Electronic Journal of Differential Equations, Vol. 2015 (2015), No. 244, pp. 1–9.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

UNIQUENESS OF SOLUTIONS TO BOUNDARY-VALUE PROBLEMS FOR THE BIHARMONIC EQUATION IN A BALL

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Abstract. In this article we study a generalized third boundary-value problem for homogeneous biharmonic equation in a unit ball with general boundary operators up to third order inclusively, containing normal derivatives and Laplacian. A uniqueness theorem for the solution is proved, and some examples are given.

1. Introduction

Mathematical modeling of deformation problems of the plane theory of elasticity is reduced in many cases to problems for the biharmonic equation under the corresponding boundary conditions. Numerous scientific researches [1, 2, 3, 6, 7, 8, 23] are devoted to the application of the biharmonic problems in mechanics and physics. The necessity of modeling of complex processes leads to problems with more general, than classical, boundary conditions.

The Dirichlet boundary-value problem is more well-researched problem for the biharmonic equation. Despite this fact many such problems have not been investigated until the last time. For example, the Green's function of the Dirichlet problem for the polyharmonic equation in the unit ball has been constructed rather recently in [11, 12].

In recent years such boundary-value problems as the problems by Riquier, Neumann, Robin for the biharmonic equation are actively studied. The questions of spectral geometry are investigated both for classical Dirichlet and Neumann problems and for the boundary-value problems of Steklov's type. Due to this fact the questions of well-posedness of the boundary-value problems with more general than classical, boundary conditions acquire relevance.

In this article a boundary-value problem with general boundary conditions for the biharmonic equation in the unit ball is considered. Let $S = \{x \in \mathbb{R}^n : |x| < 1\}$ be n-dimensional unit ball in $\mathbb{R}^n$ and $\partial S = \{x \in \mathbb{R}^n : |x| = 1\}$ be the unit sphere. Hereinafter $|x| = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$.

In the unit ball S we consider the following boundary-value problem for the biharmonic equation

$$\Delta^2 u = f(x), \quad x \in S, \tag{1.1}$$

2010 *Mathematics Subject Classification.* 35J05, 35J25, 26A33.
Key words and phrases. Biharmonic equation; boundary value problem; Laplace operator.
©2015 Texas State University - San Marcos.
Submitted August 27, 2015. Published September 22, 2015.

$$\begin{gathered} a_{00}u + a_{01}\frac{\partial}{\partial\nu}u + a_{02}\Delta u = \varphi_1(s), \quad s \in \partial S, \\ a_{10}u + a_{11}\frac{\partial}{\partial\nu}u + a_{12}\Delta u + a_{13}\frac{\partial}{\partial\nu}\Delta u = \varphi_2(s), \quad s \in \partial S \end{gathered} \tag{1.2}$$

where $\frac{\partial}{\partial\nu}$ is the exterior normal derivative. Here the coefficients a_{0j} and a_{1j} at $j = 1, 2, 3$ are real constants, and $f(x), \varphi_1(x), \varphi_2(x)$ are given sufficiently smooth functions.

As a solution of the problem (1.1)-(1.2) we call a function from the class $u \in C^4(S) \cap C^3(\overline{S})$ turning equation (1.1) and the boundary conditions (1.2) into the identity.

Note that this problem generalizes the classical Dirichlet problem [19, 20] ($a_{00} = 1$, $a_{11} = 1$, and all other coefficients are zero), the Riquier's problem [14] ($a_{00} = 1$, $a_{12} = 1$, and all other coefficients are zero), but does not generalizes the Neumann problem [16, 17, 18]

$$\begin{gathered} \Delta^2 u(x) = f(x), x \in S, \\ \frac{\partial u}{\partial\nu} = \varphi_1(s), \frac{\partial^2 u}{\partial\nu^2} = \varphi_2(s), s \in \partial S. \end{gathered}$$

When $a_{00} = 1$, $a_{12} > 0$, $a_{11} < 0$, and all other coefficients are zero, the conditions (1.2) are called the Steklov's conditions [9].

Problem (1.1)-(1.2) was considered in [22]. The necessary conditions of the solution's uniqueness are found. In particular it was shown that if

$$\begin{vmatrix} a_{00} & a_{01} + na_{02} \\ a_{10} & a_{11} + na_{12} \end{vmatrix} \neq 0, \tag{1.3}$$

then $u =$Const is not a solution of the homogeneous problem (1.1)-(1.2). In this article a criterion of the uniqueness of asolution to (1.1)-(1.2) is established.

Note that for various values of the coefficients a_{0j} and a_{1j} problem (1.1)-(1.2) coincides with the problems considered in [6, 10, 24, 5]. In [28, 4, 25, 29, 21] the existence of solutions to the Neumann problem and other boundary-value problems for the biharmonic equation with an operator of the fractional order in boundary conditions are investigated. Also note that [27, 26, 15, 13] are devoted to the study of various boundary-value problems for elliptic equations in a ball.

2. Main result

Theorem 2.1. *A solution to* (1.1)-(1.2) *is unique if and only if the polynomial*

$$\Delta(t) = \begin{vmatrix} a_{00} + ta_{01} & 2a_{01} + (2n+4t)a_{02} \\ a_{10} + ta_{11} & 2a_{11} + (2n+4t)a_{12} + t(2n+4t)a_{13} \end{vmatrix} \tag{2.1}$$

has no integer roots in $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. *If* $\Delta(m) = 0$ *for some integer nonnegative* $m \in \mathbb{N}_0$, *then the homogeneous problem* (1.1)-(1.2) *has solution*

$$u(x) = \big(C_2|x|^2 + C_1 - C_2\big)H_m(x),$$

where $H_m(x)$ *are homogeneous harmonic polynomials of degree* m, *and the constants* C_1, C_2 *are found from the system of algebraic equations*

$$\begin{pmatrix} a_{00} + ma_{01} & 2a_{01} + (2n+4m)a_{02} \\ a_{10} + ma_{11} & 2a_{11} + (2n+4m)a_{12} + m(2n+4m)a_{13} \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = 0. \tag{2.2}$$

Proof. Suppose that there exist two functions $u_1(x)$ and $u_2(x)$ which are solutions to (1.1)-(1.2). We show that the function $u(x) = u_1(x) - u_2(x)$ should equal to zero.

It is obvious that the function $u(x)$ is biharmonic and satisfies the homogeneous conditions (1.2):

$$\Delta^2 u = 0, \quad x \in S, \tag{2.3}$$

$$\begin{gathered} a_{00}u + a_{01}\frac{\partial}{\partial \nu}u + a_{02}\Delta u = 0, \quad s \in \partial S, \\ a_{10}u + a_{11}\frac{\partial}{\partial \nu}u + a_{12}\Delta u + a_{13}\frac{\partial}{\partial \nu}\Delta u = 0, \quad s \in \partial S. \end{gathered} \tag{2.4}$$

Any biharmonic in S function from the class $u(x) \in C^3(\overline{S})$ can be represented by the Almansi formula in the form (see [15]):

$$u(x) = u_0(x) + |x|^2 v_0(x) = \sum_{m=0}^{\infty}\sum_{i=1}^{h_n}\left(u_m^{(i)} + |x|^2 v_m^{(i)}\right) H_m^{(i)}(x), \quad x \in S, \tag{2.5}$$

where $h_m = \frac{2m+n-2}{n-2}\binom{m+n-3}{n-3}$, and $\{H_m^{(i)}(x), m \in \mathbb{N}_0, i = \overline{1,h_k}\}$ is a complete orthogonal on ∂S system of homogeneous harmonic polynomials [15]. Herewith a series in (2.5) is uniformly converges for $|x| \le \varepsilon < 1$, this series allows termwise differentiation of any order and the obtained series also converge uniformly.

Consider the two operators

$$\begin{gathered} L_1 = a_{00} + a_{01}\Lambda + a_{02}\Delta, \\ L_2 = a_{10} + a_{11}\Lambda + a_{12}\Delta + a_{13}\Lambda\Delta, \end{gathered}$$

where

$$\Lambda = \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i}.$$

Since $u \in C^3(\overline{S})$, then from the properties of the operator Λ (see. [15]) it follows that

$$\begin{gathered} L_1 u(x) \underset{s\in\partial S}{\rightrightarrows} a_{00}u + a_{01}\frac{\partial}{\partial \nu}u + a_{02}\Delta u = 0, \\ L_2 u(x) \underset{s\in\partial S}{\rightrightarrows} a_{10}u + a_{11}\frac{\partial}{\partial \nu}u + a_{12}\Delta\, u + a_{13}\frac{\partial}{\partial \nu}\Delta u = 0, \end{gathered} \quad x \to s \in \partial S. \tag{2.6}$$

It is easy to notice that for every fixed $j = 1, 2$ the polynomials

$$L_j\left(u_m^{(i)} + |x|^2 v_m^{(i)}\right) H_m^{(i)}(x)\Big|_{x=s}$$

are orthogonal on ∂S for all $m \in \mathbb{N}_0$ and $i = \overline{1, h_m}$.

We fix arbitrary $m \in \mathbb{N}_0$ and $i = \overline{1, h_m}$. By virtue of the uniform convergence of the series (2.5) at $|x| \le \varepsilon < 1$, we have

$$\begin{aligned} &\int_{|x|=\varepsilon} H_m^{(i)}(x) L_j u(x) ds_x \\ &= \int_{|x|=\varepsilon} H_m^{(i)}(x) L_j \sum_{p=0}^{\infty}\sum_{k=1}^{h_p}\left(u_p^{(k)} + |x|^2 v_p^{(k)}\right) H_p^{(k)}(x) ds_x \end{aligned}$$

$$= \int_{|x|=\varepsilon} H_m^{(i)}(x) L_j \left(u_m^{(i)} + |x|^2 v_m^{(i)}\right) H_m^{(i)}(x) ds_x.$$

Taking the limit $\varepsilon \to 1$ in this equality and using (2.6), we obtain

$$\int_{|x|=1} H_m^{(i)}(x) L_j \left(u_m^{(i)} + |x|^2 v_m^{(i)}\right) H_m^{(i)}(x) ds_x = 0, \quad j = 1, 2. \tag{2.7}$$

We separately calculate the integrand. For this we note that

$$\Lambda(uw) = w\Lambda u + u\Lambda w,$$
$$\Delta(|x|^2 H_m(x)) = 2nH_m(x) + 4mH_m(x) = (2n+4m)H_m(x).$$

Then on ∂S we have

$$\begin{aligned}
&L_1 \left(u_m^{(i)} + |x|^2 v_m^{(i)}\right) H_m^{(i)}(x) \\
&= (a_{00} + a_{01}\Lambda + a_{02}\Delta) \left(u_m^{(i)} + |x|^2 v_m^{(i)}\right) H_m^{(i)}(x) \\
&= (a_{00} \left(u_m^{(i)} + |x|^2 v_m^{(i)}\right) H_m^{(i)}(x) + a_{01} \left(2|x|^2 v_m^{(i)} + m u_m^{(i)} + m|x|^2 v_m^{(i)}\right) \\
&\quad + a_{02} v_m^{(i)} (2n+4m)) H_m^{(i)}(x) \\
&= \left(u_m^{(i)} (a_{00} + m a_{01}) + v_m^{(i)} (a_{00} + (m+2)a_{01} + (2n+4m)a_{02})\right) H_m^{(i)}(x)
\end{aligned}$$

and

$$\begin{aligned}
&L_2 \left(u_m^{(i)} + |x|^2 v_m^{(i)}\right) H_m^{(i)}(x) \\
&= (a_{10} + a_{11}\Lambda + a_{12}\Delta + a_{13}\Lambda\Delta) \left(u_m^{(i)} + |x|^2 v_m^{(i)}\right) H_m^{(i)}(x) \\
&= \left(a_{10} \left(u_m^{(i)} + |x|^2 v_m^{(i)}\right) + a_{11} \left(m u_m^{(i)} + (m+2)|x|^2 v_m^{(i)}\right)\right) H_m^{(i)}(x) \\
&\quad + \left(a_{12} v_m^{(i)} (2n+4m) + a_{13} v_m^{(i)} m(2n+4m)\right) H_m^{(i)}(x) \\
&= \big(u_m^{(i)} (a_{10} + m a_{11}) + v_m^{(i)} \big(a_{10} + (m+2)a_{11} \\
&\quad + (2n+4m)a_{12} + m(2n+4m)a_{13}\big)\big) H_m^{(i)}(x).
\end{aligned}$$

Therefore equation (2.7) can be rewritten in the form

$$\left(u_m^{(i)} (a_{00} + m a_{01}) + v_m^{(i)} (a_{00} + (m+2)a_{01} + (2n+4m)a_{02})\right) \|H_m^{(i)}(x)\|^2_{L_2(\partial S)} = 0,$$
$$\begin{aligned}
&\big(u_m^{(i)} (a_{10} + m a_{11}) + v_m^{(i)} \big(a_{10} + (m+2)a_{11} + (2n+4m)a_{12} \\
&+ m(2n+4m)a_{13}\big)\big) \|H_m^{(i)}(x)\|^2_{L_2(\partial S)} = 0.
\end{aligned}$$

Since $\|H_m^{(i)}(x)\|^2_{L_2(\partial S)} \neq 0$, then we obtain

$$u_m^{(i)} (a_{00} + m a_{01}) + v_m^{(i)} (a_{00} + (m+2)a_{01} + (2n+4m)a_{02}) = 0,$$
$$u_m^{(i)} (a_{10} + m a_{11}) + v_m^{(i)} (a_{10} + (m+2)a_{11} + (2n+4m)a_{12} + m(2n+4m)a_{13}) = 0,$$

or in the matrix form

$$\begin{pmatrix} a_{00} + m a_{01} & a_{00} + (m+2)a_{01} + (2n+4m)a_{02} \\ a_{10} + m a_{11} & a_{10} + (m+2)a_{11} + (2n+4m)(a_{12} + m a_{13}) \end{pmatrix} \begin{pmatrix} u_m^{(i)} \\ v_m^{(i)} \end{pmatrix} = 0. \tag{2.8}$$

It is easy to see that the determinant of this system is equal to $\Delta(m)$. Therefore because $\Delta(m) \neq 0$ the system (2.8) has the only trivial solution $\begin{pmatrix} u_m^{(i)} \\ v_m^{(i)} \end{pmatrix} = 0$. By virtue of arbitrary choice of indexes $m \in \mathbb{N}_0$ and $i = \overline{1, h_m}$, we obtain that the problem (2.3)-(2.4) has the only trivial solution.

If $\Delta(m) = 0$ for some $m \in \mathbb{N}_0$, then the algebraic system (2.2) has a nontrivial solution $\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} \neq 0$ and hence

$$\begin{pmatrix} a_{00} + ma_{01} & a_{00} + (m+2)a_{01} + (2n+4m)a_{02} \\ a_{10} + ma_{11} & a_{10} + (m+2)a_{11} + (2n+4m)(a_{12} + ma_{13}) \end{pmatrix} \begin{pmatrix} C_1 - C_2 \\ C_2 \end{pmatrix} = 0.$$

Consequently, on ∂S the equalities

$$L_1 \left(C_1 - C_2 + |x|^2 C_2\right) H_m(x) = 0,$$
$$L_2 \left(C_1 - C_2 + |x|^2 C_2\right) H_m(x) = 0$$

are true and therefore $u(x) = C_1 H_m(x) + C_2(|x|^2 - 1)H_m(x)$ is a nontrivial solution of the homogeneous problem (2.3)-(2.4). □

Corollary 2.2. *If $a_{00} = a_{10} = 0$, then solution of the problem* (1.1)-(1.2) *is not unique for all values of all other coefficients in the boundary conditions.*

Proof. As it easily follows from the representation (2.1), in this case $\Delta(0) = 0$ and therefore the system (2.2) has nontrivial solutions of the form $\begin{pmatrix} C_1 \\ 0 \end{pmatrix} \neq 0$. Consequently, the homogeneous problem (1.1)-(1.2) has a nontrivial solution of the form $u =$Const. □

Remark 2.3. If $t = 0$ from (2.1) we have

$$\Delta(0) = 2 \begin{vmatrix} a_{00} & a_{01} + na_{02} \\ a_{10} & a_{11} + na_{12} \end{vmatrix}.$$

Therefore the necessary condition (1.3) from [22] of uniqueness of the solution of the problem (1.1)-(1.2) in our terms can be written in the form

$$\Delta(0) \neq 0$$

and this condition is a particular case of our Theorem 2.1.

3. Particular cases of the problem

1. The Dirichlet problem: let $a_{00} = 1$, $a_{11} = 1$, and all other coefficients are equal to zero, then we have

$$\Delta(t) = \begin{vmatrix} 1 & 0 \\ t & 2 \end{vmatrix} = 2 \neq 0.$$

The uniqueness conditions of the solution of the Dirichlet problem (well-known result) follows from the proved Theorem 2.1.

2. The Riquier's problem [14]: let $a_{00} = 1$, $a_{12} = 1$, and all other coefficients are equal to zero, then the determinant $\Delta(t)$ has the form

$$\Delta(m) = \begin{vmatrix} 1 & 0 \\ 0 & (2n+4m) \end{vmatrix} = (2n + 4m) \neq 0.$$

From Theorem 2.1 proved by us follows the well-known result on the uniqueness of the solution of the Riquier's problem.

3. The Riquier-Neumann problem: let $a_{01} = 1$, $a_{13} = 1$, and all other coefficients are equal to zero

$$\begin{gathered} \Delta^2 u = 0,\ x \in S; \\ \frac{\partial}{\partial \nu} u = \varphi_1(x),\ \frac{\partial}{\partial \nu} \Delta u = \varphi_2(x), \quad x \in \partial S. \end{gathered} \tag{3.1}$$

The corresponding determinant of this problem has the form

$$\Delta(t) = \begin{vmatrix} t & 2 \\ 0 & t(2n+4t) \end{vmatrix} = t^2(2n+4t).$$

It is easy to see that $\Delta(0) = 0$. The corresponding system (2.2) has the form

$$\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = 0,$$

and its solution can be written in the form $C_2 = 0$, C_1 − is an arbitrary constant. By the proved Theorem 2.1 a solution of the problem (3.1) is not unique up to a constant $u(x) = C_1 H_0(x) \equiv C_1$.

4. Consider the problem (1.1)-(1.2) in a particular case when $a_{02} = 0$, $a_{10} = a_{11} = 0$:

$$\begin{gathered} \Delta^2 u = 0,\ x \in S; \\ a_{00} u + a_{01} \frac{\partial}{\partial \nu} u = \varphi_1(s), \quad a_{12} \Delta u + a_{13} \frac{\partial}{\partial \nu} \Delta u = \varphi_2(s), \quad s \in \partial S. \end{gathered} \tag{3.2}$$

The determinant $\Delta(t)$ has the form

$$\begin{aligned} \Delta(t) &= \begin{vmatrix} a_{00} + t a_{01} & 2a_{01} \\ 0 & (2n+4t)a_{12} + t(2n+4t)a_{13} \end{vmatrix} \\ &= (2n+4t)(a_{00} + t a_{01})(a_{12} + t a_{13}). \end{aligned}$$

Consequently, the solution of the problem (3.2) is unique if and only if the equation $(a_{00} + t a_{01})(a_{12} + t a_{13}) = 0$ has no integer non-negative solutions.

Let $a_{00} = -2$, $a_{01} = 1$, $a_{12} = -3$, $a_{13} = 1$ in (3.2), i.e. consider the homogeneous problem

$$\begin{gathered} \Delta^2 u = 0,\ x \in S; \\ -2u + \frac{\partial u}{\partial \nu} = 0,\ \ -3\Delta\, u + \frac{\partial \Delta u}{\partial \nu} = 0,\ x \in \partial S. \end{gathered} \tag{3.3}$$

For this problem $\Delta(t) = (2n+4t)(t-2)(t-3)$ and hence $\Delta(2) = 0$ and $\Delta(3) = 0$. If $m = 2$ the system (2.2) has the form

$$\begin{pmatrix} 0 & 1 \\ 0 & n+4) \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = 0.$$

Solutions of this system have the form $C_2 = 0$, C_1 − is an arbitrary constant. Thus the polynomial $u_2(x) = C_1 H_2(x)$ is a solution of problem (3.3).

If $m = 3$, then the system of (2.2) takes the form

$$\begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} C_3 \\ C_4 \end{pmatrix} = 0.$$

Solutions of this system have the form $C_3 = -2C_4$, C_4 – is an arbitrary constant. Hence the functions $u_3(x) = C_4(|x|^2 - 3)H_3(x)$ are solutions of the homogeneous problem (3.3) according to the proved Theorem 2.1.

Indeed, it is evident that $u_2(x)$ and $u_3(x)$ are biharmonic polynomials. Further, it is easy to calculate that

$$\begin{gathered} L_1 u_2 = -2u_2 + \Lambda u_2 = C_1(-2H_2 + \Lambda H_2) = 0, \\ L_2 u_2 = -3\Delta\, u_2 + \Lambda\Delta u_2 = 0, \\ L_1 u_3 = -2u_3 + \Lambda u_3 = C_4\left(-2(|x|^2 - 3) + (5|x|^2 - 9)\right) H_3(x) \\ = C_4\left(3|x|^2 - 3\right) H_3(x)|_{\partial S} = 0, \\ L_2 u_3 = -3\Delta\, u_2 + \Lambda\Delta u_2 = C_4\left(-3(2n+12) + 3(2n+12)\right) H_3(x) = 0, \end{gathered}$$

i.e., the boundary conditions of the problem (3.3) hold.

So, if solution of the problem (3.3) exists, then it is unique up to polynomials of the form

$$u(x) = C_1 H_2(x) + C_4(|x|^2 - 3)H_3(x)$$

with arbitrary constants C_1 and C_4.

5. The Robin problem [10]: let $a_{02} = a_{10} = a_{11} = 0$, and all other coefficients are positive:

$$\begin{gathered} \Delta^2 u = 0, \quad x \in S; \\ a_{00} u + a_{01} \frac{\partial}{\partial \nu} u = \varphi_1(s), \quad s \in \partial S, \\ a_{12} \Delta u + a_{13} \frac{\partial}{\partial \nu} \Delta u = \varphi_2(s), \quad s \in \partial S. \end{gathered}$$

In this case we have

$$\begin{aligned} \Delta(t) &= \begin{vmatrix} a_{00} + t a_{01} & 2a_{01} \\ 0 & (2n+4t)\, a_{12} + t\,(2n+4t)\, a_{13} \end{vmatrix} \\ &= (2n+4t)\,(a_{00} + t a_{01})\,(a_{12} + t a_{13}) \neq 0. \end{aligned}$$

Hence the Robin problem is unconditionally solvable.

Acknowledgements. The authors would like to thank the editor and referees for their valuable comments and remarks, which led to a great improvement of the article.

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