INVERSE SPECTRAL AND INVERSE NODAL PROBLEMS FOR ENERGY-DEPENDENT STURM-LIOUVILLE EQUATIONS WITH \( \delta \)-INTERACTION

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Abstract. In this article, we study the inverse spectral and inverse nodal problems for energy-dependent Sturm-Liouville equations with \( \delta \)-interaction. We obtain uniqueness, reconstruction and stability using the nodal set of eigenfunctions for the given problem.

1. Introduction

We consider the boundary value problem (BVP) generated by the differential equation

\[-y'' + q(x)y = \lambda^2 y, \quad x \in (0, \frac{\pi}{2}] \cup (\frac{\pi}{2}, \pi) \quad (1.1)\]

with the boundary conditions

\[U(y) := y(0) = 0, \quad V(y) := y'(\pi) = 0 \quad (1.2)\]

and at the point \( x = \frac{\pi}{2} \) satisfying

\[y(\frac{\pi}{2} + 0) = y(\frac{\pi}{2} - 0) = y(\frac{\pi}{2}), \quad y'(\frac{\pi}{2} + 0) - y'(\frac{\pi}{2} - 0) = 2\alpha \lambda y(\frac{\pi}{2}) \quad (1.3)\]

where \( q(x) \) is a nonnegative real valued function in \( L_2(0, \pi) \), \( \alpha \neq \pm 1 \) is a real number and \( \lambda \) is a spectral parameter. Without loss of generality we assume that

\[\int_0^\pi q(x)dx = 0. \quad (1.4)\]

We denote the BVP (1.1), (1.2) and (1.3) by \( L = L(q, \alpha) \).

Notice that, we can understand problem (1.1) and (1.3) as studying the equation

\[y'' + (\lambda^2 - 2\alpha p(x) - q(x))y = 0, \quad x \in (0, \pi) \quad (1.5)\]

when \( p(x) = \alpha \delta(x - \frac{\pi}{2}) \), where \( \delta(x) \) is the Dirac function (see [2]).

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We consider the inverse problems of recovering \( q(x) \) and \( \alpha \) from the given spectral and nodal characteristics. Such problems play an important role in mathematics and have many applications in natural sciences (see, for example, monographs [7, 16, 19, 24]). Inverse nodal problems consist in constructing operators from the given nodes (zeros) of eigenfunctions (see [5, 12, 15, 20, 27]). Discontinuous inverse problems (in various formulations) have been considered in [3, 8, 14, 26, 28, 29, 30].

Sturm-Liouville spectral problems with potentials depending on the spectral parameter arise in various models quantum and classical mechanics. There \( \lambda^2 \) is related to the energy of the system, this explaining the term “energy-dependent” in (1.5). The non-linear dependence of equation (1.5) on the spectral parameter \( \lambda \) should be regarded as a spectral problem for a quadratic operator pencil. The inverse spectral and nodal problems for energy-dependent Schrödinger operators with \( p(x) \in W^1_2(0,1) \) and \( q(x) \in L^2[0,1] \) and with Robin boundary conditions was discussed in [4], [10]. Such problems for separated and nonseparated boundary conditions were considered (see [1, 9, 32] and the references therein).

The inverse scattering problem for equation (1.5) with eigenparameter-dependent boundary condition on the half line solved in [17].

In this article we obtain some results on inverse spectral and inverse nodal problems and establish connections between them.

2. Inverse spectral problems

In this section we study so-called incomplete inverse problem of recovering the potential \( q(x) \) from a part of the spectrum BVP \( L \). The technique employed is similar to those used in [11, 25]. Similar problems for the Sturm-Liouville and Dirac operators were formulated and studied in [22, 23].

Let \( y(x) \) and \( z(x) \) be continuously differentiable functions on the intervals \((0,\pi/2)\) and \((\pi/2,\pi)\). Denote \( \langle y, z \rangle := y'z' - y'z \). If \( y(x) \) and \( z(x) \) satisfy the matching conditions (1.3), then

\[
\langle y, z \rangle_{x=\pi} - \langle y, z \rangle_{x=0} = \langle y, z \rangle_{x=\pi + 0}
\]

(2.1)

i.e. the function \( \langle y, z \rangle \) is continuous on \((0,\pi)\).

Let \( \varphi(x,\lambda) \) be solution of equation (1.1) satisfying the initial conditions \( \varphi(0,\lambda) = 0, \varphi'(0,\lambda) = 1 \) and the matching condition (1.3). Then \( U(\varphi) = 0 \). Denote

\[
\Delta(\lambda) := -V(\varphi) = -\varphi'(\pi,\lambda).
\]

(2.2)

By (2.1) and the Liouville’s formula (see [6, p.83]), \( \Delta(\lambda) \) does not depend on \( x \). The function \( \Delta(\lambda) \) is called characteristic function on \( L \).

Lemma 2.1. The eigenvalues of the BVP \( L \) are real, nonzero and simple.

Proof. Suppose that \( \lambda \) is an eigenvalue BVP \( L \) and that \( y(x,\lambda) \) is a corresponding eigenfunction such that \( \int_0^\pi |y(x,\lambda)|^2dx = 1 \). Multiplying both sides of (1.1) by \( y(x,\lambda) \) and integrate the result with respect to \( x \) from 0 to \( \pi \):

\[
- \int_0^\pi y''(x,\lambda)y(x,\lambda)dx + \int_0^\pi q(x)|y(x,\lambda)|^2dx = \lambda^2 \int_0^\pi |y(x,\lambda)|^2dx
\]

(2.3)

Using the formula of integration by parts and the conditions (1.2) and (1.3) we obtain

\[
\int_0^\pi y''(x,\lambda)y(x,\lambda)dx = -2\alpha\lambda|y(0,\lambda)|^2 - \int_0^\pi |y'(x,\lambda)|^2dx.
\]
It follows from this and (2.3) that
\[ \lambda^2 + B(\lambda)\lambda + C(\lambda) = 0, \] (2.4)
where
\[ B(\lambda) = -2\alpha |y(0, \lambda)|^2, \]
\[ C(\lambda) = -\int_0^\pi q(x)|y(x, \lambda)|^2dx - \int_0^\pi |y'(x, \lambda)|^2dx. \]

Thus the eigenvalue \( \lambda \) of the BVP \( L \) is a root of the quadratic equation (2.4). Therefore, \( B^2(\lambda) - 4C(\lambda) > 0 \). Consequently, the equation (2.4) has only real roots.

Let us show that \( \lambda_0 \) is a simple eigenvalue. Assume that this is not true. Suppose that \( y_1(x) \) and \( y_2(x) \) are linearly independent eigenfunctions corresponding to the eigenvalue \( \lambda_0 \). Then for a given value of \( \lambda_0 \), each solution \( y_0(x) \) of (1.5) will be given as linear combination of solutions \( y_1(x) \) and \( y_2(x) \). Moreover it will satisfy boundary conditions (1.2) and conditions (1.3) at the point \( x = \pi/2 \). However it is impossible.

Lemma 2.2. The BVP \( L \) has a countable set of eigenvalues \( \{\lambda_n\}_{n \geq 1} \). Moreover, as \( n \to \infty \),
\[ \lambda_n := n - \frac{\theta}{\pi} + \frac{1}{2(\pi n - \theta)}(w_0 + (-1)^{n-1}w_1) + o\left(\frac{1}{n}\right), \] (2.5)
where
\[ \tan \theta = \frac{1}{\alpha}, \quad w_0 = \int_0^\pi q(t)dt, \quad w_1 = \frac{\alpha}{\sqrt{1 + \alpha^2}}\left(\int_0^{\pi/2} q(t)dt - \int_{\pi/2}^\pi q(t)dt\right). \] (2.6)

Proof. Let \( \tau := \text{Im} \lambda \). For \( |\lambda| \to \infty \) uniformly in \( x \) one has (see [31, Chapter 1])
\[ \varphi(x, \lambda) = \frac{\sin \lambda x}{\lambda} - \frac{\cos \lambda x}{2\lambda^2} \int_0^x q(t)dt + o\left(\frac{1}{\lambda^2} \exp(|\tau|x)\right), \quad x < \frac{\pi}{2}, \] (2.7)
\[ \varphi(x, \lambda) = \frac{1}{\lambda^2} \left(\sqrt{1 + \alpha^2 \cos(\lambda x + \theta)} + \alpha \cos \lambda (\pi - x)\right) + \sqrt{1 + \alpha^2 \cos(\lambda x + \theta)} - \int_0^x q(t)dt \]
\[ + \alpha \frac{\sin \lambda (\pi - x)}{2\lambda^2} \left(\int_0^{\pi/2} q(t)dt - \int_{\pi/2}^x q(t)dt\right) + o\left(\frac{1}{\lambda^2} \exp(|\tau|x)\right), \quad x > \frac{\pi}{2}, \] (2.8)
\[ \varphi'(x, \lambda) = \cos \lambda x + \frac{\sin \lambda x}{2\lambda} \int_0^x q(t)dt + o\left(\frac{1}{\lambda} \exp(|\tau|x)\right), \quad x < \frac{\pi}{2} \] (2.9)
\[ \varphi'(x, \lambda) = \frac{1}{\lambda} \left(\sqrt{1 + \alpha^2 \sin(\lambda x + \theta)} + \alpha \sin \lambda (\pi - x)\right) + \sqrt{1 + \alpha^2 \sin(\lambda x + \theta)} - \int_0^x q(t)dt \]
\[ - \alpha \frac{\cos \lambda (\pi - x)}{2\lambda} \left(\int_0^{\pi/2} q(t)dt - \int_{\pi/2}^x q(t)dt\right) + o\left(\frac{1}{\lambda} \exp(|\tau|x)\right), \quad x > \frac{\pi}{2} \] (2.10)
It follows from (2.10) that as $|\lambda| \to \infty$
\[
\Delta(\lambda) = \sqrt{1 + \alpha^2 \sin(\lambda \pi + \theta)} - \sqrt{1 + \alpha^2 \cos(\lambda \pi + \theta)} \int_{0}^{\pi} q(t)dt + \frac{\alpha}{2\lambda} \left( \int_{0}^{\pi/2} q(t)dt - \int_{\pi/2}^{\pi} q(t)dt \right) + o\left(\frac{1}{\lambda}\right). \tag{2.11}
\]

Using (2.11) and Rouché's theorem, by the well-known method (see [7]) one has that as $n \to \infty$,
\[
\lambda_n := n - \frac{\theta}{\pi} + \frac{1}{2(\pi n - \theta)}(w_0 + (-1)^{n-1}w_1) + o\left(\frac{1}{n}\right). \nonumber
\]

Together with $L$ we consider a BVP $\tilde{L} = \tilde{L}(\tilde{q}, \alpha)$ of the same form but with different coefficient $\tilde{q}$. The following theorem has been proved in [13] for the Sturm-Liouville equation. We show it also holds for (1.1)-(1.3).

**Theorem 2.3.** If for any $n \in \mathbb{N} \cup \{0\}$,
\[
\lambda_n = \tilde{\lambda}_n, \quad \langle y_n, \tilde{y}_n \rangle_{x=\pi/2-0} = 0,
\]
then $q(x) = \tilde{q}(x)$ almost everywhere (a.e) on $(0, \pi)$.

**Proof.** Since
\[-g''(x, \lambda) + q(x)y(x, \lambda) = \lambda^2 y(x, \lambda), \quad -\tilde{g}''(x, \lambda) + \tilde{q}(x)\tilde{y}(x, \lambda) = \lambda^2 \tilde{y}(x, \lambda),
\]
y($0, \lambda$) = 0, $\tilde{y}'(0, \lambda) = 1$, $\tilde{y}(0, \lambda) = 0$, $\tilde{y}'(0, \lambda) = 1$,
it follows from (2.1) that
\[
\int_{0}^{\pi/2} r(x)y(x, \lambda)\tilde{y}(x, \lambda)dx = \langle y, \tilde{y} \rangle_{x=\pi/2-0} \tag{2.12}
\]
where $r(x) = q(x) - \tilde{q}(x)$. Since $\langle y_n, \tilde{y}_n \rangle_{x=\pi/2-0} = 0$ for $n \in \mathbb{N} \cup \{0\}$, it follows from (2.12) that
\[
\int_{0}^{\pi/2} r(x)y(x, \lambda_n)\tilde{y}(x, \lambda_n)dx = 0, \quad n \in \mathbb{N} \cup \{0\}. \tag{2.13}
\]
For $x \leq \pi/2$ the following representation holds (see [16] [19]):
\[
y(x, \lambda) = \frac{\sin \lambda x}{\lambda} + \int_{0}^{x} K(x, t) \frac{\sin \lambda x}{\lambda} dt,
\]
where $K(x, t)$ is a continuous function which does not depend on $\lambda$. Hence
\[
2\lambda^2 y(x, \lambda)\tilde{y}(x, \lambda) = 1 - \cos 2\lambda x - \int_{0}^{x} V(x, t) \cos 2\lambda t dt, \tag{2.14}
\]
where $V(x, t)$ is a continuous function which does not depend on $\lambda$. Substituting (2.14) into (2.13) and taking the relation (1.4) into account, we calculate
\[
\int_{0}^{\pi/2} \left( r(x) + \int_{x}^{\pi/2} V(t, x)r(x)dt \right) \cos 2\lambda_n xdx = 0, \quad n \in \mathbb{N} \cup \{0\},
\]
which implies from the completeness of the function cosine, that
\[
r(x) + \int_{x}^{\pi/2} V(t, x)r(x)dt = 0 \quad \text{a.e. on } [0, \frac{\pi}{2}].
\]
But this equation is a homogeneous Volterra integral equation and has only the zero solution, it follows that \( r(x) = 0 \) a.e. on \([0, \pi/2]\). To prove that \( q(x) = \tilde{q}(x) \) a.e. on \([\pi/2, \pi]\) we will consider the supplementary problem \( \hat{L} \):

\[
-y''(x, \lambda) + q_1(x)y(x, \lambda) = \lambda^2 y(x, \lambda), \quad q_1(x) = q(\pi - x), \quad 0 < x < \pi/2,
\]

\[
U(y) := y(0, \lambda) = 0,
\]

\[
y\bigg(\frac{\pi}{2} + 0, \lambda\bigg) = y\bigg(\frac{\pi}{2} - 0, \lambda\bigg), \quad y\bigg(\frac{\pi}{2} + 0, \lambda\bigg) - y\bigg(\frac{\pi}{2} - 0, \lambda\bigg) = 2\alpha \lambda y\bigg(\frac{\pi}{2} - 0, \lambda\bigg).
\]

It follows from (2.1) that \( \langle y_n, \tilde{y}_n \rangle_{x=\pi/2} = 0 \). A direct calculation implies that \( \tilde{y}_n(x) := y_n(\pi - x) \) is the solution to the supplementary problem \( \hat{L} \), the \( \hat{L} \) and \( \tilde{y}_n\bigg(\frac{\pi}{2} - 0\bigg) = y_n\bigg(\frac{\pi}{2} + 0\bigg) \). Thus for the supplementary problem \( \hat{L} \) the assumption conditions in Theorem 2.3 are still satisfied. If we repeat the above arguments then yields \( r(\pi - x) = 0 \) and \( 0 < x < \pi/2 \), that is \( q(x) = \tilde{q}(x) \) a.e. on \([\pi/2, \pi]\). \( \square \)

3. INVERSE NODAL PROBLEMS

In this section, we obtain uniqueness theorems and a procedure of recovering the potential \( q(x) \) on the whole interval \((0, \pi)\) from a dense subset of nodal points.

The eigenfunctions of the BVP \( L \) have the form \( y_n(x) = \varphi(x, \lambda_n) \). We note that \( y_n(x) \) are real-valued functions. Substituting (2.5) into (2.7) and (2.8) we obtain the following asymptotic formulae for \( n \to \infty \) uniformly in \( x \):

\[
\lambda_n y_n(x) = \sin(n - \frac{\theta}{\pi})x + \frac{1}{2(\pi n - \theta)} \left( -\pi \int_0^x q(t) dt + (w_0 + (-1)^{n-1}w_1)x \right)
\]

\[
\times \cos(n - \frac{\theta}{\pi})x + o\left(\frac{1}{n}\right), \quad x < \frac{\pi}{2}
\]

\[
(\lambda_n y_n(x) = \cos((n - \frac{\theta}{\pi})x + \theta)) \sqrt{1 + \alpha^2} + (-1)^n \alpha
\]

\[
+ \frac{1}{2(\pi n - \theta)} \left[ \pi \sqrt{1 + \alpha^2} \int_0^x q(t) dt + (-1)^{n-1} \alpha \pi \left( \int_0^{\pi/2} q(t) dt - \int_{\pi/2}^x q(t) dt \right) \right.
\]

\[
- \left( \sqrt{1 + \alpha^2} x + (-1)^{n-1} \alpha \right) (w_0 + (-1)^{n-1} w_1) \right]
\]

\[
\times \sin((n - \frac{\theta}{\pi})x + \theta) + o\left(\frac{1}{n}\right), \quad x > \frac{\pi}{2}
\]

(3.2)

For the BVP \( L \) an analog of Sturm’s oscillation theorem is true. More precisely, the eigenfunction \( y_n(x) \) has exactly \( (n - 1) \) (simple) zeros inside the interval \((0, \pi)\) : \( 0 < x_1^{n} < x_2^{n} < \cdots < x_{n-1}^{n} < \pi \). The set \( X_L := \{x_n\}_{n \geq 2, j=1,n-1}^{\infty} \) is called the set of nodal points of the BVP \( L \). Denote \( X_L^k := \{x_{2m-k}^{n} \}_{m \geq 1, j=1,2m-k-1}^{\infty} \), \( k = 0, 1 \). Clearly, \( X_L^0 \cup X_L^1 = X_L \). Denote \( \mu_n^0 := 0 \), \( \mu_n^1 := 1 \), \( \mu_n^k := \frac{j}{\pi n - \theta} \pi^2 \), \( \gamma_n^j := \mu_n^j - \pi^2 + 2\pi \frac{n}{\pi n - \theta} \), \( j = 1, n - 1 \).

Inverse nodal problems consist in recovering the problem \( q(x) \) from the given set \( X_L \) of nodal points or from a certain part.

Taking (3.1)-(3.2) into account, we obtain the following asymptotic formulae for nodal points as \( n \to \infty \) uniformly in \( j \):
for $x_n^j \in (0, \frac{\pi}{2})$:

$$x_n^j = \mu_n^j + \frac{\pi}{2(\pi n - \theta)} \left( \int_0^{\pi/2} q(t) dt - (w_0 + (-1)^n w_1) \mu_n^j \right) + o\left(\frac{1}{n^2}\right),$$  \tag{3.3}

for $x_n^j \in (\frac{\pi}{2}, \pi)$:

$$x_n^j = \gamma_n^j + \frac{\pi}{2(\pi n - \theta)} \left[ \int_0^{\gamma_n^j} q(t) dt - ((w_0 + (-1)^{n-1} w_1) \gamma_n^j + d_k) \right] + o\left(\frac{1}{n^2}\right),$$  \tag{3.4}

where $d_k = \sqrt{1 + \alpha^2 + (-1)^{n-1} \alpha} \left[ 2(-1)^{n-1} \alpha \pi \int_0^{\pi/2} q(t) dt + (-1)^n \alpha \pi (w_0 + (-1)^{n-1} w_1) \right]$.  \tag{3.5}

Using these formulae we arrive at the following assertion.

**Theorem 3.1.** Fix $k \in \{0, 1\}$ and $x \in [0, \pi]$. Let $\{x_n^j\} \subset X_k^k$ be chosen such that $\lim_{n \to \infty} x_n^j = x$. Then there exists a finite limit

$$g_k(x) := \lim_{n \to \infty} \frac{2(\pi n - \theta)}{\pi} \left[ (\pi n - \theta)x_n^j - \begin{cases} j\pi, & \text{if } x_n^j \in (0, \frac{\pi}{2}) \\ (j + \frac{1}{2})\pi + \theta, & \text{if } x_n^j \in (\frac{\pi}{2}, \pi) \end{cases} \right]$$  \tag{3.6}

and

$$g_k(x) = \int_0^x q(t) dt - \frac{w_0 + (-1)^{k-1} w_1}{\pi} x, \quad x \leq \frac{\pi}{2}$$  \tag{3.7}

$$g_k(x) = \int_0^x q(t) dt - \frac{w_0 + (-1)^{k-1} w_1}{\pi} x + d_k, \quad x \geq \frac{\pi}{2}$$

where $d_0$ and $d_1$ are defined by (3.5).

Let us now formulate a uniqueness theorem and provide a constructive procedure for the solution of the inverse nodal problem.

**Theorem 3.2.** Fix $k = 0 \lor 1$. Let $X \subset X_k^k$ be a subset of nodal points which is dense on $(0, \pi)$. Let $X = \tilde{X}$. Then $q(x) = \tilde{q}(x)$ a.e. on $(0, \pi)$, $\alpha = \tilde{\alpha}$. Thus the specification of $X$ uniquely determines the potential $q(x)$ on $(0, \pi)$ and the number $\alpha$. The function $q(x)$ and the number $\alpha$ can be constructed via the formulae

$$q(x) = g_k(x) + \frac{1}{\pi} (g_k(\pi) - g_k(0)),$$  \tag{3.8}

$$\alpha = \left[ \frac{2g_k(\pi) + 4g_k(\frac{\pi}{2}) - 6g_k(0)}{\pi (g_k'(x) - g_k'(x))} \right]^2 - 1^{-2}$$  \tag{3.9}

where $g_k(x)$ is calculated by (3.7).

**Proof.** Formulae (3.8), (3.9) follow from (3.7), (3.4) and (2.6). Note that by (3.7), we have

$$g_k(x) = q(x) - \frac{w_0 + (-1)^k w_1}{\pi}, \quad x \in (0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi),$$  \tag{3.10}

hence

$$g_k(\pi) - g_k(0) = \int_0^{\pi} q(x) dx - (w_0 + (-1)^{n-1} w_1), \quad w_1 = \frac{\pi}{2} [g_0'(x) - g_1'(x)].$$  \tag{3.11}
Then (3.8) can be derived directly from (3.10) and (3.11). Similarly, we can derive (3.9). Note that if \( X = \tilde{X} \), then (3.6) yields \( q_k(x) \equiv \tilde{q}_k(x) \), \( x \in [0, \pi] \). By (3.8), (3.9), we obtain \( q_k(x) = \tilde{q}_k(x) \) a.e. on \( [0, \pi] \), \( \alpha = \tilde{\alpha} \).

4. Stability of inverse problem for operator \( L \)

Finally, we also solve the stability problem. Stability is about a continuity between two metric spaces. To show this continuity, we use a homeomorphism between these two spaces. These type stability problems were studied in [15, 18, 21, 30].

**Definition 4.1.** (i) Let \( \mathbb{N}' = \mathbb{N} \setminus \{1\} \). We denote
\[
\Omega := \{ q \in L_1(0, \pi) : \int_0^\pi q(x)dx = 0 \},
\]
\( \Sigma := \) the collection of all double sequences \( X \), where
\[
X := \{ x_n^j : j = 1, n-1; n \in \mathbb{N}' \}
\]
such that \( 0 < x_1^n < x_2^n < \cdots < x_k^n < \cdots < x_n-1^n < x_1^n < x_2^n < \cdots < x_n^n < \pi \) for each \( n \).

We call \( \Omega \) the space of discontinuous Sturm-Liouville operators and \( \Sigma \) the space of all admissible sequences. Hence, when \( \overline{X} \) is the nodal set associated with \((\overline{q}, \alpha)\) and \( \overline{X} \) is close to \( X \) in \( \Sigma \), then \((\overline{q}, \alpha)\) is close to \((q, \alpha)\).

(ii) Let \( X \in \Sigma \) and define \( x_0^n = 0, x_n^n = 1 \), \( L_k^n = x_k^n - x_{k-1}^n \) and \( I_n^0 = (x_0^n, x_1^n) \) for \( j = 0, n-1 \). Note that, \( L_0^n = x_1^n \) and \( L_{n-1}^n = 1 - x_n^n \). We say \( X \) is quasinodal to some \( q \in \Omega \) if \( X \) is an admissible sequence and satisfies the conditions:

(I) As \( n \to \infty \) the limit of
\[
(\pi n - \theta)(\pi n - \theta)x_n^j - \begin{cases} 
  j\pi, & \text{if } x_n^j \in (0, \frac{\pi}{2}) \\
  (j + \frac{1}{2})\pi + \theta, & \text{if } x_n^j \in (\frac{\pi}{2}, \pi)
\end{cases}
\]
exists in \( \mathbb{R} \) for all \( j = 1, n-1 \);

(II) \( X \) has the following asymptotic uniformity for \( j \) as \( n \to \infty \),
\[
x_n^j = \begin{cases} 
  \mu_n^j + O\left(\frac{1}{n^2}\right), & \text{if } x_n^j \in (0, \frac{\pi}{2}) \\
  \gamma_n^j + O\left(\frac{1}{n^2}\right), & \text{if } x_n^j \in (\frac{\pi}{2}, \pi)
\end{cases}
\]
for \( j = 1, n-1 \).

**Definition 4.2.** Suppose that \( X, \overline{X} \in \Sigma \) with \( L^n_k \) and \( \overline{L}^n_k \) as their respective grid lengths. Let
\[
S_n(X, \overline{X}) = (\pi n - \theta)^2 \sum_{k=1}^{n-1} |L^n_k - \overline{L}^n_k|
\]
and \( d_0(X, \overline{X}) = \limsup_{n \to \infty} S_n(X, \overline{X}) \) and \( d_\Sigma(X, \overline{X}) = \limsup_{n \to \infty} \frac{S_n(X, \overline{X})}{1 + S_n(X, \overline{X})} \).

Since the function \( f(x) = \frac{1}{1+x} \) is monotonic, we have
\[
d_\Sigma(X, \overline{X}) = \frac{d_0(X, \overline{X})}{1 + d_0(X, \overline{X})} \in [0, \pi],
\]
admitting that if \( d_0(X, \overline{X}) = \infty \), then \( d_\Sigma(X, \overline{X}) = 1 \). Conversely,
\[
d_0(X, \overline{X}) = \frac{d_\Sigma(X, \overline{X})}{1 - d_\Sigma(X, \overline{X})}.
\]
After the following theorem, we can say that inverse nodal problem for operator $L$ is stable.

**Theorem 4.3.** The metric spaces $(\Omega, \| \cdot \|_1)$ and $(\Sigma/\sim, d_\Sigma)$ are homeomorphic to each other. Here, $\sim$ is the equivalence relation induced by $d_\Sigma$. Furthermore

$$\|q - \overline{q}\|_1 = \frac{2d_\Sigma(X, \overline{X})}{1 - d_\Sigma(X, \overline{X})},$$

where $d_\Sigma(X, \overline{X}) < 1$.

**Proof.** According to Theorem 3.2 using the definition of norm on $L_1$ for the potential functions, we obtain

$$\|q - \overline{q}\|_1 \leq 2(n - \frac{\theta}{\pi})^3 \int_0^{\pi} |L_n^j - \overline{L}_n^j| dx + o(1)$$

$$\leq 2(n - \frac{\theta}{\pi})^3 \int_0^{\pi} |L_n^j - \overline{L}_n^j| dx + 2(n - \frac{\theta}{\pi})^3 \int_0^{\pi} |\overline{L}_n^j - \overline{L}_n^k| dx + o(1)$$

(4.1)

Here, the integrals in the second and first terms can be written as

$$\int_0^{\pi} |\overline{L}_n^j - \overline{L}_n^k| dx = o\left(\frac{1}{n^3}\right)$$

and

$$\int_0^{\pi} |L_n^j - \overline{L}_n^k| dx = \frac{1}{(\pi n - \theta)} \sum_{k=1}^{n-1} |L_n^k - \overline{L}_k|,$$

respectively. If we consider these equalities in (4.1), we obtain

$$\|q - \overline{q}\|_1 \leq 2(n\pi - \theta)^2 \sum_{k=1}^{n-1} |L_n^k - \overline{L}_k| + o(1) = 2S_n(X, \overline{X}) + o(1).$$

(4.2)

Similarly, we can easily obtain

$$\|q - \overline{q}\|_1 \geq 2S_n(X, \overline{X}) + o(1)$$

(4.3)

The proof is complete after by taking limits in (4.2) and (4.3) as $n \to \infty$. □

**References**


