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BLOW UP AND QUENCHING FOR A PROBLEM WITH NONLINEAR BOUNDARY CONDITIONS

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Abstract. In this article, we study the blow up behavior of the heat equation $u_t = u_{xx}$ with $u_x(0,t) = u^p(0,t)$, $u_x(a,t) = u^q(a,t)$. We also study the quenching behavior of the nonlinear parabolic equation $v_t = v_{xx} + 2v_x^2/(1-v)$ with $v_x(0,t) = (1-v(0,t))^{-p+2}$, $v_x(a,t) = (1-v(a,t))^{-q+2}$. In the blow up problem, if u_0 is a lower solution then we get the blow up occurs in a finite time at the boundary $x = a$ and using positive steady state we give criteria for blow up and non-blow up. In the quenching problem, we show that the only quenching point is $x = a$ and v_t blows up at the quenching time, under certain conditions and using positive steady state we give criteria for quenching and non-quenching. These analysis is based on the equivalence between the blow up and the quenching for these two equations.

1. Introduction

In this article, we study the blow up and quenching problems with nonlinear boundary conditions.

Blow up problem. We study the behavior of solutions to the heat equation, with nonlinear boundary conditions,

$$\begin{gathered} u_t = u_{xx}, \quad 0 < x < a,\ 0 < t < T, \\ u_x(0,t) = u^p(0,t), \quad u_x(a,t) = u^q(a,t),\ 0 < t < T, \\ u(x,0) = u_0(x), \quad 0 \le x \le a, \end{gathered} \tag{1.1}$$

where p, q are positive constants and $T \le \infty$. The initial function $u_0(x)$ is a non-negative smooth function satisfying the compatibility conditions

$$u_0'(0) = u_0^p(0), \quad u_0'(a) = u_0^q(a).$$

We are interested in the occurrence of finite-time blow-up, i.e, the existence of a $T = T(u_0) < \infty$ such that

$$\sup_{x\in[0,a]} u(x,t) \to \infty \quad \text{as } t \to T.$$

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Quenching problem. If we use the transform $u = \frac{1}{1-v}$ in the problem (1.1), then we obtain the nonlinear parabolic equation, with nonlinear boundary conditions,

$$\begin{gathered}
v_t = v_{xx} + \frac{2v_x^2}{1-v}, \quad 0 < x < a,\ 0 < t < T, \\
v_x(0,t) = (1-v(0,t))^{-p+2}, \quad v_x(a,t) = (1-v(a,t))^{-q+2}, \quad 0 < t < T, \\
v(x,0) = v_0(x) = 1 - \frac{1}{u_0(x)}, \quad 0 \le x \le a,
\end{gathered} \tag{1.2}$$

where p, q are positive constants and $T \le \infty$. The initial function $v_0 : [0,a] \to (0,1)$ satisfies the compatibility conditions

$$v_0'(0) = (1-v_0(0))^{-p+2}, \quad v_0'(a) = (1-v_0(a))^{-q+2}.$$

A solution $v(x,t)$ of (1.2) is said to quench if there exists a finite time T such that

$$\lim_{t\to T^-} \max\{v(x,t) : 0 \le x \le a\} = 1.$$

For the rest of this article, we denote the quenching time of problem (1.2) with T.

Blow up problems with various boundary conditions have been studied extensively; see for example [4, 6, 7, 9, 10, 12, 16]. Lin and Wang [12] considered the problem

$$\begin{gathered}
u_t = u_{xx} + u^p, \quad 0 < x < 1,\ 0 < t < T, \\
u_x(0,t) = 0, \quad u_x(1,t) = u^q(1,t),\ 0 < t < T, \\
u(x,0) = u_0(x), \quad 0 \le x \le 1,
\end{gathered}$$

where $p, q > 0, T \le \infty$. They showed that the solutions have a finite time blow-up and obtained the exact blow-up rates for the necessary and sufficient conditions. They also proved that the blow-up will occur only at the boundary $x = 1$. Fu et al. [8] studied the same problem. Under certain conditions, they proved that the blow-up point occurs only at the boundary $x = 1$. Then, by applying the well-known method of Giga-Kohn, they derived the time asymptotic of solutions near the blow-up time. Finally, they proved that the blow-up was complete.

Since 1975, quenching problems with various boundary conditions have been studied extensively [2, 3, 5, 13, 14, 15]. Chan and Yuen [3] considered the problem

$$\begin{gathered}
u_t = u_{xx}, \quad \text{in } \Omega, \\
u_x(0,t) = (1-u(0,t))^{-p}, \quad u_x(a,t) = (1-u(a,t))^{-q}, \quad 0 < t < T, \\
u(x,0) = u_0(x), \quad 0 \le u_0(x) < 1, \quad \text{in } \overline{D},
\end{gathered}$$

where $a, p, q > 0$, $T \le \infty$, $D = (0,a)$, $\Omega = D \times (0,T)$. They showed that $x = a$ is the unique quenching point in finite time if u_0 is a lower solution, and u_t blows up at quenching time. Further, they obtained criteria for nonquenching and quenching by using the positive steady states. Ozalp and Selcuk [13] considered the problem

$$\begin{gathered}
u_t = u_{xx} + (1-u)^{-p}, \quad 0 < x < 1,\ 0 < t < T \\
u_x(0,t) = 0, \quad u_x(1,t) = (1-u(1,t))^{-q}, \quad 0 < t < T, \\
u(x,0) = u_0(x), \quad 0 < u_0(x) < 1, \quad 0 \le x \le 1.
\end{gathered}$$

They showed that $x = 1$ is the quenching point in finite time, if u_0 satisfies $u_{xx}(x,0) + (1-u(x,0))^{-p} \ge 0$ and $u_x(x,0) \ge 0$. Further they showed that u_t

blows up at quenching time. Furthermore, they obtained a quenching rate and a lower bound for the quenching time.

In Section 2, we investigate the blow up behavior of the problem (1.1). First, if u_0 is a lower solution, then we show that blow up occurs in finite time at the boundary $x = a$. Also, using positive steady state we get criteria for blow up and non-blow up. In Section 3, we investigate quenching behavior of the problem (1.2). We show that the only quenching point is $x = a$ and v_t blows up at the quenching time, under certain conditions. Also, using positive steady state we get criteria for quenching and non-quenching.

2. Blow up problem

In this section, we assume that $u_{xx}(x, 0) \geq 0$ in $(0, a)$. Now, we give some auxiliary results for the problem (1.1).

Definition 2.1. A function μ is called a lower solution of (1.1) if μ satisfies the following conditions:

$$\begin{gathered}
\mu_t - \mu_{xx} \leq 0, \quad 0 < x < a,\ 0 < t < T, \\
\mu_x(0, t) \geq \mu^p(0, t), \quad \mu_x(a, t) \leq \mu^q(a, t), \quad 0 < t < T, \\
\mu(x, 0) \leq u_0(x), \quad 0 \leq x \leq a.
\end{gathered}$$

When reverse the inequalities, we have an upper solution.

We have the following Theorem and Lemma for problem (1.1) (see [3]).

Theorem 2.2. *Let $u(x, t, u_0)$ and $h(x, t, h_0)$ be solutions of (1.1) with initial data given by $u_0(x)$ and $h_0(x)$, respectively. If $u_0 \leq h_0$, then $u(x, t, u_0) \leq h(x, t, h_0)$ on $[0, a] \times [0, T)$.*

Proof. For any $\tau < T$, let w be a solution of the problem

$$\begin{gathered}
w_{xx} - w + w_t = 0 \quad \text{in } (0, a) \times (0, \tau), \\
w(x, \tau) = g(x) \quad \text{on } [0, a], \\
w_x(0, t) = r(t)w(0, t), w_x(a, t) = s(t)w(a, t), \quad 0 < t < \tau,
\end{gathered}$$

where $g \in C^2(\overline{D})$ has compact support in $D, 0 \leq g \leq 1$, and r and s are smooth functions to be determined. By Lieberman [11], the solution w exists. By Andersen [1], there exists a constant k (depending on the length of the interval D) such that $0 \leq w \leq k$.

Now,

$$\begin{aligned}
&\int_0^a [(u(x, \tau) - h(x, \tau))w(x, \tau) - (u_0(x) - h_0(x))w(x, 0)]dx \\
&= \int_0^\tau \int_0^a \frac{\partial}{\partial\sigma}[(u(x, \sigma) - h(x, \sigma))w(x, \sigma)]\,dx\,d\sigma \\
&= \int_0^\tau \int_0^a [w(x, \sigma)\frac{\partial}{\partial\sigma}(u(x, \sigma) - h(x, \sigma)) + (u(x, \sigma) - h(x, \sigma))\frac{\partial}{\partial\sigma}w(x, \sigma)]\,dx\,d\sigma \\
&= \int_0^\tau \int_0^a \big[w(x, \sigma)\frac{\partial^2}{\partial x^2}(u(x, \sigma) - h(x, \sigma)) + (u(x, \sigma) - h(x, \sigma))\frac{\partial}{\partial\sigma}w(x, \sigma)\big]\,dx\,d\sigma \\
&= \int_0^\tau \{w(a, \sigma)\,[u^q(a, \sigma) - h^q(a, \sigma)] - w(0, \sigma)\,[u^p(0, \sigma) - h^p(0, \sigma)]
\end{aligned}$$

$$
\begin{aligned}
&- s(\sigma)\,[u(a,\sigma)-h(a,\sigma)]\,w(a,\sigma)+r(\sigma)\,[u(0,\sigma)-h(0,\sigma)]\,w(0,\sigma)\}d\sigma\\
&+\int_0^\tau\int_0^a (u(x,\sigma)-h(x,\sigma))(w_\sigma(x,\sigma)+w_{xx}(x,\sigma))\,dx\,d\sigma.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\int_0^a [(u(x,\tau)-h(x,\tau))g(x)-(u_0(x)-h_0(x))w(x,0)]dx\\
&=\int_0^\tau \{w(a,\sigma)\,[u^q(a,\sigma)-h^q(a,\sigma)-s(\sigma)\,[u(a,\sigma)-h(a,\sigma)]]\\
&\quad -w(0,\sigma)\,[u^p(0,\sigma)-h^p(0,\sigma)-r(\sigma)\,[u(0,\sigma)-h(0,\sigma)]]\}d\sigma\\
&\quad +\int_0^\tau\int_0^a (u(x,\sigma)-h(x,\sigma))w(x,\sigma)\,dx\,d\sigma.
\end{aligned}
$$

Let $r(\sigma)$ and $s(\sigma)$ be given by

$$
\begin{gathered}
r(\sigma)(u(0,\sigma)-h(0,\sigma))=u^p(0,\sigma)-h^p(0,\sigma),\\
s(\sigma)(u(a,\sigma)-h(a,\sigma))=u^q(a,\sigma)-h^q(a,\sigma).
\end{gathered}
$$

Since $u_0\le h_0$ and $w(x,0)\ge 0$, we have

$$
\int_0^a (u(x,\tau)-h(x,\tau))g(x)dx\le \int_0^\tau\int_0^a (u(x,\sigma)-h(x,\sigma))w(x,\sigma)\,dx\,d\sigma.
$$

Let

$$
(u(x,\sigma)-h(x,\sigma))^+=\max\{0,u(x,\sigma)-h(x,\sigma)\}.
$$

From, $0\le w\le k$, we obtain

$$
\int_0^a (u(x,\tau)-h(x,\tau))g(x)dx\le k\int_0^\tau\int_0^a (u(x,\sigma)-h(x,\sigma))^+\,dx\,d\sigma.
$$

Since $g\in C^2(\overline{D})$ has compact support in D and $0\le g\le 1$, we have

$$
\int_0^a (u(x,\sigma)-h(x,\sigma))^+dx\le k\int_0^\tau\int_0^a (u(x,\sigma)-h(x,\sigma))^+\,dx\,d\sigma.
$$

By the Gronwall inequality,

$$
\int_0^a (u(x,\sigma)-h(x,\sigma))^+dx\le 0,
$$

which gives $u(x,\tau)\le h(x,\tau)$ for any $\tau>0$. Thus, the theorem is proved. □

Lemma 2.3. *If $u_{xx}(x,0)\ge 0$ in $(0,a)$, then*
 (i) *$u_t>0$ in $(0,a)\times(0,T)$.*
 (ii) *$u_x>0$ in $(0,a)\times(0,T)$.*

Proof. (i) Since $u_{xx}(x,0)\ge 0$ in $(0,a)$, $u_0'(0)=u_0^p(0)$, $u_0'(a)=u_0^q(a)$ it follows from Definition 2.1 that $u_0(x)$ is a lower solution of the problem (1.1). The strong maximum principle implies that

$$
u(x,t)\ge u_0(x)\quad \text{in } (0,a)\times(0,T).
$$

Let h be a positive number less than T, and

$$
z(x,t)=u(x,t+h)-u(x,t).
$$

Then

$$
z_t=z_{xx}\quad \text{in } (0,a)\times(0,T-h),
$$

$$z(x,0) \geq 0 \quad \text{on } [0,a],$$
$$z_x(0,t) = p\xi^{p-1}(t)z(0,t), z_x(a,t) = q\eta^{q-1}(t)z(a,t), \quad 0 < t < T-h,$$

where $\xi(t)$ and $\eta(t)$ lie, respectively, between $u(0,t+h)$ and $u(0,t)$, and between $u(a,t+h)$ and $u(a,t)$. A proof similar to that of Theorem 2.2 shows that $z(x,t) \geq 0$. As $h \to 0$, we have $u_t \geq 0$ on $[0,a] \times (0,T)$.

Let $H = u_t$ in $[0,a] \times (0,T)$. Since

$$H_t - H_{xx} = 0 \quad \text{in } (0,a) \times (0,T),$$

it follows from the strong maximum principle that $H = u_t > 0$ in $(0,a) \times (0,T)$.

(ii) Since $u_x(0,t) = u^p(0,t) > 0$ and $u_{xx} = u_t > 0$ in $(0,a) \times (0,T)$. Then, u_x is an increasing function and so, $u_x(x,t) > 0$ in $(0,a) \times (0,T)$. □

Theorem 2.4. *Let u be a solution of the problem* (1.1)*, $f(u) = u^q$ and $q > 1$. We assume that*

$$\int_0^\infty \frac{ds}{f(s)} < \infty. \tag{2.1}$$

If u_0 is a lower solution, assumption (2.1) *is satisfied, $q \geq p$, then*

(a) *any positive solution of the problem* (1.1) *must blow up in a finite time T such that there exists a positive constant δ with*

$$T \leq \frac{1}{\delta}\frac{M_0^{-q+1}}{q-1},$$

where $M_0 = \max_{x\in[0,a]} u_0(x)$,

(b) *a blow up rate is obtained for t sufficiently close to T as*

$$\sup_{x\in[0,a]} u(x,t) \leq C(T-t)^{1/(-q+1)},$$

where $C = (\delta(q-1))^{1/(-q+1)}$.

Proof. Let us prove it by using [16, Theorems 1 and 2]. First, we define

$$J(x,t) = u_t(x,t) - \delta u^q(x,t) \quad \text{in } [0,a] \times [\tau,T),$$

where $\tau \in (0,T)$ and δ is a positive constant to be specified later. Then, $J(x,t)$ satisfies

$$J_t - J_{xx} = \delta q(q-1)u^{q-2}u_x^2 > 0 \quad \text{in } (0,a) \times (\tau,T),$$

since $q \geq 1$. $J(x,\tau) \geq 0$ by Lemma 2.3 (i), if δ is small enough. Further,

$$J_x(0,t) = pu^{p-1}(0,t)J(0,t) + (p-q)u^{p+q-1}(0,t) \leq pu^{p-1}(0,t)J(0,t),$$
$$J_x(a,t) = qu^{q-1}(a,t)J(a,t),$$

since $q \geq p$ and $t \in (\tau,T)$. By the maximum principle and Hopf's lemma for the parabolic equations, we obtain that $J(x,t) \geq 0$ for $(x,t) \in [0,a] \times [\tau,T)$. Thus, we get

$$u_t(x,t) \geq \delta u^q(x,t),$$

for $(x,t) \in [0,a] \times [\tau,T)$.

Integrating from t to T we obtain

$$\int_t^T \frac{u_s(x,s)}{u^q(x,s)} ds \geq \delta(T-t).$$

Let $u_0(x_0) = M_0 = \max_{x\in[0,a]} u(x,0)$. If x_0 is a blow up point and $\sup_{x_0\in[0,a]} u(x_0,T) \to \infty$ as $T \to \infty$, then

$$\int_{M_0}^{u(x_0,T)} \frac{ds}{f(s)} \geq \delta(T-t),$$

where $f(s) = s^q$. But if assumption (2.1) is satisfied, this leads to a contradiction. Therefore, any positive solution of the problem (1.1) must blow up in finite time T. Further, we get an estimate for finite blow up time as

$$T \leq \frac{1}{\delta}\frac{M_0^{-q+1}}{q-1}.$$

Furthermore, we get a blow up rate for t sufficiently close to T as

$$\sup_{x\in[0,a]} u(x,t) \leq C(T-t)^{1/(-q+1)},$$

where $C = (\delta(q-1))^{1/(-q+1)}$. □

Theorem 2.5. *If $q > 1$ and u_0 is a lower solution, then $x = a$ is the only blow up point.*

Proof. Define

$$J(x,t) = u_x - \varepsilon(x-b_1)u^q \text{ in } [b_1,b_2]\times[\tau,T),$$

where $b_1 \in [0,a)$, $b_2 \in (b_1,a]$, $\tau \in (0,T)$ and ε is a positive constant to be specified later. Then, $J(x,t)$ satisfies

$$J_t - J_{xx} = 2\varepsilon q u^{q-1}u_x + \varepsilon q(q-1)(x-b_1)u^{q-2}u_x^2 > 0$$

in $(b_1,b_2)\times[0,T)$. $J(x,\tau) \geq 0$ by Lemma 2.3 (ii), if ε is small enough. Further

$$J(b_1,t) = u_x(b_1,t) > 0,$$
$$J(b_2,t) = u_x(b_2,t) - \varepsilon(b_2-b_1)u^q > 0,$$

for $t \in (\tau,T)$. By the maximum principle, we obtain that $J(x,t) \geq 0$ for $(x,t) \in [b_1,b_2]\times[0,T)$. Namely, $u_x \geq \varepsilon(x-b_1)u^q$ for $(x,t)\in[b_1,b_2]\times[\tau,T)$. Integrating this with respect to x from b_1 to b_2, we have

$$u^{-q+1}(b_1,t) \geq u^{-q+1}(b_2,t) + \frac{\varepsilon(q-1)(b_2-b_1)^2}{2},$$
$$u(b_1,t) \leq \big[\frac{\varepsilon(q-1)(b_2-b_1)^2}{2}\big]^{\frac{1}{-q+1}} < \infty.$$

So u does not blow up in $[0,a)$. The proof is complete. □

Theorem 2.6. *If u_0 is a lower solution, $q > 1$ and $u_x(x,0) \geq xu^q(x,0)$ in $(0,a)$, then $x = a$ is the only blow up point, $a \leq 1$.*

Proof. Define $J(x,t) = u_x - xu^q$ in $[0,a]\times[0,T)$. Then, $J(x,t)$ satisfies

$$J_t - J_{xx} = 2qu^{q-1}u_x + q(q-1)xu^{q-2}u_x^2,$$

since $u_x > 0$, $J(x,t)$ cannot attain a negative interior minimum. On the other hand, $J(x,0) \geq 0$ from $u_x(x,0) \geq xu^q(x,0)$ in $(0,a)$ and

$$J(0,t) = u^p(0,t) > 0,$$
$$J(a,t) = (1-a)u^q(a,t) \geq 0,$$

if $a \leq 1$, for $t \in (0, T)$. By the maximum principle, we obtain that $J(x,t) \geq 0$, i.e. $u_x \geq x u^q$ for $(x,t) \in [0,a] \times [0,T)$. Integrating this with respect to x from x to a, we have

$$u^{-q+1}(x,t) \geq u^{-q+1}(a,t) + (q-1)\frac{a^2 - x^2}{2}$$
$$u(x,t) \leq \big[(q-1)\frac{a^2-x^2}{2}\big]^{\frac{1}{-q+1}} < \infty.$$

So u does not blow up in $[0, a)$. The proof is complete. □

Now, we first obtain criteria for the blow up and non-blow up using positive steady state. The proof of the following lemma and theorem is analogous to that of Chan and Yuen [3]. Let us consider the positive steady states of the problem (1.1):

$$U_{xx} = 0, \quad U_x(0) = U^p(0), \quad U_x(a) = U^q(a). \tag{2.2}$$

We have $U = I + nx$, where $n = I^p$, $n = (I + na)^q$. From these, we have

$$U = I + I^p x, \tag{2.3}$$

where

$$I^p = (I + I^p a)^q,$$

which gives

$$a(I) = I^{-p}(I^{p/q} - I). \tag{2.4}$$

We get

$$\lim_{I\to 0} a(I) = \lim_{I\to 0} \frac{I^{p/q} - I}{I^p} = \infty.$$

But, by using L'Hôpital's rule two times, we obtain

$$\lim_{I\to 0} a(I) = \lim_{I\to 0} \frac{(\frac{p}{q})(\frac{p}{q}-1)I^{p/q-2}}{p(p-1)I^{p-2}} = 0$$

for $p \neq 1$ and $q \neq 1$. If α is a positive number, which is very close to 0, then we get $a(\alpha) = 0$ and $a(1) = 0$. Also, If we select $p > q$, then we note that $a(I) > 0$ for $\alpha < l < 1$. Now, $a'(I) = 0$ implies

$$I = \big[\frac{q(1-p)}{p(1-q)}\big]^{\frac{q}{p-q}}. \tag{2.5}$$

We denote this value by A. From (2.4),

$$A = \big[\frac{q(1-p)}{p(1-q)}\big]^{\frac{p(1-q)}{p-q}} - \big[\frac{q(1-p)}{p(1-q)}\big]^{\frac{q(1-p)}{p-q}}.$$

Lemma 2.7. *(i) If $q \geq p$, then the steady-state problem (2.2) does not have a positive solution.*

(ii) If $p > q$, then (2.2) has a solution u if and only if $0 < a \leq A$. Furthermore, if $0 < a < A$, then there exist two positive solutions; if $a = A$, then there exists exactly one positive solution.

Proof. (i) For $a(I) > 0$, we have

$$a(I) = I^{-p+p/q} - I^{-p+1}$$

which is impossible for $q \geq p$.

(ii) Since $a(\alpha) = 0 = a(1)$ and $a(I) > 0$ for $\alpha < l < 1$, the graph of $a(I)$ is concave downwards with maximum attained at A. Thus for $p > q$, the problem

(2.2) has a solution if and only if $0 < a \le A$. To each $a \in (0, A)$, there are exactly two values of I. If $a = A$, then I is given by (2.5). $\square$

Theorem 2.8. *(a) If $p > q$ and $a \in (0, A)$, then u exists globally, provided $u_0 \le U(0)$.*

(b) Suppose that the assumptions of Theorem 2.4 hold. Then, u blows up in a finite time and $x = a$ is the only blow up point. Further, if $u_x(x, 0) \ge xu^q(x, 0)$ in $(0, a)$, then $a \le 1$.

Proof. (a) By Theorem 2.2, $u \le U$. Hence u exists globally.

(b) By Lemma 2.3 (i), $u_t > 0$ in $(0, a) \times (0, T)$. If u does not blow up in a finite time, then u converges to U which by Lemma 2.7 (i), does not exist for $q \ge p$. This contradiction and Theorem 2.4 shows that u blows up in a finite time for $q \ge p$. Further, from Theorem 2.5, $x = a$ is the only blow up point. Furthermore, from Theorem 2.6, if $u_x(x, 0) \ge xu^q(x, 0)$ in $(0, a)$, then $a \le 1$. The proof is complete.. $\square$

3. Quenching problem

The equivalence between the blow-up problem and the quenching problem is well known, for example see [5, 14]. Using transform $u = 1/(1 - v)$ in problem (1.1), we obtain the quenching problem (1.2). Then (1.2) has three heat sources for $p, q > 2$. We easily get quenching properties this difficult problem via (1.1). First, we give an auxiliary results for (1.2).

Remark 3.1. (i) Let u and h be solutions of the problem (1.1) and v and k be solutions of the problem (1.2). We let $u = \frac{1}{1-v}$, $h = \frac{1}{1-k}$. From Theorem 2.2, If $v_0 \le k_0 < 1$, then $v(x, t, v_0) \le k(x, t, k_0)$ on $[0, a] \times [0, T]$.

(ii) Let u be a solution of the problem (1.1) and v be a solution of the problem (1.2). We define $u = \frac{1}{1-v}$. If u_0 is a lower solution of the problem (1.1), then we known the following results from Lemma 2.3:

$$u_t > 0, \quad u_x > 0, \quad u_{xx} > 0 \quad \text{in } (0, a) \times (0, T).$$

Similarly, we obtain

$$v_t > 0, \quad v_x > 0 \quad \text{in } (0, a) \times (0, T).$$

Theorem 3.2. *If $u_x(x, 0) \ge u^2(x, 0)$ in $[0, a]$ and $p, q \ge 2$ in the problem (1.1), then $x = a$ is the only quenching point of problem (1.2).*

Proof. Let $M(x, t) = u_x(x, t) - u^2(x, t)$ in $[0, a] \times [0, T)$ and $M(x, t)$ satisfies

$$\begin{gathered} M_t - M_{xx} = 2u_x^2(x, t) > 0 \quad \text{in } (0, a) \times [0, T), \\ M(0, t) = u^p(0, t) - u^2(0, t) \ge 0, \quad 0 < t < T, \\ M(a, t) = u^q(a, t) - u^2(a, t) \ge 0, \quad 0 < t < T, \\ M(x, 0) \ge 0, \quad 0 \le x \le a, \end{gathered}$$

with $u_x(x, 0) \ge u^2(x, 0)$ in $[0, a]$. By the maximum principle, we obtain $u_x(x, t) \ge u^2(x, t)$ in $[0, a] \times [0, T)$. Then, we have

$$v_x(x, t) = \frac{u_x(x, t)}{u^2(x, t)} \ge 1 \quad \text{in } [0, a] \times [0, T).$$

Let $\varepsilon \in (0,a)$, integrating this with respect to x from $a-\varepsilon$ to a, we have
$$v(a-\varepsilon,t) \le v(a,t) - \varepsilon \le 1-\varepsilon.$$
So v does not quench in $[0,a)$. The proof is complete. $\square$

Theorem 3.3. *If $\lim_{t\to T} v(a,t) = 1$ for some finite time T, then v_t blows up.*

Proof. Suppose that v_t is bounded on $[0,a]\times[0,T)$. Then, there exists a positive constant M such that $v_t < M$, that is
$$v_{xx} + \frac{2v_x^2}{1-v} < M.$$
Integrating with respect to x from 0 to a, we have
$$\begin{gathered}
\int_0^a \frac{v_{xx}}{v_x}dx + \int_0^a \frac{2v_x}{1-v}dx < \int_0^a \frac{M}{v_x}dx \\
\ln\frac{v_x(a,t)}{v_x(0,t)} - 2\ln\Big(\frac{1-v(0,t)}{(1-v(a,t)}\Big) < \int_0^a \frac{M}{v_x}dx \\
\ln\frac{(1-v(0,t))^p}{(1-v(a,t))^q} < \int_0^a \frac{M}{v_x}dx.
\end{gathered}$$
As $t \to T^-$, the left-hand side tends to infinity, while the right-hand side is finite. This contradiction shows that v_t blows up at the quenching point $x = a$. $\square$

As in Section 2, let us consider the positive steady states of problem (1.2).
$$V_{xx} = -\frac{2V_x^2}{1-V}, \quad V_x(0) = (1-V(0))^{-p+2}, \quad V_x(a) = (1-V(a))^{-q+2}. \tag{3.1}$$
Dividing by V_x and integrating with respect to x, we have $V(x) = 1 - \frac{1}{cx+d}$, where $c = d^p$. From these, we have
$$V = 1 - \frac{1}{d^p x + d}, \tag{3.2}$$
where $c = (ca+d)^q$, which gives
$$a(d) = d^{-p}(d^{p/q} - d). \tag{3.3}$$
We obtain
$$\lim_{d\to 0} a(d) = \lim_{d\to 0}\frac{d^{p/q}-d}{d^p} = \infty.$$
But, by using L'Hôpital's rule two times, we obtain
$$\lim_{d\to 0} a(d) = \lim_{d\to 0}\frac{(\frac{p}{q})(\frac{p}{q}-1)d^{p/q-2}}{p(p-1)d^{p-2}} = 0$$
for $p \ne 1$ and $q \ne 1$. If β is a positive number, which is very close to 0, then we get $a(\beta) = 0$ and $a(1) = 0$. Also, If we select $p > q$, then we note that $a(d) > 0$ for $\beta < d < 1$. Now, $a'(d) = 0$ implies
$$d = \Big[\frac{q(1-p)}{p(1-q)}\Big]^{\frac{q}{p-q}}. \tag{3.4}$$
We denote this value by A. From (3.3),
$$A = \Big[\frac{q(1-p)}{p(1-q)}\Big]^{\frac{p(1-q)}{p-q}} - \Big[\frac{q(1-p)}{p(1-q)}\Big]^{\frac{q(1-p)}{p-q}}.$$

Lemma 3.4. *(i) If $q \geq p$, then the steady-state problem* (3.1) *does not have a positive solution.*

(ii) If $p > q$, then it has a solution v if and only if $0 < a \leq A$. Furthermore, if $0 < a < A$, then there exist two positive solutions; if $a = A$, then there exists exactly one positive solution.

Proof. (i) For $a(d) > 0$, we have

$$a(d) = d^{-p+p/q} - d^{-p+1}$$

which is impossible for $q \geq p$.

(ii) Since $a(\beta) = 0 = a(1)$ and $a(d) > 0$ for $\beta < d < 1$, the graph of $a(d)$ is concave downwards with maximum attained at A. Thus for $p > q$, the problem (3.1) has a solution if and only if $0 < a \leq A$. To each $a \in (0, A)$, there are exactly two values of d. If $a = A$, then d is given by (3.4). □

Theorem 3.5. *(a) If $p > q$ and $a \in (0, A)$, then v exists globally, provided $v_0 \leq V(0)$.*

(b) Suppose that the assumptions of Theorem 3.2 holds. If $q \geq p$, then $x = a$ is the only quenching point. Further, if $\lim_{t\to T} v(a, t) = 1$ for some finite time T, then v_t blows up.

Proof. (a) By Remark 3.1 (i), $v \leq V$. Hence v exists globally.

(b) By Remark 3.1 (ii), $v_t > 0$ on $(0, a) \times (0, T)$. If v does not quench in a finite time, then v converges to V which does not exist for $q \geq p$ by Lemma 3.4 (i). This contradiction shows that v quenches for $q \geq p$. Further, from Theorem 3.2, $x = a$ is the only quenching point. Furthermore, from Theorem 3.3, if $\lim_{t\to T} v(a, t) = 1$ for some finite time T, then v_t blows up. The proof is complete. □

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