\lambda_1^n, \quad n \in \mathbb{N},$$

where β_1 and β_2 are arbitrary constants.

By some calculation and using the values for a_i, b_i, c_i, d_i, for $i \in \{1,2,3,4\}$, if $(ac-b-d)^2 \neq 4bd$, we obtain

$$a_{2k+1} = a\frac{\lambda_1^{k+1} - \lambda_2^{k+1}}{\lambda_1 - \lambda_2}, \tag{2.40}$$

$$a_{2k+2} = \frac{(ac-b-\lambda_2)\lambda_1^{k+1} - (ac-b-\lambda_1)\lambda_2^{k+1}}{\lambda_1 - \lambda_2}, \tag{2.41}$$

$$b_{2k+1} = b\frac{(ac-b-\lambda_2)\lambda_1^{k} - (ac-b-\lambda_1)\lambda_2^{k}}{\lambda_1 - \lambda_2}, \tag{2.42}$$

$$b_{2k+2} = ad\frac{\lambda_1^{k+1} - \lambda_2^{k+1}}{\lambda_1 - \lambda_2}, \tag{2.43}$$

$$c_{2k+1} = c\frac{\lambda_1^{k+1} - \lambda_2^{k+1}}{\lambda_1 - \lambda_2}, \tag{2.44}$$

$$c_{2k+2} = \frac{(ac-d-\lambda_2)\lambda_1^{k+1} - (ac-d-\lambda_1)\lambda_2^{k+1}}{\lambda_1 - \lambda_2}, \tag{2.45}$$

$$d_{2k+1} = d\frac{(ac-d-\lambda_2)\lambda_1^{k} - (ac-d-\lambda_1)\lambda_2^{k}}{\lambda_1 - \lambda_2}, \tag{2.46}$$

$$d_{2k+2} = bc\frac{\lambda_1^{k+1} - \lambda_2^{k+1}}{\lambda_1 - \lambda_2}, \tag{2.47}$$

for $k \in \mathbb{N}_0$.

By using (2.40)-(2.47) into (2.16) and (2.17) we obtain that well-defined solutions to system (1.5) in this case are given by the following formulas

$$z_{2n+1} = w_0^{a\frac{\lambda_1^{n+1}-\lambda_2^{n+1}}{\lambda_1-\lambda_2}} z_{-1}^{-b\frac{(ac-b-\lambda_2)\lambda_1^n-(ac-b-\lambda_1)\lambda_2^n}{\lambda_1-\lambda_2}}, \tag{2.48}$$

$$z_{2n+2} = z_0^{\frac{(ac-b-\lambda_2)\lambda_1^{n+1}-(ac-b-\lambda_1)\lambda_2^{n+1}}{\lambda_1-\lambda_2}} w_{-1}^{-ad\frac{\lambda_1^{n+1}-\lambda_2^{n+1}}{\lambda_1-\lambda_2}}, \tag{2.49}$$

$$w_{2n+1} = z_0^{c\frac{\lambda_1^{n+1}-\lambda_2^{n+1}}{\lambda_1-\lambda_2}} w_{-1}^{-d\frac{(ac-d-\lambda_2)\lambda_1^n-(ac-d-\lambda_1)\lambda_2^n}{\lambda_1-\lambda_2}}, \tag{2.50}$$

$$w_{2n+2} = w_0^{\frac{(ac-d-\lambda_2)\lambda_1^{n+1}-(ac-d-\lambda_1)\lambda_2^{n+1}}{\lambda_1-\lambda_2}} z_{-1}^{-bc\frac{\lambda_1^{n+1}-\lambda_2^{n+1}}{\lambda_1-\lambda_2}}, \tag{2.51}$$

for $n \in \mathbb{N}_0$.

If $(ac-b-d)^2 = 4bd$, that is, if

$$\lambda_1 = \lambda_2 = \frac{ac-b-d}{2},$$

we have

$$a_{2k+1} = a(k+1)\lambda_1^k \tag{2.52}$$

$$a_{2k+2} = \big((ac-b-\lambda_1)(k+1)+\lambda_1\big)\lambda_1^k, \tag{2.53}$$

$$b_{2k+1} = b\big((ac-b-\lambda_1)k+\lambda_1\big)\lambda_1^{k-1}, \tag{2.54}$$

$$b_{2k+2} = ad(k+1)\lambda_1^k, \tag{2.55}$$

$$c_{2k+1} = c(k+1)\lambda_1^k, \tag{2.56}$$

$$c_{2k+2} = \big((ac-d-\lambda_1)(k+1)+\lambda_1\big)\lambda_1^k, \tag{2.57}$$

$$d_{2k+1} = d\big((ac-d-\lambda_1)k+\lambda_1\big)\lambda_1^{k-1}, \tag{2.58}$$

$$d_{2k+2} = bc(k+1)\lambda_1^k, \tag{2.59}$$

for $k \in \mathbb{N}_0$.

By using (2.52)-(2.59) into (2.16) and (2.17) we obtain that well-defined solutions to system (1.5) in this case are given by the following formulas

$$z_{2n+1} = w_0^{a(n+1)\lambda_1^n} z_{-1}^{-b((ac-b-\lambda_1)n+\lambda_1)\lambda_1^{n-1}}, \tag{2.60}$$

$$z_{2n+2} = z_0^{((ac-b-\lambda_1)(n+1)+\lambda_1)\lambda_1^n} w_{-1}^{-ad(n+1)\lambda_1^n}, \tag{2.61}$$

$$w_{2n+1} = z_0^{c(n+1)\lambda_1^n} w_{-1}^{-d((ac-d-\lambda_1)n+\lambda_1)\lambda_1^{n-1}}, \tag{2.62}$$

$$w_{2n+2} = w_0^{((ac-d-\lambda_1)(n+1)+\lambda_1)\lambda_1^n} z_{-1}^{-bc(n+1)\lambda_1^n}, \quad n \in \mathbb{N}_0, \tag{2.63}$$

finishing the proof of the theorem. □

From the proof of Theorem 2.1 we obtain the following corollary.

Corollary 2.2. *Consider* (1.5) *with $a, b, c, d \in \mathbb{Z}$. Assume that $z_{-1}, z_0, w_{-1}, w_0 \in \mathbb{C} \setminus \{0\}$. Then the following statements hold:*

(a) *If $b = 0$, then the general solution of system* (1.5) *is given by* (2.28) *and* (2.29).

(b) *If $d = 0$, then the general solution of system* (1.5) *is given by* (2.36) *and* (2.37).

(c) *If* $bd \neq 0$ *and* $(ac-b-d)^2 \neq 4bd$, *then the general solution of system* (1.5) *is given by* (2.48)-(2.51).
(d) *If* $bd \neq 0$ *and* $(ac-b-d)^2 = 4bd$, *then the general solution of system* (1.5) *is given by* (2.60)-(2.63).

Remark 2.3. The condition $a, b, c, d \in \mathbb{Z}$ is posed in order to avoid multi-values of powers of complex numbers, that is, we want that initial values $z_{-1}, z_0, w_{-1}, w_0 \in \mathbb{C}$ define unique solutions to system (1.5).

3. Applications

In this section we give some applications of the formulas obtained in the previous section. The long-term behavior of solutions to system (1.5) in several cases is described.

Theorem 3.1. *Consider system* (1.5). *Assume that* $a, c, d \in \mathbb{Z}$, $b = 0$, *and initial values* $z_{-1}, z_0, w_{-1}, w_0 \in \mathbb{C} \setminus \{0\}$. *Then the following statements hold:*

(a) *If* $ac = d$, *then the solution* $(z_n, w_n)_{n\geq -1}$ *is eventually constant.*
(b) *If* $ac-d = 1$, *then the solution* $(z_n, w_n)_{n\geq -1}$ *is two-periodic, while if* $ac-d = -1$, *then the solution* $(z_n, w_n)_{n\geq -1}$ *is four-periodic.*
(c) *If* $ac - d \geq 2$ *and* $0 < |w_0^a| < 1$, *then* $z_{2n+1} \to 0$, *as* $n \to \infty$.
(d) *If* $ac - d \geq 2$ *and* $|w_0^a| > 1$, *then* $z_{2n+1} \to \infty$, *as* $n \to \infty$.
(e) *If* $w_0^a = 1$, *then* $z_{2n+1} = 1$, $n \in \mathbb{N}_0$.
(f) *If* $ac \neq d$, $a \neq 0$, *and* $w_0 = e^{i\theta}$, $\theta = p\pi/q$, $q \in \mathbb{N}$ *and* $p \in \mathbb{Z}$, *then* z_{2n+1} *is periodic with period* $T \leq 2q$.
(g) *If* $ac - d \leq -2$ *and* $0 < |w_0^a| < 1$, *then* $z_{4n+1} \to 0$, *as* $n \to \infty$, *while if* $|w_0^a| > 1$, *then* $z_{4n+1} \to \infty$, *as* $n \to \infty$.
(h) *If* $ac - d \leq -2$ *and* $0 < |w_0^a| < 1$, *then* $z_{4n+3} \to \infty$, *as* $n \to \infty$, *while if* $|w_0^a| > 1$, *then* $z_{4n+3} \to 0$, *as* $n \to \infty$.
(i) *If* $ac - d \geq 2$ *and* $0 < |z_0^{ac}/w_{-1}^{ad}| < 1$, *then* $z_{2n+2} \to 0$, *as* $n \to \infty$.
(j) *If* $ac - d \geq 2$ *and* $|z_0^{ac}/w_{-1}^{ad}| > 1$, *then* $z_{2n+2} \to \infty$, *as* $n \to \infty$.
(k) *If* $z_0^{ac} = w_{-1}^{ad}$, *then* $z_{2n+2} = 1$, $n \in \mathbb{N}_0$.
(l) *If* $ac \neq d$, $a \neq 0$, $c \neq 0$ *or* $d \neq 0$, *and* $z_0^{ac} = w_{-1}^{ad} e^{i\theta}$, $\theta = p\pi/q$, $q \in \mathbb{N}$ *and* $p \in \mathbb{Z}$, *then* z_{2n+2} *is periodic with period* $T \leq 2q$.
(m) *If* $ac - d \leq -2$ *and* $0 < |z_0^{ac}/w_{-1}^{ad}| < 1$, *then* $z_{4n+2} \to 0$, *as* $n \to \infty$, *while if* $|z_0^{ac}/w_{-1}^{ad}| > 1$, *then* $z_{4n+2} \to \infty$, *as* $n \to \infty$.
(n) *If* $ac - d \leq -2$ *and* $0 < |z_0^{ac}/w_{-1}^{ad}| < 1$, *then* $z_{4n} \to \infty$, *as* $n \to \infty$, *while if* $|z_0^{ac}/w_{-1}^{ad}| > 1$, *then* $z_{4n} \to 0$, *as* $n \to \infty$.
(o) *If* $ac - d \geq 2$ *and* $0 < |z_0^c/w_{-1}^d| < 1$, *then* $w_{2n+1} \to 0$, *as* $n \to \infty$.
(p) *If* $ac - d \geq 2$, *and* $|z_0^c/w_{-1}^d| > 1$, *then* $w_{2n+1} \to \infty$, *as* $n \to \infty$.
(q) *If* $z_0^c = w_{-1}^d$, *then* $w_{2n+1} = 1$, $n \in \mathbb{N}_0$.
(r) *If* $ac \neq d$, *and* $z_0^c = w_{-1}^d e^{i\theta}$, $\theta = p\pi/q$, $q \in \mathbb{N}$ *and* $p \in \mathbb{Z}$, *then* w_{2n+1} *is periodic with period* $T \leq 2q$.
(s) *If* $ac - d \leq -2$, $0 < |z_0^c/w_{-1}^d| < 1$, *then* $w_{4n+1} \to 0$, *as* $n \to \infty$, *while if* $|z_0^c/w_{-1}^d| > 1$, *then* $w_{4n+1} \to \infty$, *as* $n \to \infty$.
(t) *If* $ac - d \leq -2$, *and* $0 < |z_0^c/w_{-1}^d| < 1$, *then* $w_{4n+3} \to \infty$, *as* $n \to \infty$, *while if* $|z_0^c/w_{-1}^d| > 1$, *then* $w_{4n+3} \to 0$, *as* $n \to \infty$.
(u) *If* $ac - d \geq 2$ *and* $0 < |w_0| < 1$, *then* $w_{2n+2} \to 0$, *as* $n \to \infty$.
(v) *If* $ac - d \geq 2$, *and* $|w_0| > 1$, *then* $w_{2n+2} \to \infty$, *as* $n \to \infty$.

(w) *If* $w_0 = 1$, *then* $w_{2n+2} = 1$, $n \in \mathbb{N}_0$.
(x) *If* $ac \neq d$, *and* $w_0 = e^{i\theta}$, $\theta = p\pi/q$, $q \in \mathbb{N}$ *and* $p \in \mathbb{Z}$, *then* w_{2n+2} *is periodic with period* $T \leq 2q$.
(y) *If* $ac - d \leq -2$, $|w_0| > 1$, *then* $w_{4n+2} \to 0$, *as* $n \to \infty$, *while if* $0 < |w_0| < 1$, *then* $w_{4n+2} \to \infty$, *as* $n \to \infty$.
(z) *If* $ac - d \leq -2$, *and* $0 < |w_0| < 1$, *then* $w_{4n+4} \to 0$, *as* $n \to \infty$, *while if* $|w_0| > 1$, *then* $w_{4n+4} \to \infty$, *as* $n \to \infty$.

Proof. (a) Using the condition $ac = d$ in (2.28) and (2.29) we obtain

$$z_{2n+1} = z_{2n+2} = w_{2n+1} = w_{2n+2} = 1, \quad n \in \mathbb{N},$$

from which the statement follows.

(b) Using the condition $ac - d = 1$ in (2.28) and (2.29) we obtain

$$z_{2n+1} = w_0^a, \quad z_{2n+2} = \frac{z_0^{ac}}{w_{-1}^{ad}}, \quad w_{2n+1} = \frac{z_0^c}{w_{-1}^d}, \quad w_{2n+2} = w_0, \quad n \in \mathbb{N}_0,$$

which means that $(z_n, w_n)_{n \geq -1}$ is two-periodic.

Using the condition $ac - d = -1$ in (2.28) and (2.29) we obtain

$$z_{2n+1} = w_0^{a(-1)^n}, \quad z_{2n+2} = \frac{z_0^{ac(-1)^n}}{w_{-1}^{ad(-1)^n}}, \quad w_{2n+1} = \frac{z_0^{c(-1)^n}}{w_{-1}^{d(-1)^n}}, \quad w_{2n+2} = w_0^{(-1)^{n+1}},$$

for $n \in \mathbb{N}_0$. From this we have

$$z_{4n+1} = w_0^a, \quad z_{4n+2} = \frac{z_0^{ac}}{w_{-1}^{ad}}, \quad w_{4n+1} = \frac{z_0^c}{w_{-1}^d}, \quad w_{4n+2} = \frac{1}{w_0}, \tag{3.1}$$

$$z_{4n+3} = \frac{1}{w_0^a}, \quad z_{4n+4} = \frac{w_{-1}^{ad}}{z_0^{ac}}, \quad w_{4n+3} = \frac{w_{-1}^d}{z_0^c}, \quad w_{4n+4} = w_0, \tag{3.2}$$

for $n \in \mathbb{N}_0$, which means that $(z_n, w_n)_{n \geq -1}$ is four-periodic.

(c), (d) If $ac - d \geq 2$, then $(ac - d)^n \to +\infty$ as $n \to +\infty$. Hence, if $0 < |w_0^a| < 1$, by using the formula

$$z_{2n+1} = w_0^{a(ac-d)^n}, \quad n \in \mathbb{N}_0, \tag{3.3}$$

we obtain that $z_{2n+1} \to 0$, as $n \to \infty$, while if $|w_0^a| > 1$, then we obtain that $z_{2n+1} \to \infty$, as $n \to \infty$.

(e) The statement directly follows by using the condition $w_0^a = 1$ in (3.3).

(f) Using the conditions $w_0 = e^{i\theta}$, $\theta = p\pi/q$, $q \in \mathbb{N}$ and $p \in \mathbb{Z}$, in (3.3) we have

$$z_{2n+1} = e^{i\pi \frac{pa(ac-d)^n}{q}}, \quad n \in \mathbb{N}_0. \tag{3.4}$$

Now note that among the numbers

$$pa, \quad pa(ac - d), \quad pa(ac - d)^2, \dots, pa(ac - d)^{2q},$$

there are two which have the same reminder by dividing by $2q$, say, $pa(ac-d)^i$ and $pa(ac - d)^j$, $0 \leq i < j \leq 2q$. This means that there is a $k_0 \in \mathbb{N}$ such that

$$pa(ac - d)^j - pa(ac - d)^i = 2k_0 q,$$

from which it follows that

$$pa(ac - d)^{m+j} - pa(ac - d)^{m+i} = 2k_0 q(ac - d)^m,$$

for every $m \in \mathbb{N}_0$. This means that the sequence $pa(ac - d)^n \pmod{2q}$, $n \in \mathbb{N}_0$, is eventually periodic with period $T = j - i \leq 2q$. Using this fact in (3.4) the statement easily follows.

(g), (h) Since $ac-d \le -2$ from (3.3) we have

$$z_{4n+1} = w_0^{a(ac-d)^{2n}}, \quad z_{4n+3} = w_0^{-a|ac-d|^{2n+1}}, \quad n \in \mathbb{N}_0. \tag{3.5}$$

From (3.5) and the posed conditions these two statements easily follow.

(i)-(l) The proofs are the same as those ones of (c)-(f), when w_0^a is replaced by z_0^{ac}/w_{-1}^{ad}, and z_{2n+1} is replaced by z_{2n+2}. Hence, we omit the detail.

(m), (n) From (2.28) we have

$$z_{4n+2} = \Big(\frac{z_0^{ac}}{w_{-1}^{ad}}\Big)^{(ac-d)^{2n}}, \quad z_{4n+4} = \Big(\frac{z_0^{ac}}{w_{-1}^{ad}}\Big)^{-|ac-d|^{2n+1}}, \tag{3.6}$$

for $n \in \mathbb{N}_0$. From (3.6) and the posed conditions these two statements easily follow.

(o)-(r) Using the formula

$$w_{2n+1} = \Big(\frac{z_0^{c}}{w_{-1}^{d}}\Big)^{(ac-d)^{n}}, \quad n \in \mathbb{N}_0, \tag{3.7}$$

statements (o)-(q) easily follow, while (r) is proved similarly to (f).

(s), (t) From (3.7) and since $ac-d \le -2$, it follows that

$$w_{4n+1} = \Big(\frac{z_0^{c}}{w_{-1}^{d}}\Big)^{(ac-d)^{2n}}, \quad w_{4n+3} = \Big(\frac{z_0^{c}}{w_{-1}^{d}}\Big)^{-|ac-d|^{2n+1}}, \quad n \in \mathbb{N}_0. \tag{3.8}$$

Using the formulas in (3.8) these two statements easily follow.

(u)-(x) Using the formula

$$w_{2n+2} = w_0^{(ac-d)^{n+1}}, \quad n \in \mathbb{N}_0, \tag{3.9}$$

statements (u)-(w) easily follow, while (x) is proved similar to (f).

(y), (z) From (3.9) and since $ac-d \le -2$, it follows that

$$w_{4n+2} = w_0^{-|ac-d|^{2n+1}}, \quad w_{4n+4} = w_0^{(ac-d)^{2n+2}}, \quad n \in \mathbb{N}_0. \tag{3.10}$$

Using the formulas in (3.10) these two statements easily follow. $\square$

Remark 3.2. The long-term behavior of solutions to system (1.5) in the cases $w_0^a = e^{i\theta}$ or $z_0^{ac}/w_{-1}^{ad} = e^{i\theta}$ or $z_0^c/w_{-1}^d = e^{i\theta}$ or $w_0 = e^{i\theta}$, when $\theta/\pi \notin \mathbb{Q}$, is more complex and will be not treated here in detail. We can only mention here that in some cases the sequence $\theta(ab-d)^n \pmod{2\pi}$, $n \in \mathbb{N}_0$, can have a set of accumulation points which is nowhere dense in the interval $[0, 2\pi]$, but is some other cases it can be everywhere dense in the interval.

Theorem 3.3. *Consider system* (1.5). *Assume that $a, b, c \in \mathbb{Z}$, $d = 0$, and initial values $z_{-1}, z_0, w_{-1}, w_0 \in \mathbb{C} \setminus \{0\}$. Then the following statements hold:*

- (a) *If $ac = b$, then the solution $(z_n, w_n)_{n \ge -1}$ is eventually constant.*
- (b) *If $ac-b = 1$, then the solution $(z_n, w_n)_{n \ge -1}$ is two-periodic, while if $ac-b = -1$, then the solution $(z_n, w_n)_{n \ge -1}$ is four-periodic.*
- (c) *If $ac-b \ge 2$ and $0 < |w_0^a/z_{-1}^b| < 1$, then $z_{2n+1} \to 0$, as $n \to \infty$.*
- (d) *If $ac-b \ge 2$ and $|w_0^a/z_{-1}^b| > 1$, then $z_{2n+1} \to \infty$, as $n \to \infty$.*
- (e) *If $w_0^a = z_{-1}^b$, then $z_{2n+1} = 1$, $n \in \mathbb{N}_0$.*
- (f) *If $ac \ne b$ and $w_0^a = z_{-1}^b e^{i\theta}$, $\theta = p\pi/q$, $q \in \mathbb{N}$ and $p \in \mathbb{Z}$, then z_{2n+1} is periodic with period $T \le 2q$.*
- (g) *If $ac-b \le -2$ and $0 < |w_0^a/z_{-1}^b| < 1$, then $z_{4n+1} \to 0$, as $n \to \infty$, while if $|w_0^a/z_{-1}^b| > 1$, then $z_{4n+1} \to \infty$, as $n \to \infty$.*

(h) *If* $ac-b\le -2$ *and* $0<|w_0^a/z_{-1}^b|<1$, *then* $z_{4n+3}\to\infty$, *as* $n\to\infty$, *while if* $|w_0^a/z_{-1}^b|>1$, *then* $z_{4n+3}\to 0$, *as* $n\to\infty$.
(i) *If* $ac-b\ge 2$ *and* $0<|z_0|<1$, *then* $z_{2n+2}\to 0$, *as* $n\to\infty$.
(j) *If* $ac-b\ge 2$ *and* $|z_0|>1$, *then* $z_{2n+2}\to\infty$, *as* $n\to\infty$.
(k) *If* $ac=b$ *or* $z_0=1$, *then* $z_{2n+2}=1$, $n\in\mathbb{N}_0$.
(l) *If* $ac\ne b$ *and* $z_0=e^{i\theta}$, $\theta=p\pi/q$, $q\in\mathbb{N}$ *and* $p\in\mathbb{Z}$, *then* z_{2n+2} *is periodic with period* $T\le 2q$.
(m) *If* $ac-b\le -2$ *and* $0<|z_0|<1$, *then* $z_{4n+2}\to\infty$, *as* $n\to\infty$, *while if* $|z_0|>1$, *then* $z_{4n+2}\to 0$, *as* $n\to\infty$.
(n) *If* $ac-b\le -2$ *and* $0<|z_0|<1$, *then* $z_{4n}\to 0$, *as* $n\to\infty$, *while if* $|z_0|>1$, *then* $z_{4n}\to\infty$, *as* $n\to\infty$.
(o) *If* $ac-b\ge 2$ *and* $0<|z_0^c|<1$, *then* $w_{2n+1}\to 0$, *as* $n\to\infty$.
(p) *If* $ac-b\ge 2$ *and* $|z_0^c|>1$, *then* $w_{2n+1}\to\infty$, *as* $n\to\infty$.
(q) *If* $z_0^c=1$, *then* $w_{2n+1}=1$, $n\in\mathbb{N}_0$.
(r) *If* $ac\ne b$, *and* $z_0^c=e^{i\theta}$, $\theta=p\pi/q$, $q\in\mathbb{N}$ *and* $p\in\mathbb{Z}$, *then* w_{2n+1} *is periodic with period* $T\le 2q$.
(s) *If* $ac-b\le -2$ *and* $0<|z_0^c|<1$, *then* $w_{4n+1}\to 0$, *as* $n\to\infty$, *while if* $|z_0^c|>1$, *then* $w_{4n+1}\to\infty$, *as* $n\to\infty$.
(t) *If* $ac-b\le -2$ *and* $0<|z_0^c|<1$, *then* $w_{4n+3}\to\infty$, *as* $n\to\infty$, *while if* $|z_0^c|>1$, *then* $w_{4n+3}\to 0$, *as* $n\to\infty$.
(u) *If* $ac-b\ge 2$ *and* $0<|w_0^{ac}/z_{-1}^{bc}|<1$, *then* $w_{2n+2}\to 0$, *as* $n\to\infty$.
(v) *If* $ac-b\ge 2$ *and* $|w_0^{ac}/z_{-1}^{bc}|>1$, *then* $w_{2n+2}\to\infty$, *as* $n\to\infty$.
(w) *If* $w_0^{ac}=z_{-1}^{bc}$, *then* $w_{2n+2}=1$, $n\in\mathbb{N}_0$.
(x) *If* $ac\ne b$ *and* $w_0^{ac}=z_{-1}^{bc}e^{i\theta}$, $\theta=p\pi/q$, $q\in\mathbb{N}$ *and* $p\in\mathbb{Z}$, *then* w_{2n+2} *is periodic with period* $T\le 2q$.
(y) *If* $ac-b\le -2$ *and* $0<|w_0^{ac}/z_{-1}^{bc}|<1$, *then* $w_{4n+2}\to 0$, *as* $n\to\infty$, *while if* $|w_0^{ac}/z_{-1}^{bc}|>1$, *then* $w_{4n+2}\to\infty$, *as* $n\to\infty$.
(z) *If* $ac-b\le -2$ *and* $0<|w_0^{ac}/z_{-1}^{bc}|<1$, *then* $w_{4n+4}\to\infty$, *as* $n\to\infty$, *while if* $|w_0^{ac}/z_{-1}^{bc}|>1$, *then* $w_{4n+4}\to 0$, *as* $n\to\infty$.

Proof. (a) Using the condition $ac=b$ in (2.36) and (2.37) we obtain

$$z_{2n+1}=z_{2n+2}=w_{2n+1}=w_{2n+2}=1,\quad n\in\mathbb{N},$$

from which the statement follows.

(b) Using the condition $ac-b=1$ in (2.36) and (2.37) we obtain

$$z_{2n+1}=\frac{w_0^a}{z_{-1}^b},\quad z_{2n+2}=z_0,\quad w_{2n+1}=z_0^c,\quad w_{2n+2}=\frac{w_0^{ac}}{z_{-1}^{bc}},\quad n\in\mathbb{N}_0,$$

which means that $(z_n,w_n)_{n\ge -1}$ is two-periodic.

Using the condition $ac-b=-1$ in (2.36) and (2.37) we obtain

$$z_{2n+1}=\frac{w_0^{a(-1)^n}}{z_{-1}^{b(-1)^n}},\quad z_{2n+2}=z_0^{(-1)^{n+1}},\quad w_{2n+1}=z_0^{c(-1)^n},\quad w_{2n+2}=\frac{w_0^{ac(-1)^n}}{z_{-1}^{bc(-1)^n}},$$

for $n\in\mathbb{N}_0$. From this we have

$$z_{4n+1}=\frac{w_0^a}{z_{-1}^b},\quad z_{4n+2}=\frac{1}{z_0},\quad w_{4n+1}=z_0^c,\quad w_{4n+2}=\frac{w_0^{ac}}{z_{-1}^{bc}} \tag{3.11}$$

$$z_{4n+3}=\frac{z_{-1}^b}{w_0^a},\quad z_{4n+4}=z_0,\quad w_{4n+3}=\frac{1}{z_0^c},\quad w_{4n+4}=\frac{z_{-1}^{bc}}{w_0^{ac}}, \tag{3.12}$$

for $n \in \mathbb{N}_0$, which means that $(z_n, w_n)_{n\ge -1}$ is four-periodic.

(c), (d) If $ac-b \ge 2$, then $(ac-b)^n \to +\infty$ as $n \to +\infty$. Hence, if $0 < |w_0^a/z_{-1}^b| < 1$, by using the formula

$$z_{2n+1} = \Big(\frac{w_0^a}{z_{-1}^b}\Big)^{(ac-b)^n}, \quad n \in \mathbb{N}_0, \tag{3.13}$$

we obtain that $z_{2n+1} \to 0$, as $n \to \infty$, while if $|w_0^a/z_{-1}^b| > 1$, then we obtain that $z_{2n+1} \to 0$, as $n \to \infty$.

(e) The statement directly follows by using the condition $w_0^a = z_{-1}^b$ in (3.13).

(f) Using the conditions $w_0 = e^{i\theta}$, $\theta = p\pi/q$, $q \in \mathbb{N}$ and $p \in \mathbb{Z}$, in (3.13) we have

$$z_{2n+1} = e^{i\pi \frac{p(ac-d)^n}{q}}, \quad n \in \mathbb{N}_0. \tag{3.14}$$

The rest of the proof is similar to the one of statement (f) in Theorem 3.1, so is omitted.

(g), (h) Since $ac - b \le -2$ from (3.13) we have

$$z_{4n+1} = \Big(\frac{w_0^a}{z_{-1}^b}\Big)^{(ac-b)^{2n}}, \quad z_{4n+3} = \Big(\frac{w_0^a}{z_{-1}^b}\Big)^{-|ac-b|^{2n+1}}, \quad n \in \mathbb{N}_0. \tag{3.15}$$

From (3.15) and the posed conditions these two statements easily follow.

(i)-(l) The proofs are similar to those ones of (c)-(f), when w_0^a/z_{-1}^b is replaced by z_0, and z_{2n+1} is replaced by z_{2n+2}. Hence, we omit the detail.

(m), (n) From (2.36) and since $ac - b \le -2$, we have

$$z_{4n+2} = z_0^{-|ac-b|^{2n+1}}, \quad z_{4n+4} = z_0^{(ac-b)^{2n+2}}, \tag{3.16}$$

for $n \in \mathbb{N}_0$. From (3.16) and the posed conditions these two statements easily follow.

(o)-(r) Using the formula

$$w_{2n+1} = (z_0^c)^{(ac-b)^n}, \quad n \in \mathbb{N}_0, \tag{3.17}$$

statements (o)-(q) easily follow, while (r) is proved similar to (f).

(s), (t) From (3.17) and since $ac - d \le -2$, it follows that

$$w_{4n+1} = (z_0^c)^{(ac-b)^{2n}}, \quad w_{4n+3} = (z_0^c)^{-|ac-b|^{2n+1}}, \quad n \in \mathbb{N}_0. \tag{3.18}$$

Using the formulas in (3.18) these two statements easily follow.

(u)-(x) Using the formula

$$w_{2n+2} = \Big(\frac{w_0^{ac}}{z_{-1}^{bc}}\Big)^{(ac-b)^n}, \quad n \in \mathbb{N}_0, \tag{3.19}$$

statements (u)-(w) easily follow, while (x) is proved similar to (f).

(y), (z) From (3.19) and since $ac - b \le -2$, it follows that

$$w_{4n+2} = \Big(\frac{w_0^{ac}}{z_{-1}^{bc}}\Big)^{(ac-b)^{2n}}, \quad w_{4n+4} = \Big(\frac{w_0^{ac}}{z_{-1}^{bc}}\Big)^{-|ac-b|^{2n+1}}, \quad n \in \mathbb{N}_0. \tag{3.20}$$

Using the formulas in (3.20) these two statements easily follow. □

Theorem 3.4. *Consider system* (1.5). *Assume that* $a, b, c, d \in \mathbb{Z}$, $(ac-b-d)^2 = 4bd$ *and* $ac - b - d = 2$, *and initial values* $z_{-1}, z_0, w_{-1}, w_0 \in \mathbb{C} \setminus \{0\}$. *Then the following statements hold:*

(a) *If* $0 < |w_0^a/z_{-1}^{b(ac-b-1)}| < 1$, *then* $z_{2n+1} \to 0$, *as* $n \to \infty$.

(b) *If* $|w_0^a/z_{-1}^{b(ac-b-1)}| > 1$, *then* $z_{2n+1} \to \infty$, *as* $n \to \infty$.
(c) *If* $w_0^a = z_{-1}^{b(ac-b-1)}$, *then* $z_{2n+1} = w_0^a/z_{-1}^b$, *as* $n \to \infty$.
(d) *If* $w_0^a = z_{-1}^{b(ac-b-1)}e^{i\theta}$, $\theta = p\pi/q$, $q \in \mathbb{N}$ *and* $p \in \mathbb{Z}$, *then* z_{2n+1} *is periodic with period* $T \le 2q$.
(e) *If* $0 < |z_0^{ac-b-1}/w_{-1}^{ad}| < 1$, *then* $z_{2n+2} \to 0$, *as* $n \to \infty$.
(f) *If* $|z_0^{ac-b-1}/w_{-1}^{ad}| > 1$, *then* $z_{2n+2} \to \infty$, *as* $n \to \infty$.
(g) *If* $z_0^{ac-b-1} = w_{-1}^{ad}$, *then* $z_{2n+2} = z_0^{ac-b}/w_{-1}^{ad}$, *as* $n \to \infty$.
(h) *If* $z_0^{ac-b-1} = w_{-1}^{ad}e^{i\theta}$, $\theta = p\pi/q$, $q \in \mathbb{N}$ *and* $p \in \mathbb{Z}$, *then* z_{2n+2} *is periodic with period* $T \le 2q$.
(i) *If* $0 < |z_0^c/w_{-1}^{d(ac-d-1)}| < 1$, *then* $w_{2n+1} \to 0$, *as* $n \to \infty$.
(j) *If* $|z_0^c/w_{-1}^{d(ac-d-1)}| > 1$, *then* $w_{2n+1} \to \infty$, *as* $n \to \infty$.
(k) *If* $z_0^c = w_{-1}^{d(ac-d-1)}$, *then* $w_{2n+1} = z_0^c/w_{-1}^d$, *as* $n \to \infty$.
(l) *If* $z_0^c = w_{-1}^{d(ac-d-1)}e^{i\theta}$, $\theta = p\pi/q$, $q \in \mathbb{N}$ *and* $p \in \mathbb{Z}$, *then* w_{2n+1} *is periodic with period* $T \le 2q$.
(m) *If* $0 < |w_0^{ac-d-1}/z_{-1}^{bc}| < 1$, *then* $w_{2n+2} \to 0$, *as* $n \to \infty$.
(n) *If* $|w_0^{ac-d-1}/z_{-1}^{bc}| > 1$, *then* $w_{2n+2} \to \infty$, *as* $n \to \infty$.
(o) *If* $w_0^{ac-d-1} = z_{-1}^{bc}$, *then* $w_{2n+2} = w_0^{ac-d}/z_{-1}^{bc}$, *as* $n \to \infty$.
(p) *If* $w_0^{ac-d-1} = z_{-1}^{bc}e^{i\theta}$, $\theta = p\pi/q$, $q \in \mathbb{N}$ *and* $p \in \mathbb{Z}$, *then* z_{2n+2} *is periodic with period* $T \le 2q$.

Proof. First note that in this case the characteristic roots of polynomial (2.38) are $\lambda_1 = \lambda_2 = 1$. Using it into (2.60)-(2.63) we obtain the following formulas

$$z_{2n+1} = \frac{w_0^a}{z_{-1}^b}\Big(\frac{w_0^a}{z_{-1}^{b(ac-b-1)}}\Big)^n, \quad z_{2n+2} = \frac{z_0^{ac-b}}{w_{-1}^{ad}}\Big(\frac{z_0^{ac-b-1}}{w_{-1}^{ad}}\Big)^n,$$

$$w_{2n+1} = \frac{z_0^c}{w_{-1}^d}\Big(\frac{z_0^c}{w_{-1}^{d(ac-d-1)}}\Big)^n, \quad w_{2n+2} = \frac{w_0^{ac-d}}{z_{-1}^{bc}}\Big(\frac{w_0^{ac-d-1}}{z_{-1}^{bc}}\Big)^n,$$

for $n \in \mathbb{N}_0$, from which all the statements of the theorem easily follow. $\square$

Theorem 3.5. *Consider system* (1.5)*. Assume that* $a, b, c, d \in \mathbb{Z}$, $(ac-b-d)^2 = 4bd$ *and* $ac-b-d = -2$, *and initial values* $z_{-1}, z_0, w_{-1}, w_0 \in \mathbb{C}\setminus\{0\}$*. Then the following statements hold:*

(a) *If* $0 < |w_0^a z_{-1}^{b(ac-b+1)}| < 1$, *then* $z_{4n+1} \to 0$ *and* $z_{4n+3} \to \infty$, *as* $n \to \infty$.
(b) *If* $|w_0^a z_{-1}^{b(ac-b+1)}| > 1$, *then* $z_{4n+1} \to \infty$ *and* $z_{4n+3} \to 0$, *as* $n \to \infty$.
(c) *If* $w_0^a z_{-1}^{b(ac-b+1)} = 1$, *then* $z_{4n+1} = w_0^a/z_{-1}^b = 1/z_{4n+3}$, *as* $n \in \mathbb{N}_0$.
(d) *If* $w_0^a z_{-1}^{b(ac-b+1)} = e^{i\theta}$, $\theta = p\pi/q$, $q \in \mathbb{N}$ *and* $p \in \mathbb{Z}$, *then* z_{4n+1} *and* z_{4n+3} *are periodic with period* $T \le 2q$.
(e) *If* $0 < |z_0^{ac-b+1}/w_{-1}^{ad}| < 1$, *then* $z_{4n+2} \to 0$ *and* $z_{4n+4} \to \infty$, *as* $n \to \infty$.
(f) *If* $|z_0^{ac-b+1}/w_{-1}^{ad}| > 1$, *then* $z_{4n+2} \to \infty$ *and* $z_{4n+4} \to 0$, *as* $n \to \infty$.
(g) *If* $z_0^{ac-b+1} = w_{-1}^{ad}$, *then* $z_{4n+2} = z_0^{ac-b}/w_{-1}^{ad} = 1/z_{4n+4}$, *as* $n \in \mathbb{N}_0$.
(h) *If* $z_0^{ac-b+1} = w_{-1}^{ad}e^{i\theta}$, $\theta = p\pi/q$, $q \in \mathbb{N}$ *and* $p \in \mathbb{Z}$, *then* z_{4n+2} *and* z_{4n+4} *are periodic with period* $T \le 2q$.
(i) *If* $0 < |z_0^c w_{-1}^{d(ac-d+1)}| < 1$, *then* $w_{4n+1} \to 0$ *and* $w_{4n+3} \to \infty$, *as* $n \to \infty$.
(j) *If* $|z_0^c w_{-1}^{d(ac-d+1)}| > 1$, *then* $w_{4n+1} \to \infty$ *and* $w_{4n+3} \to 0$, *as* $n \to \infty$.
(k) *If* $z_0^c w_{-1}^{d(ac-d+1)} = 1$, *then* $w_{4n+1} = z_0^c/w_{-1}^d = 1/w_{4n+3}$, *as* $n \in \mathbb{N}_0$.

(l) *If* $z_0^c w_{-1}^{d(ac-d+1)} = e^{i\theta}$, $\theta = p\pi/q$, $q \in \mathbb{N}$ *and* $p \in \mathbb{Z}$, *then* w_{4n+1} *and* w_{4n+3} *are periodic with period* $T \le 2q$.

(m) *If* $0 < |w_0^{ac-d+1}/z_{-1}^{bc}| < 1$, *then* $w_{4n+2} \to 0$ *and* $w_{4n+4} \to \infty$, *as* $n \to \infty$.

(n) *If* $|w_0^{ac-d+1}/z_{-1}^{bc}| > 1$, *then* $w_{4n+2} \to \infty$ *and* $w_{4n+4} \to 0$, *as* $n \to \infty$.

(o) *If* $w_0^{ac-d+1} = z_{-1}^{bc}$, *then* $w_{4n+2} = w_0^{ac-d}/z_{-1}^{bc} = 1/w_{4n+4}$, *as* $n \in \mathbb{N}_0$.

(p) *If* $w_0^{ac-d+1} = z_{-1}^{bc} e^{i\theta}$, $\theta = p\pi/q$, $q \in \mathbb{N}$ *and* $p \in \mathbb{Z}$, *then* w_{4n+2} *and* w_{4n+4} *are periodic with period* $T \le 2q$.

Proof. First note that in this case the characteristic roots of polynomial (2.38) are $\lambda_1 = \lambda_2 = -1$. Using it into (2.60)-(2.63) we obtain the following formulas

$$z_{2n+1} = \frac{w_0^{a(-1)^n}}{z_{-1}^{b(-1)^n}} \Big(\frac{w_0^a}{z_{-1}^{-b(ac-b+1)}} \Big)^{n(-1)^n}, \quad z_{2n+2} = \frac{z_0^{(ac-b)(-1)^n}}{w_{-1}^{ad(-1)^n}} \Big(\frac{z_0^{ac-b+1}}{w_{-1}^{ad}} \Big)^{n(-1)^n},$$

$$w_{2n+1} = \frac{z_0^{c(-1)^n}}{w_{-1}^{d(-1)^n}} \Big(\frac{z_0^c}{w_{-1}^{-d(ac-d+1)}} \Big)^{n(-1)^n}, \quad w_{2n+2} = \frac{w_0^{(ac-d)(-1)^n}}{z_{-1}^{bc(-1)^n}} \Big(\frac{w_0^{ac-d+1}}{z_{-1}^{bc}} \Big)^{n(-1)^n},$$

for $n \in \mathbb{N}_0$, from which it follows that

$$z_{4n+1} = \frac{w_0^a}{z_{-1}^b} \Big(\frac{w_0^a}{z_{-1}^{-b(ac-b+1)}} \Big)^{2n}, \quad z_{4n+3} = \frac{z_{-1}^b}{w_0^a} \Big(\frac{z_{-1}^{-b(ac-b+1)}}{w_0^a} \Big)^{2n+1}, \tag{3.21}$$

$$z_{4n+2} = \frac{z_0^{ac-b}}{w_{-1}^{ad}} \Big(\frac{z_0^{ac-b+1}}{w_{-1}^{ad}} \Big)^{2n}, \quad z_{4n+4} = \frac{w_{-1}^{ad}}{z_0^{ac-b}} \Big(\frac{w_{-1}^{ad}}{z_0^{ac-b+1}} \Big)^{2n+1}, \tag{3.22}$$

$$w_{4n+1} = \frac{z_0^c}{w_{-1}^d} \Big(\frac{z_0^c}{w_{-1}^{-d(ac-d+1)}} \Big)^{2n}, \quad w_{4n+3} = \frac{w_{-1}^d}{z_0^c} \Big(\frac{w_{-1}^{-d(ac-d+1)}}{z_0^c} \Big)^{2n+1}, \tag{3.23}$$

$$w_{4n+2} = \frac{w_0^{ac-d}}{z_{-1}^{bc}} \Big(\frac{w_0^{ac-d+1}}{z_{-1}^{bc}} \Big)^{2n}, \quad w_{4n+4} = \frac{z_{-1}^{bc}}{w_0^{ac-d}} \Big(\frac{z_{-1}^{bc}}{w_0^{ac-d+1}} \Big)^{2n+1}, \tag{3.24}$$

for $n \in \mathbb{N}_0$. Using formulas (3.21)-(3.24) all the statements of the theorem easily follow. $\square$

Theorem 3.6. *Consider system* (1.5). *Assume that* $a, b, c, d \in \mathbb{Z}$, $bd \ne 0$, $(ac-b-d)^2 = 4bd$ *and* $ac - b - d > 2$, *and initial values* $z_{-1}, z_0, w_{-1}, w_0 \in \mathbb{C} \setminus \{0\}$. *Then the following statements hold:*

(a) *If* $0 < |w_0^{a\lambda_1}/z_{-1}^{b(ac-b-\lambda_1)}| < 1$, *then* $z_{2n+1} \to 0$, *as* $n \to \infty$.

(b) *If* $|w_0^{a\lambda_1}/z_{-1}^{b(ac-b-\lambda_1)}| > 1$, *then* $z_{2n+1} \to \infty$, *as* $n \to \infty$.

(c) *If* $w_0^{a\lambda_1} = z_{-1}^{b(ac-b-\lambda_1)}$ *and* $0 < |w_0^a/z_{-1}^b| < 1$, *then* $z_{2n+1} \to 0$, *as* $n \to \infty$.

(d) *If* $w_0^{a\lambda_1} = z_{-1}^{b(ac-b-\lambda_1)}$ *and* $|w_0^a/z_{-1}^b| > 1$, *then* $z_{2n+1} \to \infty$, *as* $n \to \infty$.

(e) *If* $w_0^{a\lambda_1} = z_{-1}^{b(ac-b-\lambda_1)}$ *and* $w_0^a = z_{-1}^b$, *then* $z_{2n+1} = 1$, *as* $n \in \mathbb{N}_0$.

(f) *If* $w_0^{a\lambda_1} = z_{-1}^{b(ac-b-\lambda_1)}$ *and* $w_0^a = z_{-1}^b e^{i\theta}$, $\theta = p\pi/q$, $q \in \mathbb{N}$ *and* $p \in \mathbb{Z}$, *then* z_{2n+1} *is periodic with period* $T \le 2q$.

(g) *If* $0 < |z_0^{ac-b-\lambda_1}/w_{-1}^{ad}| < 1$, *then* $z_{2n+2} \to 0$, *as* $n \to \infty$.

(h) *If* $|z_0^{ac-b-\lambda_1}/w_{-1}^{ad}| > 1$, *then* $z_{2n+2} \to \infty$, *as* $n \to \infty$.

(i) *If* $z_0^{ac-b-\lambda_1} = w_{-1}^{ad}$ *and* $0 < |z_0^{ac-b}/w_{-1}^{ad}| < 1$, *then* $z_{2n+2} \to 0$, *as* $n \to \infty$.

(j) *If* $z_0^{ac-b-\lambda_1} = w_{-1}^{ad}$ *and* $|z_0^{ac-b}/w_{-1}^{ad}| > 1$, *then* $z_{2n+2} \to \infty$, *as* $n \to \infty$.

(k) *If* $z_0^{ac-b-\lambda_1} = w_{-1}^{ad}$ *and* $z_0^{ac-b} = w_{-1}^{ad}$, *then* $z_{2n+2} = 1$, *as* $n \in \mathbb{N}_0$.

(l) *If* $z_0^{ac-b-\lambda_1} = w_{-1}^{ad}$ *and* $z_0^{ac-b} = w_{-1}^{ad}e^{i\theta}$, $\theta = p\pi/q$, $q \in \mathbb{N}$ *and* $p \in \mathbb{Z}$, *then* z_{2n+2} *is periodic with period* $T \le 2q$.

(m) *If* $0 < |z_0^{c\lambda_1}/w_{-1}^{d(ac-d-\lambda_1)}| < 1$, *then* $w_{2n+1} \to 0$, *as* $n \to \infty$.

(n) *If* $|z_0^{c\lambda_1}/w_{-1}^{d(ac-d-\lambda_1)}| > 1$, *then* $w_{2n+1} \to \infty$, *as* $n \to \infty$.

(o) *If* $z_0^{c\lambda_1} = w_{-1}^{d(ac-d-\lambda_1)}$ *and* $0 < |z_0^c/w_{-1}^d| < 1$, *then* $w_{2n+1} \to 0$, *as* $n \to \infty$.

(p) *If* $z_0^{c\lambda_1} = w_{-1}^{d(ac-d-\lambda_1)}$ *and* $|z_0^c/w_{-1}^d| > 1$, *then* $w_{2n+1} \to \infty$, *as* $n \to \infty$.

(q) *If* $z_0^{c\lambda_1} = w_{-1}^{d(ac-d-\lambda_1)}$ *and* $z_0^c = w_{-1}^d$, *then* $w_{2n+1} = 1$, *as* $n \in \mathbb{N}_0$.

(r) *If* $z_0^{c\lambda_1} = w_{-1}^{d(ac-d-\lambda_1)}$ *and* $z_0^c = w_{-1}^d e^{i\theta}$, $\theta = p\pi/q$, $q \in \mathbb{N}$ *and* $p \in \mathbb{Z}$, *then* w_{2n+1} *is periodic with period* $T \le 2q$.

(s) *If* $0 < |w_0^{ac-d-\lambda_1}/z_{-1}^{bc}| < 1$, *then* $w_{2n+2} \to 0$, *as* $n \to \infty$.

(t) *If* $|w_0^{ac-d-\lambda_1}/z_{-1}^{bc}| > 1$, *then* $w_{2n+2} \to \infty$, *as* $n \to \infty$.

(u) *If* $w_0^{ac-d-\lambda_1} = z_{-1}^{bc}$ *and* $0 < |w_0^{ac-d}/z_{-1}^{bc}| < 1$, *then* $w_{2n+2} \to 0$, *as* $n \to \infty$.

(v) *If* $w_0^{ac-d-\lambda_1} = z_{-1}^{bc}$ *and* $|w_0^{ac-d}/z_{-1}^{bc}| > 1$, *then* $w_{2n+2} \to \infty$, *as* $n \to \infty$.

(w) *If* $w_0^{ac-d-\lambda_1} = z_{-1}^{bc}$ *and* $w_0^{ac-d} = z_{-1}^{bc}$, *then* $w_{2n+2} = 1$, *as* $n \in \mathbb{N}_0$.

(x) *If* $w_0^{ac-d-\lambda_1} = z_{-1}^{bc}$ *and* $w_0^{ac-d} = z_{-1}^{bc}e^{i\theta}$, $\theta = p\pi/q$, $q \in \mathbb{N}$ *and* $p \in \mathbb{Z}$, *then* w_{2n+2} *is periodic with period* $T \le 2q$.

Proof. First note that in this case the characteristic roots of polynomial (2.38) are such that

$$\lambda_1 = \lambda_2 = \sqrt{|bd|} > 1.$$

Note that they are natural numbers, since $2\sqrt{|bd|} = |ac - b - d| \in \mathbb{N}$. Equations (2.60)-(2.63) can be written in the form

$$z_{2n+1} = \Big(\frac{w_0^a}{z_{-1}^b}\Big)^{\lambda_1^n}\Big(\frac{w_0^{a\lambda_1}}{z_{-1}^{b(ac-b-\lambda_1)}}\Big)^{n\lambda_1^{n-1}}, \tag{3.25}$$

$$z_{2n+2} = \Big(\frac{z_0^{ac-b}}{w_{-1}^{ad}}\Big)^{\lambda_1^n}\Big(\frac{z_0^{ac-b-\lambda_1}}{w_{-1}^{ad}}\Big)^{n\lambda_1^n}, \tag{3.26}$$

$$w_{2n+1} = \Big(\frac{z_0^c}{w_{-1}^d}\Big)^{\lambda_1^n}\Big(\frac{z_0^{c\lambda_1}}{w_{-1}^{d(ac-d-\lambda_1)}}\Big)^{n\lambda_1^{n-1}}, \tag{3.27}$$

$$w_{2n+2} = \Big(\frac{w_0^{ac-d}}{z_{-1}^{bc}}\Big)^{\lambda_1^n}\Big(\frac{w_0^{ac-d-\lambda_1}}{z_{-1}^{bc}}\Big)^{n\lambda_1^n}, \tag{3.28}$$

for $n \in \mathbb{N}_0$. Using formulas (3.25)-(3.28) all the statement easily follow. $\square$

Theorem 3.7. *Consider system* (1.5). *Assume that* $a, b, c, d \in \mathbb{Z}$, $bd \ne 0$, $(ac - b - d)^2 = 4bd$ *and* $ac - b - d < -2$, *and initial values* $z_{-1}, z_0, w_{-1}, w_0 \in \mathbb{C} \setminus \{0\}$. *Then the following statements hold:*

(a) *If* $0 < |w_0^{a\lambda_1}/z_{-1}^{b(ac-b-\lambda_1)}| < 1$, *then* $z_{4n+1} \to \infty$ *and* $z_{4n+3} \to 0$, *as* $n \to \infty$.

(b) *If* $|w_0^{a\lambda_1}/z_{-1}^{b(ac-b-\lambda_1)}| > 1$, *then* $z_{4n+1} \to 0$ *and* $z_{4n+3} \to \infty$, *as* $n \to \infty$.

(c) *If* $w_0^{a\lambda_1} = z_{-1}^{b(ac-b-\lambda_1)}$ *and* $0 < |w_0^a/z_{-1}^b| < 1$, $z_{4n+1} \to 0$ *and* $z_{4n+3} \to \infty$, *as* $n \to \infty$.

(d) *If* $w_0^{a\lambda_1} = z_{-1}^{b(ac-b-\lambda_1)}$ *and* $|w_0^a/z_{-1}^b| > 1$, $z_{4n+1} \to \infty$ *and* $z_{4n+3} \to 0$, *as* $n \to \infty$.

(e) *If* $w_0^{a\lambda_1} = z_{-1}^{b(ac-b-\lambda_1)}$ *and* $w_0^a = z_{-1}^b$, *then* $z_{2n+1} = 1$, *as* $n \in \mathbb{N}_0$.

(f) *If* $w_0^{a\lambda_1} = z_{-1}^{b(ac-b-\lambda_1)}$ *and* $w_0^a = z_{-1}^b e^{i\theta}$, $\theta = p\pi/q$, $q \in \mathbb{N}$ *and* $p \in \mathbb{Z}$, *then* z_{4n+1} *and* z_{4n+3} *are periodic with period* $T \le 2q$.

(g) *If* $0 < |z_0^{ac-b-\lambda_1}/w_{-1}^{ad}| < 1$, *then* $z_{4n+2} \to 0$ *and* $z_{4n+4} \to \infty$, *as* $n \to \infty$.

(h) *If* $|z_0^{ac-b-\lambda_1}/w_{-1}^{ad}| > 1$, *then* $z_{4n+2} \to \infty$ *and* $z_{4n+4} \to 0$, *as* $n \to \infty$.

(i) *If* $z_0^{ac-b-\lambda_1} = w_{-1}^{ad}$ *and* $0 < |z_0^{ac-b}/w_{-1}^{ad}| < 1$, *then* $z_{4n+2} \to 0$ *and* $z_{4n+4} \to \infty$, *as* $n \to \infty$.

(j) *If* $z_0^{ac-b-\lambda_1} = w_{-1}^{ad}$ *and* $|z_0^{ac-b}/w_{-1}^{ad}| > 1$, *then* $z_{4n+2} \to \infty$ *and* $z_{4n+4} \to 0$, *as* $n \to \infty$.

(k) *If* $z_0^{ac-b-\lambda_1} = w_{-1}^{ad}$ *and* $z_0^{ac-b} = w_{-1}^{ad}$, *then* $z_{2n+2} = 1$, *as* $n \in \mathbb{N}_0$.

(l) *If* $z_0^{ac-b-\lambda_1} = w_{-1}^{ad}$ *and* $z_0^{ac-b} = w_{-1}^{ad} e^{i\theta}$, $\theta = p\pi/q$, $q \in \mathbb{N}$ *and* $p \in \mathbb{Z}$, *then* z_{4n+2} *and* z_{4n+4} *are periodic with period* $T \le 2q$.

(m) *If* $0 < |z_0^{c\lambda_1}/w_{-1}^{d(ac-d-\lambda_1)}| < 1$, $w_{4n+1} \to \infty$ *and* $w_{4n+3} \to 0$, *as* $n \to \infty$.

(n) *If* $|z_0^{c\lambda_1}/w_{-1}^{d(ac-d-\lambda_1)}| > 1$, $w_{4n+1} \to 0$ *and* $w_{4n+3} \to \infty$, *as* $n \to \infty$.

(o) *If* $z_0^{c\lambda_1} = w_{-1}^{d(ac-d-\lambda_1)}$ *and* $0 < |z_0^c/w_{-1}^d| < 1$, *then* $w_{4n+1} \to 0$ *and* $w_{4n+3} \to \infty$, *as* $n \to \infty$.

(p) *If* $z_0^{c\lambda_1} = w_{-1}^{d(ac-d-\lambda_1)}$ *and* $|z_0^c/w_{-1}^d| > 1$, $w_{4n+1} \to \infty$ *and* $w_{4n+3} \to 0$, *as* $n \to \infty$.

(q) *If* $z_0^{c\lambda_1} = w_{-1}^{d(ac-d-\lambda_1)}$ *and* $z_0^c = w_{-1}^d$, *then* $w_{2n+1} = 1$, *as* $n \in \mathbb{N}_0$.

(r) *If* $z_0^{c\lambda_1} = w_{-1}^{d(ac-d-\lambda_1)}$ *and* $z_0^c = w_{-1}^d e^{i\theta}$, $\theta = p\pi/q$, $q \in \mathbb{N}$ *and* $p \in \mathbb{Z}$, *then* w_{4n+1} *and* w_{4n+3} *are periodic with period* $T \le 2q$.

(s) *If* $0 < |w_0^{ac-d-\lambda_1}/z_{-1}^{bc}| < 1$, *then* $w_{4n+2} \to 0$ *and* $w_{4n+4} \to \infty$, *as* $n \to \infty$.

(t) *If* $|w_0^{ac-d-\lambda_1}/z_{-1}^{bc}| > 1$, *then* $w_{4n+2} \to \infty$ *and* $w_{4n+4} \to 0$, *as* $n \to \infty$.

(u) *If* $w_0^{ac-d-\lambda_1} = z_{-1}^{bc}$ *and* $0 < |w_0^{ac-d}/z_{-1}^{bc}| < 1$, *then* $w_{4n+2} \to 0$ *and* $w_{4n+4} \to \infty$, *as* $n \to \infty$.

(v) *If* $w_0^{ac-d-\lambda_1} = z_{-1}^{bc}$ *and* $|w_0^{ac-d}/z_{-1}^{bc}| > 1$, *then* $w_{4n+2} \to \infty$ *and* $w_{4n+4} \to 0$, *as* $n \to \infty$.

(w) *If* $w_0^{ac-d-\lambda_1} = z_{-1}^{bc}$ *and* $w_0^{ac-d} = z_{-1}^{bc}$, *then* $w_{2n+2} = 1$, *as* $n \in \mathbb{N}_0$.

(x) *If* $w_0^{ac-d-\lambda_1} = z_{-1}^{bc}$ *and* $w_0^{ac-d} = z_{-1}^{bc} e^{i\theta}$, $\theta = p\pi/q$, $q \in \mathbb{N}$ *and* $p \in \mathbb{Z}$, *then* w_{4n+2} *and* w_{4n+4} *are periodic with period* $T \le 2q$.

Proof. First note that in this case the characteristic roots of polynomial (2.38) are such that

$$\lambda_1 = \lambda_2 = -\sqrt{|bd|} < 1.$$

Note that they are negative integers, since $2\sqrt{|bd|} = |ac - b - d| \in \mathbb{N}$. Equations (3.25)-(3.28) can be written in the form

$$z_{2n+1} = \Big(\frac{w_0^a}{z_{-1}^b}\Big)^{(-|\lambda_1|)^n} \Big(\frac{w_0^{a\lambda_1}}{z_{-1}^{b(ac-b-\lambda_1)}}\Big)^{n(-|\lambda_1|)^{n-1}}, \tag{3.29}$$

$$z_{2n+2} = \Big(\frac{z_0^{ac-b}}{w_{-1}^{ad}}\Big)^{(-|\lambda_1|)^n} \Big(\frac{z_0^{ac-b-\lambda_1}}{w_{-1}^{ad}}\Big)^{n(-|\lambda_1|)^n}, \tag{3.30}$$

$$w_{2n+1} = \Big(\frac{z_0^c}{w_{-1}^d}\Big)^{(-|\lambda_1|)^n} \Big(\frac{z_0^{c\lambda_1}}{w_{-1}^{d(ac-d-\lambda_1)}}\Big)^{n(-|\lambda_1|)^{n-1}}, \tag{3.31}$$

$$w_{2n+2} = \Big(\frac{w_0^{ac-d}}{z_{-1}^{bc}}\Big)^{(-|\lambda_1|)^n} \Big(\frac{w_0^{ac-d-\lambda_1}}{z_{-1}^{bc}}\Big)^{n(-|\lambda_1|)^n}, \tag{3.32}$$

for $n \in \mathbb{N}_0$.

From (3.29)-(3.32) it follows that

$$z_{4n+1}=\Big(\frac{w_0^a}{z_{-1}^b}\Big)^{\lambda_1^{2n}}\Big(\frac{w_0^{a\lambda_1}}{z_{-1}^{b(ac-b-\lambda_1)}}\Big)^{-2n|\lambda_1|^{2n-1}}, \tag{3.33}$$

$$z_{4n+3}=\Big(\frac{w_0^a}{z_{-1}^b}\Big)^{-|\lambda_1|^{2n+1}}\Big(\frac{w_0^{a\lambda_1}}{z_{-1}^{b(ac-b-\lambda_1)}}\Big)^{(2n+1)\lambda_1^{2n}}, \tag{3.34}$$

$$z_{4n+2}=\Big(\frac{z_0^{ac-b}}{w_{-1}^{ad}}\Big)^{\lambda_1^{2n}}\Big(\frac{z_0^{ac-b-\lambda_1}}{w_{-1}^{ad}}\Big)^{2n\lambda_1^{2n}}, \tag{3.35}$$

$$z_{4n+4}=\Big(\frac{z_0^{ac-b}}{w_{-1}^{ad}}\Big)^{-|\lambda_1|^{2n+1}}\Big(\frac{z_0^{ac-b-\lambda_1}}{w_{-1}^{ad}}\Big)^{-(2n+1)|\lambda_1|^{2n+1}}, \tag{3.36}$$

$$w_{4n+1}=\Big(\frac{z_0^c}{w_{-1}^d}\Big)^{\lambda_1^{2n}}\Big(\frac{z_0^{c\lambda_1}}{w_{-1}^{d(ac-d-\lambda_1)}}\Big)^{-2n|\lambda_1|^{2n-1}}, \tag{3.37}$$

$$w_{4n+3}=\Big(\frac{z_0^c}{w_{-1}^d}\Big)^{-|\lambda_1|^{2n+1}}\Big(\frac{z_0^{c\lambda_1}}{w_{-1}^{d(ac-d-\lambda_1)}}\Big)^{(2n+1)|\lambda_1|^{2n}}, \tag{3.38}$$

$$w_{4n+2}=\Big(\frac{w_0^{ac-d}}{z_{-1}^{bc}}\Big)^{\lambda_1^{2n}}\Big(\frac{w_0^{ac-d-\lambda_1}}{z_{-1}^{bc}}\Big)^{2n\lambda_1^{2n}}, \tag{3.39}$$

$$w_{4n+4}=\Big(\frac{w_0^{ac-d}}{z_{-1}^{bc}}\Big)^{-|\lambda_1|^{2n+1}}\Big(\frac{w_0^{ac-d-\lambda_1}}{z_{-1}^{bc}}\Big)^{-(2n+1)|\lambda_1|^{2n+1}}, \tag{3.40}$$

for $n\in\mathbb{N}_0$.

Using formulas (3.33)-(3.40) all the statements easily follow. □

Formulations and the proofs of the results on the long-term behavior of solutions to system (1.5) for the case $a,b,c,d\in\mathbb{Z}$, $bd\neq 0$, $(ac-b-d)^2\neq 4bd$, $z_{-1},z_0,w_{-1},w_0\in\mathbb{C}\setminus\{0\}$, we leave them for the reader as an exercise.

Acknowledgements. This project was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah, under grant no. (36-130-33-HiCi). The authors, therefore, acknowledge technical and financial support of KAU.

References

[1] M. Aloqeili; Dynamics of a kth order rational difference equation, *Appl. Math. Comput.*, **181** (2006), 1328-1335.

[2] M. Aloqeili; Dynamics of a rational difference equation, *Appl. Math. Comput.*, **176** (2006), 768-774.

[3] A. Andruch-Sobilo, M. Migda; Further properties of the rational recursive sequence $x_{n+1}=ax_{n-1}/(b+cx_nx_{n-1})$, *Opuscula Math.*, **26** (3) (2006), 387-394.

[4] A. Andruch-Sobilo, M. Migda; On the rational recursive sequence $x_{n+1}=ax_{n-1}/(b+cx_nx_{n-1})$, *Tatra Mt. Math. Publ.*, **43** (2009), 1-9.

[5] I. Bajo, E. Liz; Global behaviour of a second-order nonlinear difference equation, *J. Differ. Equations Appl.*, **17** (10) (2011), 1471-1486.

[6] K. Berenhaut, J. Foley, S. Stević; Boundedness character of positive solutions of a max difference equation, *J. Differ. Equations Appl.*, **12** (12) (2006), 1193-1199.

[7] K. Berenhaut, J. Foley, S. Stević; The global attractivity of the rational difference equation $y_n=1+(y_{n-k}/y_{n-m})$, *Proc. Amer. Math. Soc.* **135** (2007), 1133-1140.

[8] K. Berenhaut, S. Stević; The behaviour of the positive solutions of the difference equation $x_n=A+(x_{n-2}/x_{n-1})^p$, *J. Differ. Equations Appl.*, **12** (9) (2006), 909-918.

[9] L. Berg, S. Stević; Periodicity of some classes of holomorphic difference equations, *J. Difference Equ. Appl.*, **12** (8) (2006), 827-835.

[10] L. Berg, S. Stević; On some systems of difference equations, *Appl. Math. Comput.*, **218** (2011), 1713-1718.
[11] F. Brauer and C. Castillo-Chavez; *Mathematical Models in Population Biology and Epidemiology*, Springer, 2012.
[12] C. Cinar; On the positive solutions of difference equation, *Appl. Math. Comput.* **150** (1) (2004), 21-24.
[13] M. Ghergu, V. Rădulescu; *Nonlinear PDEs. Mathematical Models in Biology, Chemistry and Population Genetics,* Springer Monographs in Mathematics, Springer, Heidelberg, 2012.
[14] E. A. Grove, G. Ladas; Periodicities in Nonlinear Difference Equations, Chapman & Hall, CRC Press, Boca Raton, 2005.
[15] B. Iričanin, S. Stević; Eventually constant solutions of a rational difference equation, *Appl. Math. Comput.*, **215** (2009), 854-856.
[16] C. M. Kent, M. Kustesky M, A. Q. Nguyen, B. V. Nguyen; Eventually periodic solutions of $x_{n+1} = \max\{A_n/x_n, B_n/x_{n-1}\}$ when the parameters are two cycles, *Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal.* **10** (1-3) (2003), 33-49.
[17] G. Papaschinopoulos, V. Hatzifilippidis; On a max difference equation, *J. Math. Anal. Appl.*, **258** (2001), 258-268.
[18] G. Papaschinopoulos, C. J. Schinas; On a system of two nonlinear difference equations, *J. Math. Anal. Appl.*, **219** (2) (1998), 415-426.
[19] G. Papaschinopoulos, C. J. Schinas; On the behavior of the solutions of a system of two nonlinear difference equations, *Comm. Appl. Nonlinear Anal.*, **5** (2) (1998), 47-59.
[20] G. Papaschinopoulos, C. J. Schinas; Invariants for systems of two nonlinear difference equations, *Differential Equations Dynam. Systems*, **7** (2) (1999), 181-196.
[21] G. Papaschinopoulos, C. J. Schinas; Invariants and oscillation for systems of two nonlinear difference equations, *Nonlinear Anal. TMA*, **46** (7) (2001), 967-978.
[22] G. Papaschinopoulos, C. J. Schinas, G. Stefanidou; On the nonautonomous difference equation $x_{n+1} = A_n + (x_{n-1}^p/x_n^q)$, *Appl. Math. Comput.*, **217** (2011), 5573-5580.
[23] G. Papaschinopoulos, G. Stefanidou; Asymptotic behavior of the solutions of a class of rational difference equations, *Inter. J. Difference Equations*, **5** (2) (2010), 233-249.
[24] V. Rădulescu, D. Repovš; *Partial Differential Equations with Variable Exponents: Variational Methods and Qualitative Analysis,* CRC Press, Taylor and Francis Group, Boca Raton FL, 2015.
[25] G. Stefanidou, G. Papaschinopoulos, C. Schinas; On a system of max difference equations, *Dynam. Contin. Discrete Impuls. Systems Ser. A*, **14** (6) (2007), 885-903.
[26] G. Stefanidou, G. Papaschinopoulos, C. J. Schinas; On a system of two exponential type difference equations, *Commun. Appl. Nonlinear Anal.*, **17** (2) (2010), 1-13.
[27] S. Stević; A global convergence results with applications to periodic solutions, *Indian J. Pure Appl. Math.*, **33** (1) (2002), 45-53.
[28] S. Stević; On the recursive sequence $x_{n+1} = A/\prod_{i=0}^{k} x_{n-i} + 1/\prod_{j=k+2}^{2(k+1)} x_{n-j}$, *Taiwanese J. Math.*, **7** (2) (2003), 249-259.
[29] S. Stević; More on a rational recurrence relation, *Appl. Math. E-Notes*, **4** (2004), 80-85.
[30] S. Stević; On the recursive sequence $x_{n+1} = \alpha + (x_{n-1}^p/x_n^p)$, *J. Appl. Math. & Computing*, **18** (1-2) (2005) 229-234.
[31] S. Stević; A short proof of the Cushing-Henson conjecture, *Discrete Dyn. Nat. Soc.*, Vol. 2006, Article ID 37264, (2006), 5 pages.
[32] S. Stević; Asymptotics of some classes of higher order difference equations, *Discrete Dyn. Nat. Soc.*, Vol. 2007, Article ID 56813, (2007), 20 pages.
[33] S. Stević; On a generalized max-type difference equation from automatic control theory, *Nonlinear Anal. TMA*, **72** (2010), 1841-1849.
[34] S. Stević; Periodicity of max difference equations, *Util. Math.* **83** (2010), 69-71.
[35] S. Stević; On a nonlinear generalized max-type difference equation, *J. Math. Anal. Appl.*, **376** (2011), 317-328.
[36] S. Stević; On a system of difference equations, *Appl. Math. Comput.* **218** (2011), 3372-3378.
[37] S. Stević; On the difference equation $x_n = x_{n-2}/(b_n + c_n x_{n-1} x_{n-2})$, *Appl. Math. Comput.*, **218** (2011), 4507-4513.
[38] S. Stević; Periodicity of a class of nonautonomous max-type difference equations, *Appl. Math. Comput.*, **217** (2011), 9562-9566.

[39] S. Stević; On a solvable rational system of difference equations, *Appl. Math. Comput.*, **219** (2012), 2896-2908.
[40] S. Stević; On a third-order system of difference equations, *Appl. Math. Comput.*, **218** (2012), 7649-7654.
[41] S. Stević; On some periodic systems of max-type difference equations, *Appl. Math. Comput.*, **218** (2012), 11483-11487.
[42] S. Stević; On some solvable systems of difference equations, *Appl. Math. Comput.* **218** (2012), 5010-5018.
[43] S. Stević; On the difference equation $x_n = x_{n-k}/(b + cx_{n-1} \cdots x_{n-k})$, *Appl. Math. Comput.*, **218** (2012), 6291-6296.
[44] S. Stević; Solutions of a max-type system of difference equations, *Appl. Math. Comput.*, **218** (2012), 9825-9830.
[45] S. Stević; Domains of undefinable solutions of some equations and systems of difference equations, *Appl. Math. Comput.*, **219** (2013), 11206-11213.
[46] S. Stević; On a system of difference equations which can be solved in closed form, *Appl. Math. Comput.*, **219** (2013), 9223-9228.
[47] S. Stević; On a system of difference equations of odd order solvable in closed form, *Appl. Math. Comput.*, **219** (2013) 8222-8230.
[48] S. Stević; On the system $x_{n+1} = y_n x_{n-k}/(y_{n-k+1}(a_n + b_n y_n x_{n-k}))$, $y_{n+1} = x_n y_{n-k}/(x_{n-k+1}(c_n + d_n x_n y_{n-k}))$, *Appl. Math. Comput.*, **219** (2013), 4526-4534.
[49] S. Stević; Representation of solutions of bilinear difference equations in terms of generalized Fibonacci sequences, *Electron. J. Qual. Theory Differ. Equ.*, Vol. 2014, Article No. 67, (2014), 15 pages.
[50] S. Stević, M. A. Alghamdi, A. Alotaibi, N. Shahzad; On a higher-order system of difference equations, *Electron. J. Qual. Theory Differ. Equ.*, Atr. No. 47, (2013), 18 pages.
[51] S. Stević, M. A. Alghamdi, A. Alotaibi, N. Shahzad; Boundedness character of a max-type system of difference equations of second order, *Electron. J. Qual. Theory Differ. Equ.*, Vol. 2015, Article No. 45, (2014), 12 pages.
[52] S. Stević, M. A. Alghamdi, D. A. Maturi, N. Shahzad; On a class of solvable difference equations, *Abstr. Appl. Anal.*, Vol. 2013, Article ID 157943, (2013), 7 pages.
[53] S. Stević, J. Diblik, B. Iričanin, Z. Šmarda; On a third-order system of difference equations with variable coefficients, *Abstr. Appl. Anal.*, Vol. 2012, Article ID 508523, (2012), 22 pages.
[54] S. Stević, J. Diblik, B. Iričanin, Z. Šmarda; On a solvable system of rational difference equations, *J. Difference Equ. Appl.*, **20** (5-6) (2014), 811-825.
[55] S. Stević, J. Diblik, B. Iričanin, Z. Šmarda; Solvability of nonlinear difference equations of fourth order, *Electron. J. Differential Equations*, Vol. 2014, Article No. 264, (2014), 14 pages.
[56] D. T. Tollu, Y. Yazlik, N. Taskara; On fourteen solvable systems of difference equations, *Appl. Math. Comput.*, **233** (2014), 310-319.

Stevo Stević
Mathematical Institute of the Serbian Academy of Sciences, Knez Mihailova 36/III, 11000 Beograd, Serbia.
Operator Theory and Applications Research Group, Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia
E-mail address: sstevic@ptt.rs

Mohammed A. Alghamdi
Operator Theory and Applications Research Group, Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia
E-mail address: proff-malghamdi@hotmail.com

Abdullah Alotaibi
Operator Theory and Applications Research Group, Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia
E-mail address: aalotaibi@kau.edu.sa

Elsayed M. Elsayed
Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia.
Mathematics Department, Faculty of Science, Mansoura University, Mansoura 35516, Egypt
E-mail address: emmelsayed@yahoo.com