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MULTIPLE SOLUTIONS FOR KIRCHHOFF TYPE PROBLEM NEAR RESONANCE

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Abstract. Based on Ekeland's variational principle and the mountain pass theorem, we show the existence of three solutions to the Kirchhoff type problem

$$-\Big(a+b\int_\Omega |\nabla u|^2dx\Big)\Delta u = b\mu u^3+f(x,u)+h(x), \quad \text{in } \Omega,$$
$$u=0, \quad \text{on } \partial\Omega.$$

Where the parameter μ is sufficiently close, from the left, to the first nonlinear eigenvalue.

1. Introduction and statement of main result

The articles shows the existence of multiple solutions for the Kirchhoff type problem with Dirichlet boundary condition,

$$\begin{gathered}-\Big(a+b\int_\Omega |\nabla u|^2dx\Big)\Delta u = b\mu u^3+f(x,u)+h(x), \quad \text{in } \Omega,\\ u=0, \quad \text{on } \partial\Omega,\end{gathered} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ $(N=1,2,3)$ with a smooth boundary $\partial\Omega$, $a\ge 0$, $b>0$ are real constants and μ is a nonnegative parameter. Assume that $f\in C(\bar\Omega\times\mathbb{R},\mathbb{R})$ satisfies the sublinear growth condition:

(F1) $\lim_{|t|\to\infty}\frac{f(x,t)}{bt^3}=0$, uniformly for $x\in\Omega$.

Problem (1.1) can be looked on as a perturbed problem which was first studied by Mawhin and Schmit[6], related to the two-point boundary value equation

$$-u''-\lambda u = f(x,u)+h, \quad u(0)=u(\pi)=0. \tag{1.2}$$

Specifically, on the assumption: $\lambda<\lambda_1$ is sufficiently near to λ_1 (λ_1 is the first eigenvalue of the corresponding linear problem) and f is bounded and satisfies a sign condition, the existence of three solutions to equation (1.2) was proved in [6]. Later, various papers related to the result appeared. We mention for example, [1, 3, 4, 5, 8]. Ma, Ramos and Sanchez [3] considered the boundary-value problem for

$$\Delta u+\lambda u+f(x,u)=h(x)$$

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defined on a bounded open set $\Omega \subset \mathbb{R}^N$. As the parameter λ is sufficiently close to λ_1 from the left, there exist three solutions on both Dirchlet boundary conditions and Neumann boundary conditions. In addition, similar to the results in the linear case, the existence of three solutions was proved to the perturbed p-Laplacian equation in a bounded domain. Further consideration to the perturbed p-Laplacian equation in a bounded domain can be found in [4]. As for extension to the the perturbed p-Laplacian equation in the whole space $\mathbb{R}^N$, we refer to [5]. These results were also extended to some elliptic systems with the Dirichlet boundary conditions, refer to [8]. More recently, the authors in [1] extended these conclusions to some degenerate quasilinear elliptic systems with the Dirichlet boundary conditions. By analogy to the results mentioned above, we expect that problem (1.1) has at least three solutions as the parameter $\mu < \mu_1$ is sufficiently close to μ_1. Here μ_1 is the first eigenvalue of the eigenvalue problem

$$\begin{gathered} -\|u\|^2 \Delta u = \mu u^3, \quad \text{in } \Omega, \\ u = 0, \quad \text{on } \partial\Omega. \end{gathered}$$

Let $H = H_0^1(\Omega)$ be the Hilbert space equipped with the norm $\|u\| = (\int_\Omega |\nabla u|^2 dx)^{1/2}$ and $\|u\|_{L^s} = (\int_\Omega |u|^s dx)^{\frac{1}{s}}$ denote the norm of $L^s(\Omega)$. As shown in [9] and [13], the first nonlinear eigenvalue $\mu_1 > 0$ is simple and has a eigenfunction $\psi_1 > 0$ with $\|\psi_1\|_{L^4} = 1$. Specifically, μ_1 can be characterized by

$$\mu_1 = \inf \big\{ \|u\|^4 : \ u \in H, \ \int_\Omega |u|^4 dx = 1 \big\}. \tag{1.3}$$

Now we are in a position to state our result.

Theorem 1.1. *Suppose f satisfies* (F1) *and the following conditions:*

(F2)
$$\lim_{|t| \to \infty} \int_\Omega F(x, t\psi_1) - \frac{a}{2} \sqrt{\mu_1} t^2 = +\infty, \quad \text{uniformly for } x \in \Omega,$$
where $F(x,t) = \int_0^t f(x,s)ds$.

(H1) *$h \in L^2(\Omega)$ and $\int_\Omega h(x)\psi_1(x)dx = 0$.*

Then (1.1) *has at least three solutions if $\mu < \mu_1$ is sufficiently close to μ_1.*

Many authors have studied the Kirchhoff type equation in a bounded domain by applying variational methods. For example, they consider Kirchhoff type problem

$$\begin{gathered} -\Big(a + b \int_\Omega |\nabla u|^2 dx\Big) \Delta u = g(x,u), \quad \text{in } \Omega, \\ u = 0, \quad \text{on } \partial\Omega, \end{gathered} \tag{1.4}$$

assuming that

$$\lim_{|t| \to \infty} \frac{4G(x,t)}{bt^4} = \mu, \quad \text{uniformly in } x \in \Omega \tag{1.5}$$

where $G(x,t) = \int_0^t g(x,s)ds$. For the case $\mu < \mu_1$ in (1.5), the Euler functional corresponding to (1.4) is coercive. For the case $\mu = \mu_1$ in (1.5), that is, problem (1.4) is resonance at the first nonlinear eigenvalue μ_1, the Euler functional corresponding to (1.4) is still coercive, together with the assumption that $\lim_{|t| \to \infty} [g(x,t)t - 4G(x,t)] = +\infty$. So, the existence of weak solution for equation (1.4) is obtained based on the Least Action Principle (refer to [11, 12, 13]).

Furthermore, provided g with some conditions at zero, positive solution was obtained based on the topological degree argument (refer to [2]), multiple solutions are found by means of invariant sets of descent flow method (refer to [11, 13]), or the Local Linking Theorem (refer to [12]). Our result is different from the results in [2, 11, 12, 13] since we deal with the perturbation problem near to μ_1 and all hypotheses on f are just at infinity.

2. Proof of main result

We begin with some standard facts upon the variational formulation of problem (1.1). Let $I_\mu : H \mapsto \mathbb{R}$ be the functional defined by

$$I_\mu(u) = \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{b\mu}{4}\int_\Omega |u|^4 dx - \int_\Omega F(x,u)dx - \int_\Omega hudx.$$

Since f satisfies the sublinear growth condition (F1), it is not difficult to verify that $I_\mu \in C^1(H, \mathbb{R})$. Furthermore, finding weak solutions of (1.1) is equivalent to finding critical points of functional I_μ in H.

Since Ω is a bounded domain in $\mathbb{R}^N$ $(N = 1, 2, 3)$, the embedding $H \hookrightarrow L^s(\Omega)$ is continuous for $s \in [1, 2^*]$, compact for $s \in [1, 2^*)$,

$$2^* = \begin{cases} \frac{2N}{N-2}, & N = 3, \\ +\infty, & N = 1, 2. \end{cases}$$

Hence, for $s \in [1, 2^*]$, there exists $\tau_s > 0$ such that

$$\|u\|_{L^s} \le \tau_s \|u\|, \quad \forall u \in H. \tag{2.1}$$

We will prove the result by using Ekeland's variational principle [7, Theorem 4.1] and a mountain pass theorem [10]. For the convenience of readers, we state the mountain pass theorem as follows.

Theorem 2.1 ([10, Corollary 1])**.** *Consider a real Banach space X and a function $I \in C^1(X, \mathbb{R})$. If the (PS) condition holds and if I has two different local mininum points, then I possesses a third critical point.*

To prove our theorem, using critical point theory, we need the Palais-Smale compactness.

Lemma 2.2. *Assume that* (F1) *holds. Then any bounded (PS) sequence of I_μ has a convergent subsequence in H.*

Proof. Let $\{u_n\} \subset H$ be a bounded (PS) sequence of I_μ; that is,

$$\|u_n\| \le c, \quad |I_\mu(u_n)| \le c, \quad \|I'_\mu(u_n)\| \to 0, \tag{2.2}$$

where c denotes positive constant. By the reflexivity of H, we can assume that there exists $u \in H$ such that

$$u_n \rightharpoonup u \quad \text{weakly in } H, \tag{2.3}$$

$$u_n \to u \quad \text{strongly in } L^p(\Omega)\ (1 \le p < 2^*). \tag{2.4}$$

It follows from (F1) that for any $\varepsilon > 0$, there exits $M_\varepsilon > 0$ such that

$$|f(x,t)| \le b\varepsilon|t|^3 + M_\varepsilon, \quad \forall (x,t) \in \Omega \times \mathbb{R}. \tag{2.5}$$

We can now put together the results in (2.1), (2.2), (2.4) and (2.5) to conclude that

$$\begin{aligned} \Big|\int_\Omega f(x,u_n)(u-u_n)dx\Big| &\le \int_\Omega |f(x,u_n)||u-u_n|dx \\ &\le \int_\Omega (b\varepsilon|u_n|^3+M_\varepsilon)|u-u_n|dx \\ &\le b\varepsilon\|u_n\|_{L^4}^3\|u-u_n\|_{L^4}+M_\varepsilon|\Omega|^{1/2}\|u-u_n\|_{L^2} \\ &\le b\tau_4^3\varepsilon\|u_n\|^3\|u-u_n\|_{L^4}+M_\varepsilon|\Omega|^{1/2}\|u-u_n\|_{L^2} \\ &\le c(\|u-u_n\|_{L^4}+\|u-u_n\|_{L^2})\to 0, \quad \text{as } n\to\infty, \end{aligned} \tag{2.6}$$

where $c=\max\{b\tau_4^3\varepsilon, M_\varepsilon|\Omega|^{1/2}\}$, and $|\Omega|$ is the measure of Ω. Similarly, we may deduce that

$$\int_\Omega (|u_n|^2u_n(u-u_n)-|u|^2u(u-u_n))dx\to 0, \quad \text{as } n\to\infty. \tag{2.7}$$

From (2.2) and (2.4), we have

$$\langle I_\mu'(u_n)-I_\mu'(u), u-u_n\rangle\to 0, \quad \text{as } n\to\infty,$$

which combining with (2.6), (2.7), implies $\|u_n\|\to\|u\|$ as $n\to\infty$. It follows from (2.3) that $u_n\to u$ in H. $\square$

Set

$$V=\big\{v\in H:\int_\Omega \psi_1^3 v dx=0\big\}.$$

From the simplicity of μ_1 we have $H=\operatorname{span}\{\psi_1\}\oplus V$. We introduce the quantity

$$\mu_V=\inf\big\{\|u\|^4: u\in V, \|u\|_{L^4}^4=1\big\}.$$

Then

$$\|u\|^4\ge\mu_V\|u\|_{L^4}^4, \quad \forall u\in V, \tag{2.8}$$

and we have the following result.

Lemma 2.3. $\mu_1<\mu_V$.

Proof. It is evident from (1.3) that $\mu_1\le\mu_V$. Assume, by contradiction, that $\mu_1=\mu_V$. Then there exists a sequence $\{u_n\}\subseteq V$ such that $\|u_n\|_{L^4}=1$ for all $n\ge 1$, and $\|u_n\|^4\to\mu_V=\mu_1$. Since the sequence $\{u_n\}$ is bounded in H, we may assume that

$$u_n\rightharpoonup u \quad \text{weakly in } H, \quad u_n\to u \quad \text{strongly in } L^4(\Omega). \tag{2.9}$$

Thus, one has

$$\|u\|_{L^4}=\lim_{n\to+\infty}\|u_n\|_{L^4}=1,$$
$$\mu_1\le\|u\|^4\le\liminf_{n\to\infty}\|u_n\|^4=\lim_{n\to\infty}\|u_n\|^4=\mu_1.$$

So, $\|u\|_{L^4}=1$ and $\|u\|^4=\mu_1$. This implies $u=\pm\psi_1$.

On the other hand, from $\{u_n\}\subseteq V$ it follows that $\int_\Omega\psi_1^3u_n=0$ for all $n\ge 1$. Combining this with (2.9) and Hölder's inequality, we have

$$\begin{aligned} \Big|\int_\Omega \psi_1^3 u dx\Big| = \Big|\int_\Omega\psi_1^3udx-\int_\Omega\psi_1^3u_ndx\Big| &= \Big|\int_\Omega\psi_1^3(u-u_n)dx\Big| \\ &\le \int_\Omega \big|\psi_1^3(u-u_n)\big|\,dx \end{aligned}$$

$$\leq \|\psi_1\|_{L^4}^3 \|u_n - u\|_{L^4} \to 0, \quad \text{as } n \to \infty.$$

This is in direct contradiction to the fact $u = \pm\psi_1$. Hence $\mu_1 < \mu_V$. $\square$

Proof of Theorem 1.1. We shall divide the proof into four steps.

Step 1. The functional I_μ is bounded below in H and V and even coercive in H and V. More specifically, there is a constant α, independent of μ, such that $\inf_V I_\mu \geq \alpha$. From (2.5), we obtain

$$|F(x,t)| \leq \frac{b\varepsilon}{4}|t|^4 + M_\varepsilon|t|, \quad \forall (x,t) \in \Omega \times \mathbb{R}. \tag{2.10}$$

It follows from (2.1), (2.10) and Hölder inequality that

$$\begin{aligned} I_\mu(u) &\geq \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{b(\mu+\varepsilon)}{4}\int_\Omega u^4 dx - (M_\varepsilon|\Omega|^{1/2} + \|h\|_{L^2})\|u\|_{L^2} \\ &\geq \frac{b}{4}(1 - \frac{\mu+\varepsilon}{\mu_1})\|u\|^4 - \tau_2(M_\varepsilon|\Omega|^{1/2} + \|h\|_{L^2})\|u\|, \quad \forall u \in H\,. \end{aligned}$$

Note that $\mu < \mu_1$. Then, for $0 < \varepsilon < \mu_1 - \mu$, I_μ is bounded below and even coercive in H. Similarly, for $0 < \varepsilon < \mu_V - \mu_1$, (2.1), (2.8), (2.10) and Hölder inequality lead to

$$\begin{aligned} I_{\mu_1}(v) &\geq \frac{a}{2}\|v\|^2 + \frac{b}{4}\|v\|^4 - \frac{b(\mu_1+\varepsilon)}{4}\int_\Omega v^4 dx - (M_\varepsilon|\Omega|^{1/2} + \|h\|_{L^2})\|v\|_{L^2} \\ &\geq \frac{b}{4}(1 - \frac{\mu_1+\varepsilon}{\mu_V})\|v\|^4 - \tau_2(M_\varepsilon|\Omega|^{1/2} + \|h\|_{L^2})\|v\|, \quad \forall v \in V, \end{aligned}$$

which implies that I_{μ_1} is bounded below and coercive in V. Noting that $I_\mu \geq I_{\mu_1}$ for all $\mu < \mu_1$, we deduce I_μ is coercive in V and

$$\inf_V I_\mu \geq \alpha := \inf_V I_{\mu_1}.$$

Step 2. If $\mu < \mu_1$ is sufficiently close to μ_1, there exist two constants t^-, t^+ with $t^- < 0 < t^+$ such that $I_\mu(t^\pm\psi_1) < \alpha$. Noting $\|\psi_1\|_{L^4} = 1, \|\psi_1\|^4 = \mu_1$ and then combining this with (H1), for $t \in \mathbb{R}$, we have

$$\begin{aligned} I_\mu(t\psi_1) &= \frac{at^2}{2}\|\psi_1\|^2 + \frac{bt^4}{4}\|\psi_1\|^4 - \frac{b\mu t^4}{4}\int_\Omega \psi_1^4 dx - \int_\Omega F(x,t\psi_1)dx \\ &= \frac{b(\mu_1-\mu)}{4}t^4 - \Big(\int_\Omega F(x,t\psi_1)dx - \frac{a}{2}\sqrt{\mu_1}t^2\Big). \end{aligned}$$

From (F2), taking a constant t^+ with $t^+ > 0$ large enough, we obtain

$$\int_\Omega F(x,t^+\psi_1)dx - \frac{a}{2}\sqrt{\mu_1}(t^+)^2 > -\alpha + 1.$$

The above inequality reduces to

$$I_\mu(t^+\psi_1) \leq \frac{b(\mu_1-\mu)}{4}(t^+)^4 + \alpha - 1.$$

Consequently, for $-\frac{4\mu_1}{b(t^+)^4} < \mu < \mu_1$, we obtain $I_\mu(t^+\psi_1) < \alpha$. The same conclusion holds for a constant t^- with $t^- < 0$.

Step 3. Two solutions are obtained based on the coerciveness of I_μ and Ekeland's variational principle. Set

$$\Theta^\pm = \{u \in H : u = \pm t\psi_1 + v \text{ with } \; t > 0, v \in V\}.$$

When $\mu < \mu_1$ is sufficiently close to μ_1, from step 1 and step 2, I_μ is bounded below in Θ^+ with

$$-\infty < c^+ := \inf_{\Theta^+} I_\mu < \alpha.$$

In Θ^+, if we apply Ekeland's variational principle to I_μ, there exists a sequence $\{u_n\} \subset \Theta^+$ such that $I_\mu(u_n) \to c^+$ and $I'_\mu(u_n) \to 0$ as $n \to \infty$. By the coerciveness of I_μ in H, we deduce that $\{u_n\}$ is bounded. So, $\{u_n\}$ is a sequence satisfying (2.2) so that Lemma 2.2 implies $\{u_n\}$ has a convergent subsequence, say $\{u_n\}$ itself. Noting that $V = \partial\Theta^+$ and $\inf_V I_\mu \geq \alpha$ (step 1), we conclude that $\{u_n\}$ converges to an interior point $u^+ \in \Theta^+$, that is, the infimum is attained in Θ^+. Therefore, I_μ has a critical point u^+ as a local minimum in Θ^+. Similarly, we obtain a critical point u^- of I_μ as a local minimum in Θ^-. Note that $\Theta^+ \cap \Theta^- = \emptyset$ which implies $u^+ \neq u^-$, that is, I_μ has two different local minimum points.

Step 4. It follows from Theorem 2.1 that I_μ has a third solution. By Lemma 2.2, we see I_μ satisfies (PS) condition. It follows from step 3 that u^+, u^- are two different local minimum points. Consequently, Theorem 2.1 shows that I_μ has a third critical point. □

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References

[1] Y. C. An, X. Lu, H. M. Suo; *Existence and multiplicity results for a degenerate quasilinear elliptic system near resonance.* Bound. Value Probl. 2014, 2014:184.

[2] Z .P. Liang, F. Y. Li, J. P. Shi; *Positive solutions to Kirchhoff type equations with nonlinearity having prescribed asymptotic behavior.* Ann. Inst. H. Poincaré Anal. Non Linéaire 31 (2014), no. 1, 155–167.

[3] T. F. Ma, M. Ramos, L. Sanchez; *Multiple solutions for a class of nonlinear boundary value problems near resonance: a variational approach. Proceedings of the Second World Congress of Nonlinear Analysts*, Part 6 (Athens, 1996). Nonlinear Anal. 30 (1997), no. 6, 3301–3311.

[4] T. F. Ma, M. Ramos; *Three solutions of a quasilinear elliptic problem near resonance.* Math. Slovaca 47(1997), 451-457.

[5] T. F. Ma, M. Pelicer; *Perturbations near resonance for the p-Laplacian in* $\mathbb{R}^N$. Abstr. Appl. Anal. 7 (2002), no. 6, 323–334.

[6] J. Mawhin, K. Schmitt; *Nonlinear eigenvalue problems with the parameter near resonance.* Ann. Polon. Math. 51 (1990), 241–248.

[7] J. Mawhin, M. Willem; *Critical point theory and Hamiltonian systems.* Applied Mathematical Sciences, 74. Springer-Verlag, New York, 1989.

[8] Z. Q. Ou, C. L. Tang; *Existence and multiplicity results for some elliptic systems at resonance.* Nonlinear Anal. 71 (2009), no. 7-8, 2660–2666.

[9] K. Perera, Z. T. Zhang; *Nontrivial solutions of Kirchhoff-type problems via the Yang index.* J. Differential Equations 221 (2006), no. 1, 246–255.

[10] P. Pucci, J. Serrin; *A mountain pass theorem.* J. Differential Equations 60 (1985), no. 1, 142–149.

[11] Y. Yang, J. H. Zhang; *Positive and negative solutions of a class of nonlocal problems.* Nonlinear Anal. 73 (2010), no. 1, 25–30.

[12] Y. Yang, J. H. Zhang; *Nontrivial solutions of a class of nonlocal problems via local linking theory.* Appl. Math. Lett. 23 (2010), no. 4, 377–380.

[13] Z. T. Zhang, K. Perera; *Sign changing solutions of Kirchhoff type problems via invariant sets of descent flow.* J. Math. Anal. Appl. 317 (2006), no. 2, 456–463.

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