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TWO TYPES OF GROUND STATE SOLUTIONS FOR A PERIODIC SCHRÖDINGER EQUATIONS WITH ZERO ON THE BOUNDARY OF THE SPECTRUM

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Abstract. This article concerns the Schrödinger equation

$$-\Delta u + V(x)u = f(x,u), \quad \text{for } x \in \mathbb{R}^N,$$
$$u(x) \to 0, \quad \text{as } |x| \to \infty\,.$$

Assuming that V and f are periodic in x, and 0 is a boundary point of the spectrum $\sigma(-\Delta + V)$, two types of ground state solutions are obtained with some super-quadratic conditions.

1. Introduction and main results

We consider the Schrödinger equation

$$\begin{gathered} -\Delta u + V(x)u = f(x,u), \quad \text{for } x \in \mathbb{R}^N, \\ u(x) \to 0, \quad \text{as } |x| \to \infty, \end{gathered} \tag{1.1}$$

where $V : \mathbb{R}^N \to \mathbb{R}$ and $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is superlinear as $|u| \to \infty$. (1.1) has been widely investigated in the literature over the past several decades for both its importance in applications and mathematical interest, see, e.g., [3, 4, 5, 7, 8, 9, 12, 13, 14, 15, 16, 18, 19, 20, 21, 22, 23, 24, 25, 26, 28, 29, 30, 31, 33, 34, 35, 36, 37, 38, 39, 40] and the references therein. In this paper, we mainly study the case that V and f are periodic in x and 0 is a boundary point of the spectrum $\sigma(-\Delta + V)$, i.e.

- (V1) $V \in C(\mathbb{R}^N, \mathbb{R})$ is 1-periodic in each of $x_1, x_2, \dots, x_N$, $0 \in \sigma(-\Delta + V)$, and there exists a $b_0 > 0$ such that $(0, b_0] \cap \sigma(-\Delta + V) = \emptyset$;

Compared to the situation that 0 lies in a gap of $\sigma(-\Delta + V)$, this case is very difficult because $H^1(\mathbb{R}^N)$ is no longer the working space on which the variational functional associated with (1.1) defines. Indeed, the working space is only a Banach space, not a Hilbert space. In particular, there are considerably fewer results, see [3, 21, 24, 25, 33, 37, 39].

Bartsch and Ding [3] obtained a weak solution with a stronger version of the classic condition

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(AR) there exist constants $\mu > 2$ such that

$$0 < \mu F(x,t) \leq tf(x,t), \quad \forall (x,t) \in \mathbb{R}^N \times (\mathbb{R} \setminus \{0\}).$$

Their main idea is to use an approximation argument to construct some kind of Palais-Smale sequence and show that after translations a subsequence converges in certain sense to a weak solution u of (1.1). Later, this result was improved by Willem and Zou [37] by using an generalized weak link theorem. In a recent publication, Yang et al.[39] established existence of one weak solution of (1.1) with the following Nehari type assumption:

(Ne) $t \mapsto \frac{f(x,t)}{|t|}$ is strictly increasing on $(-\infty, 0) \cup (0, \infty)$.

Their main technique is the same as that used in [3, 37]. Later, this result was improved by authors's recent paper [25] by using a generalized linking theorem established in [15, 18]. The following condition seems to be necessary to obtain the existence of one weak solution of (1.1) in [3, 37, 39].

(F0) there exist constants $c_0 > 0$, $2 < \varrho < 2^*$ such that

$$tf(x,t) \geq c_0 |t|^{\varrho}, \quad \forall (x,t) \in \mathbb{R}^N \times \mathbb{R}.$$

However, (F0) it is a severe restriction, since it strictly controls the growth of $f(x,t)$ as $|t| \to \infty$. There are functions which are superlinear at both zero and infinity, but do not satisfy the condition (F0). For example $f(x,t) = at|t|^{\alpha-2}\ln(1+|t|^{1/N})$ with $a > 0$ and $\alpha \in (2, 2^* - 1/N)$. Recently, with the aid of a proper variational framework, Tang [33] weakened (AR), (F0) and (Ne) to the following assumptions:

(F1) $f \in C(\mathbb{R}^{N+1}, \mathbb{R})$ is 1-periodic in each of $x_1, x_2, \dots, x_N$, and there exist constants $c_1, c_2 > 0$ and $2 < \varrho \leq p < 2^*$ such that

$$c_1 \min\{|t|^{\varrho}, |t|^2\} \leq tf(x,t) \leq c_2(|t|^{\varrho} + |t|^p), \quad \forall (x,t) \in \mathbb{R}^N \times \mathbb{R};$$

(F2) $\lim_{|t|\to\infty} F(x,t)/t^2 = \infty$, a.e. $x \in \mathbb{R}^N$.

(DL) $\mathcal{F}(x,t) := \frac{1}{2}tf(x,t) - F(x,t) > 0$ for all $x \in \mathbb{R}^N, t \in \mathbb{R} \setminus \{0\}$, and there exist $r_0 > 0$, $c_0 > 0$ and $\sigma > \max\{1, N/2\}$ such that

$$|f(x,t)|^{\sigma} \leq c_0 |t|^{\sigma} \mathcal{F}(x,t), \quad \forall (x,t) \in \mathbb{R}^N \times \mathbb{R},\ |t| \geq r_0;$$

(Ta) there exists a $\theta_0 \in (0,1)$ such that

$$\frac{1-\theta^2}{2} tf(x,t) \geq \int_{\theta t}^{t} f(x,s)\mathrm{d}s = F(x,t) - F(x,\theta t), \quad \forall \theta \in [0,\theta_0],\ (x,t) \in \mathbb{R}^N \times \mathbb{R}.$$

Under basic assumptions (V1), (F1) and (F2), assuming moreover (DL) or (Ta) holds, *least energy solutions* were obtained by Tang [33], i.e. a nontrivial solution $u_0 \in E$ such that $\Phi(u_0) = \inf_{\mathcal{M}} \Phi$, where

$$\mathcal{M} = \{u \in E \setminus \{0\} : \Phi'(u) = 0\}, \tag{1.2}$$

E is the working space on which the energy functional Φ associated with (1.1) defines. In recent paper [21], Mederski studied (1.1) under (Ne) by using the generalized Nehari manifold method due to Szulkin and Weth [28, 29], moreover (F0) was weakened to a similar version of (F1) there. For multiple results of (1.1), we refer to papers [3, 21, 24].

Condition (DL) was introduced by Ding and Lee [7], and it is commonly used instead of condition (AR), see for instance [9, 23, 31, 32, 33] and the references therein. Clearly, (AR) is much stronger than (DL), (F2) and (Ta). On the other hand, (Ta) was introduced by Tang [30] and is weaker than the following mild version of (Ne):

(WN) $t \mapsto \frac{f(x,t)}{|t|}$ is non-decreasing on $(-\infty,0)\cup(0,\infty)$.

In addition, (F1) weakens (F0) greatly and is satisfied by many functions, such as $f(x,t) = at|t|^{\alpha-2}\ln(1+|t|^{1/N})$ with $a>0$ and $\alpha \in (2, 2^*-1/N)$. We point out that Liu [16] first uses (WN) to replace of (Ne) for the case that 0 lies in a gap of $\sigma(-\Delta+V)$, and obtains a least energy solution. Later, this result is improved by Tang in [30] by taking advantage of (Ta).

It is well known that for periodic potential V, the operator $\mathcal{A} := -\Delta + V$ has purely continuous spectrum $\sigma(\mathcal{A})$ which is bounded below and consists of closed disjoint intervals (see [27, Theorem XIII.100]). Before stating our main results, we first present the following weaker version of (DL) which was introduced by Tang in recent paper [35]:

(F3) $\mathcal{F}(x,t) \geq 0$, and there exist constants $c_3 > 0$, $\delta_0 \in (0,\Lambda_0)$ and $\sigma > \max\{1, N/2\}$ such that

$$\frac{f(x,t)}{t} \geq (\Lambda_0 - \delta_0) \Longrightarrow \big(\frac{f(x,t)}{t}\big)^{\sigma} \leq c_3\mathcal{F}(x,t),$$

where $\Lambda_0 := \inf[\sigma(-\Delta+V)\cap(0,\infty)]$.

Clearly, $\Lambda_0 \geq b_0$ by (V1), and (DL) *implies* (F3) *under* (F1). More precisely, (F3) holds under (DL) and basic conditions that $f \in C(\mathbb{R}^{N+1},\mathbb{R})$ is 1-periodic in each of $x_1, x_2, \dots, x_N$ and $f(x,t) = o(|t|)$ as $|t| \to 0$ uniformly in $x \in \mathbb{R}^N$. Thus (F3) weakens (DL), and there are some functions satisfying (F3), but not (DL), see Example 1.6 and 1.7. Moreover, it is more convenient to use, see Lemma 3.6.

In this article, we continue to study problem (1.1), and construct two types of ground state solutions of (1.1), i.e. *the least energy solution* and *the Nehari-Pankov type*. We first weaken (Ne) to (WN), and establish the existence of a ground state solution of Nehari-Pankov type. The *generalized Nehari manifold method* introduced by Szulkin and Weth [28, 29] can not be adopted due to the lack of strict monotonicity, see Remark 1.3. So a new method is looked forward to being introduced which is the right issue this paper intends to address. Inspired by the works [3, 20, 22, 26, 28, 32, 33, 34, 35], a more direct approach is used in the present paper. The main ingredient in our approach is the observation that a minimizing Cerami sequence for the energy functional can be found outside the Nehari-Pankov manifold $\mathcal{N}^-$ by using the diagonal method, see Lemma 3.5, part of which derives from recent papers of Tang [32, 34]. Moreover, under weaker condition (F3), a least energy solution is obtained with the aid of a generalized linking theorem established in [33].

Let E, E^- be the Banach space defined in Section 2. Under assumptions (V1) and (F1), the functional

$$\Phi(u) = \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2)\mathrm{d}x - \int_{\mathbb{R}^N} F(x,u)\mathrm{d}x, \tag{1.3}$$

is well defined for all $u \in E$, moreover $\Phi \in C^1(E,\mathbb{R})$, see Lemma 2.2. Denote the Nehari-Pankov manifold by

$$\mathcal{N}^- = \{u \in E \setminus E^- : \langle \Phi'(u), u\rangle = \langle \Phi'(u), v\rangle = 0,\ \forall v \in E^-\}. \tag{1.4}$$

The set $\mathcal{N}^-$ was introduced by Pankov [22], which is a subset of the Nehari manifold

$$\mathcal{N} = \{u \in E \setminus \{0\} : \langle \Phi'(u), u\rangle = 0\}. \tag{1.5}$$

Now, we are ready to state the main results of this article.

Theorem 1.1. *Let* (V1), (F1), (F2), (WN) *be satisfied, then* (1.1) *has a solution* $u_0 \in E$ *such that* $\Phi(u_0) = \inf_{\mathcal{N}^-} \Phi \geq \kappa$, *where* κ *is a positive constant.*

Theorem 1.2. *Let* (V1), (F1), (F2), (F3) *be satisfied, then* (1.1) *has a solution* $u_0 \in E$ *such that* $\Phi(u_0) = \inf_{\mathcal{M}} \Phi$, *where* $\mathcal{M}$ *is defined by* (1.2).

Note that, $u \in \mathcal{N}^-$ if $u \neq 0$ and $\Phi'(u) = 0$. Hence $\mathcal{N}^-$ contains all nontrivial critical points of Φ, i.e. $\mathcal{M}$ is a very small subset of $\mathcal{N}^-$. In general, $\mathcal{N}^-$ contains infinitely many elements of E. In fact, for any $u \in E^+ \setminus \{0\}$, there exist $t = t(u) > 0$ and $w = w(u) \in E^-$ such that $w + tu \in \mathcal{N}^-$ which is the global maximum of $\Phi|_{E^-\oplus\mathbb{R}^+u}$, see Corollary 2.6 and Lemma 3.3. As a consequence of Theorem 1.1, the least energy value $m := \inf_{\mathcal{N}^-} \Phi$ has a minimax characterization given by

$$m = \Phi(u_0) = \inf_{u\in E^+\setminus\{0\}} \max_{v\in E^-\oplus\mathbb{R}^+u} \Phi(v).$$

Note that this minimax principle is much simpler than the usual characterizations related to the concept of linking. Since u_0 is a solution at which Φ has least "energy" in set $\mathcal{N}^-$, it was called a ground state solution of Nehari-Pankov type in [32, 34].

We remark that Theorems 1.1 and 1.2 generalize and improve the results in [3, 21, 24, 33, 37, 39].

As a motivation we recall the generalized Nehari manifold method introduced by Szulkin and Weth [28, 29]. For the case that 0 lies in a gap of $\sigma(-\Delta + V)$, they obtained a ground state solution of Nehari-Pankov type under (Ne) and some additional conditions. The generalized Nehari manifold method developed there is based on a direct and simple reduction of the strongly indefinite problem to a definite one. More precisely, a homeomorphism between the Nehari-Pankov manifold $\mathcal{N}^-$ and a unit sphere S^+ in E^+ is established which allows to find a minimizing sequence on the sphere and hence on the Nehari-Pankov manifold. We point out that the assumption "strictly increasing" in (Ne) is very crucial in the argument of Szulkin and Weth [28, 29]. In fact, the starting point of their approach is to show that for each $u \in E \setminus E^-$, the Nehari-Pankov manifold $\mathcal{N}^-$ intersects $\hat{E}(u) := E^- \oplus \mathbb{R}^+u$ in exactly one point $\hat{m}(u)$. The uniqueness of $\hat{m}(u)$ enables one to define a map $u \mapsto \hat{m}(u)$, which is crucial to construct the homeomorphism between $\mathcal{N}^-$ and S^+, see [29, Chapter 4].

Remark 1.3. The ground state solution of Nehari-Pankov type can not be established by using the generalized Nehari manifold method if (Ne) was weakened to (WN). Indeed, without the strict monotonicity, the uniqueness of $\hat{m}(u)$ can not be guaranteed unless some additional conditions on the nonlinearity are assumed, and so the homeomorphism between the Nehari-Pankov manifold and the sphere can not be established (see [28, (A.2) and Prop. 2.3] and [40]). Thus, it is infeasible to find a minimizing sequence on the Nehari-Pankov manifold by reducing the problem on the sphere which is a definite case. Compared to the generalized Nehari manifold method, the approach used in this paper seems more direct and simpler.

Before proceeding to the proof of Theorems 1.1 and 1.2, we give some examples to illustrate the assumptions.

Example 1.4. $F(x,t) = h(x)|t|^2 \ln(1 + |t|^{1/N})$, where $h \in C(\mathbb{R}^N, (0, +\infty))$ is 1-periodic in each of $x_1, x_2, \dots, x_N$. It is not difficult to show that if $\mathcal{F}(x,t) = 1/(2N)|t|^{2+1/N}(1 + |t|^{1/N})^{-1} \geq 0$, then F satisfies (WN) and (F1)–(F3) with $\sigma > \max\{1, N/2\}$, but it does not satisfies (AR) and (F0).

Example 1.5. $F(x,t) = h(x)\min\{|t|^{\varrho_1}, |t|^{\varrho_2}\}$, where $2 < \varrho_1 < \varrho_2 < 2^*$ and $h \in C(\mathbb{R}^N, (0, +\infty))$ is 1-periodic in each of $x_1, x_2, \dots, x_N$. Clearly, (F1)–(F3) and (WN) hold for F with $\sigma = \rho_1/(\rho_1 - 2) > \max\{1, N/2\}$, but (F0) fails.

Example 1.6. $F(x,t) = 2\sum_{i=1}^m |t|^{\beta_i} \sin^2(2\pi x_1)$, where $2^* > \beta_1 > \beta_2 > \dots > \beta_m \geq 2$. It is easy to see that $\mathcal{F}(x,t) = \sum_{i=1}^m (\beta_i - 2)|t|^{\beta_i} \sin^2(2\pi x_1) \geq 0$. Then F does not satisfies (AR) and (DL), but it satisfies (F3) with $\sigma = \beta_1/(\beta_1 - 2) > \max\{1, N/2\}$.

Example 1.7. $F(x,t) = a(8/5|t|^{13/4} - 4|t|^{11/4} + 9/2|t|^{9/4})$, where $a > 0$ and $N \leq 4$. By simple computation, one has $\mathcal{F}(x,t) = a|t|^{9/4}(\sqrt{|t|} - 3/4)^2 \geq 0$. Then F does not satisfies (AR) and (DL), but it satisfies (F3) with $\sigma = 12/5$ if $a \in (0, 8\Lambda_0/81)$.

The remaining of this article is organized as follows. In Section 2, some preliminary results are presented. The proofs of main results will be given in the last Section.

2. Variational setting and preliminaries

In this section, as in [33], we introduce the variational framework associated with problem (1.1). Throughout this paper, we denote by $\|\cdot\|_s$ the usual $L^s(\mathbb{R}^N)$ norm for $s \in [1, \infty)$ and C_i, $i \in \mathbb{N}$ for different positive constants. Let $\mathcal{A} = -\Delta + V$, then $\mathcal{A}$ is self-adjoint in $L^2(\mathbb{R}^N)$ with domain $\mathfrak{D}(\mathcal{A}) = H^2(\mathbb{R}^N)$. Let $\{\mathcal{E}(\lambda) : -\infty \leq \lambda \leq +\infty\}$ be the spectral family of $\mathcal{A}$, and $|\mathcal{A}|^{1/2}$ be the square root of $|\mathcal{A}|$. Set $\mathcal{U} = id - \mathcal{E}(0) - \mathcal{E}(0-)$. Then $\mathcal{U}$ commutes with $\mathcal{A}$, $|\mathcal{A}|$ and $|\mathcal{A}|^{1/2}$, and $\mathcal{A} = \mathcal{U}|\mathcal{A}|$ is the polar decomposition of $\mathcal{A}$ (see [10, Theorem 4.3.3]). Let $E_* = \mathfrak{D}(|\mathcal{A}|^{1/2})$, the domain of $|\mathcal{A}|^{1/2}$, then $\mathcal{E}(\lambda)E_* \subset E_*$ for all $\lambda \in \mathbb{R}$. On E_* define an inner product

$$(u,v)_0 = (|\mathcal{A}|^{1/2}u, |\mathcal{A}|^{1/2}v)_{L^2} + (u,v)_{L^2}, \quad \forall u, v \in E_*,$$

and the norm

$$\|u\|_0 = \sqrt{(u,v)_0}, \quad \forall u \in E_*,$$

where and in the sequel, $(\cdot,\cdot)_{L^2}$ denotes the usual $L^2(\mathbb{R}^N)$ inner product.

By (V1), we can choose $a_0 > 0$ such that

$$V(x) + a_0 > 0, \quad \forall x \in \mathbb{R}^N. \tag{2.1}$$

For $u \in C_0^\infty(\mathbb{R}^N)$, one has

$$\begin{aligned}
\|u\|_0^2 &= (|\mathcal{A}|u, u)_{L^2} + \|u\|_2^2 \\
&= ((\mathcal{A} + a_0)\mathcal{U}u, u)_{L^2} - a_0(\mathcal{U}u, u)_{L^2} + \|u\|_2^2 \\
&\leq \|\mathcal{U}(\mathcal{A} + a_0)^{1/2}u\|_2 \|(\mathcal{A} + a_0)^{1/2}u\|_2 + a_0\|\mathcal{U}u\|_2\|u\|_2 + \|u\|_2^2 \\
&\leq \|(\mathcal{A} + a_0)^{1/2}u\|_2^2 + (a_0 + 1)\|u\|_2^2 \\
&\leq (1 + 2a_0 + M)\|u\|_{H^1(\mathbb{R}^N)}^2
\end{aligned} \tag{2.2}$$

and

$$\begin{aligned} \|u\|_{H^1(\mathbb{R}^N)}^2 &\le ((\mathcal{A}+a_0+1)u,u)_{L^2} \\ &= (\mathcal{A}u,u)_{L^2} + (a_0+1)\|u\|_2^2 \\ &= (\mathcal{U}|\mathcal{A}|^{1/2}u, |\mathcal{A}|^{1/2}u)_{L^2} + (a_0+1)\|u\|_2^2 \\ &\le \||\mathcal{A}|^{1/2}u\|_2^2 + (a_0+1)\|u\|_2^2 \le (1+a_0)\|u\|_0^2, \end{aligned} \tag{2.3}$$

where $M = \sup_{x\in\mathbb{R}^N} |V(x)|$. Since $C_0^\infty(\mathbb{R}^N)$ is dense in $(E_*, \|\cdot\|_0)$ and $H^1(\mathbb{R}^N)$, thus

$$\frac{1}{1+a_0}\|u\|_{H^1(\mathbb{R}^N)}^2 \le \|u\|_0^2 \le (1+2a_0+M)\|u\|_{H^1(\mathbb{R}^N)}^2, \quad \forall u \in E_* = H^1(\mathbb{R}^N). \tag{2.4}$$

Let

$$E_*^- = \mathcal{E}(0)E_*, \quad E^+ = [\mathcal{E}(+\infty) - \mathcal{E}(0)]E_*,$$

and

$$(u,v)_* = \big(|\mathcal{A}|^{1/2}u, |\mathcal{A}|^{1/2}v\big)_{L^2}, \quad \|u\|_* = \sqrt{(u,u)_*}, \quad \forall u,v \in E_*. \tag{2.5}$$

Lemma 2.1 ([33, Lemma 3.1]). *Suppose that* (V1) *is satisfied. Then $E_* = E_*^- \oplus E^+$,*

$$(u,v)_* = (u,v)_{L^2} = 0, \quad \forall u \in E_*^-,\ v \in E^+, \tag{2.6}$$

$$\|u^+\|_*^2 \ge \Lambda_0\|u^+\|_2^2, \quad \|u^-\|_*^2 \le a_0\|u^-\|_2^2, \quad \forall u = u^- + u^+ \in E_* = E_*^- \oplus E^+, \tag{2.7}$$

where b_0 is given by (V1) *and a_0 by* (2.1).

It is easy to see that $\|\cdot\|_*$ and $\|\cdot\|_{H^1(\mathbb{R}^N)}$ are equivalent norms on E^+, and if $u \in E_*$ then $u \in E^+ \Leftrightarrow \mathcal{E}(0)u = 0$. Thus E^+ is a closed subset of $(E_*, \|\cdot\|_0) = H^1(\mathbb{R}^N)$. We introduce a new norm on E_*^- by setting

$$\|u\|_- = (\|u\|_*^2 + \|u\|_\varrho^2)^{1/2}, \quad \forall u \in E_*^-. \tag{2.8}$$

Let E^- be the completion of E_*^- with respect to $\|\cdot\|_-$. Then E^- is separable and reflexive, $E^- \cap E^+ = \{0\}$ and $(u,v)_* = 0$, $\forall u \in E^-$, $v \in E^+$. Let $E = E^- \oplus E^+$ and define norm $\|\cdot\|$ as follows

$$\|u\| = (\|u^-\|_-^2 + \|u^+\|_*^2)^{1/2}, \quad \forall u = u^- + u^+ \in E = E^- \oplus E^+. \tag{2.9}$$

It is easy to verify that $(E, \|\cdot\|)$ is a Banach space, and

$$\sqrt{\Lambda_0}\|u^+\|_2 \le \|u^+\|_* = \|u^+\|, \quad \|u^+\|_s \le \gamma_s\|u^+\|, \quad \forall u \in E,\ s \in [2, 2^*], \tag{2.10}$$

where $\gamma_s \in (0, +\infty)$ is imbedding constant.

Lemma 2.2 ([33, Lemma 3.2]). *() Suppose that* (V1) *is satisfied. Then the following conclusions hold.*

(i) *$E^- \hookrightarrow L^s(\mathbb{R}^N)$ for $\varrho \le s \le 2^*$;*
(ii) *$E^- \hookrightarrow H^1_{\rm loc}(\mathbb{R}^N)$ and $E^- \hookrightarrow\hookrightarrow L^s_{\rm loc}(\mathbb{R}^N)$ for $2 \le s < 2^*$;*
(iii) *For $\varrho \le s \le 2^*$, there exists a constant $C_s > 0$ such that*

$$\|u\|_s^s \le C_s\Big[\|u\|_*^s + \Big(\int_\Omega |u|^\varrho \mathrm{d}x\Big)^{s/\varrho} + \Big(\int_{\Omega^c} |u|^2 \mathrm{d}x\Big)^{s/2}\Big], \quad \forall u \in E^-, \tag{2.11}$$

where $\Omega \subset \mathbb{R}^N$ is any measurable set, $\Omega^c = \mathbb{R}^N \setminus \Omega$.

The following linking theorem is an extension of [15] (see also [36, Theorem 6.10]), which plays an important role in proving our main results.

Proposition 2.3 ([33, Theorem 2.4]). *Let X be real Banach space with $X = Y \oplus Z$, where Y and Z are subspaces of X, Y is separable and reflexive, and there exists a constant $\zeta_0 > 0$ such that the following inequality holds*

$$\|P_1 u\| + \|P_2 u\| \leq \zeta_0 \|u\|, \quad \forall u \in X, \tag{2.12}$$

where $P_1 : X \to Y$, $P_2 : X \to Z$ are the projections. Let $\{\mathfrak{f}_k\}_{k \in \mathbb{N}} \subset Y^*$ be the dense subset with $\|\mathfrak{f}_k\|_{Y^*} = 1$, and the τ-topology on X be generated by the norm

$$\|u\|_\tau := \max\Big\{\|P_2 u\|, \sum_{k=1}^{\infty} \frac{1}{2^k} |\langle \mathfrak{f}_k, P_1 u\rangle|\Big\}, \quad \forall u \in X. \tag{2.13}$$

Suppose that the following assumptions are satisfied:

- (H1) *$\varphi \in C^1(X, \mathbb{R})$ is τ-upper semi-continuous and $\varphi' : (\varphi_a, \|\cdot\|_\tau) \to (X^*, \mathcal{T}_{w^*})$ is continuous for every $a \in \mathbb{R}$;*
- (H2) *there exists $r > \rho > 0$ and $e \in Z$ with $\|e\| = 1$ such that*

$$\kappa := \inf \varphi(S_\rho) > 0 \geq \sup \varphi(\partial Q),$$

where

$$S_\rho = \{u \in Z : \|u\| = \rho\}, \quad Q = \{v + se : v \in Y,\ s \geq 0,\ \|v + se\| \leq r\}.$$

Then there exist $c \in [\kappa,\ \sup_Q \varphi]$ and a sequence $\{u_n\} \subset X$ satisfying

$$\varphi(u_n) \to c, \quad \|\varphi'(u_n)\|_{X^*}(1 + \|u_n\|) \to 0. \tag{2.14}$$

Such a sequence is called a Cerami sequence on the level c, or a $(C)_c$-sequence.

Let $X = E$, $Y = E^-$ and $Z = E^+$. Then (2.12) is obvious true by (2.9). Since E^- is separable and reflective subspace of E, then $(E^-)^*$ is also separable. Thus we can choose a dense subset $\{\mathfrak{f}_k\}_{k \in \mathbb{N}} \subset (E^-)^*$ with $\|\mathfrak{f}_k\|_{(E^-)^*} = 1$. Hence, it follows from (2.13) that

$$\|u\|_\tau := \max\Big\{\|u^+\|, \sum_{k=1}^{\infty} \frac{1}{2^k} |\langle \mathfrak{f}_k, u^-\rangle|\Big\}, \quad \forall u \in E. \tag{2.15}$$

It is clear that

$$\|u^+\| \leq \|u\|_\tau \leq \|u\|, \quad \forall u \in E. \tag{2.16}$$

By Lemma 2.2, it is easy to see that the functional Φ defined by (1.3) is of class C^1, moreover

$$\langle \Phi'(u), v\rangle = \int_{\mathbb{R}^N} (\nabla u \nabla v + V(x) uv) \mathrm{d}x - \int_{\mathbb{R}^N} f(x,u) v \mathrm{d}x, \ \ \forall u, v \in E. \tag{2.17}$$

This shows that critical points of Φ are the solutions of (1.1). Furthermore

$$\Phi(u) = \frac{1}{2}(\|u^+\|^2 - \|u^-\|_*^2) - \int_{\mathbb{R}^N} F(x,u) \mathrm{d}x, \quad \forall u = u^+ + u^- \in E^- \oplus E^+ = E, \tag{2.18}$$

$$\langle \Phi'(u), v\rangle = (u^+, v)_* - (u^-, v)_* - \int_{\mathbb{R}^N} f(x,u) v \mathrm{d}x, \quad \forall u, v \in E. \tag{2.19}$$

Lemma 2.4 ([33, Lemma 3.3]). *Suppose that* (V1), (F1) *are satisfied. Then $\Phi \in C^1(E, \mathbb{R})$ is τ-upper semi-continuous and $\Phi' : (\Phi_a, \|\cdot\|_\tau) \to (E^*, \mathcal{T}_{w^*})$ is continuous for every $a \in \mathbb{R}$.*

Lemma 2.5. *Suppose that* (V1), (F1), (WN) *are satisfied. Then*

$$\Phi(u) \geq \Phi(tu+w) + \frac{1}{2}\|w\|_*^2 + \frac{1-t^2}{2}\langle \Phi'(u), u\rangle - t\langle \Phi'(u), w\rangle,$$

for all $u \in E$, $w \in E^-$, $t \geq 0$.

The proof of the above lemma is the same as one of [34, Lemma 2.4], we omit it here. From Lemma 2.5, we have the following two corollaries.

Corollary 2.6. *Suppose that* (V1), (F1), (WN) *are satisfied, assume moreover* $u \in \mathcal{N}^-$. *Then*

$$\Phi(u) \geq \Phi(tu+w), \quad \forall w \in E^-,\ t \geq 0. \tag{2.20}$$

Corollary 2.7. *Suppose that* (V1), (F1), (WN) *are satisfied. Then*

$$\Phi(u) \geq \Phi(tu^+) + \frac{t^2\|u^-\|_*^2}{2} + \frac{1-t^2}{2}\langle \Phi'(u), u\rangle + t^2\langle \Phi'(u), u^-\rangle, \quad \forall u \in E,\ t \geq 0.$$

3. Proof of main results

Lemma 3.1. *Suppose that* (V1), (F1) *are satisfied. Then*

(i) *there exists* $\rho > 0$ *such that*

$$m := \inf_{\mathcal{N}^-} \Phi \geq \kappa := \inf\{\Phi(u) : u \in E^+, \|u\| = \rho\} > 0;$$

(ii) $\|u^+\| \geq \max\{\|u^-\|_*, \sqrt{2m}\}$ *for all* $u \in \mathcal{N}^-$;

Proof. (i) The first inequality is a direct consequence of Corollary 2.6, since for every $u \in \mathcal{N}^-$ there is $t > 0$ such that $\|tu^+\| = \rho$. For any $\varepsilon > 0$, (F1) implies the existence of $C_\varepsilon > 0$ such that

$$|F(x,t)| \leq \varepsilon|t|^2 + C_\varepsilon|t|^p, \quad \forall (x,t) \in \mathbb{R}^N \times \mathbb{R}, \tag{3.1}$$

which, together with (2.18) and Lemma 2.2, yields

$$\begin{aligned} \Phi(u) &= \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N} F(x,u)\mathrm{d}x \\ &\geq \frac{1}{2}\|u\|^2 - C_1(\varepsilon\|u\|^\varrho + C_\varepsilon\|u\|^p), \quad \forall u \in E^+. \end{aligned} \tag{3.2}$$

Choosing an appropriate ε we see that the second inequality holds for some $\rho > 0$.

(ii) For $u \in \mathcal{N}^-$, it follows from (i), (F1) and (2.18) that

$$m \leq \frac{1}{2}(\|u^+\|^2 - \|u^-\|_*^2) - \int_{\mathbb{R}^N} F(x,u)\mathrm{d}x \leq \frac{1}{2}(\|u^+\|^2 - \|u^-\|_*^2),$$

hence, $\|u^+\| \geq \max\{\|u^-\|_*, \sqrt{2m}\}$. □

Lemma 3.2 ([33, Lemma 4.2])**.** *Suppose that* (V1), (F1), (F2) *are satisfied. Let* $e \in E^+$ *with* $\|e\| = 1$. *Then there is a* $r_1 > 0$ *such that* $\sup \Phi(\partial Q) \leq 0$ *for* $r \geq r_1$, *where*

$$Q = \{w + se : w \in E^-, s \geq 0, \|w + se\| \leq r\}. \tag{3.3}$$

Lemma 3.3. *Suppose that* (V1), (F1), (F2), (WN) *are satisfied. Then for any* $u \in E^+ \setminus \{0\}$, $\mathcal{N}^- \cap (E^- \oplus \mathbb{R}^+ u) \neq \emptyset$, *i.e., there exist* $t(u) > 0$ *and* $w(u) \in E^-$ *such that* $t(u)u + w(u) \in \mathcal{N}^-$.

Proof. By Lemma 3.2, there exists $R > 0$ such that $\Phi(v) \le 0$ for $v \in (E^- \oplus \mathbb{R}^+ u) \setminus B_R(0)$. By Lemma 3.1 (i), $\Phi(tu) > 0$ for small $t > 0$. Thus, $0 < \sup \Phi(E^- \oplus \mathbb{R}^+ u) < \infty$. It is easy to see that Φ is weakly upper semi-continuous on $E^- \oplus \mathbb{R}^+ u$, therefore, $\Phi(u_0) = \sup \Phi(E^- \oplus \mathbb{R}^+ u)$ for some $u_0 \in E^- \oplus \mathbb{R}^+ u$. This u_0 is a critical point of $\Phi|_{E^- \oplus \mathbb{R} u}$, so $\langle \Phi'(u_0), u_0\rangle = \langle \Phi'(u_0), v\rangle = 0$ for all $v \in E^- \oplus \mathbb{R}\ u$. Consequently, $u_0 \in \mathcal{N}^- \cap (E^- \oplus \mathbb{R}^+ u)$. $\square$

Lemma 3.4. *Suppose that* (V1), (F1), (F2) *are satisfied. Then there exist a constant* $c_0 \in [\kappa, \sup \Phi(Q)]$ *and a sequence* $\{u_n\} \subset E$ *satisfying*

$$\Phi(u_n) \to c_0, \quad \|\Phi'(u_n)\|(1 + \|u_n\|) \to 0, \tag{3.4}$$

where Q *is defined by* (3.3)*.*

The above lemma is a direct corollary of Lemmas 2.4, 3.1 (i), 3.2 and Proposition 2.3.

Lemma 3.5. *Suppose that* (V1), (F1), (F2), (WN) *are satisfied. Then there exist a constant* $c_* \in [\kappa,\ m]$ *and a sequence* $\{u_n\} \subset E$ *satisfying*

$$\Phi(u_n) \to c_*, \quad \|\Phi'(u_n)\|(1 + \|u_n\|) \to 0. \tag{3.5}$$

Using the diagonal method and taking into account Lemmas 3.1, 3.2, 3.4 and Corollary 2.6, one can prove the above lemma by the same argument as in the proof of [32, Lemma 3.8].

Lemma 3.5 shows that a minimizing Cerami sequence for the energy functional can be found outside the Nehari-Pankov manifold, from which one can easily demonstrate a ground state solution of Nehari-Pankov type for problem (1.1).

Lemma 3.6. *Suppose that* (V1), (F1), (F2), (F3) *are satisfied. Then any sequence* $\{u_n\} \subset E$ *satisfying*

$$\Phi(u_n) \to c \ge 0, \quad \langle \Phi'(u_n), u_n^{\pm}\rangle \to 0, \tag{3.6}$$

is bounded in E*.*

Proof. The following argument is essentially contained in [35, Lemma 3.5], for the reader convenience we choose to write it in detail. It follows from (F3) and (3.6) that

$$C_2 \ge \Phi(u_n) - \frac{1}{2}\langle \Phi'(u_n), u_n\rangle = \int_{\mathbb{R}^N} \mathcal{F}(x, u_n)\mathrm{d}x \ge 0. \tag{3.7}$$

First we prove that $\{\|u_n\|_*\}$ is bounded. To this end, arguing by contradiction, suppose that $\|u_n\|_* \to \infty$. Let $v_n = u_n/\|u_n\|_*$, then $\|v_n\|_* = 1$. If

$$\delta := \limsup_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B(y,1)} |v_n^+|^2 \mathrm{d}x = 0,$$

then by Lions's concentration compactness principle [17] or [36, Lemma 1.21], $v_n^+ \to 0$ in $L^s(\mathbb{R}^N)$ for $2 < s < 2^*$. Set $\sigma' = \sigma/(\sigma - 1)$ and

$$\Omega_n := \big\{x \in \mathbb{R}^N :\ \frac{f(x, u_n)}{u_n} \le (\Lambda_0 - \delta_0)\big\}. \tag{3.8}$$

Clearly, $2\sigma' \in (2, 2^*)$ by the fact $\sigma > \max\{1, N/2\}$. It follows from (F1) and (2.10) that

$$\int_{\Omega_n} \frac{f(x, u_n)}{u_n}(v_n^+)^2 \mathrm{d}x \le (\Lambda_0 - \delta_0)\|v_n^+\|_2^2 \le (1 - \frac{\delta_0}{\Lambda_0})\|v_n^+\|_*^2 \le 1 - \frac{\delta_0}{\Lambda_0}. \tag{3.9}$$

On the other hand, by virtue of (F3), (3.7) and Hölder inequality, one can get that

$$\begin{aligned}\int_{\mathbb{R}^N\setminus\Omega_n}\frac{f(x,u_n)}{u_n}(v_n^+)^2\mathrm{d}x &\le \Big[\int_{\mathbb{R}^N\setminus\Omega_n}(\frac{f(x,u_n)}{u_n})^\sigma \mathrm{d}x\Big]^{1/\sigma}\|v_n^+\|_{2\sigma'}^2\\ &\le \Big[c_3\int_{\mathbb{R}^N\setminus\Omega_n}\mathcal{F}(x,u_n)\mathrm{d}x\Big]^{1/\sigma}\|v_n^+\|_{2\sigma'}^2\\ &\le (c_3C_2)^{1/\sigma}\|v_n^+\|_{2\sigma'}^2=o(1).\end{aligned}\tag{3.10}$$

Combining (3.9) and (3.10), and using (F1), (2.19) and (3.6), one has

$$\begin{aligned}1+o(1)&=\frac{\|u_n\|_*^2-\langle\Phi'(u_n),u_n^+-u_n^-\rangle}{\|u_n\|_*^2}\\ &=\int_{u_n\neq 0}\frac{f(x,u_n)}{u_n}[(v_n^+)^2-(v_n^-)^2]\mathrm{d}x\\ &\le\int_{\Omega_n}\frac{f(x,u_n)}{u_n}(v_n^+)^2\mathrm{d}x+\int_{\mathbb{R}^N\setminus\Omega_n}\frac{f(x,u_n)}{u_n}(v_n^+)^2\mathrm{d}x\\ &\le 1-\frac{\delta_0}{\Lambda_0}+o(1).\end{aligned}\tag{3.11}$$

This contradiction shows that $\delta>0$. The rest of the argument is the same as in the proof of [33, Lemma 4.4]. □

Note that condition (WN) is stronger than (Ta), then one gets directly the following lemma from [33, Lemma 4.4].

Lemma 3.7. *Suppose that* (V1), (F1), (F2), (WN) *are satisfied. Then any sequence* $\{u_n\}\subset E$ *satisfying* (3.6) *is bounded in* E.

Lemma 3.8 ([3, Corollary 2.3])**.** *Suppose that* (V1) *is satisfied. If* $u\subset E$ *is a weak solution of the Schrödinger equation*

$$-\Delta u+V(x)u=f(x,u),\quad x\in\mathbb{R}^N,\tag{3.12}$$

i.e.

$$\int_{\mathbb{R}^N}(\nabla u\nabla\psi+V(x)u\psi)\mathrm{d}x=\int_{\mathbb{R}^N}f(x,u)\psi\mathrm{d}x,\quad\forall\psi\in C_0^\infty(\mathbb{R}^N),\tag{3.13}$$

then $u_n\to 0$ *as* $|x|\to\infty$.

Proof of Theorem 1.1. Lemma 3.5 and 3.7 imply the existence of a bounded sequence $\{u_n\}\subset E$ satisfying (3.5). By (2.10) and Lemma 2.2 (i), $\|u_n\|_\varrho^\varrho+\|u_n\|_p^p$ is also bounded. If

$$\delta:=\limsup_{n\to\infty}\sup_{y\in\mathbb{R}^N}\int_{B(y,1)}|u_n^+|^2\mathrm{d}x=0,$$

then by Lions's concentration compactness principle, $u_n^+\to 0$ in $L^s(\mathbb{R}^N)$ for $2<s<2^*$. From (F1), (2.18), (2.19) and (3.5), one sees that

$$\begin{aligned}2c_*+o(1)&=\|u_n^+\|^2-\|u_n^-\|_*^2-2\int_{\mathbb{R}^N}F(x,u_n)\mathrm{d}x\\ &\le\|u_n^+\|^2=\int_{\mathbb{R}^N}f(x,u_n)u_n^+\mathrm{d}x+\langle\Phi'(u_n),u_n^+\rangle\\ &\le c_2\int_{\mathbb{R}^N}(|u_n|^{\varrho-1}+|u_n|^{p-1})|u_n^+|\mathrm{d}x+o(1)\end{aligned}$$

$$\le c_2(\|u_n\|_\varrho^{\varrho-1}\|u_n^+\|_\varrho + \|u_n\|_p^{p-1}\|u_n^+\|_p)\mathrm{d}x + o(1) = o(1)$$

which is a contradiction. Thus $\delta > 0$.

Going if necessary to a subsequence, we may assume the existence of $k_n \in \mathbb{Z}^N$ such that

$$\int_{B(k_n,1+\sqrt{N})} |u_n^+|^2\mathrm{d}x > \frac{\delta}{2}.$$

Let us define $v_n(x) = u_n(x+k_n)$ so that

$$\int_{B(0,1+\sqrt{N})} |v_n^+|^2\mathrm{d}x > \frac{\delta}{2}. \tag{3.14}$$

Since $V(x)$ and $f(x,t)$ are periodic in x, we have $\|v_n\| = \|u_n\|$ and

$$\Phi(v_n) \to c_* \in [\kappa,\ m], \quad \|\Phi'(v_n)\|(1+\|v_n\|) \to 0. \tag{3.15}$$

Passing to a subsequence, we have $v_n \rightharpoonup v_0$ in E, $v_n \to v_0$ in $L^s_{\rm loc}(\mathbb{R}^N)$ for $2 \le s < 2^*$ and $v_n \to v_0$ a.e. on $\mathbb{R}^N$. (3.14) implies that $v_0 \ne 0$. By a standard argument, one can prove that $\langle \Phi'(v_0), \psi\rangle = 0$ for any $\psi \in C_0^\infty(\mathbb{R}^N)$. It follows that $\Phi'(v_0) = 0$ since $C_0^\infty(\mathbb{R}^N)$ is dense in E. Then $v_0 \in \mathcal{N}^-$ and so $\Phi(v_0) \ge m$. On the other hand, by (3.15), (WN) and Fatou's Lemma, we have

$$\begin{aligned}
m \ge c_* &= \lim_{n\to\infty}\big[\Phi(v_n) - \frac{1}{2}\langle \Phi'(v_n), v_n\rangle\big]\\
&= \lim_{n\to\infty}\int_{\mathbb{R}^N}\big[\frac{1}{2}f(x,v_n) - F(x,v_n)\big]\mathrm{d}x\\
&\ge \int_{\mathbb{R}^N}\lim_{n\to\infty}\big[\frac{1}{2}f(x,v_n) - F(x,v_n)\big]\mathrm{d}x\\
&= \int_{\mathbb{R}^N}\big[\frac{1}{2}f(x,v_0) - F(x,v_0)\big]\mathrm{d}x\\
&= \Phi(v_0) - \frac{1}{2}\langle \Phi'(v_0), v_0\rangle = \Phi(v_0).
\end{aligned}$$

Then $\Phi(v_0) \le m$ and so $\Phi(v_0) = m = \inf_{\mathcal{N}^-}\Phi \ge \kappa$ by Lemma 3.1 (i). It follows from Lemma 3.8 that v_0 is a ground state solution of problem (1.1). $\square$

Proof of Theorem 1.2. Applying Lemmas 3.4 and 3.6, there exists a bounded sequence $\{u_n\} \subset E$ satisfying (3.4). Similar to the argument as in the proof of Theorem 1.1, we can show that $\Phi'(\bar{u}) = 0$ for some $\bar{u} \in E \setminus \{0\}$, i.e. $\mathcal{M} \ne 0$. Let $\hat{c} := \inf_{\mathcal{M}}\Phi$. By (F3), for any $u \in \mathcal{M}$, one has

$$\Phi(u) = \Phi(u) - \frac{1}{2}\langle \Phi'(u), u\rangle = \int_{\mathbb{R}^N}\mathcal{F}(x,u)\mathrm{d}x \ge 0;$$

therefore $\hat{c} \ge 0$. Let $\{u_n\} \subset \mathcal{M}$ such that $\Phi(u_n) \to \hat{c}$. Then $\langle \Phi'(u_n), v\rangle = 0$ for any $v \in E$. It follows from Lemma 3.6 that $\{u_n\}$ is bounded in E. The rest of the argument is the same as in the proof of Theorem 1.1 by using (F3) instead of (WN). $\square$

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