Electronic Journal of Differential Equations, Vol. 2015 (2015), No. 258, pp. 1–8.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

L^p ESTIMATES FOR DIRICHLET-TO-NEUMANN OPERATOR AND APPLICATIONS

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Abstract. In this article, we consider the time dependent linear elliptic problem with dynamic boundary condition. We recall the corresponding Dirichlet-to-Neumann operator on Γ denoted by $-\Lambda_\gamma$. Then we show that when $\gamma = 1$ near the boundary, $\Lambda_\gamma - \Lambda_1$ is bounded by $\gamma - 1$ in $L^p(\Omega)$ norm. This result is a generalization of the bound with the $L^\infty(\Omega)$ norm and is applicable for comparing the Dirichlet to Neumann semigroup and the Lax semigroup. Finally, we present numerical experiments for validation of our results.

1. Introduction

Let $\Omega \subset \mathbb{R}^2$ be a bounded open set of class C^2, with boundary Γ, and let $]0, T[$ to denote an interval in $\mathbb{R}$ where $T \in (0, +\infty)$ is a fixed final time. We denote by $n(x)$ the unit outward normal vector at $x \in \Gamma$. We intend to work with the following time dependent linear elliptic problem with dynamic boundary condition:

$$\begin{gathered} -\operatorname{div} \gamma(x)\nabla u(t,x) = 0 \quad \text{in }]0,T[\times\Omega, \\ \frac{\partial u}{\partial t}(t,x) + \gamma(x) n(x)\cdot\nabla u(t,x) = 0 \quad \text{on }]0,T[\times\Gamma, \\ u(0,x) = u_0 \quad \text{on } \Gamma, \end{gathered} \tag{1.1}$$

where $\gamma \in L^\infty_+(\Omega)$ and $u_0 \in H^{1/2}(\Gamma)$, and we suppose that there exists a real positive number β such that

$$\beta^{-1} \le \gamma(x) \le \beta \quad \forall x \in \overline{\Omega}.$$

The unknown is u while u_0 is the initial condition at time $t = 0$.

The trace value of the solution $u(t,x)$ on Γ is directly related to the elliptic Dirichlet-to-Neumann map. In fact, for a given f, u^γ solves the Dirichlet problem

$$\begin{gathered} \operatorname{div}(\gamma\nabla u^\gamma) = 0 \quad \text{in } \Omega, \\ u^\gamma = f \quad \text{on } \Gamma. \end{gathered} \tag{1.2}$$

For any $f \in H^{1/2}(\Gamma)$, it is well known that the Dirichlet problem (1.2) is uniquely solvable in $H^1(\Omega)$. We denote by $u^\gamma = L_\gamma f$ where the function u^γ is called the γ-harmonic lifting of f and the operator L_γ is called the γ-harmonic lifting operator.

2010 *Mathematics Subject Classification.* 47D06, 47A99, 35J15, 35M13, 65M60.
Key words and phrases. Dynamic boundary condition; Dirichlet-to-Neumann operator; L^p estimation; finite element method.

Submitted September 5, 2015. Published October 2, 2015.

If u^γ and γ are smooth, the Dirichlet-to-Neumann operator is defined by

$$\Lambda_\gamma f = (n.\gamma\nabla u^\gamma)|_\Gamma\,. \tag{1.3}$$

In another words $\Lambda_\gamma = n\cdot\gamma\nabla L_\gamma$ (see for instance [5]).

We can extend Λ_γ uniquely to an operator $\Lambda_\gamma \in \mathcal{L}(H^{1/2}(\Gamma), H^{-\frac{1}{2}}(\Gamma))$. If we denote its part in $L^2(\Gamma)$ again by Λ_γ, we define the Dirichlet-to-Neumann operator as an unbounded operator with domain

$$D(\Lambda_\gamma) = \{f \in H^{1/2}(\Gamma); \Lambda_\gamma f \in L^2(\Gamma)\}. \tag{1.4}$$

The Dirichlet-to-Neumann operator Λ_γ is positive, self adjoint and a first order pseudo-analytic operator (see for instance [11] and [12]). By Lummer-Phillips theorem, $-\Lambda_\gamma$ generates a C_0 semigroup denoted by $e^{-t\Lambda_\gamma}$ in $L^2(\Gamma)$ (see [13]).

For the existence and the uniqueness of the solution of problem (1.1), we refer to [13, Theorem 1.1, page 169].

Theorem 1.1. *If Γ is of class C^2, γ is of class C^α $(\alpha > 2)$, and for each $u_0 \in L^2(\Gamma)$, problem* (1.1) *has a unique solution $u : [0,+\infty) \to H^1(\Omega)$ satisfying:*

(1) $u \in C([0,+\infty); H^1(\Omega)) \cap L^2([0,+\infty); H^1(\Omega))$;
(2) $u|_\Gamma \in C([0,+\infty); L^2(\Gamma)) \cap C^1([0,+\infty); L^2(\Gamma))$;
(3) $n.\nabla u \in C([0,+\infty); L^2(\Gamma))$.

By taking the trace of the solution to (1.1) and denoting it by $u(t,.)|_\Gamma$, the Dirichlet-to-Neumann semigroup $e^{-t\Lambda_\gamma}u_0$ is defined by

$$(e^{-t\Lambda_\gamma}u_0)(x) = u(t,x)|_\Gamma, \quad x \in \Gamma\,. \tag{1.5}$$

Remark 1.2. Lax introduced an explicit representation for the Dirichlet-to-Neumann semigroup for $\gamma = 1$ and $\Omega = B(0,1)$. The Lax semigroup is defined by

$$(e^{-t\Lambda_1}u_0)(x) = u^1(e^{-t}x) \quad \text{for } x \in \partial B(0,1), \tag{1.6}$$

where $u^1 = L_1 f$ is the harmonic lifting of f (see [7]).

For $\Omega \neq B(0,1)$ there is no explicit representation of the Dirichlet to Neumann semigroup (see [5]). This motivate several authors to construct families of approximation via Chernoff's theorem (see [5, 1]). Here an important question arises: what is the effect of the support of γ on the comparison of the general Dirichlet-to-Neumann semigroup $e^{-t\Lambda_\gamma}$ and the Lax semigroup?

In [2], the authors showed that for $\gamma = 1$ near the boundary, the distance $\|\Lambda_\gamma - \Lambda_1\|_{\mathcal{L}(H^{1/2}(\Gamma),H^s(\Gamma))}$ is bounded by $\|\gamma - 1\|_{L^\infty(\Omega)}$ for any $s \in \mathbb{R}$. The assumption $\gamma = 1$ near the boundary has multiple physical applications, in particular it is usually used in the EIT (electrical Impedance Tomography) community (see [10]).

In this article, we compare the general Dirichlet-to-Neumann semigroup $e^{-t\Lambda_\gamma}$ to the Lax semigroup. We start by comparing Λ_γ to Λ_1 for $\gamma = 1$ near the boundary. In particular we show that $\|\Lambda_\gamma - \Lambda_1\|_{\mathcal{L}(H^{1/2}(\Gamma),H^s(\Gamma))}$ is bounded by $\|\gamma - 1\|_{L^p(\Omega)}$ for all $s \in \mathbb{R}$ and $p > 2$. As a straightforward consequence, we show that for the particular case where $\Omega = B(0,1)$, $\|e^{-t\Lambda_\gamma}u_0 - e^{-t\Lambda_1}u_0\|_{L^2(\Gamma)}$ is also bounded by $\|\gamma - 1\|_{L^p(\Omega)}$. At the end we give a numerical example which justify our theoretical results.

We suppose that $u_0 \in H^{1/2}(\Gamma)$ and introduce the following variational problem in the sense of distributions on $]0,T[$: Find $u(t,.) \in H^1(\Omega)$ such that,

$$\begin{gathered} u(0) = u_0 \quad \text{on } \Gamma, \\ \int_\Omega \gamma(x)\nabla u(t,x)\nabla v(x)\,dx + \frac{d}{dt}\Big(\int_\Gamma u(t,s)v(s)\,ds\Big) = 0, \quad \forall v \in H^1(\Omega). \end{gathered} \tag{1.7}$$

Theorem 1.3 ([4]). *If $u \in L^2(0,T;H^1(\Omega))$ and $u|_\Gamma \in L^\infty(0,T;L^2(\Gamma))$, then problem* (1.1) *is equivalent to the variational problem* (1.7)*. Furthermore, we have the bound*

$$\|\nabla u\|^2_{L^2(0,\tau,L^2(\Omega)^2)} + \|u(\tau,.)\|^2_{L^2(\Gamma)} \le c\|u_0\|^2_{L^2(\Gamma)},$$

where c is a positive constant and $\tau \in]0,T]$.

2. Main result

To avoid the complexity of notations, we denote by $\|\cdot\|_{1/2,s} := \|\cdot\|_{\mathcal{L}(H^{1/2}(\Gamma),H^s(\Gamma))}$. As it was proved in [1], the distance between the General Dirichlet-to-Neumann semigroup $e^{-t\Lambda_\gamma}$ and the Lax semigroup $e^{-t\Lambda_1}$ with respect to the $L^2(\Gamma)$ topology depends directly on the distance γ to 1 with respect to the $L^\infty(\Omega)$ topology. However, as it was proved in [3], the support of $\gamma - 1$ plays an important role in the comparison of the Dirichlet-to-Neumann maps.

In this section, we show that when $\|\gamma - 1\|_{L^p(\Omega)}$, $p > 2$, tends to zero and $\gamma = 1$ near Γ, the general Dirichlet-to-Neumann semigroup $e^{-t\Lambda_\gamma}$ tends to the Lax semigroup $e^{-t\Lambda_1}$. In particular for $t \in]0,T]$, the following estimate holds,

$$\|e^{-t\Lambda_\gamma}u_0 - e^{-t\Lambda_1}u_0\|_{L^2(\Gamma)} \le C(T)\|\gamma - 1\|_{L^p(\Omega)}\|u_0\|_{H^{1/2}(\Gamma)}. \tag{2.1}$$

Like the L^∞ estimate (see [1]), it is clear that this estimate is a straightforward consequence of the following lemma.

Lemma 2.1. *Let $\gamma \in L^\infty_+(\Omega)$ be a positive conductivity satisfying $\gamma = 1$ near Γ. Then for $p > 2$ and for all $s \in \mathbb{R}$, the following estimate holds:*

$$\|\Lambda_\gamma - \Lambda_1\|_{1/2,s} \le C_2\|\gamma - 1\|_{L^p(\Omega)} \tag{2.2}$$

where the constant C_2 depends on s, Ω and β.

Proof. For $\gamma = 1$ near the boundary, the operator $\Lambda_\gamma - \Lambda_1$ is a smoothing operator, i.e. it acts from $H^{1/2}(\Gamma)$ to $H^s(\Gamma)$ for all values of $s \in \mathbb{R}$. Depending on the values of s, the proof is divided into three steps.

Step 1: $s \le -\frac{1}{2}$. Since $H^{-1/2}(\Gamma)$ is continuously embedded in $H^s(\Gamma)$,

$$\|(\Lambda_\gamma - \Lambda_1)f\|_{H^s(\Gamma)} \le C\|(\Lambda_\gamma - \Lambda_1)f\|_{H^{-1/2}(\Gamma)}. \tag{2.3}$$

As shown in [3], the following estimate holds for $p > 1$,

$$\|(\Lambda_\gamma - \Lambda_1)f\|_{H^{-1/2}(\Gamma)} \le C\|\gamma - 1\|_{L^{2p}(\Omega)}\|f\|_{H^{1/2}(\Gamma)}. \tag{2.4}$$

The estimate (2.2) follows by combining (2.3) and (2.4).

Step 2: $s \ge \frac{3}{2}$. First, we recall the following estimate (proved in [2] for $m = \frac{1}{2}$):

$$\|(\Lambda_\gamma - \Lambda_1)f\|_{H^{3/2}(\Gamma)} \le C\|u^\gamma - u^1\|_{H^1(\Omega)}. \tag{2.5}$$

Since

$$\begin{gathered} \operatorname{div}(\gamma\nabla u^\gamma) = 0 \quad \text{in } \Omega, \\ \Delta u^1 = 0 \quad \text{in } \Omega, \end{gathered}$$

$$u^\gamma = u^1 = f \quad \text{on } \Gamma.$$

It is clear that $(u^\gamma - u^1) \in H_0^1(\Omega)$ solves the homogenous Dirichlet problem

$$\begin{gathered}\operatorname{div}(\gamma\nabla(u^\gamma - u^1)) = -\operatorname{div}((\gamma-1)\nabla u^1) \quad \text{in } \Omega,\\ u^\gamma - u^1 = 0 \quad \text{on } \Gamma.\end{gathered}$$

Since $u^1 \in H^1(\Omega)$ and $(\gamma - 1) \in L^\infty_+(\Omega)$, it follows that $\operatorname{div}((\gamma-1)\nabla u^1) \in H^{-1}(\Omega)$. From standard estimates for linear elliptic boundary-value problems, the following estimate holds

$$\|u^\gamma - u^1\|_{H^1(\Omega)} \le C\|\operatorname{div}((\gamma-1)\nabla u^1)\|_{H^{-1}(\Omega)}. \tag{2.6}$$

By denoting $\rho = \operatorname{supp}(\gamma - 1)$ and using the divergence theorem, one gets

$$\begin{aligned}
&\|\operatorname{div}((\gamma-1)\nabla u^1)\|_{H^{-1}(\Omega)}\\
&= \sup_{v\in H_0^1;\|v\|_{H^1(\Omega)}\le 1} |\langle \operatorname{div}((\gamma-1)\nabla u^1), v\rangle|\\
&= \sup_{v\in H_0^1;\|v\|_{H^1(\Omega)}\le 1} \Big|\int_\rho (\gamma-1)\nabla u^1 \nabla v dx\Big|\\
&\le \sup_{v\in H_0^1;\|v\|_{H^1(\Omega)}\le 1} \Big(\int_\rho (\gamma-1)^2|\nabla u^1|^2 dx\Big)^{1/2}\Big(\int_\rho |\nabla v|^2\Big)^{1/2}.
\end{aligned}$$

Since $\|v\|_{H^1(\Omega)} \le 1$ we get (see [3])

$$\begin{gathered}\int_\rho |\nabla v|^2 dx \le 1\,,\\ \int_\rho |\nabla u^1|^{2q'} dx < \infty \quad \text{for } q' > 1.\end{gathered}$$

Now we are able to apply the Holder inequality and we deduce that for $(p', q') \in]1,\infty[^2$ such that $1/p' + 1/q' = 1$,

$$\|\operatorname{div}((\gamma-1)\nabla u^1)\|_{H^{-1}(\Omega)} \le \Big(\int_\rho (\gamma-1)^{2p'}\Big)^{\frac{1}{2p'}}\Big(\int_\rho |\nabla u^1|^{2q'}\Big)^{\frac{1}{2q'}}. \tag{2.7}$$

In [3], the following estimate was proved,

$$\Big(\int_\rho |\nabla u^1|^{2q'}\Big)^{\frac{1}{2q'}} \le C\|u^1\|_{H^1(\Omega)}. \tag{2.8}$$

By denoting $p = 2p'$, combining the energy estimate $\|u^1\|_{H^1(\Omega)} \le C\|f\|_{H^{1/2}(\Gamma)}$ and (2.8), we deduce

$$\|\operatorname{div}((\gamma-1)\nabla u^1)\|_{H^{-1}(\Omega)} \le C\|\gamma-1\|_{L^p(\Omega)}\|f\|_{H^{1/2}(\Gamma)}.$$

Finally

$$\|(\Lambda_\gamma - \Lambda_1)f\|_{\frac{3}{2}} \le C\|\gamma-1\|_{L^p(\Omega)}\|f\|_{H^{1/2}(\Gamma)}.$$

Step 3: $-1/2 < s \le 3/2$. In this case we have $s = (1-\theta)(-\frac{1}{2}) + \theta(3/2)$ for $\theta \in]0,1]$; so the space $H^s(\Gamma)$ is an interpolation space of $H^{-1/2}(\Gamma)$ and $H^{3/2}(\Gamma)$. In other words, $H^s(\Gamma) = [H^{-1/2}(\Gamma), H^{3/2}(\Gamma)]_\theta$ (See [8]). By applying the interpolation inequality we deduce

$$\|(\Lambda_\gamma - \Lambda_1)f\|_{H^s(\Gamma)} \le C\|(\Lambda_\gamma - \Lambda_1)f\|^\theta_{H^{-\frac{1}{2}}(\Gamma)}\|(\Lambda_\gamma - \Lambda_1)f\|^{1-\theta}_{H^{3/2}(\Gamma)}$$

Finally, by using the estimates of step 1 and step 2, we deduce (2.2) for $-1/2 < s \leq 3/2$. □

Theorem 2.2. *For $\gamma = 1$ near Γ such that $\gamma \in C^2(\Omega)$, and $u_0 \in H^{1/2}(\Gamma)$, there exists a constant $C(T)$ depending on β, u_0, and T such that :*

$$\|e^{-t\Lambda_\gamma}u_0 - e^{-t\Lambda_1}u_0\|_{L^2(\Gamma)} \leq C(T)\|\gamma - 1\|_{L^p(\Omega)}. \tag{2.9}$$

The estimate in the above theorem follows directly from (2.2), see [1]. We omit its proof.

3. The discrete problem

For the rest of this article, we assume that $\partial\Omega$ is a polyhedron. To describe the time discretization with an adaptive choice of local time steps, we introduce a partition of the interval $[0, T]$ into equal subintervals $I_n = [t_{n-1}, t_n]$, $1 \leq n \leq N$, such that $0 = t_0 \leq t_1 \leq \dots \leq t_N = T$. We denote by τ the length of the subintervals I_n.

Now, we describe the space discretization. Let $(\mathcal{T}_h)_h$ be a regular triangulation of Ω. $(\mathcal{T}_h)_h$ is a set of non degenerate elements which satisfies:

- for each h, $\bar{\Omega}$ is the union of all elements of $\mathcal{T}_h$;
- the intersection of two distinct elements of $\mathcal{T}_h$, is either empty, a common vertex, or an entire common edge;
- the ratio of the diameter of an element κ in $\mathcal{T}_h$ to the diameter of its inscribed circle is bounded by a constant independent of n and h.

As usual, h denotes the maximal diameter of the elements of all $\mathcal{T}_h$. For each κ in $\mathcal{T}_h$, we denote by $P_1(\kappa)$ the space of restrictions to κ of polynomials with two variables and total degree at most one.

For a given triangulation $\mathcal{T}_h$, we define by X_h a finite dimensional space of functions such that their restrictions to any element κ of $\mathcal{T}_h$ belong to a space of polynomials of degree one. In other words,

$$X_h = \{v_n^h \in C^0(\overline{\Omega}),\ v_h h|_\kappa \text{ is affine for all } \kappa \in \mathcal{T}_h\}.$$

We note that for each h, $X_h \subset H^1(\Omega)$.

The full discrete implicit scheme associated with the problem (1.7) is as follows: Given $u_h^{n-1} \in X_h$, find u_h^n with values in X_h such that for all $v_h \in X_h$ we have:

$$\int_\Omega \gamma(x)\nabla u_h^n \nabla v_h dx + \int_\Gamma \frac{u_h^n - u_h^{n-1}}{\tau_n} v_h d\sigma = 0. \tag{3.1}$$

by assuming that u_h^0 is an approximation of $u(0)$ in X_h.

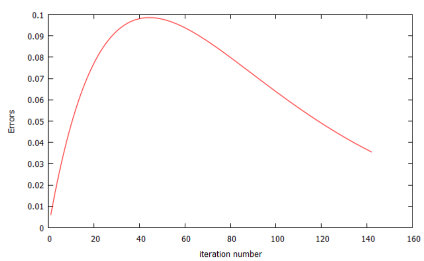
Remark 3.1. It is a simple exercise to prove existence and uniqueness of the solution of problem (3.1) as a consequence of discrete problem of Poisson's equation with a Robin condition.

Theorem 3.2. *For each $m = 1, \dots, N$, the solution u_h^m of the problem* (3.1) *satisfies*

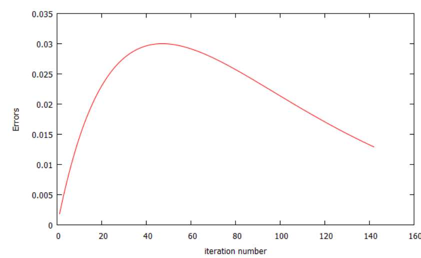
$$\|u_h^m\|_{0,\Gamma}^2 + \sum_{n=1}^{m} \tau_n |u_h^n|_{1,\Omega}^2 \leq c\|u_h^0\|_{0,\Gamma}^2, \tag{3.2}$$

Remark 3.3. In [4], we establish optimal *a priori* and *a posteriori* error estimates for the problem (3.1) an shown numerical results of validation.

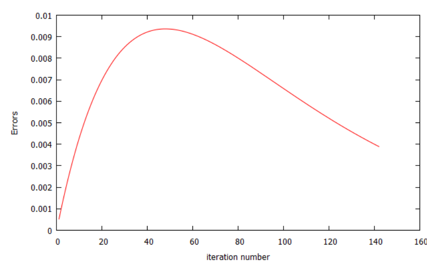
4. Numerical results



(a) Err_u^n with respect to the iteration numbers for $\gamma^1_{5,3/4}$, ($\mathrm{Err}_\gamma^4 = 0.86$)



(b) Err_u^n with respect to the iteration numbers for $\gamma^2_{10,3/4}$, ($\mathrm{Err}_\gamma^4 = 1.14$)



(c) Err_u^n with respect to the iteration numbers for $\gamma^3_{e^8,1/2}$, ($\mathrm{Err}_\gamma^4 = 0.84$)

Figure 1. Err_u^n with respect to the iteration numbers for different functions $\gamma^i_{\alpha,\rho}$, $i = 1, 2, 3$.

To validate the theoretical results, we present several numerical simulations using the FreeFem++ software (see [6]). We choose $T = 3$,

$$u(0,x,y) = \frac{x^2 - y^2}{2} + y + \frac{1}{2},$$

and the function γ as (see [9])

$$\gamma^i_{\alpha,\rho}(x) = (\alpha F_{i,\rho}(|x|) + 1)^2, i = 1, 2, 3, \tag{4.1}$$

where the function $F_{i,\rho} \in C^4(\mathbb{R})$ satisfies $F_{i,\rho}(x) = 0$ for $|x| > \rho$ and for $|x| \le \rho$ takes one of the following three forms:

$$F_{1,\rho}(x) = (x^2 - \rho^2)^4 (1.5 - \cos\frac{3\pi x}{2\rho}), \tag{4.2}$$

$$F_{2,\rho}(x) = (x^2 - \rho^2)^4 \cos\frac{3\pi x}{2\rho}, \tag{4.3}$$

$$F_{3,\rho}(x) = e^{-\frac{2(x^2+\rho^2)}{(x+\rho)^2(x-\rho)^2}}. \tag{4.4}$$

We consider the two-dimensional unit circle. In fact, the mesh corresponding to Ω is a polygon and we introduce here a geometrical approximation. Nevertheless, the numerical results given in the end of this section show that this approximation has not a major influence. The considered mesh contains 15542 triangles with $m = 300$ segments on the boundary Γ. Thus, the mesh step size is $h = \frac{2\pi}{m}$. We choose a time step $\tau = h$ and we consider the numerical scheme (3.1).

We denote by $u^n_{h,\gamma}$ the solution of problem (3.1) for a given γ and $u^n_{h,1}$ the solution of the same problem for $\gamma = 1$. We define the errors

$$
\begin{gathered}
\mathrm{Err}^n_u = \|u^n_{h,\gamma} - u^n_{h,1}\|_{L^2(\Gamma)}, \\
\mathrm{Err}_u = \max_{1 \le i \le N} \mathrm{Err}^i_u, \\
\mathrm{Err}^p_\gamma = \|\gamma - 1\|_{L^p(\Omega)}.
\end{gathered}
$$

We choose $p = 4$ and followed [9] for the choice of ρ and α. Figures 1(a)-(c) show the evolution of Err^n_u with respect to the iteration numbers for the three cases of γ. It is easy to check that all this curves are bounded and smaller than the corresponding Err^4_γ. For example, Figure 1(b) represents the error Err^u_γ for the second function $\gamma^2_{10,3/4}$ with a maximum of 0.0309 which is smaller the corresponding $\mathrm{Err}^4_u = 1.14$.

To show the dependency of this errors with ρ, in an other word where it equals to 1 in a neighborhood of Γ (the neighborhood depends on ρ), table 1 shows Err_u and Err^4_γ with respect to ρ for the functions $\gamma^1_{5,\rho}$ and $\gamma^2_{10,\rho}$, and for $T = 1$ and $p = 4$. We remark that Err_u is always smaller than Err^4_γ in all the considered cases.

Table 1. Err_u and Err^4_γ with respect to ρ for the three cases of γ: $\gamma^1_{5,\rho}$ and $\gamma^2_{10,\rho}$.

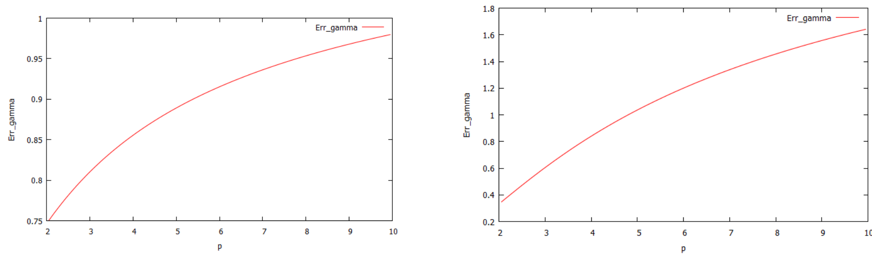
$\gamma^1_{5,\rho}$										
ρ	0.5	0.55	0.6	0.65	0.7	0.75	0.8	0.85	0.9	0.95
Err_u	0.002	0.005	0.012	0.026	0.053	0.098	0.169	0.267	0.391	0.537
Err_γ	0.022	0.051	0.109	0.2232	0.44	0.855	1.65	3.23	6.37	12.72

$\gamma^2_{10,\rho}$										
ρ	0.5	0.55	0.6	0.65	0.7	0.75	0.8	0.85	0.9	0.95
Err_u	0.0002	0.0007	0.0018	0.0047	0.01182	0.02999	0.077	0.200	0.4833	0.5680
Err_γ	0.0263	0.0601	0.1301	0.2716	0.5574	1.1440	2.3757	5.0105	10.6903	22.8861

To show the dependency with p, we consider for example the functions $\gamma^1_{5,3/4}$ and $\gamma^3_{e^8,1/2}$ and we study the errors for different values of $p > 2$. Figures 2(a) and 2(b) show Err^p_γ with respect to p. We remark that the corresponding curves increase with p starting from 0.75 for Figure 2(a) and from 0.34 for the Figure 2(b), whereas the values of Err_u are $0,03$ for the first case $\gamma^1_{5,3/4}$ and 0.08 for the third one $\gamma^3_{e^8,1/2}$.

We remark that all the numerical results validate the theoretical estimates.

Acknowledgments. The authors want to thank the anonymous referees for their careful reading of the orignal manuscript and for their suggestions.



(a) Err_γ^p with respect to p for $\gamma^1_{5,3/4}$ (b) Err_γ^p with respect to p $\gamma^3_{e^8,1/2}$

Figure 2. Err_γ^p with respect to p for the first and the third function $\gamma^i_{\alpha,\rho}, i = 1, 3$.

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