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L^p-CONTINUITY OF SOLUTIONS TO PARABOLIC FREE BOUNDARY PROBLEMS

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Abstract. In this article, we consider a class of parabolic free boundary problems. We establish some properties of the solutions, including L^∞-regularity in time and a monotonicity property, from which we deduce strong L^p-continuity in time.

1. Introduction

In this work, we study the following weak formulation which describes a class of nonstationary free boundary problems:

Problem (p). Find $(u,\chi) \in L^2(0,T;H^1(\Omega)) \times L^\infty(Q)$ such that

(i) $u \geq 0$, $0 \leq \chi \leq 1$, $u(1-\chi) = 0$ a.e. in Q;
(ii) $u = \phi$ on Σ_2;
(iii)

$$\int_Q \big[\big(a(x)\nabla u + \chi H(x)\big)\cdot\nabla\xi - (\alpha u + \chi)\xi_t\big]\,dx\,dt \leq \int_\Omega (\chi_0(x) + \alpha u_0(x))\xi(x,0)\,dx$$

for all $\xi \in H^1(Q)$, $\xi = 0$ on $\Sigma_3, \xi \geq 0$ on Σ_4, $\xi(x,T) = 0$ for a.e. $x \in \Omega$,

where α, T are positive numbers, Ω is a bounded domain in $\mathbb{R}^n (n \geq 2)$ with Lipschitz boundary $\partial\Omega = \Gamma_1 \cup \Gamma_2$, $Q = \Omega \times (0,T)$, $\Sigma_1 = \Gamma_1 \times (0,T)$, $\Sigma_2 = \Gamma_2 \times (0,T)$, $\Sigma_3 = \Sigma_2 \cap \{\phi > 0\}$ and $\Sigma_4 = \Sigma_2 \cap \{\phi = 0\}$, with ϕ a nonnegative Lipschitz continuous function defined in $\overline{Q}$. For a.e. $x \in \Omega$, $a(x) = (a_{ij}(x))_{ij}$ is an $n \times n$ matrix, $H : \Omega \to \mathbb{R}^n$ is a vector function satisfying for some positive constants λ, Λ and $\overline{H}$:

$$\forall \xi \in \mathbb{R}^n, \quad \text{a.e. } x \in \Omega \quad \lambda|\xi|^2 \leq a(x)\xi\cdot\xi, \tag{1.1}$$

$$\forall \xi \in \mathbb{R}^n, \quad \text{a.e. } x \in \Omega \quad |a(x)\xi| \leq \Lambda|\xi|, \tag{1.2}$$

$$|H(x)| \leq \overline{H} \quad \text{a.e. } x \in \Omega. \tag{1.3}$$

Moreover, we assume that

$$\operatorname{div}(H(x)) \in L^2(\Omega), \tag{1.4}$$

and the functions $u_0, \chi_0 : \Omega \to \mathbb{R}$ satisfying

$$u_0, \chi_0 \in L^\infty(\Omega), \tag{1.5}$$

$$u_0(x) \geq 0 \quad \text{for a.e. } x \in \Omega, \tag{1.6}$$

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$$0 \le \chi_0(x) \le 1 \quad \text{for a.e. } x \in \Omega. \tag{1.7}$$

Note that problem (p) describes in particular the weak formulation of the non-steady state dam problem [1, 2, 3, 7, 9]. For the heterogeneous stationary dam problem, we refer for example to [5, 11]. Another free boundary problem described by the above formulation is the one-phase Stefan problem (see for example [15, 16]).

Under assumptions (1.1)-(1.7), existence of a solution is proved in [18]. The proof is based on the Tychonoff fixed theorem and combines technics from [1, 9], where existence was established for the unsteady filtration problem in a homogeneous porous medium respectively in the incompressible and compressible cases. Another approach with quasi-variational inequalities was adopted in [17] for rectangular domains.

Uniqueness of the solution was proved for dams with general geometry and rectangular dams respectively in [2] and [7] with different methods. Extensions to a quasilinear operator modeling incompressible fluid flow governed by a generalized nonlinear Darcy's law with Dirichlet, Neuman, or generalized boundary conditions were considered in [4, 12, 13, 14].

In this article, we are concerned with the $L^p(\Omega)$-continuity in time of the functions u and χ. We recall that regularity of the solution was investigated in [3, 2], when $a(x) = I_n$ and $H(x) = e = (0, \dots, 0, 1) \in \mathbb{R}^n$, where it was proved that $\chi \in C^0([0,T], L^p(\Omega))$ for all $p \ge 1$ in both incompressible and compressible cases, and that $u \in C^0([0,T], L^p(\Omega))$ for all $1 \le p \le 2$, in the compressible case. Extensions to the quasilinear case were obtained in [12, 13, 14] in both homogeneous and nonhomogeneous frameworks.

2. Properties

We shall denote by (u, χ) a solution of the problem (p).

Proposition 2.1. *We have*

$$\alpha u + \chi \in C^0([0,T]; V'), \quad \text{where } V = \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_2\}.$$

For a proof of the above proposition see [18].

Proposition 2.2. *If $\alpha > 0$, then we have*

$$u \in L^\infty(0,T; L^2(\Omega)). \tag{2.1}$$

Proof. Let ζ be a smooth function such that $d(\operatorname{supp}(\zeta), \Sigma_2) > 0$ and $\operatorname{supp}(\zeta) \subset \mathbb{R}^n \times (0,T)$. Then there exists $0 < \tau_0 < T$ such that:

$$\forall \tau \in (-\tau_0, \tau_0), \quad (x,t) \mapsto \pm\zeta(x, t-\tau) \text{ are test functions for (p).}$$

Then we have that for all $\tau \in (-\tau_0, \tau_0)$,

$$\begin{aligned} \int_Q \Big[&(a(x)\nabla u(x,t) + \chi(x,t)H(x)).\nabla\zeta(x,t-\tau) \\ &- (\alpha u(x,t) + \chi(x,t))\zeta_t(x,t-\tau) \Big]\,dx\,dt = 0 \end{aligned}$$

which can be written as

$$\begin{aligned} &\int_Q (a(x)\nabla u(x,t+\tau) + \chi(x,t+\tau)H(x)).\nabla\zeta(x,t)\,dx\,dt \\ &= -\frac{\partial}{\partial\tau}\Big(\int_Q (\alpha u(x,t+\tau) + \chi(x,t+\tau))\zeta(x,t)\,dx\,dt\Big) \quad \forall \tau \in (-\tau_0, \tau_0). \end{aligned} \tag{2.2}$$

Moreover (2.2) remains true for all $\zeta \in L^2(0,T;H^1(\Omega))$ such that $\zeta = 0$ on Σ_2 and $\zeta = 0$ on $\Omega \times ((0,\tau_0) \cup (T-\tau_0,T))$. Therefore for $\xi \in \mathcal{D}(\overline{\Omega} \times (\tau_0, T-\tau_0))$ with $\xi \geq 0$, (2.2) is true for the function

$$\zeta(x,t) = (u(x,t+\tau) - \phi(x,t+\tau))\xi(x,t)$$

and we have that for all $\tau \in (-\tau_0,\tau_0)$,

$$\begin{aligned}
&\int_Q (a(x)\nabla u(x,t+\tau) + \chi(x,t+\tau)H(x)).\nabla((u(x,t+\tau) - \phi(x,t+\tau))\xi(x,t))\,dx\,dt \\
&= -\frac{\partial}{\partial\tau}\Big(\int_Q (\alpha u(x,t+\tau) + \chi(x,t+\tau))(u(x,t+\tau) - \phi(x,t+\tau))\xi(x,t)\,dx\,dt\Big).
\end{aligned} \tag{2.3}$$

Since

$$\begin{aligned}
&\int_Q (a(x)\nabla u(x,t+\tau) + \chi(x,t+\tau)H(x)).\nabla((u(x,t+\tau) - \phi(x,t+\tau))\xi(x,t))\,dx\,dt \\
&= \int_Q (a(x)\nabla u(x,t) + \chi(x,t)H(x)).\nabla((u(x,t) - \phi(x,t))\xi(x,t-\tau))\,dx\,dt
\end{aligned}$$

the integral in the left hand side of (2.3) is continuous in $(-\tau_0,\tau_0)$. We deduce that the function

$$G(\tau) = \int_Q (\alpha u(x,t+\tau) + \chi(x,t+\tau))(u(x,t+\tau) - \phi(x,t+\tau))\xi(x,t)\,dx\,dt$$

belongs to $C^1(-\tau_0,\tau_0)$. Hence for $\tau = 0$ we obtain

$$\int_Q (a(x)\nabla u(x,t) + \chi(x,t)H(x)).\nabla((u(x,t) - \phi(x,t))\xi(x,t))\,dx\,dt = -G'(0). \tag{2.4}$$

Note that

$$\begin{aligned}
G(\tau) &= \int_Q (\alpha u(x,t+\tau) + \chi(x,t+\tau))(u(x,t+\tau) - \phi(x,t+\tau))\xi(x,t)\,dx\,dt \\
&= \int_Q (\alpha u(x,t) + \chi(x,t))(u(x,t) - \phi(x,t))\xi(x,t-\tau)\,dx\,dt
\end{aligned}$$

and then

$$G'(0) = -\int_Q (\alpha u(x,t) + \chi(x,t))(u(x,t) - \phi(x,t))\xi_t(x,t)\,dx\,dt. \tag{2.5}$$

It follows from (2.4) and (2.5) that

$$\int_Q (a(x)\nabla u + \chi H(x)).\nabla((u-\phi)\xi)\,dx\,dt = \int_Q (\alpha u + \chi)(u-\phi)\xi_t\,dx\,dt. \tag{2.6}$$

Now

$$\begin{aligned}
\int_Q (\alpha u + \chi)(u-\phi)\xi_t\,dx\,dt &= \int_Q (\alpha u^2 - \alpha\phi + u - \chi\phi)\xi_t\,dx\,dt \\
&= \int_Q \alpha\Big(u^2 + \frac{1-\alpha\phi}{\alpha}u - \frac{\chi\phi}{\alpha}\Big)\xi_t\,dx\,dt \\
&= \int_Q \alpha\Big(u + \frac{1-\alpha\phi}{2\alpha}\Big)^2 - \alpha\Big(\frac{1-\alpha\phi}{2\alpha}\Big)^2 - \frac{\chi\phi}{\alpha}\Big)\xi_t\,dx\,dt \\
&= \int_Q \Big[\alpha\Big(u + \frac{1-\alpha\phi}{2\alpha}\Big)^2 - \frac{(1-\alpha\phi)^2}{4\alpha} - \chi\phi\Big]\xi_t\,dx\,dt\,.
\end{aligned}$$

From (2.6) we obtain

$$\begin{aligned}&\int_Q (a(x)\nabla u + \chi H(x)).\nabla((u-\phi)\xi)\,dx\,dt =\\ &= \int_Q \Big[\alpha\Big(u + \frac{1-\alpha\phi}{2\alpha}\Big)^2 - \frac{(1-\alpha\phi)^2}{4\alpha} - \chi\phi\Big]\xi_t\,dx\,dt\end{aligned}$$

or by taking $\xi \in \mathcal{D}(0,T)$,

$$\begin{aligned}&\int_0^T \xi dt \int_\Omega (a(x)\nabla u + \chi H(x)).\nabla(u-\phi)dx\\ &= \int_0^T \xi_t dt \int_\Omega \Big[\alpha\Big(u + \frac{1-\alpha\phi}{2\alpha}\Big)^2 - \frac{(1-\alpha\phi)^2}{4\alpha} - \chi\phi\Big]dx\end{aligned}$$

which leads in the distributional sense in $\mathcal{D}'(0,T)$ to

$$\begin{aligned}&\frac{d}{dt}\int_\Omega \Big[\alpha\Big(u + \frac{1-\alpha\phi}{2\alpha}\Big)^2 - \frac{(1-\alpha\phi)^2}{4\alpha} - \chi\phi\Big]dx\\ &= -\int_\Omega (a(x)\nabla u + \chi H(x)).\nabla(u-\phi)dx.\end{aligned}$$

Therefore, the function

$$t \mapsto \int_\Omega \Big[\alpha\Big(u + \frac{1-\alpha\phi}{2\alpha}\Big)^2 - \frac{(1-\alpha\phi)^2}{4\alpha} - \chi\phi\Big]dx$$

is in $\in W^{1,1}(0,T) \subset C^0([0,T])$. Given that $\chi, \phi \in L^\infty(Q)$ and $\alpha > 0$, we conclude that $u \in L^\infty(0,T;L^2(\Omega))$, which is (2.1). $\square$

The following result will be used to establish a monotonicity property of χ which is the key point to prove the main result of the paper.

Proposition 2.3. *We have*

$$\operatorname{div}(\chi H(x)) - \chi_{\{u>0\}}\operatorname{div}(H(x)) - \chi_t \le 0 \quad \text{in } \mathcal{D}'(Q). \tag{2.7}$$

Proof. Arguing as in the beginning of the proof of Proposition 2.2, we have for any $\zeta \in L^2(0,T;H^1(\Omega))$ such that $\zeta = 0$ on Σ_2 and $\zeta = 0$ on $\Omega \times ((0,\tau_0) \cup (T-\tau_0,T))$, with $\tau_0 > 0$

$$\begin{aligned}&\int_Q (a(x)\nabla u(x,t+\tau) + \chi(x,t+\tau)H(x)).\nabla\zeta(x,t)\,dx\,dt\\ &= -\frac{\partial}{\partial\tau}\Big(\int_Q (\alpha u(x,t+\tau) + \chi(x,t+\tau))\zeta(x,t)\,dx\,dt\Big) \quad \forall \tau \in (-\tau_0,\tau_0).\end{aligned}\tag{2.8}$$

Now, let us consider $\epsilon > 0$, $\xi \in \mathcal{D}(\Omega \times (\tau_0, T-\tau_0))$ such that $\xi \ge 0$, and choose $\zeta(x,t) = \min\big(\frac{u(x,t+\tau)}{\epsilon}, 1\big)\xi$ in (2.8). We obtain

$$\begin{aligned}&\int_Q (a(x)\nabla u(x,t+\tau) + \chi(x,t+\tau)H(x)).\nabla\Big(\min\Big(\frac{u(x,t+\tau)}{\epsilon},1\Big)\xi(x,t)\Big)\,dx\,dt\\ &= -\frac{\partial}{\partial\tau}\Big(\int_Q (\alpha u(x,t+\tau) + \chi(x,t+\tau))\min\Big(\frac{u(x,t+\tau)}{\epsilon},1\Big)\xi(x,t)\,dx\,dt\Big)\end{aligned}\tag{2.9}$$

for all $\tau \in (-\tau_0, \tau_0)$. Obviously the integral at the left hand side of (2.9) is continuous in $(-\tau_0, \tau_0)$. Consequently the function

$$G(\tau) = \int_Q (\alpha u(x, t+\tau) + \chi(x, t+\tau)) \min\Big(\frac{u(x, t+\tau)}{\epsilon}, 1\Big)\xi(x, t)\, dx\, dt$$

is a C^1 function in $(-\tau_0, \tau_0)$. For $\tau = 0$, we obtain

$$\int_Q (a(x)\nabla u + \chi H(x)).\nabla\Big(\min\Big(\frac{u}{\epsilon}, 1\Big)\xi\Big)\, dx\, dt = -G'(0). \tag{2.10}$$

Since

$$\begin{aligned} G(\tau) &= \int_Q (\alpha u(x,t) + \chi(x,t)) \min\Big(\frac{u(x,t)}{\epsilon}, 1\Big)\xi(x, t-\tau)\, dx\, dt \\ &= \int_Q (\alpha u(x,t) + 1) \min\Big(\frac{u(x,t)}{\epsilon}, 1\Big)\xi(x, t-\tau)\, dx\, dt, \end{aligned}$$

we obtain

$$G'(0) = -\int_Q (\alpha u + 1) \min\Big(\frac{u}{\epsilon}, 1\Big)\xi_t\, dx\, dt. \tag{2.11}$$

Hence from (2.10) and (2.11) we obtain

$$\begin{aligned} &\int_Q (a(x)\nabla u + \chi H(x)) \cdot \nabla\Big(\min\Big(\frac{u}{\epsilon}, 1\Big)\xi\Big)\, dx\, dt \\ &= \int_Q (\alpha u + 1) \min\Big(\frac{u}{\epsilon}, 1\Big)\xi_t\, dx\, dt \end{aligned}$$

which leads to

$$\begin{aligned} &\int_Q a(x)\nabla u \cdot \nabla\Big(\min\Big(\frac{u}{\epsilon}, 1\Big)\xi\Big) - \alpha u \min\Big(\frac{u}{\epsilon}, 1\Big)\xi_t\, dx\, dt \\ &= -\int_Q \chi H(x).\nabla\Big(\min\Big(\frac{u}{\epsilon}, 1\Big)\xi\Big)\, dx\, dt + \alpha \int_Q u \min\Big(\frac{u}{\epsilon}, 1\Big)\xi_t\, dx\, dt \\ &= -\int_Q H(x).\nabla\Big(\min\Big(\frac{u}{\epsilon}, 1\Big)\xi\Big)\, dx\, dt + \alpha \int_Q u \min\Big(\frac{u}{\epsilon}, 1\Big)\xi_t\, dx\, dt. \\ &= \int_Q \operatorname{div}(H(x)).\min\Big(\frac{u}{\epsilon}, 1\Big)\xi\, dx\, dt + \alpha \int_Q u \min\Big(\frac{u}{\epsilon}, 1\Big)\xi_t\, dx\, dt \end{aligned}$$

or

$$\begin{aligned} &\int_Q \min\Big(\frac{u}{\epsilon}, 1\Big)a(x)\nabla u.\nabla\xi - \alpha u \min\Big(\frac{u}{\epsilon}, 1\Big)\xi_t\, dx\, dt \\ &= \int_Q \operatorname{div}(H(x)).\min\Big(\frac{u}{\epsilon}, 1\Big)\xi\, dx\, dt + \alpha \int_Q u \min\Big(\frac{u}{\epsilon}, 1\Big)\xi_t\, dx\, dt \\ &\quad - \int_{Q\cap\{u<\epsilon\}} \xi a(x)\nabla u.\nabla u\, dx\, dt \\ &\le \int_Q \operatorname{div}(H(x)).\min\Big(\frac{u}{\epsilon}, 1\Big)\xi\, dx\, dt + \alpha \int_Q u \min\Big(\frac{u}{\epsilon}, 1\Big)\xi_t\, dx\, dt. \end{aligned} \tag{2.12}$$

Letting $\epsilon \to 0$ in (2.12), we obtain

$$\int_Q a(x)\nabla u.\nabla\xi - \alpha u\xi_t\, dx\, dt \le \int_Q \chi_{\{u>0\}} \operatorname{div}(H(x))\xi\, dx\, dt$$

or

$$\operatorname{div}(a(x)\nabla u) + \chi_{\{u>0\}} \operatorname{div}(H(x)) - \alpha u_t \geq 0 \text{ in } \mathcal{D}'(Q). \tag{2.13}$$

Now using $\pm\xi$ as a test function in (p), we obtain

$$\operatorname{div}(a(x)\nabla u + \chi H(x)) - \alpha u_t - \chi_t = 0 \quad \text{in } \mathcal{D}'(Q). \tag{2.14}$$

Taking into account (2.13) and (2.14), we obtain

$$\begin{aligned}&\operatorname{div}(\chi H(x)) - \chi_{\{u>0\}} \operatorname{div}(H(x)) - \chi_t \\ &= -\operatorname{div}(a(x)\nabla u) - \chi_{\{u>0\}} \operatorname{div}(H(x)) + \alpha u_t \leq 0 \quad \text{in } \mathcal{D}'(Q),\end{aligned}$$

which is (2.7). □

3. Monotonicity property

In all what follows, we shall assume that

$$H(x) = (h_1(x), \dots, h_n(x)) \in C^{0,1}(\overline{\Omega}, \mathbb{R}^n) \tag{3.1}$$

$$\operatorname{div}(H(x)) \geq 0 \quad \text{a.e. } x \in \Omega \tag{3.2}$$

and for two positive constants $\underline{h}$ and $\overline{h}$,

$$0 < \underline{h} \leq h_n(x) \leq \overline{h}, \quad |h_i(x)| \leq \overline{h} \quad \forall x \in \overline{\Omega},\ i = 1, \dots, n-1. \tag{3.3}$$

Since $H \in C^{0,1}(\overline{\Omega})$, there exists by Kirszbraun's theorem (see [8, Theorem 2.10.43 p. 210]) an extension $\widetilde{H} \in C^{0,1}(\mathbb{R}^n)$ of H with the same Lipschitz constant. Then the function $\overline{H} = (\overline{H}_1, \dots, \overline{H}_{n-1}, \overline{H}_n)$ defined by

$$\begin{gathered}\overline{H}_i = \min(\bar{h}, \max(\widetilde{H}_i, -\bar{h})) \quad i = 1, \dots, n-1 \\ \overline{H}_n = \min(\bar{h}, \max(\widetilde{H}_n, \underline{h}))\end{gathered}$$

satisfies $\overline{H} \in C^{0,1}(\mathbb{R}^n)$, $\overline{H}_{/\overline{\Omega}} = H$, and

$$0 < \underline{h} \leq \overline{H}_n(x) \leq \overline{h}, \quad |\overline{H}_i(x)| \leq \overline{h} \quad \forall x \in \mathbb{R}^n,\ i = 1, \dots, n-1.$$

For simplicity, we will denote $\overline{H}$ by H.

Let $h_0 \in \mathbb{R}$ such that Ω is located strictly above the hyperplane $x_n = h_0$. We consider for each $\omega \in \mathbb{R}^{n-1}$ the differential equation

$$\begin{gathered}X'(s, \omega) = H(X(s, \omega)) \\ X(0, \omega) = (\omega, h_0).\end{gathered} \tag{3.4}$$

Then we have the following proposition.

Proposition 3.1. *There exists a unique maximal solution $x(\cdot, \omega)$ of (3.4) defined on $(-\infty, \infty)$. Moreover x is of class $C^{0,1}$ with respect to ω, $C^{1,1}$ with respect to s, and we have*

$$\lim_{s \to \pm\infty} x_n(s, \omega) = \pm\infty. \tag{3.5}$$

Proof. By the classical theory of ordinary differential equations there exists a unique maximal solution $x(\cdot, \omega)$ of (3.4) defined on $(\alpha_-(\omega), \alpha_+(\omega))$. Moreover since H is of class $C^{0,1}$, x is of class $C^{0,1}$ with respect to ω, $C^{1,1}$ with respect to s. For (3.5), we refer to the proof of (2.4) in [14]. □

Theorem 3.2. *The mappings* $\mathcal{T}:\mathbb{R}^n\to\mathbb{R}^n$ *defined by* $\mathcal{T}(s,\omega)=x(s,\omega)$ *is a* $C^{0,1}$*-homeomorphism from* $\mathbb{R}^n$ *to* $\mathbb{R}^n$. *Moreover*

$$Y(s,\omega)=\mathcal{J}\mathcal{T}(s,\omega)=(-1)^{n+1}h_n(\omega,h_0)\exp\Big(\int_0^s(divH)(x(\sigma,\omega))\,d\sigma\Big)\neq 0,$$

where $\mathcal{J}$ *denotes the Jacobian.*

Proof. We refer to the proof of [6, Theorem 2.2] and to the proof of [14, Theorem 2.1]. □

Remark 3.3. Let $\mathcal{O}=\mathcal{T}^{-1}(\Omega)$. Then $\mathcal{O}$ is a domain of $\mathbb{R}^n$ and $\mathcal{T}:\mathcal{O}\to\Omega$ is a $C^{0,1}$-homeomorphism.

Let $f(s,\omega,t)=\chi(T(s,\omega),t)$. In the following theorem we show that f satisfies a monotonicity result similar to the one in [6, Theorem 2.1] for the stationary case and to [14, Theorem 2.2] for the nonstationary case. This extends the well known monotonicity in the homogeneous case i.e. $\chi_n-\chi_t\geq 0$ in $\mathcal{D}'(Q)$ when $a(x)=I_n$ (see [2, 3]). This result will be the key point for the proof of the L^p-continuity of χ and u.

Theorem 3.4. *Let* (u,χ) *be a solution of* (p). *We have*

$$(\frac{\partial}{\partial s}-\frac{\partial}{\partial t})f\leq 0\quad \text{in } \mathcal{D}'(\mathcal{O}\times(0,T)). \tag{3.6}$$

Proof. Let $\phi\in\mathcal{D}(\mathcal{O}\times(0,T))$, $\phi\geq 0$. Since $\mathcal{T}^{-1}\in C^{0,1}(\Omega)$, by approximation we can use $\phi\circ\mathcal{T}^{-1}$ as a test function in (2.7). So we have

$$\begin{aligned}\int_{\mathcal{T}(\mathcal{O})\times(0,T)}&\Big\{\chi H(x).\nabla(\phi\circ\mathcal{T}^{-1})+\chi_{\{u>0\}}\operatorname{div}(H(x)).\phi\circ\mathcal{T}^{-1}\\&-\chi(\phi\circ\mathcal{T}^{-1})_t\Big\}\,dx\,dt\geq 0.\end{aligned}\tag{3.7}$$

Since $\mathcal{T}$ is a $C^{0,1}$-homeomorphism from $\mathcal{O}$ to Ω, we can use the change of variables formula [19, p. 52] to obtain, from (3.7),

$$\int_{\mathcal{O}\times(0,T)}\Big(\chi\circ\mathcal{T}.\frac{\partial\phi}{\partial s}+\chi_{\{u\circ\mathcal{T}>0\}}(\operatorname{div}(H))\circ\mathcal{T}.\phi-\chi\circ\mathcal{T}.\frac{\partial\phi}{\partial t}\Big)|Y|\,ds\,d\omega\,dt\geq 0$$

which, given that $\frac{\partial|Y|}{\partial s}=|Y|.(\operatorname{div}(H))\circ\mathcal{T}$, leads to

$$\begin{aligned}&\int_{\mathcal{O}\times(0,T)}\Big(\chi\circ\mathcal{T}.\frac{\partial(|Y|.\phi)}{\partial s}-\chi\circ\mathcal{T}.\frac{\partial(|Y|.\phi)}{\partial t}\Big)\,ds\,d\omega\,dt\\&=\int_{\mathcal{O}\times(0,T)}\Big(\chi\circ\mathcal{T}.\frac{\partial\phi}{\partial s}|Y|+\chi\circ\mathcal{T}.(\operatorname{div}(H))\circ\mathcal{T}.\phi|Y|-\chi\circ\mathcal{T}.\frac{\partial\phi}{\partial t}|Y|\Big)\,ds\,d\omega\,dt\\&\geq\int_{\mathcal{O}\times(0,T)}\Big(\chi\circ\mathcal{T}.\frac{\partial\phi}{\partial s}+\chi_{\{u\circ\mathcal{T}>0\}}.(\operatorname{div}(H))\circ\mathcal{T}.\phi-\chi\circ\mathcal{T}.\frac{\partial\phi}{\partial t}\Big)|Y|\,ds\,d\omega\,dt\\&\geq 0.\end{aligned}\tag{3.8}$$

By approximation, (3.8) holds for any nonnegative function ϕ with compact support such that $\phi_s,\phi_t\in L^1(\mathcal{O}\times(0,T))$. Since $Y,Y_s\in L^\infty(\mathcal{O}\times(0,T))$, one can choose $\phi=\frac{\psi}{|Y|}$, with $\psi\in\mathcal{D}(\mathcal{O}\times(0,T))$ and $\psi\geq 0$. Thus we get the result. □

4. Continuity of χ and αu

The main result of the this article is the following theorem.

Theorem 4.1. *Let* (u, χ) *be a solution of problem* (p). *Then we have*

$$\chi \in C^0([0,T]; L^p(\Omega)) \quad \forall p \in [1, \infty), \tag{4.1}$$

$$\textit{If } \alpha > 0, \ \textit{then } u \in C^0([0,T]; L^p(\Omega)) \quad \forall p \in [1, 2]. \tag{4.2}$$

Proof. Let $v = uo\mathcal{T}^{-1}$. Since $\mathcal{T}$ is a $C^{0,1}$-homeomorphism, we get from Propositions 2.1 and 2.2

$$f + \alpha v \in C^0([0,T]; H^{-1}(\mathcal{O})), \tag{4.3}$$

$$v \in L^\infty([0,T]; L^2(\mathcal{O})). \tag{4.4}$$

Taking into account (4.3)-(4.4), the monotonicity of f in (3.6), and arguing as in the proof [2, Theorem 2.4], we obtain

$$f \in C^0([0,T]; L^p(\mathcal{O})) \quad \forall p \in [1, \infty),$$

which by using the change of variables $\mathcal{T}$ leads to

$$\chi \in C^0([0,T]; L^p(\mathcal{T}(\mathcal{O}))) = C^0\big([0,T], L^p(\Omega)\big) \quad \forall p \in [1, \infty). \tag{4.5}$$

Assume that $\alpha > 0$. Since $\chi, \phi \in C^0([0,T], L^2(\Omega))$, we deduce from the last part of the proof of Proposition 2.2 that $u \in C^0(0,T; L^2(\Omega))$, and since Ω is bounded (4.2) follows. $\square$

Remark 4.2. If $\alpha > 0$ and $u \in L^\infty(0,T; L^p(\Omega))$ for some $p > 2$, we have, $u \in C^0([0,T]; L^p(\Omega))$. In particular, if $u \in L^\infty(Q)$, we have

$$u \in C^0([0,T]; L^p(\Omega)) \quad \forall p \geq 1.$$

If $\alpha = 0$, in general, $u \notin C^0([0,T]; L^p(\Omega))$ (see [3, Remark 3.9]).

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