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LIMIT OF NONLINEAR ELLIPTIC EQUATIONS WITH CONCENTRATED TERMS AND VARYING DOMAINS: THE NON UNIFORMLY LIPSCHITZ CASE

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Abstract. In this article, we analyze the limit of the solutions of nonlinear elliptic equations with Neumann boundary conditions, when nonlinear terms are concentrated in a region which neighbors the boundary of domain and this boundary presents a highly oscillatory behavior which is non uniformly Lipschitz. More precisely, if the Neumann boundary conditions are nonlinear and the nonlinearity in the boundary is dissipative, then we obtain a limit equation with homogeneous Dirichlet boundary conditions. Moreover, if the Neumann boundary conditions are homogeneous, then we obtain a limit equation with nonlinear Neumann boundary conditions, which captures the behavior of the concentration's region. We also prove the upper semicontinuity of the families of solutions for both cases.

1. Introduction

In this article we analyze the limit of the solutions of nonlinear elliptic equations with terms concentrating on the boundary and Neumann boundary conditions for a family of domains Ω_ϵ when the boundary $\partial\Omega_\epsilon$ presents a behavior which is not uniformly Lipschitz, as the parameter $\epsilon \to 0$, although $\Omega_\epsilon \to \Omega$ and $\partial\Omega_\epsilon \to \partial\Omega$. This fact can be understood considering that in each point $x \in \partial\Omega$ the measure $|\partial\Omega_\epsilon \cap B(x,r)| \to \infty$ when $\epsilon \to 0$, where $B(x,r)$ is an open ball centre in x with radius r and $|\cdot|$ is the $(N-1)$-dimensional measure. For instance, our case can treat the family Ω_ϵ such that part of $\partial\Omega_\epsilon$ is parameterized by $\psi_\epsilon(\theta) = (r_\epsilon(\theta)\cos(\theta), r_\epsilon(\theta)\sin(\theta))$ and $r_\epsilon(\theta) = r_0(\theta)\epsilon\rho(\frac{\theta}{\epsilon^\alpha})$, for $\theta \in [0, 2\pi]$, where $\alpha > 1$, $r_0(\cdot)$ is a continuous function and $\rho(\cdot)$ is a periodic function. In this example, the period of oscillations is much smaller than its amplitude.

We consider two types of boundary conditions. In the first case, we study the behavior of the solutions of a concentrated elliptic equation with nonlinear Neumann boundary conditions of the type

$$\begin{gathered} -\Delta u_\epsilon + u_\epsilon = \frac{1}{\epsilon}\mathcal{X}_{\omega_\epsilon} f(x, u_\epsilon) + h(x, u_\epsilon), \quad \text{in } \Omega_\epsilon \\ \frac{\partial u_\epsilon}{\partial n} + g(x, u_\epsilon) = 0, \quad \text{on } \partial\Omega_\epsilon \end{gathered} \tag{1.1}$$

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and after the case that $g \equiv 0$, that is, the case with homogeneous Neumann boundary conditions of the type

$$\begin{aligned} -\Delta u_\epsilon + u_\epsilon &= \frac{1}{\epsilon}\mathcal{X}_{\omega_\epsilon} f(x,u_\epsilon) + h(x,u_\epsilon), \quad \text{in } \Omega_\epsilon \\ \frac{\partial u_\epsilon}{\partial n} &= 0, \quad \text{on } \partial\Omega_\epsilon. \end{aligned} \tag{1.2}$$

To describe the problem, we consider a family of bounded smooth domains $\Omega_\epsilon \subset \mathbb{R}^N$, with $N \geq 2$ and $0 \leq \epsilon \leq \epsilon_0$, for some $\epsilon_0 > 0$ fixed. We assume that $\Omega \equiv \Omega_0 \subset \Omega_\epsilon$ and we refer to Ω as the unperturbed domain and Ω_ϵ as the perturbed domains. We also assume that the nonlinearities $f, g, h : U \times \mathbb{R} \to \mathbb{R}$ are continuous in both variables and C^2 in the second one, where U is a fixed and smooth bounded domain containing all $\overline{\Omega}_\epsilon$, for all $0 \leq \epsilon \leq \epsilon_0$. For sufficiently small ϵ, ω_ϵ is the region between the boundaries of $\partial\Omega$ and $\partial\Omega_\epsilon$. Note that ω_ϵ shrinks to $\partial\Omega$ as $\epsilon \to 0$ and we use the characteristic function $\mathcal{X}_{\omega_\epsilon}$ of the region ω_ϵ to express the concentration in ω_ϵ. Figure 1 illustrates the oscillating set $\omega_\epsilon \subset \overline{\Omega}_\epsilon$.

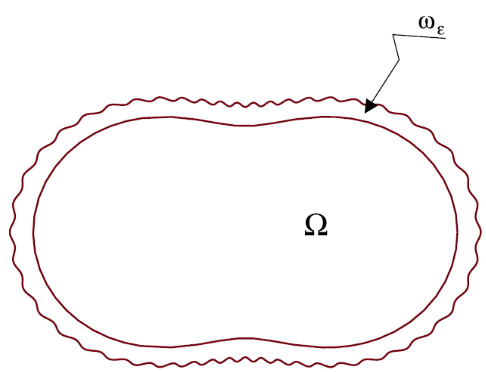

FIGURE 1. The set ω_ϵ.

The existing literature analyses separately concentrated terms and non uniformly Lipschitz deformation. We consider a problem where these two issues interact. In [5] the authors consider non uniformly Lipschitz deformation without concentrated terms. It is proved that the interaction's effect of non uniformly Lipschitz deformation with a strongly dissipative nonlinear Neumann boundary condition results in a limit problem with homogeneous Dirichlet boundary condition. On the other hand, the behavior of the solutions of elliptic and parabolic problems with reaction and potential terms concentrated in a neighborhood of the boundary of the domain was initially studied in [7, 9], when the neighborhood is a strip of width ϵ and has a base in the boundary, without oscillatory behavior and inside of Ω. In [7, 9] the domain Ω is C^2 in $\mathbb{R}^N$.

Also, considering only perturbation of domain, in [6] it was proved that the homogeneous Neumann boundary condition is preserved in the limit problem for a large class of perturbations of domains in which the non uniformly Lipschitz deformation is included.

In [2] some results of [7] were adapted to a nonlinear elliptic problem posed on an open square Ω in $\mathbb{R}^2$, considering $\omega_\epsilon \subset \Omega$ and with highly oscillatory behavior in the boundary inside of Ω. Later, [3] it proved the continuity of attractors for a nonlinear parabolic problem posed on a C^2 domain Ω in $\mathbb{R}^2$, when some terms are concentrated in a neighborhood of the boundary and the "inner boundary" of this neighborhood presents a highly oscillatory behavior.

It is important to note that these previous works with terms concentrating in a neighborhood of the boundary treat with non varying domain and since ω_ϵ is inside of Ω then all the equations are defined in the same domain.

We consider the concentration and varying domains simultaneously, therefore it is necessary to investigate how these two effects interact and what is the combined result. In this line, in [1] we consider varying domains and the region of concentration ω_ϵ is outside of Ω in which the main assumption was that $\partial\Omega_\epsilon$ is expressed in local charts as a Lipschitz deformation of $\partial\Omega$ with the Lipschitz constant uniformly bounded in ϵ. In [1] it was proved that the limiting equation of (1.1), with $h \equiv 0$, is given by

$$\begin{gathered} -\Delta u + u = 0, \quad \text{in } \Omega \\ \frac{\partial u}{\partial n} + \gamma(x) g(x,u) = \beta(x) f(x,u), \quad \text{on } \partial\Omega \end{gathered}$$

where the function $\gamma \in L^\infty(\partial\Omega)$ is related to the behavior of the measure $(N-1)$-dimensional of the $\partial\Omega_\epsilon$ and $\beta \in L^\infty(\partial\Omega)$ is related to the behavior of the measure N-dimensional of the region of concentration ω_ϵ. Since ω_ϵ shrinks to $\partial\Omega$ as $\epsilon \to 0$, it is reasonable to expect that the family of solutions of (1.1) will converge to a solution of an equation with a nonlinear boundary condition on $\partial\Omega$ that inherits the information about the region ω_ϵ. Moreover, the oscillations at the boundary amplify the effect of the nonlinearity $g(x,u)$ at the point $x \in \partial\Omega$ by a factor $\gamma(x)$. Hence, if $g(x,u)$ is strongly dissipative so that energy is lost through the boundary, then the oscillations increase the energy loss. While if the effect of the nonlinearity is to drive energy into the system through the boundary, the oscillations increase the intake of energy.

In this work, we continue the analysis initiated in [1] when $\partial\Omega_\epsilon$ is expressed in local charts as a Lipschitz deformation of $\partial\Omega$ with the Lipschitz constant non uniformly bounded in ϵ. In this case, if the nonlinearity $g(x,u)$ is strongly dissipative, we prove that the family of solutions of (1.1) will converge to a solution of an equation with most dissipative boundary condition, which is the homogeneous Dirichlet boundary condition $u = 0$, and that it does not inherit information about the region of concentration ω_ϵ. More precisely, we will show that the limiting equation of (1.1) is given by

$$\begin{gathered} -\Delta u + u = h(x,u), \quad \text{in } \Omega \\ u = 0, \quad \text{on } \partial\Omega. \end{gathered} \tag{1.3}$$

Also, we show that the limiting equation of (1.2) is an equation with nonlinear Neumann boundary condition that inherits the information about the region ω_ϵ which is given by

$$\begin{gathered} -\Delta u + u = h(x,u), \quad \text{in } \Omega \\ \frac{\partial u}{\partial n} = \beta(x) f(x,u), \quad \text{on } \partial\Omega \end{gathered} \tag{1.4}$$

where $\beta \in L^\infty(\partial\Omega)$ is related to the behavior of the measure N-dimensional of the region ω_ϵ. In both cases, we will prove the upper semicontinuity of the families of solutions of (1.1) and (1.2) in $H^1(\Omega_\epsilon)$.

This paper is organized as follows: in Section 2, we define the domain perturbation and state our main results (Theorems 2.4 and 2.5). In Section 3, we analyze the limit of concentrated integrals and interior integrals. In Section 4, we prove the upper semicontinuity of the family of solutions of (1.1) in $H^1(\Omega_\epsilon)$. In Section 5, we prove the upper semicontinuity of the family of solutions of (1.2) in $H^1(\Omega_\epsilon)$. In the last section, we state additional results. We leave out some the proofs, but they can be obtained upon request to the authors.

2. Setting of the problem and main results

We consider a family of smooth bounded domains $\Omega_\epsilon \subset \mathbb{R}^N$, with $N \geq 2$ and $0 \leq \epsilon \leq \epsilon_0$, for some $\epsilon_0 > 0$ fixed, and we regard Ω_ϵ as a perturbation of the fixed domain $\Omega \equiv \Omega_0$. We consider the following hypothesis on the domains

(H1) There exists a finite open cover $\{U_i\}_{i=0}^m$ of Ω such that $\overline{U}_0 \subset \Omega$, $\partial\Omega \subset \cup_{i=1}^m U_i$ and for each $i = 1, \dots, m$, there exists a Lipschitz diffeomorphism $\Phi_i : Q_N \to U_i$, where $Q_N = (-1,1)^N \subset \mathbb{R}^N$, such that

$$\Phi_i(Q_{N-1} \times (-1,0)) = U_i \cap \Omega \quad \text{and} \quad \Phi_i(Q_{N-1} \times \{0\}) = U_i \cap \partial\Omega.$$

We assume that $\overline{\Omega}_\epsilon \subset \cup_{i=0}^m U_i \equiv U$. For each $i = 1, \dots, m$, there exists a Lipschitz function $\rho_{i,\epsilon} : Q_{N-1} \to (-1,1)$ such that $\rho_{i,\epsilon}(x') \to 0$ as $\epsilon \to 0$, uniformly in Q_{N-1}. Moreover, we assume that $\Phi_i^{-1}(U_i \cap \partial\Omega_\epsilon)$ is the graph of $\rho_{i,\epsilon}$ this means

$$U_i \cap \partial\Omega_\epsilon = \Phi_i(\{(x', \rho_{i,\epsilon}(x')) : x' = (x_1, \dots, x_{N-1}) \in Q_{N-1}\}).$$

We consider the following mappings: $T_{i,\epsilon} : Q_N \to Q_N$ defined by

$$T_{i,\epsilon}(x', s) = \begin{cases} (x', s + s\rho_{i,\epsilon}(x') + \rho_{i,\epsilon}(x')), & \text{for } s \in (-1,0) \\ (x', s - s\rho_{i,\epsilon}(x') + \rho_{i,\epsilon}(x')), & \text{for } s \in [0,1). \end{cases}$$

Also,

$$\Phi_{i,\epsilon} := \Phi_i \circ T_{i,\epsilon} : Q_N \to U_i;$$
$$\Psi_{i,\epsilon} := \Phi_i \circ T_{i,\epsilon} \circ \Phi_i^{-1} : U_i \cap \partial\Omega \to U_i \cap \partial\Omega_\epsilon.$$

We also denote

$$\begin{aligned} \psi_{i,\epsilon} : Q_{N-1} &\to U_i \cap \partial\Omega_\epsilon \\ x' &\mapsto \Phi_{i,\epsilon}(x', 0) \end{aligned} \qquad \text{and} \qquad \begin{aligned} \psi_i : Q_{N-1} &\to U_i \cap \partial\Omega \\ x' &\mapsto \Phi_i(x', 0). \end{aligned}$$

Notice that $\psi_{i,\epsilon}$ and ψ_i are local parameterizations of $\partial\Omega_\epsilon$ and $\partial\Omega$, respectively. Furthermore, observe that all the maps above are Lipschitz, although the Lipschitz constant may not be bounded as $\epsilon \to 0$. Figure 2 illustrates the parameterizations.

With the notation above, we define

$$\omega_\epsilon = \cup_{i=1}^m \Phi_i\left(\left\{(x', x_N) \in \mathbb{R}^N : 0 \leq x_N < \rho_{i,\epsilon}(x') \text{ and } x' \in Q_{N-1}\right\}\right),$$

for $0 < \epsilon \leq \epsilon_0$.

To state the hypothesis to deal with the concentration in ω_ϵ, and to analyze the behavior of the solutions of (1.1) and (1.2), as $\epsilon \to 0$, we need the following definition

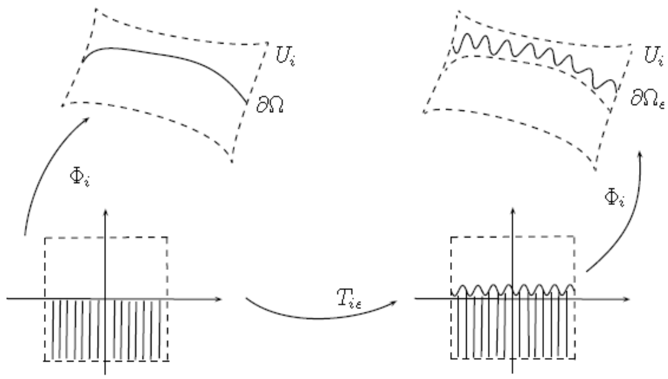

FIGURE 2. The parameterizations.

Definition 2.1. Let $\eta : A \subset \mathbb{R}^{N-1} \to \mathbb{R}^N$ almost everywhere differentiable, we define the $(N-1)$-dimensional Jacobian of η as

$$J_{N-1}\eta \equiv \Big|\frac{\partial \eta}{\partial x_1} \wedge \ldots \wedge \frac{\partial \eta}{\partial x_{N-1}}\Big| = \sqrt{\sum_{j=1}^{N} (\det(\mathrm{Jac}\,\eta)_j)^2},$$

where $v_1 \wedge \ldots \wedge v_{N-1}$ is the exterior product of the $(N-1)$ vectors $v_1, \ldots, v_{N-1} \in \mathbb{R}^N$ and $(\mathrm{Jac}\,\eta)_j$ is the $(N-1)$-dimensional matrix obtained by deleting the j-th row of the Jacobian matrix of η.

We use J_N for the absolute value of the N-dimensional Jacobian determinant. Now, we are ready to give the hypothesis.

(H2) For each $i = 1, \ldots, m$, $\rho_{i,\epsilon}(x')$ is $O(\epsilon)$ as $\epsilon \to 0$, uniformly in Q_{N-1}, that means $\|\frac{\rho_{i,\epsilon}}{\epsilon}\|_{L^\infty(Q_{N-1})} \le C$, with $C > 0$ independent of ϵ, $i = 1, \ldots, m$. And there exists a function $\tilde{\beta}_i \in L^\infty(Q_{N-1})$ such that

$$\frac{\rho_{i,\epsilon}}{\epsilon} \rightharpoonup \tilde{\beta}_i \quad \text{in } L^1(Q_{N-1}), \text{ as } \epsilon \to 0.$$

Definition 2.2. For $x \in U_i \cap \partial\Omega$, let $(x', 0) = \Phi_i^{-1}(x) \in Q_N$, we define $\beta : \partial\Omega \to \mathbb{R}$ as

$$\beta(x) = \frac{\tilde{\beta}_i(x')(J_N \Phi_i)(x', 0)}{J_{N-1}\psi_i(x')}.$$

The function β is independent of the charts U_i and the maps Φ_i and $\rho_{i,\epsilon}$. This was proved in [1, Corollary 3.7].

Now we give an example of the function $\rho_{i,\epsilon}$ satisfying the hypothesis (H2) and its correspondent function $\tilde{\beta}_i$.

Example 2.3. For each $i = 1, \ldots, m$, let $\rho_i : \mathbb{R}^{N-1} \to \mathbb{R}^+$ be a Y-periodic Lipschitz function, where $Y = (0, l_1) \times \cdots \times (0, l_{N-1}) \in \mathbb{R}^{N-1}$ with $l_1, \ldots, l_{N-1} \in \mathbb{R}^+$ (a function ρ_i is called Y-periodic if and only if $\rho_i(x' + kl_j e_j) = \rho_i(x')$ on Q_{N-1}, for all $k \in \mathbb{Z}$ and all $j \in \{1, \ldots, N-1\}$, where $x' = (x_1, \ldots, x_{N-1}) \in \mathbb{R}^{N-1}$ and $\{e_1, \ldots, e_{N-1}\}$ is the canonical basis of $\mathbb{R}^{N-1}$), and we define $\rho_{i,\epsilon} : Q_{N-1} \to [0, 1)$

by

$$\rho_{i,\epsilon}(x') = \epsilon\varphi(x')\rho_i\Big(\frac{x'}{\epsilon^\alpha}\Big),$$

for $x' \in Q_{N-1}$ and sufficiently small ϵ, say $0 < \epsilon \le \epsilon_0$, where $\alpha > 0$ and $\varphi : Q_{N-1} \to \mathbb{R}$ is a continuous function. From [8, Theorem 2.6], we obtain

$$\frac{\rho_{i,\epsilon}}{\epsilon} \rightharpoonup \varphi M_Y(\rho_i) = \tilde{\beta}_i \quad \text{in } L^1(Q_{N-1}), \text{ as } \epsilon \to 0,$$

where $M_Y(\rho_i)$ is the mean value of ρ_i over Y given by

$$M_Y(\rho_i) = \frac{1}{|Y|}\int_Y \rho_i(x')dx'.$$

The behavior of $J_{N-1}\psi_{i,\epsilon}$, as $\epsilon \to 0$, will be very important to decide the behavior of the solutions of (1.1), as $\epsilon \to 0$. Then, we will consider the hypothesis

(H3) For each $t > 1$, the set $\{x' \in Q_{N-1} : J_{N-1}\psi_{i,\epsilon}(x') \le t\}$ satisfies that its $(N-1)$-dimensional measure goes to zero as $\epsilon \to 0$, for all $i = 1, \dots, m$.

Now, with respect to the equations, we will be interested in studying the behavior of the solutions of the elliptic equations (1.1) and (1.2) where, as we mentioned in the introduction, the nonlinearities $f, g, h : U \times \mathbb{R} \to \mathbb{R}$ are continuous in both variables and C^2 in the second one, where U is a bounded domain containing $\overline{\Omega}_\epsilon$, for all $0 \le \epsilon \le \epsilon_0$.

Consider the family of spaces $H^1(\Omega_\epsilon)$ and $H^1(\Omega)$ with their usual norms. Since we will need to compare functions defined in Ω_ϵ with functions defined in the unperturbed domain $\Omega \equiv \Omega_0$, we will need a tool to compare functions which are defined in different spaces. The appropriate notion for this is the concept of E-convergence and a key ingredient for this will be the use of the extension operator $E_\epsilon : H^1(\Omega) \to H^1(\Omega_\epsilon)$, which is defined as $E_\epsilon = R_\epsilon \circ P$, where $P : H^1(\Omega) \to H^1(\mathbb{R}^N)$ is an extension operator and R_ϵ is the restriction operator from functions defined in $\mathbb{R}^N$ to functions defined in Ω_ϵ, $R_\epsilon w = w_{|\Omega_\epsilon}$. Observe that we also have $E_\epsilon : L^p(\Omega) \to L^p(\Omega_\epsilon)$ and $E_\epsilon : W^{1,p}(\Omega) \to W^{1,p}(\Omega_\epsilon)$, for all $1 \le p \le \infty$. Considering $X_\epsilon = H^1(\Omega_\epsilon)$ or $L^p(\Omega_\epsilon)$ or $W^{1,p}(\Omega_\epsilon)$, for $\epsilon \ge 0$, from [4] we have

$$\|E_\epsilon u\|_{X_\epsilon} \to \|u\|_{X_0}.$$

The concept of E-convergence is defined as follows: $u_\epsilon \xrightarrow{E} u$ if $\|u_\epsilon - E_\epsilon u\|_{H^1(\Omega_\epsilon)} \to 0$, as $\epsilon \to 0$. We also have a notion of weak E-convergence, which is defined as follows: $u_\epsilon \overset{E}{\rightharpoonup} u$, if $(u_\epsilon, w_\epsilon)_{H^1(\Omega_\epsilon)} \to (u, w)_{H^1(\Omega)}$, as $\epsilon \to 0$, for any sequence $w_\epsilon \xrightarrow{E} w$, where $(\cdot,\cdot)_{H^1(\Omega_\epsilon)}$ and $(\cdot,\cdot)_{H^1(\Omega)}$ denote the inner product in $H^1(\Omega_\epsilon)$ and $H^1(\Omega)$, respectively. More details about E-convergence can be found in [4, Subsection 3.2].

Since $\Omega \subset \Omega_\epsilon$, Ω_ϵ is an exterior perturbation of Ω, we consider the restriction operator $R_\Omega : H^1(\Omega_\epsilon) \to H^1(\Omega)$ given by $R_\Omega(u) = u_{|\Omega}$.

Our main results are stated in the following theorems

Theorem 2.4. *Assume that* (H1)–(H3) *are satisfied and that the nonlinearity g satisfies a dissipative condition:*

$$\exists\, b > 0,\ d \ge 1,\ \text{s.t. } g(x,s)s \ge b|s|^{d+1}, \quad \forall |s| \le R+1 \quad \text{and} \quad \forall x \in U. \tag{2.1}$$

Let $\{u^*_\epsilon\}$, $0 < \epsilon \le \epsilon_0$, be a family of the solutions of problem (1.1) *satisfying $\|u^*_\epsilon\|_{L^\infty(\Omega_\epsilon)} \le R$, for some constant $R > 0$ independent of ϵ. Then, there exist a subsequence $\{u^*_{\epsilon_k}\}$ and a function $u^*_0 \in H^1_0(\Omega)$, with $\|u^*_0\|_{L^\infty(\Omega)} \le R$, solution of* (1.3) *satisfying $u^*_{\epsilon_k} \xrightarrow{E} u^*_0$.*

Theorem 2.5. *Assume that* (H1)–(H2) *are satisfied. Let* $\{u_\epsilon^*\}$, $0 < \epsilon \le \epsilon_0$, *be a family of the solutions of problem* (1.2) *satisfying* $\|u_\epsilon^*\|_{L^\infty(\Omega_\epsilon)} \le R$, *for some constant* $R > 0$ *independent of* ϵ. *Then, there exist a subsequence* $\{u_{\epsilon_k}^*\}$ *and a function* $u_0^* \in H^1(\Omega)$, *with* $\|u_0^*\|_{L^\infty(\Omega)} \le R$, *solution of* (1.4) *satisfying* $u_{\epsilon_k}^* \xrightarrow{E} u_0^*$.

Remark 2.6. Since in Theorems 2.4 and 2.5 we are concerned with solutions that are uniformly bounded in $L^\infty(\Omega_\epsilon)$, we may perform a cut-off in the nonlinearities f, g and h outside the region $|u| \le R$, without modifying any of these solutions, in such a way that

$$\begin{aligned} &|f(x,u)| + |\partial_u f(x,u)| \le M, \quad \forall x \in U,\ \forall u \in \mathbb{R} \\ &|g(x,u)| + |\partial_u g(x,u)| \le M, \quad \forall x \in U,\ \forall u \in \mathbb{R} \\ &|h(x,u)| + |\partial_u h(x,u)| \le M, \quad \forall x \in U,\ \forall u \in \mathbb{R} \end{aligned} \tag{2.2}$$

and we may also assume that the cut-off is performed so that the following also holds

$$g(x,s)s \ge b|s|, \quad \forall |s| \ge R+1,\ \forall x \in U. \tag{2.3}$$

3. Concentrated integrals and interior integrals

In this section, we will analyze how the concentrated integrals converge to boundary integrals and the convergence of the interior integrals, as $\epsilon \to 0$. These convergence results will be needed to analyze the limit of the solutions of (1.1) and (1.2), as $\epsilon \to 0$. Initially, we have

Lemma 3.1. *Assume that* (H1)–(H2) *are satisfied. Suppose that* $v_\epsilon \in W^{1,q}(\Omega_\epsilon)$ *with* $\frac{1}{q} < s \le 1$. *Then, for small* ϵ_0, *there exist constants* $L, \tilde{L} > 0$ *independents of* ϵ *and* v_ϵ *such that for any* $0 < \epsilon \le \epsilon_0$, *we have*

$$\frac{1}{\epsilon}\int_{\omega_\epsilon} |v_\epsilon|^q d\xi \le L\epsilon^{q/q'} \|v_\epsilon\|^q_{W^{1,q}(\Omega_\epsilon)} + \tilde{L}\|v_\epsilon\|^q_{H^{s,q}(\Omega)}, \quad \text{where } \frac{1}{q} + \frac{1}{q'} = 1. \tag{3.1}$$

Proof. Consider the finite cover $\{U_i\}_{i=0}^m$ such that $\overline{\Omega}_\epsilon \subset \cup_{i=0}^m U_i \equiv U$ given in (H1). We have

$$\begin{aligned} \frac{1}{\epsilon}\int_{\omega_\epsilon \cap U_i} |v_\epsilon|^q d\xi &= \frac{1}{\epsilon}\int_{Q_{N-1}} \int_0^{\rho_{i,\epsilon}(x')} |v_\epsilon(\Phi_i(x', x_N))|^q J_N\Phi_i(x', x_N) dx_N\, dx' \\ &\le \big\|J_N\Phi_i\big\|_{L^\infty(Q_N)} \int_{Q_{N-1}} \int_0^1 |v_\epsilon(\Phi_i(x', s\rho_{i,\epsilon}(x')))|^q \frac{\rho_{i,\epsilon}(x')}{\epsilon} ds\, dx' \\ &\le 2^q \big\|J_N\Phi_i\big\|_{L^\infty(Q_N)} \big\|\frac{\rho_{i,\epsilon}}{\epsilon}\big\|_{L^\infty(Q_{N-1})} \\ &\quad \times \Big[\int_{Q_{N-1}} \int_0^1 |v_\epsilon(\Phi_i(x', s\rho_{i,\epsilon}(x'))) - v_\epsilon(\Phi_i(x', 0))|^q ds\, dx' \\ &\quad + \int_{Q_{N-1}} \int_0^1 |v_\epsilon(\Phi_i(x', 0))|^q ds\, dx'\Big], \end{aligned}$$

where we changed the variable using $x_N = s\rho_{i,\epsilon}(x')$ and, by hypothesis (H2), $\|\frac{\rho_{i,\epsilon}}{\epsilon}\|_{L^\infty(Q_{N-1})} \le C$, with $C > 0$ independent of ϵ, $i = 1, \dots, m$.

Using the trace Theorem for a fixed domain, we obtain

$$
\begin{aligned}
&\int_{Q_{N-1}}\int_0^1 |v_\epsilon(\Phi_i(x',0))|^q ds\,dx' \\
&= \int_{Q_{N-1}} |v_\epsilon(\psi_i(x'))|^q dx' \\
&\le \frac{1}{\inf_{z\in Q_{N-1}} J_{N-1}\psi_i(z)} \int_{Q_{N-1}} |v_\epsilon(\psi_i(x'))|^q J_{N-1}\psi_i(x')\,dx' \\
&\le \tilde{C}\|v_\epsilon\|^q_{L^q(\partial\Omega)} \le \hat{C}\|v_\epsilon\|^q_{H^{s,q}(\Omega)}.
\end{aligned}
\tag{3.2}
$$

Now, we estimate

$$
\begin{aligned}
&\int_{Q_{N-1}}\int_0^1 |v_\epsilon(\Phi_i(x',s\rho_{i,\epsilon}(x'))) - v_\epsilon(\Phi_i(x',0))|^q ds\,dx' \\
&= \int_{Q_{N-1}}\int_0^1 \Big|\int_0^{s\rho_{i,\epsilon}(x')} \frac{\partial(v_\epsilon\circ\Phi_i)}{\partial\tilde{x}_N}(x',\tilde{x}_N)\,d\tilde{x}_N\Big|^q ds\,dx' \\
&\le \int_{Q_{N-1}}\int_0^1 |s\rho_{i,\epsilon}(x')|^{q/q'} \int_0^{s\rho_{i,\epsilon}(x')} \Big|\frac{\partial(v_\epsilon\circ\Phi_i)}{\partial\tilde{x}_N}(x',\tilde{x}_N)\Big|^q d\tilde{x}_N\,ds\,dx' \\
&\le \epsilon^{q/q'}C^{q/q'}\int_{Q_{N-1}}\int_0^1\int_0^{\rho_{i,\epsilon}(x')} |\nabla(v_\epsilon\circ\Phi_i)(x',\tilde{x}_N)|^q d\tilde{x}_N\,ds\,dx' \\
&\le \epsilon^{q/q'}N^{q+1}C^{q/q'}\|D\Phi_i\|^q_{L^\infty(Q_N)}\int_{Q_{N-1}}\int_0^{\rho_{i,\epsilon}(x')} |\nabla v_\epsilon(\Phi_i(x',\tilde{x}_N))|^q d\tilde{x}_N\,dx' \\
&\le \frac{\epsilon^{q/q'}N^{q+1}C^{q/q'}\|D\Phi_i\|^q_{L^\infty(Q_N)}}{\inf_{z\in Q_N} J_N\Phi_i(z)} \\
&\quad\times \int_{Q_{N-1}}\int_0^{\rho_{i,\epsilon}(x')} |\nabla v_\epsilon(\Phi_i(x',\tilde{x}_N))|^q J_N\Phi_i(x',\tilde{x}_N)\,d\tilde{x}_N\,dx' \\
&= \frac{\epsilon^{q/q'}N^{q+1}C^{q/q'}\|D\Phi_i\|^q_{L^\infty(Q_N)}}{\inf_{z\in Q_N} J_N\Phi_i(z)}\int_{\omega_\epsilon\cap U_i}|\nabla v_\epsilon|^q\,d\xi \\
&\le \frac{N^{q+1}C^{q/q'}\|D\Phi_i\|^q_{L^\infty(Q_N)}}{\inf_{z\in Q_N} J_N\Phi_i(z)}\epsilon^{q/q'}\|v_\epsilon\|^q_{W^{1,q}(\Omega_\epsilon)}.
\end{aligned}
$$

From the above inequality and (3.2) we obtain (3.1). □

In [1, Lemma 3.6] we proved that concentrated integrals converge to boundary integrals, as $\epsilon \to 0$. We note that this result of convergence is still true in the case of non uniformly Lipschitz deformation since we did not use the hypothesis of $\|\nabla\rho_{i,\epsilon}\|_{L^\infty(Q_{N-1})}$ is uniformly bounded for ϵ. This result is stated in the following lemma

Lemma 3.2. *Assume that* (H1)–(H2) *are satisfied. Then, for any functions $h,\varphi \in H^s(U)$ with $\frac{1}{2} < s \le 1$, we obtain*

$$
\lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_{\omega_\epsilon} h\varphi\,d\xi = \int_{\partial\Omega}\beta h\varphi\,dS.
$$

Proceeding as in [1, Proposition 4.3], we have the following lemma.

Lemma 3.3. *Assume that* (H1)–(H2) *are satisfied. Let* $\{u_\epsilon\}$ *and* $\{z_\epsilon\}$ *be bounded sequences in* $H^1(\Omega_\epsilon)$ *such that* $u_\epsilon \xrightarrow{E} u$ *and* $z_\epsilon \xrightarrow{E} z$*. Then*

$$\frac{1}{\epsilon}\int_{\omega_\epsilon} f(x,u_\epsilon)z_\epsilon \to \int_{\partial\Omega} \beta f(x,u)z, \quad \int_{\Omega_\epsilon} h(x,u_\epsilon)z_\epsilon \to \int_{\Omega} h(x,u)z,$$

as $\epsilon \to 0$.

Proof. From [5, Lemma 3.1 (iii)] we obtain the convergence of the interior integrals. Now, using (2.2), Cauchy-Schwarz and Lemma 3.1, we have

$$\begin{aligned}
&\Big|\frac{1}{\epsilon}\int_{\omega_\epsilon} f(x,u_\epsilon)z_\epsilon - \int_{\partial\Omega}\beta f(x,u)z\Big| \\
&\le \tilde{C}\Big(\frac{1}{\epsilon}\int_{\omega_\epsilon}|u_\epsilon - E_\epsilon u|^2\Big)^{1/2}\Big(\frac{1}{\epsilon}\int_{\omega_\epsilon}|z_\epsilon|^2\Big)^{1/2} + C\Big(\frac{1}{\epsilon}\int_{\omega_\epsilon}|z_\epsilon - E_\epsilon z|^2\Big)^{1/2} \\
&\quad + \Big|\frac{1}{\epsilon}\int_{\omega_\epsilon} f(x,E_\epsilon u)E_\epsilon z - \int_{\partial\Omega}\beta f(x,u)z\Big| \\
&\le \tilde{K}\Big(\epsilon\|u_\epsilon - E_\epsilon u\|^2_{H^1(\Omega_\epsilon)} + \|u_\epsilon - E_\epsilon u\|^2_{H^s(\Omega)}\Big)^{1/2}\|z_\epsilon\|_{H^1(\Omega_\epsilon)} \\
&\quad + K\Big(\epsilon\|z_\epsilon - E_\epsilon z\|^2_{H^1(\Omega_\epsilon)} + \|z_\epsilon - E_\epsilon z\|^2_{H^s(\Omega)}\Big)^{1/2} \\
&\quad + \Big|\frac{1}{\epsilon}\int_{\omega_\epsilon} f(x,E_\epsilon u)E_\epsilon z - \int_{\partial\Omega}\beta f(x,u)z\Big| \to 0, \quad \text{as } \epsilon \to 0,
\end{aligned}$$

where $s \in \mathbb{R}$ such that $\frac{1}{2} < s < 1$. Using Lemma 3.2, we obtain that the last term goes to 0. Since $\{u_\epsilon\}$ and $\{z_\epsilon\}$ are bounded sequences in $H^1(\Omega_\epsilon)$ such that $u_\epsilon \xrightarrow{E} u$ and $z_\epsilon \xrightarrow{E} z$, considering subsequences if necessary, we have that $R_\Omega(u_\epsilon) \xrightarrow{\epsilon\to 0} u$ and $R_\Omega(z_\epsilon) \xrightarrow{\epsilon\to 0} z$ in $H^1(\Omega)$. Hence, using compact embedding for a fixed domain, we have that $\|u_\epsilon - E_\epsilon u\|_{H^s(\Omega)} \to 0$ and $\|z_\epsilon - E_\epsilon z\|_{H^s(\Omega)} \to 0$ and we complete the proof. □

4. Upper semicontinuity of solutions for the nonlinear boundary conditions problem (1.1)

In this section, we will provide a proof of Theorem 2.4. Initially, we prove a result that implies boundedness of the solutions of (1.1) and will be used to obtain the homogeneous Dirichlet boundary condition in the limiting equation of (1.1).

Lemma 4.1. *Assume that* (H1)–(H2) *are satisfied. If* $\{z_\epsilon\}$, $0 < \epsilon \le \epsilon_0$, *is a family of the solutions of* (1.1) *satisfying* $\|z_\epsilon\|_{L^\infty(\Omega_\epsilon)} \le R$, *for some constant* $R > 0$ *independent of* ϵ, *then there exists* $C > 0$ *independent of* ϵ *such that*

$$\|\nabla z_\epsilon\|^2_{L^2(\Omega_\epsilon)} + \|z_\epsilon\|^2_{L^2(\Omega_\epsilon)} + \int_{\partial\Omega_\epsilon}|z_\epsilon|^{d(z_\epsilon(x))} \le C, \quad 0 < \epsilon \le \epsilon_0, \tag{4.1}$$

for some sufficiently small ϵ_0, *where*

$$d(s) = \begin{cases} d+1, & \text{if } |s| \le R+1 \\ 1, & \text{if } |s| \ge R+1 \end{cases} \tag{4.2}$$

where d *and* R *are defined in* (2.1).

Proof. Multiplying the equation (1.1) by z_ϵ and integrating by parts, we obtain

$$\int_{\Omega_\epsilon} |\nabla z_\epsilon|^2 + \int_{\Omega_\epsilon} |z_\epsilon|^2 + \int_{\partial\Omega_\epsilon} g(x, z_\epsilon) z_\epsilon = \frac{1}{\epsilon}\int_{\omega_\epsilon} f(x, z_\epsilon) z_\epsilon + \int_{\Omega_\epsilon} h(x, z_\epsilon) z_\epsilon.$$

By Cauchy-Schwarz and Young inequalities, (2.2) and Lemma 3.1 with $s = 1$, we have

$$\begin{aligned} \frac{1}{\epsilon}\int_{\omega_\epsilon} f(x, z_\epsilon) z_\epsilon &\le \frac{\delta}{\epsilon}\|z_\epsilon\|^2_{L^2(\omega_\epsilon)} + \frac{C_\delta}{\epsilon}\|f(\cdot, z_\epsilon(\cdot))\|^2_{L^2(\omega_\epsilon)} \\ &\le \delta K_2 \|z_\epsilon\|^2_{H^1(\Omega_\epsilon)} + C_\delta K_1, \end{aligned} \tag{4.3}$$

where K_1 and K_2 are independent of ϵ, with $0 < \epsilon \le \epsilon_0$ for some sufficiently small ϵ_0. Again, using Cauchy-Schwarz and Young inequalities and (2.2), we obtain

$$\int_{\Omega_\epsilon} h(x, z_\epsilon) z_\epsilon \le \delta\|z_\epsilon\|^2_{L^2(\Omega_\epsilon)} + C_\delta\|h(\cdot, z_\epsilon(\cdot))\|^2_{L^2(\Omega_\epsilon)} \le \delta\|z_\epsilon\|^2_{H^1(\Omega_\epsilon)} + C_\delta K_3. \tag{4.4}$$

Now, using (2.1), (2.3) and (4.2), we obtain

$$\int_{\partial\Omega_\epsilon} g(x, z_\epsilon) z_\epsilon \ge b\int_{\partial\Omega_\epsilon} |z_\epsilon|^{d(z_\epsilon(x))}. \tag{4.5}$$

Therefore, using (4.3), (4.4) and (4.5) and taking δ such that $\delta(K_2 + 1) < 1$, we obtain

$$\min\{1 - \delta(K_2 + 1), b\}\Big(\|\nabla z_\epsilon\|^2_{L^2(\Omega_\epsilon)} + \|z_\epsilon\|^2_{L^2(\Omega_\epsilon)} + \int_{\partial\Omega_\epsilon} |z_\epsilon|^{d(z_\epsilon(x))}\Big) \le C_\delta(K_1 + K_3).$$

This shows (4.1). □

Now, we can prove the upper semicontinuity of the solutions of (1.1).

Proof of Theorem 2.4. Let $\{u^*_\epsilon\}$, $0 < \epsilon \le \epsilon_0$, be a family of the solutions of (1.1) satisfying $\|u^*_\epsilon\|_{L^\infty(\Omega_\epsilon)} \le R$, for some constant $R > 0$ independent of ϵ. Applying Lemma 4.1, the sequence $\{u^*_\epsilon\}$ is bounded in $H^1(\Omega_\epsilon)$. By [5, Lemma 3.1 (i)], there exist a subsequence $\{u^*_{\epsilon_k}\}$ and a function $u^*_0 \in H^1(\Omega)$ such that $u^*_{\epsilon_k} \xrightarrow{E} u^*_0$ and $\|u^*_{\epsilon_k} - E_{\epsilon_k} u^*_0\|_{L^2(\Omega_{\epsilon_k})} \to 0$.

We give a brief proof of $u^*_0 \in H^1_0(\Omega)$ in the case $\Omega \subset \Omega_\epsilon$ for $0 < \epsilon \le \epsilon_0$. The complete proof is in [5, Proposition 4.2].

The trace operator from $H^1(\Omega)$ to $L^2(U_i \cap \partial\Omega)$, $i = 1, 2, \dots, n$, is continuous and compact, then $u^*_{\epsilon_k | U_i \cap \partial\Omega}$ converges to $u^*_{0 | U_i \cap \partial\Omega}$ in $L^2(U_i \cap \partial\Omega)$. Hence, given $\beta > 0$ small, there exists ϵ_0 such that

$$\int_{U_i \cap \partial\Omega} |u^*_{\epsilon_k} - u^*_0| \le \beta, \quad \text{for } 0 < \epsilon \le \epsilon_0. \tag{4.6}$$

Using [5, Lemma 3.2] for $\eta = 0$, we obtain that for each $\beta > 0$ fixed, we can choose an even smaller ϵ_0 such that

$$\int_{U_i \cap \partial\Omega} |u^*_{\epsilon_k} \circ \Psi_{i,\epsilon_k} - u^*_{\epsilon_k}| \le \beta, \quad \text{for } 0 < \epsilon \le \epsilon_0. \tag{4.7}$$

Putting together (4.6) and (4.7), we obtain that for $0 < \epsilon \le \epsilon_0$,

$$\int_{U_i \cap \partial\Omega} |u^*_0| \le 2\beta + \int_{U_i \cap \partial\Omega} |u^*_{\epsilon_k} \circ \Psi_{i,\epsilon_k}|. \tag{4.8}$$

For each $t > 1$, we consider the sets $A_t^{\epsilon_k} = \{x' \in Q_{N-1} : J_{N-1}\psi_{i,\epsilon_k}(x') \le t\}$ and $B_t^{\epsilon_k} = \{x' \in Q_{N-1} : J_{N-1}\psi_{i,\epsilon_k}(x') > t\}$ so that $Q_{N-1} = A_t^{\epsilon_k} \cup B_t^{\epsilon_k}$, $A_t^{\epsilon_k} \cap B_t^{\epsilon_k} = \emptyset$ and, by (H3), $|A_t^{\epsilon_k}| \to 0$ as $\epsilon_k \to 0$. Moreover,

$$\int_{U_i \cap \partial\Omega} |u_{\epsilon_k}^* \circ \Psi_{i,\epsilon_k}| = \int_{\psi_i(A_t^{\epsilon_k})} |u_{\epsilon_k}^* \circ \Psi_{i,\epsilon_k}| + \int_{\psi_i(B_t^{\epsilon_k})} |u_{\epsilon_k}^* \circ \Psi_{i,\epsilon_k}|. \tag{4.9}$$

We analyze separately the two integrals in (4.9). Initially, for all $1 < p < \infty$, we have

$$\int_{\psi_i(A_t^{\epsilon_k})} |u_{\epsilon_k}^* \circ \Psi_{i,\epsilon_k}| \le \Big(\int_{U_i \cap \partial\Omega} |u_{\epsilon_k}^* \circ \Psi_{i,\epsilon_k}|^p\Big)^{1/p} [\mathcal{H}_{N-1}(\psi_i(A_t^{\epsilon_k}))]^{1/p'},$$

where $\frac{1}{p} + \frac{1}{p'} = 1$ and $\mathcal{H}_{N-1}$ is the $(N-1)$-dimensional Hausdorff measure.

Taking into account that $\|u_{\epsilon_k}^*\|_{H^1(\Omega)} \le C$ and using [5, Lemma 3.2] for $\eta = 0$ and trace Theorems, we have for $1 < p$ small,

$$\begin{aligned}
\Big(\int_{U_i \cap \partial\Omega} |u_{\epsilon_k}^* \circ \Psi_{i,\epsilon_k}|^p\Big)^{1/p} &\le \Big(\int_{U_i \cap \partial\Omega} |u_{\epsilon_k}^* \circ \Psi_{i,\epsilon_k} - u_{\epsilon_k}^*|^p\Big)^{1/p} + \Big(\int_{U_i \cap \partial\Omega} |u_{\epsilon_k}^*|^p\Big)^{1/p}\\
&\le C.
\end{aligned}$$

Since $\mathcal{H}_{N-1}(\psi_i(A_t^{\epsilon_k})) \le C|A_t^{\epsilon_k}| \to 0$ as $\epsilon_k \to 0$ by (H3), we have

$$\int_{\psi_i(A_t^{\epsilon_k})} |u_{\epsilon_k}^* \circ \Psi_{i,\epsilon_k}| \to 0, \quad \text{as } \epsilon_k \to 0. \tag{4.10}$$

Now, using $J_{N-1}\psi_i \le C$, we observe that

$$\begin{aligned}
\int_{\psi_i(B_t^{\epsilon_k})} |u_{\epsilon_k}^* \circ \Psi_{i,\epsilon_k}| &= \int_{B_t^{\epsilon_k}} |u_{\epsilon_k}^* \circ \Psi_{i,\epsilon_k} \circ \psi_i(x')| J_{N-1}\psi_i(x') dx'\\
&\le C \int_{B_t^{\epsilon_k}} |u_{\epsilon_k}^* \circ \psi_{i,\epsilon_k}(x')| dx'.
\end{aligned}$$

Consider the decomposition of the set $B_t^{\epsilon_k} = C_t^{\epsilon_k} \cup D_t^{\epsilon_k}$ where $C_t^{\epsilon_k} = \{x' \in B_t^{\epsilon_k} : |u_{\epsilon_k}^*(\psi_{i,\epsilon_k}(x'))| \le R+1\}$ and $D_t^{\epsilon_k} = \{x' \in B_t^{\epsilon_k} : |u_{\epsilon_k}^*(\psi_{i,\epsilon_k}(x'))| > R+1\}$, with R given as in the Lemma 4.1, so that

$$\int_{B_t^{\epsilon_k}} |u_{\epsilon_k}^* \circ \psi_{i,\epsilon_k}(x')| dx' = \int_{C_t^{\epsilon_k}} |u_{\epsilon_k}^* \circ \psi_{i,\epsilon_k}(x')| dx' + \int_{D_t^{\epsilon_k}} |u_{\epsilon_k}^* \circ \psi_{i,\epsilon_k}(x')| dx'.$$

Using Holder inequality, $\frac{J_{N-1}\psi_{i,\epsilon_k}}{t} > 1$ on $B_t^{\epsilon_k}$ and (4.1), we obtain

$$\begin{aligned}
&\int_{C_t^{\epsilon_k}} |u_{\epsilon_k}^* \circ \psi_{i,\epsilon_k}(x')| dx'\\
&\le \frac{|C_t^{\epsilon_k}|^{\frac{d}{d+1}}}{t^{\frac{1}{d+1}}} \Big(\int_{C_t^{\epsilon_k}} |u_{\epsilon_k}^* \circ \psi_{i,\epsilon_k}(x')|^{d+1} J_{N-1}\psi_{i,\epsilon_k}(x') dx'\Big)^{\frac{1}{d+1}}\\
&\le \frac{|C_t^{\epsilon_k}|^{\frac{d}{d+1}}}{t^{\frac{1}{d+1}}} \Big(\int_{U_i \cap \partial\Omega_{\epsilon_k}} |u_{\epsilon_k}^*|^{d(u_{\epsilon_k}^*)}\Big)^{\frac{1}{d+1}} \le C t^{-\frac{1}{d+1}}
\end{aligned}$$

and

$$\begin{aligned}
\int_{D_t^{\epsilon_k}} |u_{\epsilon_k}^* \circ \psi_{i,\epsilon_k}(x')| dx' &\le \frac{1}{t} \int_{D_t^{\epsilon_k}} |u_{\epsilon_k}^* \circ \psi_{i,\epsilon_k}(x')| J_{N-1}\psi_{i,\epsilon_k}(x') dx'\\
&\le \frac{1}{t} \int_{U_i \cap \partial\Omega_{\epsilon_k}} |u_{\epsilon_k}^*|^{d(u_{\epsilon_k}^*)}
\end{aligned}$$

$$\leq Ct^{-1} \leq Ct^{-\frac{1}{d+1}}.$$

Since t can be chosen arbitrarily large in the inequalities above and using (4.8), (4.9) and (4.10), we obtain

$$\int_{\partial\Omega\cap U_i} |u_0^*| = 0, \quad i = 1, 2, \dots, n,$$

which implies that $u_0^* \in H_0^1(\Omega)$.

To show that u_0^* is a weak solution of (1.3), we consider $\theta \in \mathcal{C}_c^\infty(\Omega)$. Multiplying (1.1) by $E_{\epsilon_k}\theta$ and integrating by parts, we obtain

$$\int_{\Omega_{\epsilon_k}} \nabla u_{\epsilon_k}^* \nabla E_{\epsilon_k}\theta + \int_{\Omega_{\epsilon_k}} u_{\epsilon_k}^* E_{\epsilon_k}\theta = \int_{\Omega_{\epsilon_k}} h(x, u_{\epsilon_k}^*) E_{\epsilon_k}\theta.$$

Taking the limit as $\epsilon_k \to 0$ and using that $u_{\epsilon_k}^* \xrightarrow{E} u_0^*$ in $H^1(\Omega_{\epsilon_k})$ and Lemma 3.3, we obtain that u_0^* satisfies

$$\int_\Omega \nabla u_0^* \nabla\theta + \int_\Omega u_0^*\theta = \int_\Omega h(x, u_0^*)\theta.$$

Therefore, u_0^* is a weak solution of (1.3).

Now, we prove that $u_{\epsilon_k}^* \xrightarrow{E} u_0^*$. In order to do this, we prove the convergence of the norms $\|u_{\epsilon_k}^*\|_{H^1(\Omega_{\epsilon_k})} \to \|u_0^*\|_{H^1(\Omega)}$. In fact, multiplying the equation (1.1) by $u_{\epsilon_k}^*$ and integrating by parts, we obtain

$$\begin{aligned}
\|u_{\epsilon_k}^*\|_{H^1(\Omega_{\epsilon_k})}^2 &= \frac{1}{\epsilon_k}\int_{\omega_{\epsilon_k}} f(x, u_{\epsilon_k}^*)u_{\epsilon_k}^* + \int_{\Omega_{\epsilon_k}} h(x, u_{\epsilon_k}^*)u_{\epsilon_k}^* - \int_{\partial\Omega_{\epsilon_k}} g(x, u_{\epsilon_k}^*)u_{\epsilon_k}^* \\
&\leq \frac{1}{\epsilon_k}\int_{\omega_{\epsilon_k}} f(x, u_{\epsilon_k}^*)u_{\epsilon_k}^* + \int_{\Omega_{\epsilon_k}} h(x, u_{\epsilon_k}^*)u_{\epsilon_k}^*,
\end{aligned}$$

where we have used that $g(x, u)u \geq 0$. Using Lemma 3.3 and $u_0^* \in H_0^1(\Omega)$, we obtain that $\lim_{\epsilon_k\to 0} \|u_{\epsilon_k}^*\|_{H^1(\Omega_{\epsilon_k})}^2 \leq \|u_0^*\|_{H^1(\Omega)}^2$. By [4, Proposition 3.2], we obtain $u_{\epsilon_k}^* \xrightarrow{E} u_0^*$. This completes the proof. □

Remark 4.2. If $h(x, u) = 0$ in (1.1) then the limit problem of (1.1) is given by

$$\begin{gathered}
-\Delta u + u = 0, \quad \text{in } \Omega \\
u = 0, \quad \text{on } \partial\Omega.
\end{gathered} \tag{4.11}$$

By Lax-Milgram Theorem, the unique solution in $H^1(\Omega)$ of (4.11) is given by $u \equiv 0$. Hence, in Theorem 2.4, $u_{\epsilon_k}^* \xrightarrow{E} 0$. Moreover, by uniqueness of solutions of (4.11), we obtain the E-convergence for the whole family $\{u_\epsilon^*\}$ of solution of (1.1), that is, $u_\epsilon^* \xrightarrow{E} 0$.

5. Upper semicontinuity of solutions for the homogeneous boundary conditions (1.2)

In this section, we provide a proof of Theorem 2.5. Initially, we prove boundedness of the solutions of (1.2).

Lemma 5.1. *Assume that* (H1)–(H2) *are satisfied. If* $\{z_\epsilon\}$, $0 < \epsilon \le \epsilon_0$, *is a family of the solutions of* (1.2) *satisfying* $\|z_\epsilon\|_{L^\infty(\Omega_\epsilon)} \le R$, *for some constant* $R > 0$ *independent of* ϵ, *then there exists* $C > 0$ *independent of* ϵ *such that*

$$\|z_\epsilon\|_{H^1(\Omega_\epsilon)} \le C, \quad 0 < \epsilon \le \epsilon_0, \tag{5.1}$$

for some sufficiently small ϵ_0.

Proof. Multiplying (1.2) by z_ϵ and integrating by parts, we obtain

$$\|z_\epsilon\|^2_{H^1(\Omega_\epsilon)} = \int_{\Omega_\epsilon} |\nabla z_\epsilon|^2 + \int_{\Omega_\epsilon} |z_\epsilon|^2 = \frac{1}{\epsilon}\int_{\omega_\epsilon} f(x, z_\epsilon) z_\epsilon + \int_{\Omega_\epsilon} h(x, z_\epsilon) z_\epsilon.$$

Therefore, using (4.3) and (4.4), we obtain

$$[1 - \delta(K_2 + 1)]\|z_\epsilon\|^2_{H^1(\Omega_\epsilon)} \le C_\delta(K_1 + K_3).$$

Now, taking δ such that $\delta(K_2 + 1) < 1$, we obtain (5.1). □

Now, we can prove the upper semicontinuity of the solutions of (1.2).

Proof of Theorem 2.5. Let $\{u^*_\epsilon\}$, $0 < \epsilon \le \epsilon_0$, be a family of the solutions of (1.2) satisfying $\|u^*_\epsilon\|_{L^\infty(\Omega_\epsilon)} \le R$, for some constant $R > 0$ independent of ϵ. Applying Lemma 5.1, the sequence $\{u^*_\epsilon\}$ is bounded in $H^1(\Omega_\epsilon)$. By [5, Lemma 3.1 (i)], there exist a subsequence $\{u^*_{\epsilon_k}\}$ and a function $u^*_0 \in H^1(\Omega)$ such that $u^*_{\epsilon_k} \xrightharpoonup{E} u^*_0$ and $\|u^*_{\epsilon_k} - E_{\epsilon_k} u^*_0\|_{L^2(\Omega_{\epsilon_k})} \to 0$.

To show that u^*_0 is a weak solution of (1.4), we consider $\theta \in H^1(\Omega)$. Multiplying (1.2) by $E_{\epsilon_k}\theta$ and integrating by parts, we obtain

$$\int_{\Omega_{\epsilon_k}} \nabla u^*_{\epsilon_k} \nabla E_{\epsilon_k}\theta + \int_{\Omega_{\epsilon_k}} u^*_\epsilon E_{\epsilon_k}\theta = \frac{1}{\epsilon_k}\int_{\omega_{\epsilon_k}} f(x, u^*_{\epsilon_k}) E_{\epsilon_k}\theta + \int_{\Omega_{\epsilon_k}} h(x, u^*_{\epsilon_k}) E_{\epsilon_k}\theta.$$

Taking the limit as $\epsilon_k \to 0$ and using that $u^*_{\epsilon_k} \xrightharpoonup{E} u^*_0$ in $H^1(\Omega_{\epsilon_k})$ and Lemma 3.3, we obtain that u^*_0 is a weak solution of (1.4).

Now, we prove that $u^*_{\epsilon_k} \xrightarrow{E} u^*_0$. To do this, we prove the convergence of the norms $\|u^*_{\epsilon_k}\|_{H^1(\Omega_{\epsilon_k})} \to \|u^*_0\|_{H^1(\Omega)}$. In fact, multiplying the equation (1.2) by $u^*_{\epsilon_k}$, integrating by parts and again using Lemma 3.3, we obtain

$$\begin{aligned}\|u^*_{\epsilon_k}\|^2_{H^1(\Omega_{\epsilon_k})} &= \frac{1}{\epsilon_k}\int_{\omega_{\epsilon_k}} f(x, u^*_{\epsilon_k}) u^*_{\epsilon_k} + \int_{\Omega_{\epsilon_k}} h(x, u^*_{\epsilon_k}) u^*_{\epsilon_k} \\ &\to \int_{\partial\Omega} \beta f(x, u^*_0) u^*_0 + \int_\Omega h(x, u^*_0) u^*_0 = \|u^*_0\|^2_{H^1(\Omega)}, \quad \text{as } \epsilon_k \to 0.\end{aligned}$$

Hence, we obtain that $\lim_{\epsilon_k \to 0} \|u^*_{\epsilon_k}\|^2_{H^1(\Omega_{\epsilon_k})} \le \|u^*_0\|^2_{H^1(\Omega)}$. By [4, Proposition 3.2], we obtain $u^*_{\epsilon_k} \xrightarrow{E} u^*_0$. □

Conclusion. With the results obtained in this work and proceeding analogously to [5] and [6], we can prove the lower semicontinuity of the families of solutions of (1.1) and (1.2) in $H^1(\Omega_\epsilon)$, in the case where the solutions of the limit problems (1.3) and (1.4) are hyperbolic, and then the convergence of the eigenvalues and eigenfunctions of the linearizations around the solutions. These results are important for understanding the behavior of the dynamics of the parabolic equations associated to the problems (1.1) and (1.2).

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References

[1] G. S. Aragão, S. M. Bruschi; *Concentrated terms and varying domains in elliptic equations: Lipschitz case*, Submitted for publication, arXiv:1412.5850.

[2] G. S. Aragão, A. L. Pereira, M. C. Pereira; *A nonlinear elliptic problem with terms concentrating in the boundary*, Mathematical Methods in the Applied Sciences **35** (2012), no. 9, 1110-1116.

[3] G. S. Aragão, A. L. Pereira, M. C. Pereira; *Attractors for a nonlinear parabolic problem with terms concentrating on the boundary*, Journal of Dynamics and Differential Equations **26** (2014), no. 4, 871-888.

[4] J. M. Arrieta, S. M. Bruschi; *Rapidly varying boundaries in equations with nonlinear boundary conditions. The case of a Lipschitz deformation*, Mathematical Models and Methods in Applied Sciences **17** (2007), no. 10, 1555-1585.

[5] J. M. Arrieta, S. M. Bruschi; *Very rapidly varying boundaries in equations with nonlinear boundary conditions. The case of a non uniformly Lipschitz deformation*, Discrete and Continuous Dynamical Systems Series B **14** (2010), no. 2, 327-351.

[6] J. M. Arrieta, A. N. Carvalho; *Spectral convergence and nonlinear dynamics of reaction-diffusion equations under perturbations of domain*, Journal of Differential Equations **199** (2004), no. 1, 143-178.

[7] J. M. Arrieta, A. Jiménez-Casas, A. Rodríguez-Bernal; *Flux terms and Robin boundary conditions as limit of reactions and potentials concentrating at the boundary*, Revista Matemática Iberoamericana **24** (2008), no. 1, 183-211.

[8] D. Cioranescu, P. Donato; *An introduction to homogenization*, Oxford Lecture Series in Mathematics and its Applications, vol.17, Oxford University Press, New York, 1999.

[9] A. Jiménez-Casas, A. Rodríguez-Bernal; *Aymptotic behaviour of a parabolic problem with terms concentrated in the boundary*, Nonlinear Analysis: Theory, Methods & Applications **71** (2009), 2377-2383.

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