MULTIPLE SOLUTIONS FOR KIRCHHOFF TYPE PROBLEM NEAR RESONANCE

SHU-ZHI SONG, CHUN-LEI TANG, SHANG-JIE CHEN

ABSTRACT. Based on Ekeland’s variational principle and the mountain pass theorem, we show the existence of three solutions to the Kirchhoff type problem

\[-\left(a + b \int_{\Omega} |\nabla u|^2 dx\right) \Delta u = b\mu u^3 + f(x, u) + h(x), \quad \text{in } \Omega,\]

\[u = 0, \quad \text{on } \partial \Omega.\]

Where the parameter \(\mu\) is sufficiently close, from the left, to the first nonlinear eigenvalue.

1. INTRODUCTION AND STATEMENT OF MAIN RESULT

The articles shows the existence of multiple solutions for the Kirchhoff type problem with Dirichlet boundary condition,

\[-\left(a + b \int_{\Omega} |\nabla u|^2 dx\right) \Delta u = b\mu u^3 + f(x, u) + h(x), \quad \text{in } \Omega,\]

\[u = 0, \quad \text{on } \partial \Omega,\]

where \(\Omega\) is a bounded domain in \(\mathbb{R}^N (N = 1, 2, 3)\) with a smooth boundary \(\partial \Omega\), \(a \geq 0, b > 0\) are real constants and \(\mu\) is a nonnegative parameter. Assume that \(f \in C(\bar{\Omega} \times \mathbb{R}, \mathbb{R})\) satisfies the sublinear growth condition:

\[(F1) \lim_{|t| \to \infty} \frac{f(x, t)}{t^3} = 0, \quad \text{uniformly for } x \in \Omega.\]

Problem \((1.1)\) can be looked on as a perturbed problem which was first studied by Mawhin and Schmit[6], related to the two-point boundary value equation

\[-u'' - \lambda u = f(x, u) + h, \quad u(0) = u(\pi) = 0.\]  \((1.2)\)

Specifically, on the assumption: \(\lambda < \lambda_1\) is sufficiently near to \(\lambda_1\) (\(\lambda_1\) is the first eigenvalue of the corresponding linear problem) and \(f\) is bounded and satisfies a sign condition, the existence of three solutions to equation \((1.2)\) was proved in \([6]\).

Later, various papers related to the result appeared. We mention for example, \([1, 3, 4, 5, 8]\). Ma, Ramos and Sanchez [3] considered the boundary-value problem for

\[\Delta u + \lambda u + f(x, u) = h(x)\]

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defined on a bounded open set $\Omega \subset \mathbb{R}^N$. As the parameter $\lambda$ is sufficiently close to $\lambda_1$ from the left, there exist three solutions on both Dirichlet boundary conditions and Neumann boundary conditions. In addition, similar to the results in the linear case, the existence of three solutions was proved to the perturbed $p$-Laplacian equation in a bounded domain. Further consideration to the perturbed $p$-Laplacian equation in a bounded domain can be found in [4]. As for extension to the the perturbed $p$-Laplacian equation in the whole space $\mathbb{R}^N$, we refer to [5]. These results were also extended to some elliptic systems with the Dirichlet boundary conditions, refer to [8]. More recently, the authors in [1] extended these conclusions to some degenerate quasilinear elliptic systems with the Dirichlet boundary conditions. By analogy to the results mentioned above, we expect that problem (1.1) has at least three solutions as the parameter $\mu < \mu_1$ is sufficiently close to $\mu_1$. Here $\mu_1$ is the first eigenvalue of the eigenvalue problem
\begin{equation}
-\|u\|^2 \Delta u = \mu u^3, \quad \text{in } \Omega,
\end{equation}
\begin{equation}
u = 0, \quad \text{on } \partial \Omega.
\end{equation}

Let $H = H^1_0(\Omega)$ be the Hilbert space equipped with the norm $\|u\| = (\int_\Omega |\nabla u|^2 dx)^{1/2}$ and $\|u\|_{L^s} = (\int_\Omega |u|^s dx)^{1/s}$ denote the norm of $L^s(\Omega)$. As shown in [9] and [13], the first nonlinear eigenvalue $\mu_1 > 0$ is simple and has a eigenfunction $\psi_1 > 0$ with $\|\psi_1\|_{L^4} = 1$. Specifically, $\mu_1$ can be characterized by
\begin{equation}
\mu_1 = \inf \{ \|u\|^4 : u \in H, \int_\Omega |u|^4 dx = 1 \}.
\end{equation}

Now we are in a position to state our result.

**Theorem 1.1.** Suppose $f$ satisfies (F1) and the following conditions:
\begin{enumerate}
\item[(F2)]
\begin{equation}
\lim_{|t| \to \infty} \int_\Omega F(x,t\psi_1) - \frac{a}{2} \sqrt{\bar{\mu}_1} t^2 = +\infty, \quad \text{uniformly for } x \in \Omega,
\end{equation}
\end{enumerate}
where $F(x,t) = \int_0^t f(x,s) ds$.
\begin{enumerate}
\item[(H1)]
$h \in L^2(\Omega)$ and $\int_\Omega h(x)\psi_1(x) dx = 0$.
\end{enumerate}
Then (1.1) has at least three solutions if $\mu < \mu_1$ is sufficiently close to $\mu_1$.

Many authors have studied the Kirchhoff type equation in a bounded domain by applying variational methods. For example, they consider Kirchhoff type problem
\begin{equation}
-(a + b \int_\Omega |\nabla u|^2 dx) \Delta u = g(x,u), \quad \text{in } \Omega,
\end{equation}
\begin{equation}
u = 0, \quad \text{on } \partial \Omega,
\end{equation}
assuming that
\begin{equation}
\lim_{|t| \to \infty} \frac{4G(x,t)}{bt^4} = \mu, \quad \text{uniformly in } x \in \Omega.
\end{equation}
where $G(x,t) = \int_0^t g(x,s) ds$. For the case $\mu < \mu_1$ in (1.5), the Euler functional corresponding to (1.4) is coercive. For the case $\mu = \mu_1$ in (1.5), that is, problem (1.4) is resonance at the first nonlinear eigenvalue $\mu_1$, the Euler functional corresponding to (1.4) is still coercive, together with the assumption that $\lim_{|t| \to \infty} [g(x,t) t - 4G(x,t)] = +\infty$. So, the existence of weak solution for equation (1.4) is obtained based on the Least Action Principle (refer to [11, 12, 13]).
Furthermore, provided \( g \) with some conditions at zero, positive solution was obtained based on the topological degree argument (refer to [2]), multiple solutions are found by means of invariant sets of descent flow method (refer to [11, 13]), or the Local Linking Theorem (refer to [12]). Our result is different from the results in [2, 11, 12, 13] since we deal with the perturbation problem near to \( \mu_1 \) and all hypotheses on \( f \) are just at infinity.

2. Proof of main result

We begin with some standard facts upon the variational formulation of problem (1.1). Let \( I_\mu : H \rightarrow \mathbb{R} \) be the functional defined by

\[
I_\mu(u) = \frac{a}{2} \|u\|^2 + \frac{b}{4} \|u\|^4 - \frac{b\mu}{4} \int_\Omega |u|^4 \, dx - \int_\Omega F(x, u) \, dx - \int_\Omega hudx.
\]

Since \( f \) satisfies the sublinear growth condition (F1), it is not difficult to verify that \( I_\mu \in C^1(H, \mathbb{R}) \). Furthermore, finding weak solutions of (1.1) is equivalent to finding critical points of functional \( I_\mu \) in \( H \).

Since \( \Omega \) is a bounded domain in \( \mathbb{R}^N \) \((N = 1, 2, 3)\), the embedding \( H \hookrightarrow \rightarrow L^s(\Omega) \) is continuous for \( s \in [1, 2^*) \), compact for \( s \in [1, 2^*) \),

\[
2^* = \begin{cases} \frac{2N}{N-2}, & N = 3, \\ +\infty, & N = 1, 2. \end{cases}
\]

Hence, for \( s \in [1, 2^*) \), there exists \( \tau_s > 0 \) such that

\[
\|u\|_{L^s} \leq \tau_s \|u\|, \quad \forall u \in H. \tag{2.1}
\]

We will prove the result by using Ekeland’s variational principle [7, Theorem 4.1] and a mountain pass theorem [10]. For the convenience of readers, we state the mountain pass theorem as follows.

**Theorem 2.1** ([10, Corollary 1]). Consider a real Banach space \( X \) and a function \( I \in C^1(X, \mathbb{R}) \). If the (PS) condition holds and if \( I \) has two different local minimum points, then \( I \) possesses a third critical point.

To prove our theorem, using critical point theory, we need the Palais-Smale compactness.

**Lemma 2.2.** Assume that (F1) holds. Then any bounded (PS) sequence of \( I_\mu \) has a convergent subsequence in \( H \).

**Proof.** Let \( \{u_n\} \subset H \) be a bounded (PS) sequence of \( I_\mu \); that is,

\[
\|u_n\| \leq c, \quad |I_\mu(u_n)| \leq c, \quad \|I_\mu'(u_n)\| \rightarrow 0, \tag{2.2}
\]

where \( c \) denotes positive constant. By the reflexivity of \( H \), we can assume that there exists \( u \in H \) such that

\[
u_n \rightharpoonup u \quad \text{weakly in } H, \quad u_n \rightarrow u \quad \text{strongly in } L^p(\Omega) \quad (1 \leq p < 2^*). \tag{2.3}
\]

It follows from (F1) that for any \( \varepsilon > 0 \), there exists \( M_\varepsilon > 0 \) such that

\[
|f(x, t)| \leq b|t|^3 + M_\varepsilon, \quad \forall (x, t) \in \Omega \times \mathbb{R}. \tag{2.5}
\]
We can now put together the results in (2.1), (2.2), (2.4) and (2.5) to conclude that
\[
\left| \int_{\Omega} f(x, u_n)(u - u_n) dx \right| \leq \int_{\Omega} |f(x, u_n)| |u - u_n| dx
\]
\[
\leq \int_{\Omega} (bc|u_n|^3 + M_{\varepsilon}) |u - u_n| dx
\]
\[
\leq b\varepsilon \|u_n\|_{L^4}^3 \|u - u_n\|_{L^4} + M_{\varepsilon} \|\Omega\|^{1/2} \|u - u_n\|_{L^2}
\]
\[
\leq b\varepsilon \|u_n\|_{L^4}^3 \|u - u_n\|_{L^4} + M_{\varepsilon} \|\Omega\|^{1/2} \|u - u_n\|_{L^2}
\]
\[
\leq c(\|u - u_n\|_{L^4} + \|u - u_n\|_{L^2}) \to 0, \quad \text{as } n \to \infty,
\]
where \(c = \max\{b\varepsilon, M_{\varepsilon} \|\Omega\|^{1/2}\}\), and \(|\Omega|\) is the measure of \(\Omega\). Similarly, we may deduce that
\[
\int_{\Omega} (|u_n|^2 u_n(u - u_n) - |u|^2 u(u - u_n)) dx \to 0, \quad \text{as } n \to \infty. \quad (2.7)
\]

From (2.2) and (2.4), we have
\[
\langle I'_\mu(u_n) - I'_\mu(u), u - u_n \rangle \to 0, \quad \text{as } n \to \infty,
\]
which combining with (2.6), implies \(\|u_n\| \to \|u\|\) as \(n \to \infty\). It follows from (2.3) that \(u_n \to u\) in \(H\).

Set
\[
V = \{ v \in H : \int_{\Omega} \psi_1^3 v dx = 0 \}.
\]

From the simplicity of \(\mu_1\) we have \(H = \text{span}\{\psi_1\} \oplus V\). We introduce the quantity
\[
\mu_V = \inf \left\{ \|u\|^4 : u \in V, \|u\|_{L^4}^4 = 1 \right\}.
\]

Then
\[
\|u\|^4 \geq \mu_V \|u\|_{L^4}^4, \quad \forall u \in V, \quad (2.8)
\]
and we have the following result.

**Lemma 2.3.** \(\mu_1 < \mu_V\).

**Proof.** It is evident from (1.3) that \(\mu_1 \leq \mu_V\). Assume, by contradiction, that \(\mu_1 = \mu_V\). Then there exists a sequence \(\{v_n\} \subseteq V\) such that \(\|u_n\|_{L^4} = 1\) for all \(n \geq 1\), and \(\|u_n\|^4 \to \mu_V = \mu_1\). Since the sequence \(\{u_n\}\) is bounded in \(H\), we may assume that
\[
u_n \to u \quad \text{weakly in } H, \quad u_n \to u \quad \text{strongly in } L^4(\Omega). \quad (2.9)
\]

Thus, one has
\[
\|u\|_{L^4} = \lim_{n \to +\infty} \|u_n\|_{L^4} = 1,
\]
\[
\mu_1 \leq \|u\|^4 \leq \liminf_{n \to -\infty} \|u_n\|^4 = \lim_{n \to -\infty} \|u_n\|^4 = \mu_1.
\]

So, \(\|u\|_{L^4} = 1\) and \(\|u\|^4 = \mu_1\). This implies \(u = \pm \psi_1\).

On the other hand, from \(\{u_n\} \subseteq V\) it follows that \(\int_{\Omega} \psi_1^3 u_n = 0\) for all \(n \geq 1\). Combining this with (2.9) and Hölder’s inequality, we have
\[
\left| \int_{\Omega} \psi_1^3 u dx \right| = \left| \int_{\Omega} \psi_1^3 u dx - \int_{\Omega} \psi_1^3 u_n dx \right| = \left| \int_{\Omega} \psi_1^3 (u - u_n) dx \right|
\]
\[
\leq \int_{\Omega} \left| \psi_1^3 (u - u_n) \right| dx.
\]
This is in direct contradiction to the fact \( u = \pm \psi_1 \). Hence \( \mu_1 < \mu_V \).

**Proof of Theorem 1.1.** We shall divide the proof into four steps.

**Step 1.** The functional \( I_\mu \) is bounded below in \( H \) and \( V \) and even coercive in \( H \) and \( V \). More specifically, there is a constant \( \alpha \), independent of \( \mu \), such that \( \inf_V I_\mu \geq \alpha \). From (2.5), we obtain

\[
|F(x, t)| \leq \frac{b_\varepsilon}{4} |t|^4 + M_L |t|, \quad \forall (x, t) \in \Omega \times \mathbb{R}.
\]

(2.10)

It follows from (2.1), (2.10) and Hölder inequality that

\[
I_\mu(u) \geq \frac{a}{2} \|u\|^2 + \frac{b}{4} \|u\|^4 - \frac{b(\mu + \varepsilon)}{4} \int_{\Omega} u^4 dx - (M_\varepsilon |\Omega|^{1/2} + \|h\|_{L^2}) \|u\|_{L^2}
\]

\[
\geq \frac{b}{4}(1 - \frac{\mu + \varepsilon}{\mu_1}) \|u\|^4 - \tau_2(M_\varepsilon |\Omega|^{1/2} + \|h\|_{L^2}) \|u\|, \quad \forall u \in H.
\]

Note that \( \mu < \mu_1 \). Then, for \( 0 < \varepsilon < \mu_1 - \mu \), \( I_\mu \) is bounded below and even coercive in \( H \). Similarly, for \( 0 < \varepsilon < \mu_V - \mu_1 \), (2.1), (2.8), (2.10) and Hölder inequality lead to

\[
I_{\mu_1}(v) \geq \frac{a}{2} \|v\|^2 + \frac{b}{4} \|v\|^4 - \frac{b(\mu + \varepsilon)}{4} \int_{\Omega} v^4 dx - (M_\varepsilon |\Omega|^{1/2} + \|h\|_{L^2}) \|v\|_{L^2}
\]

\[
\geq \frac{b}{4}(1 - \frac{\mu_1 + \varepsilon}{\mu_V}) \|v\|^4 - \tau_2(M_\varepsilon |\Omega|^{1/2} + \|h\|_{L^2}) \|v\|, \quad \forall v \in V,
\]

which implies that \( I_{\mu_1} \) is bounded below and coercive in \( V \). Noting that \( I_\mu \geq I_{\mu_1} \) for all \( \mu < \mu_1 \), we deduce \( I_\mu \) is coercive in \( V \) and

\[
\inf_V I_\mu \geq \alpha := \inf_V I_{\mu_1}.
\]

**Step 2.** If \( \mu < \mu_1 \) is sufficiently close to \( \mu_1 \), there exist two constants \( t^-, t^+ \) with \( t^- < 0 < t^+ \) such that \( I_\mu(t^\pm \psi_1) < \alpha \). Noting \( \|\psi_1\|_{L^4} = 1, \|\psi_1\|^4 = \mu_1 \) and then combining this with (H1), for \( t \in \mathbb{R} \), we have

\[
I_\mu(t\psi_1) = \frac{at^2}{2} \|\psi_1\|^2 + \frac{bt^4}{4} \|\psi_1\|^4 - \frac{b\mu t^4}{4} \int_{\Omega} \psi_1^4 dx - \int_{\Omega} F(x, t\psi_1) dx
\]

\[
= \frac{b(\mu_1 - \mu)}{4} t^4 - \left( \int_{\Omega} F(x, t\psi_1) dx - \frac{a}{2} \sqrt{\mu_1 t^2} \right).
\]

From (F2), taking a constant \( t^+ \) with \( t^+ > 0 \) large enough, we obtain

\[
\int_{\Omega} F(x, t^+ \psi_1) dx - \frac{a}{2} \sqrt{\mu_1 (t^+)^2} > - \alpha + 1.
\]

The above inequality reduces to

\[
I_\mu(t^+ \psi_1) \leq \frac{b(\mu_1 - \mu)}{4} (t^+)^4 + \alpha - 1.
\]

Consequently, for

\[
-\frac{b\mu}{b(t^+)^2} < \mu < \mu_1,
\]

we obtain \( I_\mu(t^+ \psi_1) < \alpha \). The same conclusion holds for a constant \( t^- \) with \( t^- < 0 \).

**Step 3.** Two solutions are obtained based on the coerciveness of \( I_\mu \) and Ekeland’s variational principle. Set

\[
\Theta^\pm = \{ u \in H : u = \pm \psi_1 + v \text{ with } t > 0, v \in V \}.
\]
When \( \mu < \mu_1 \) is sufficiently close to \( \mu_1 \), from step 1 and step 2, \( I_\mu \) is bounded below in \( \Theta^+ \) with
\[
-\infty < c^+ := \inf_{\Theta^+} I_\mu < \alpha.
\]

In \( \Theta^+ \), if we apply Ekeland’s variational principle to \( I_\mu \), there exists a sequence \( \{u_n\} \subset \Theta^+ \) such that \( I_\mu(u_n) \to c^+ \) and \( I'_\mu(u_n) \to 0 \) as \( n \to \infty \). By the coerciveness of \( I_\mu \) in \( H \), we deduce that \( \{u_n\} \) is bounded. So, \( \{u_n\} \) is a sequence satisfying (2.2) so that Lemma 2.2 implies \( \{u_n\} \) has a convergent subsequence, say \( \{u_n\} \).

Noting that \( V = \partial \Theta^+ \) and \( \inf_V I_\mu \geq \alpha \) (step 1), we conclude that \( \{u_n\} \) converges to an interior point \( u^+ \in \Theta^+ \), that is, the infimum is attained in \( \Theta^+ \). Therefore, \( I_\mu \) has a critical point \( u^+ \) as a local minimum in \( \Theta^+ \). Similarly, we obtain a critical point \( u^- \) of \( I_\mu \) as a local minimum in \( \Theta^- \). Note that \( \Theta^+ \cap \Theta^- = \emptyset \) which implies \( u^+ \neq u^- \), that is, \( I_\mu \) has two different local minimum points.

**Step 4.** It follows from Theorem 2.1 that \( I_\mu \) has a third solution. By Lemma 2.2, we see \( I_\mu \) satisfies (PS) condition. It follows from step 3 that \( u^+, u^- \) are two different local minimum points. Consequently, Theorem 2.1 shows that \( I_\mu \) has a third critical point. \( \square \)

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**References**


Shu-Zhi Song  
School of Mathematics and Statistics, Southwest University, Chongqing 400715, China  
E-mail address: sjrdj@163.com

Chun-Lei Tang (corresponding author)  
School of Mathematics and Statistics, Southwest University, Chongqing 400715, China  
E-mail address: tangcl@swu.edu.cn, Tel +8613883159865

Shang-Jie Chen  
School of Mathematics and Statistics, Chongqing Technology and Business University, Chongqing 400067, China  
E-mail address: chensj@ctbu.edu.cn, 11183356@qq.com