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# DIFFERENTIAL INCLUSIONS AND EXACT PENALTIES

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Abstract. The article considers differential inclusion with a given set-valued mapping and initial point. It is required to find a solution of this differential inclusion that minimizes an integral functional. Some classical results about the maximum principle for differential inclusions are obtained using the support and exact penalty functions. This is done for differentiable and for non-differentiable set-valued mappings in phase variables.

## 1. Introduction

Nowadays, differential inclusions (or differential equations with a multivalued right-hand side) are almost indispensable in mathematical modeling of systems with incomplete description [4] and analyzing behavior of discontinuous systems [9]. Applications of differential inclusions to the problem of constructing Lyapunov functions and optimization are known. The problem of finding solutions for a differential inclusion is important for applications [5, 6, 10], [14]–[16]. As a rule, it is possible to obtain an analytical solution of a differential inclusion only in special cases but in the other cases we have to use numerical methods for this purpose.

It should be noted that the conditions known of the existence of solutions [1, 4] usually contain either a continuity requirement, or both the semi-continuities and the convexity of the corresponding multivalued mapping.

**Definition 1.1.** A differential inclusion is a relation of the form

$$\dot{x}(t) \in F(x(t), t) \tag{1.1}$$

with respect to the unknown function $x : I \to \mathbb{R}^n$, where $I \subset \mathbb{R}$ is an interval, $F : R \times \mathbb{R}^n \subset 2^{\mathbb{R}^n}$ ($2^M$ hereinafter means the set of all subsets of $M$).

**Definition 1.2.** A function $x : I \to \mathbb{R}^n$ is called a solution of differential inclusion (1.1) on the interval $I$ ($I \subset \mathbb{R}$) if it is absolutely continuous in $I$ and almost everywhere in $I$ satisfies relation (1.1). If the gradient of $x$ has only discontinuities of the first kind, such a solution is called the proper one.

**Definition 1.3.** A multivalued mapping $F : X \to 2^Y$ is called lower semicontinuous in $x_0$ if for every $y_0 \in F(x_0)$ and for every neighborhood $U(y_0)$ of the point $y_0$ there

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exists such a neighborhood $U(x_0)$ of the point $x_0$ that $F(x) \cap U(_0) \neq 0$ for every $x \in U(x_0)$.

**Definition 1.4.** A support function of multivalued mapping $F$ from $X$ to $Y$ is the function

$$c(F(x), p) := \sup_{y \in F(x)} (y, p) \quad \forall x \in X, \ \forall p \in Y^*,$$

where $Y^*$ is a dual space, $(\cdot, \cdot)$ is a scalar product of vectors. It describes all closed semispaces which contain $F(x)$.

**Definition 1.5.** A subdifferential of the lower semicontinuous convex function $V$ on Hilbert space $X$ with the values in $\mathbb{R} \cup \{+\infty\}$ is the set

$$\partial V(x) = \{p \in X^* : (p, x) - V(x) = \max_{y \in X} [(p, y) - V(y)]\}.$$

It is a closed convex subset of $X^*$. If $V$ has the gradient $\nabla V(x) \in X^*$ at the point $x$, then $\partial V(x) = \{\nabla V(x)\}$.

If the function $U(x)$ is a concave one, then the set

$$\bar{\partial} U(x) = \{u : U(y) - U(x) \leq (u, y - x) \ \forall y \in X\}$$

is called a superdifferential of the function $U$ at the point $x$.

Differential inequalities, implicit differential equations, differential equations with restrictions on the phase coordinates may be represented in the form of the differential inclusion $\dot{x} \in F(x, t)$. So a differential inclusion is generalization of ordinary differential equations and since it has a whole family of trajectories which come out from the initial point $x_0$, then it is natural to state the problem of picking out the solutions with definite properties, for example, those which minimize a certain functional.

In [4], [9] some classical results are given which extend the known Pontryagin's maximum principle for differential inclusions. The maximum principle is obtained under sufficiently stringent assumptions, in particular, provided that the support function $c(F(t, x), \psi)$ of the multivalued mapping $F(t, x)$ is continuously differentiable in the vector of phase coordinates. This article studies the problem in this case using the apparatus of support functions [13] and exact penalty functions [12]. With the help of this apparatus it is comparatively easy to obtain the known Blagodatskih's maximum principle. The case is additionally investigated when differentiability in $x$ of the support function $c(F(t, x), \psi)$ is not assumed.

## 2. Statement of the problem

Consider the differential inclusion

$$\dot{x} \in F(x, t) \tag{2.1}$$

with the initial condition

$$x(0) = x_0. \tag{2.2}$$

Here $F(x, t)$, $t \in [0, T]$, is a given multivalued mapping, which is supposed to be upper semicontinuous, $x(t)$ is a $n$-dimensional vector-function of the phase coordinates, which is supposed to be continuous with partially continuous in $[0, T]$ gradient, $T > 0$ is a given moment of time. We assume that the function $F(x, t)$ puts in correspondence a certain convex compact set from $\mathbb{R}^n$ for every moment of time $t \in [0, T]$ and for every phase point $x \in \mathbb{R}^n$. It is required to find such a vector-function $x^*(t) \in C_n[0,T]$, which is the solution of inclusion (2.1), satisfies initial condition (2.2) and minimizes the functional

$$I(x) = \int_0^T f_0(x,t)dt, \tag{2.3}$$

where $f_0$ is a given real scalar function which is supposed to be continuous in both arguments and continuously differentiable in $x$.

## 3. Equivalent statement of the Problem

Further, for brevity, we sometimes write $F$ instead of $F(x,t)$. Since for all $t \in [0,T]$ and all $x \in \mathbb{R}^n$ the multivalued mapping $F(x,t)$ is a convex, closed and bounded set, inclusion (2.1) may be rewritten as follows [2]

$$(\dot{x}, \psi) \le c(F, \psi) \quad \forall t \in [0,T],$$

where $\psi \in \mathbb{R}^n$, $\|\psi\| = 1$. Denote $z(t) = \dot{x}(t)$, then from (2.2) we have

$$x(t) = x_0 + \int_0^t z(\tau)d\tau.$$

We introduce the functions

$$l(\psi, z, t) = (z, \psi) - c(F, \psi), \tag{3.1}$$

$$h(\psi, z, t) = \max\{0, l(\psi, z, t)\} \tag{3.2}$$

and construct the functional

$$\varphi(\psi, z) = \Big(\int_0^T h^2(\psi, z, t)dt\Big)^{1/2}. \tag{3.3}$$

We consider the sets

$$\Omega = [z \in P_n[0,T] : \varphi(\psi, z) = 0], \quad \Omega_\delta = [z \in P_n[0,T] \mid \varphi(\psi, z) < \delta].$$

Then

$$\Omega_\delta / \Omega = [z \in P_n[0,T] : 0 < \varphi(\psi, z) < \delta].$$

One may easily check that for functional (3.3) the following relation holds

$$z \begin{cases} \in \Omega, & \text{if } (\dot{x}, \psi) \le c(F, \psi) \ \forall t \in [0,T], \\ \notin \Omega, & \text{if not.} \end{cases}$$

Let us write the functional

$$\Phi(\psi, z) = I(z) + \lambda \varphi(\psi, z), \tag{3.4}$$

in which

$$I(z) = I\Big(x_0 + \int_0^t z(\tau)d\tau\Big),$$

where $\lambda$ is a sufficiently big positive number which is called a penalty parameter. It will be shown that under some additional assumptions it is an exact penalty function. Then the problem of minimization of functional (2.3) under constraints (2.1), (2.2) may be reduced to unconstrained minimization of functional (3.4).

## 4. Differential properties of $\varphi(z)$ and $I(z)$

Further we assume that the vector-function $f(x,t)$ is continuous in both of its arguments and continuously differentiable in $x$ and that for all $t \in [0,T]$ and for all $x \in \mathbb{R}^n$ the inclusion $f(x,t) \in F(x,t)$ takes place. In the paper we sometimes write $f$ instead of $f(x,t)$. Further we consider the functions $l(\psi,z,t)$, $h(\psi,z,t)$, $\varphi(\psi,z)$ and $\Phi(\psi,z)$ for a fixed value $\psi$, so we write $l(z,t)$, $h(z,t)$, $\varphi(z)$ and $\Phi(z)$ respectively instead of them.

Consider the functional $\varphi(z)$. Let $v \in P_n[0,T]$. Put

$$z_\alpha(t) = z(t) + \alpha v(t).$$

We calculate

$$l(z_\alpha,t) = l(z,t) + \alpha H_1(z_\alpha,t) + o(\alpha,t),$$

where

$$\frac{o(\alpha,t)}{\alpha} \to 0 \quad \text{as } \alpha \downarrow 0,$$

$$H_1(z_\alpha,t) = (\psi, v(t)) - \max_{f\in F}\Big(\frac{\partial f}{\partial x}\int_0^t v(\tau)d\tau, \psi\Big).$$

Here the definition of a support function and the property of additivity of a support function in the first argument are used [4]. Using (3.1) and (3.2), we find

$$h(z_\alpha,v) = h(z,t) + \alpha H(z_\alpha,t) + o(\alpha,t),$$

where

$$\frac{o(\alpha,t)}{\alpha} \to 0 \quad \text{as } \alpha \downarrow 0,$$

$$H(z_\alpha,t) = H_1(z_\alpha,t), \quad l(z,t) > 0,$$

$$H(z_\alpha,t) = 0, \quad l(z,t) < 0,$$

$$H(z_\alpha,t) = \max\{0, H_1(z_\alpha,t)\}, \quad l(z,t) = 0.$$

We introduce the sets

$$T_+(z) = [t \in [0,T] : l(z,t) > 0],$$

$$T_-(z) = [t \in [0,T] : l(z,t) < 0],$$

$$T_0(z) = [t \in [0,T] : l(z,t) = 0].$$

At first, consider the case $z \notin \Omega$.

**Lemma 4.1.** *If $z \notin \Omega$, then the functional $\varphi(z)$ is superdifferentiable* [8] *and its superdifferential at a point $z$ is expressed as follows*

$$\overline{\partial}\varphi(z) = \frac{h(z,t)}{\varphi(z)}\psi + \int_t^T -\Big(\frac{\partial f}{\partial x}\Big)'\frac{h(z,\tau)}{\varphi(z)}\psi\, d\tau, \quad f \in F,$$

*where $'$ means the transpose operation.*

*Proof.* From (3.3) we have

$$\varphi(z_\alpha) = \varphi(z) + \alpha\int_0^T \frac{h(z,t)}{\varphi(z)} H(z_\alpha,t)dt + o(\alpha),$$

where

$$\frac{o(\alpha)}{\alpha} \to 0 \quad \text{as } \alpha \downarrow 0.$$

Since $z \notin \Omega$, we have $H = H_1$. Then, using the expression for $H_1$, the theorem of the integral of a support function and positive homogeneity of a support function in the second argument [2], one obtains

$$\begin{aligned}\varphi'(z,v) &= \lim_{\alpha\downarrow 0}\frac{\varphi(z+\alpha v)-\varphi(z)}{\alpha}\\ &= \int_0^T \Big(\frac{h(z,t)}{\varphi(z)}\psi, v(t)\Big)dt - \max_{f\in F}\int_0^T \Big(\frac{\partial f}{\partial x}\int_0^t v(\tau)d\tau, \frac{h(z,t)}{\varphi(z)}\psi\Big)dt.\end{aligned}$$

Integrating by parts in the last summand, we obtain

$$\begin{aligned}&\varphi'(z,v)\\ &= \int_0^T \Big(\frac{h(z,t)}{\varphi(z)}\psi, v(t)\Big)dt - \max_{f\in F}\int_0^T \Big(v(t), \int_t^T (\frac{\partial f}{\partial x})'\frac{h(z,\tau)}{\varphi(z)}\psi\, d\tau\Big)dt\\ &= \int_0^T \Big(\frac{h(z,t)}{\varphi(z)}\psi, v(t)\Big)dt + \min_{f\in F}\int_0^T \Big(v(t), \int_t^T -(\frac{\partial f}{\partial x})'\frac{h(z,\tau)}{\varphi(z)}\psi\, d\tau\Big)dt\\ &= \min_{V\in\overline{\partial}\varphi(z)}(V,v),\end{aligned} \tag{4.1}$$

where

$$\overline{\partial}\varphi(z) = \Big\{V \in P_n[0,T] : V(t) = \frac{h(z,t)}{\varphi(z)}\psi + \int_t^T -(\frac{\partial f}{\partial x})'\frac{h(z,\tau)}{\varphi(z)}\psi\, d\tau,\ f\in F\Big\}.$$

We denote

$$\kappa(z,t) = \frac{h(z,t)}{\varphi(z)}, \quad \kappa(z)\in P[0,T]\,.$$

Then $\kappa(z,t)\ge 0$ for all $t\in[0,T]$, $\|\kappa(z)\|=1$, where $\|\cdot\|$ is the norm in $L_2[0,T]$. From (4.1) it is clear that the functional $\varphi(z)$ is superdifferentiable and its subdifferential is of the form

$$\overline{\partial}\varphi(z) = \kappa(z,t)\psi + \int_t^T -(\frac{\partial f}{\partial x})'\kappa(z,\tau)\psi\, d\tau, \quad f\in F. \tag{4.2}$$

The proof is complete. □

Now consider the case $z\in\Omega$.

**Lemma 4.2.** *If $z\in\Omega$, then the functional $\varphi(z)$ is Dini differentiable in any direction $v\in P_n[0,T]$ and its $D$-derivative in the direction $v$ at a point $z$ is expressed as follows*

$$\varphi'(z,v) = \max_{\|w\|\le 1}\Big[\int_0^T \big(w(t)\psi, v(t)\big)dt + \min_{f\in F}\int_0^T \Big(\int_t^T -(\frac{\partial f}{\partial x})'w(\tau)\psi\, d\tau, v(t)\Big)dt\Big],$$

*where $w\in P[0,T]$, $\|w\|\le 1$.*

*Proof.* Since $z\in\Omega$, we have $\varphi(z)=0$. Then from expression (3.3) we obtain

$$\varphi'(z,v) = \lim_{\alpha\downarrow 0}\frac{\varphi(z+\alpha v)-\varphi(z)}{\alpha} = \|H(z_\alpha)\| = \max_{\|w\|\le 1}\int_0^T H(z_\alpha,t)w(t)dt.$$

Under the assumptions of this theorem we have $T_+(z)=\emptyset$, so

$$\varphi'(z,v) = \max_{\|w\|\le 1}\int_{T_0\cup T_-} w(t)\max_{\overline{w}(t)\in[0,1]}(\overline{w}(t)H_1(z_\alpha,t))dt$$

$$= \max_{w \in W_1} \int_0^T H_1(z_\alpha, t) w(t) dt,$$

where

$$W_1 = \{w \in P[0,T] : \|w\| \le 1,\ w(t) \ge 0 \,\forall t \in T_0,\ w(t) = 0 \,\forall t \in T_-\}. \tag{4.3}$$

From the expression for $H_1$ we obtain

$$\begin{aligned} &\varphi'(z,v) \\ &= \max_{w \in W_1} \Big[ \int_0^T (w(t)\psi, v(t)) dt + \min_{f \in F} \int_0^T \Big( \int_t^T -\big(\frac{\partial f}{\partial x}\big)' w(\tau)\psi \, d\tau, v(t) \Big) dt \Big]. \end{aligned} \tag{4.4}$$

The proof is complete. □

Finding the derivative of the functional $I(z)$ in the direction $v \in P_n[0,T]$, we show that $I$ is Gateaux differentiable

$$I'(z,v) = \int_0^T \Big( \int_t^T \frac{\partial f_0}{\partial x} d\tau, v(t) \Big) dt,$$

and its gradient on the set $P_n[0,T]$ is expressed by the formula

$$\nabla I(z) = \int_t^T \frac{\partial f_0}{\partial x} d\tau. \tag{4.5}$$

**Lemma 4.3.** *If the support function $c(F,\psi)$ of the multivalued mapping $F(x,t)$ is continuously differentiable in the phase variable $x$, then:*

- *if $z \notin \Omega$, then the functional $\varphi(z)$ is Gateaux differentiable and its gradient on the set $P_n[0,T]$ may be found by the formula*
$$\nabla \varphi(z) = \frac{h(z,t)}{\varphi(z)} \psi + \int_t^T -\frac{h(z,\tau)}{\varphi(z)} \frac{\partial c(F(x,\tau),\psi)}{\partial x} d\tau;$$
- *if $z \in \Omega$, then the functional $\varphi(z)$ is subdifferentiable* [8] *and its subdifferential at a point $z$ may be found by the formula*
$$\partial \varphi(z) = \big\{ W \in P_n[0,T] \mid W(t) = w(t)\psi + \int_t^T -w(\tau) \frac{\partial c(F(x,\tau),\psi)}{\partial x} d\tau,\ w \in W_1 \big\},$$
*where the set $W_1$ is defined by the formula* (4.3).

*Proof.* By the definition of the support function

$$c(F,\psi) = \max_{f \in F} (f,\psi).$$

It is clear that this function is subdifferentiable, and its subdifferential is expressed by the formula

$$\partial c(F,\psi) = \operatorname{co} \big\{ \big(\frac{\partial f}{\partial x}\big)' \psi \big\}, \quad f \in R = \{f \in F : c(F,\psi) = (f,\psi)\}.$$

Hence $c(F,\psi)$ is differentiable in $x$ if and only if the set $R$ consists of the only element. Let us denote it $f^*$. Thus, in this case we can assume that

$$c(F,\psi) = (f^*,\psi),$$

and the following relations hold

$$\big(\frac{\partial f^*}{\partial x}\big)' \psi = \frac{\partial (f^*,\psi)}{\partial x} = \frac{\partial c(F,\psi)}{\partial x}. \tag{4.6}$$

Let $z \notin \Omega$. From expression (4.2) we conclude that the superdifferential of the functional $\varphi(z)$ consists of the only element, therefore $\varphi(z)$ is Gateaux differentiable. Its gradient on the set $P_n[0,T]$ is as follows

$$\nabla\varphi(z) = \frac{h(z,t)}{\varphi(z)}\psi + \int_t^T -\frac{h(z,\tau)}{\varphi(z)}\big(\frac{\partial f^*}{\partial x}\big)'\psi\, d\tau. \tag{4.7}$$

Using (4.6), from (4.7) we finally get the expression

$$\nabla\varphi(z) = \frac{h(z,t)}{\varphi(z)}\psi + \int_t^T -\frac{h(z,\tau)}{\varphi(z)}\frac{\partial c(F(x,\tau),\psi)}{\partial x} d\tau,$$

which proves the first part of the lemma.

Let $z \in \Omega$. From expression (4.4) we conclude that the derivative of the functional $\varphi(z)$ in the direction $v$ may be expressed as

$$\begin{aligned}\varphi'(z,v) &= \max_{\|w\|\le 1}\Big[\int_0^T (w(t)\psi, v(t))dt + \Big(\int_t^T -\big(\frac{\partial f^*}{\partial x}\big)'w(\tau)\psi\, d\tau, v(t)\Big)dt\Big] \\ &= \max_{W\in\partial\varphi(z)}(W,v),\end{aligned} \tag{4.8}$$

where

$$\partial\varphi(z) = \big\{W \in P_n[0,T] \ \big|\ W(t) = w(t)\psi + \int_t^T -w(\tau)\big(\frac{\partial f^*}{\partial x}\big)'\psi\, d\tau,\ w \in W_1\big\},$$

and $W_1$ is defined in (4.3). Using (4.6), from (4.8) we finally get the expression

$$\begin{aligned}\partial\varphi(z) = \Big\{&W \in P_n[0,T] : W(t) = w(t)\psi \\ &+ \int_t^T -w(\tau)\frac{\partial c(F(x,\tau),\psi)}{\partial x}d\tau,\ w \in W_1\Big\},\end{aligned} \tag{4.9}$$

which proves the second part of the lemma. Note that in this case the following equality also holds

$$\gamma w(t)\frac{\partial c(F(x,t),\psi)}{\partial x} = \frac{\partial c(F(x,t),\gamma w(t)\psi)}{\partial x} \quad \forall t \in [0,T],\ \forall\gamma > 0. \tag{4.10}$$

□

## 5. Necessary minimum conditions

**Theorem 5.1.** *Let* $\inf_{z\in\Omega} I(z) = I(z^*) > -\infty$ *and there exists such a positive number* $\lambda_0 < \infty$ *that* $\forall\lambda > \lambda_0$ *there exists* $z(\lambda) \in P_n[0,T]$*, for which*

$$\Phi_\lambda(z(\lambda)) = \inf_{z\in P_n[0,T]} \Phi_\lambda(z).$$

*Let the functional* $I(z)$ *be locally Lipschitz on the set* $\Omega_\delta/\Omega$*. Then functional* (3.4) *will be an exact penalty function.*

*Proof.* It is sufficient [7] to show that there exist such numbers $a > 0$, $\delta > 0$ that

$$\varphi^\downarrow(z) = \liminf_{y\to z}\frac{\varphi(y)-\varphi(z)}{\rho(z,y)} < -a < 0 \quad \forall z \in \Omega_\delta/\Omega. \tag{5.1}$$

Here $y \in P_n[0,T]$, $\rho(z,y)$ is the following metric on the set $P_n[0,T]$

$$\rho(z,y) = \max_{t\in[0,T]}\Big|\int_0^t z(t) - y(t)dt\Big|.$$

Put

$$y(t) = z(t) + \alpha v^*(t), \ v^*(t) = -\Big(\frac{h(z,t)}{\varphi(z)}\psi + \int_t^T -\big(\frac{\partial f}{\partial x}\big)' \frac{h(z,\tau)}{\varphi(z)}\psi \, d\tau\Big).$$

Then using Lemma 4.1, one gets

$$\begin{aligned}\varphi'(z, v^*) = \min_{f\in F} \int_0^T -\Big(\frac{h(z,t)}{\varphi(z)}\psi + \int_t^T -\big(\frac{\partial f}{\partial x}\big)' \frac{h(z,\tau)}{\varphi(z)}\psi \, d\tau, \\ \frac{h(z,t)}{\varphi(z)}\psi + \int_t^T -\big(\frac{\partial f}{\partial x}\big)' \frac{h(z,\tau)}{\varphi(z)}\psi \, d\tau\Big) dt.\end{aligned}$$

Let us show that

$$\frac{h(z,t)}{\varphi(z)}\psi + \int_t^T -\big(\frac{\partial f}{\partial x}\big)' \frac{h(z,\tau)}{\varphi(z)}\psi \, d\tau \neq \mathbf{0}, \quad f \in F,$$

identically on the interval $[0, T]$. Assume the contrary. Then $\frac{h(z,t)}{\varphi(z)}\psi = \mathbf{0}$ for all $t \in [0, T]$ which contradicts the constraints on $\psi$ and $\frac{h(z,t)}{\varphi(z)}$. Let $\varphi(y) \to \varphi(z)$ if $y \to z$, i. e. there exists sequence $\{z_k\} \in \Omega_\delta/\Omega$ such that $\varphi'(z_k, v^*) \to 0$ if $k \to \infty$, where

$$\begin{gathered}\frac{\partial f}{\partial x} = \frac{\partial f(x_k, t)}{\partial x}, \quad f \in F, \\ x_k(t) = x_0 + \int_0^t z_k(\tau) d\tau.\end{gathered}$$

Therefore,

$$\|\frac{h(z_k,t)}{\varphi(z_k)}\psi + \int_t^T -\Big(\frac{\partial f}{\partial x}\Big)' \frac{h(z_k,\tau)}{\varphi(z_k)}\psi \, d\tau\| \to 0, \quad \text{as } k \to \infty, \ f \in F,$$

hence

$$\|\frac{h(z_k,t)}{\varphi(z_k)}\psi\| \to 0 \quad \text{as } k \to \infty,$$

that also contradicts the assumptions

$$\|\psi\| = 1, \quad \|\frac{h(z)}{\varphi(z)}\| = 1.$$

Then we conclude that

$$\varphi(y) - \varphi(z) = \alpha\varphi'(z, v^*) + o(\alpha) < 0 \quad \forall z \in \Omega_\delta/\Omega.$$

Now find that

$$\rho(z, y) = \alpha \max_{t\in[0,T]} \Big| \int_0^t v^*(t) dt \Big| > 0.$$

From the last two inequalities follows (5.1). The proof is complete. □

**Theorem 5.2.** *Let the conditions of Theorem 5.1 be satisfied. Let the support function of the multivalued mapping $F(x, t)$ from* (2.1) *be continuously differentiable in $x$. For the point*

$$x^* = x_0 + \int_0^t z^*(\tau) d\tau$$

*to satisfy inclusion* (2.1) *and condition* (2.2) *and to minimize functional* (2.3)*, the existence of such a vector-function* $\Psi(t)$ *that for all* $t \in [0,T]$ *the following relations hold*

$$\dot{\Psi}(t) = -\frac{\partial c(F(x^*,t),\Psi(t))}{\partial x} + \frac{\partial f_0(x^*,t)}{\partial x}, \tag{5.2}$$

$$(\dot{x}^*,\Psi(t)) - c(F(x^*,t),\Psi(t)) = 0, \tag{5.3}$$

$$\Psi(T) = \mathbf{0} \tag{5.4}$$

*is necessary.*

*Proof.* In Theorem 5.1 it has been shown that the functional (3.4) is an exact penalty function, hence there exists such a number $\lambda^* > 0$ that for all $\lambda > \lambda^*$ functional (2.3) minimization problem under constraints (2.1), (2.2) is equivalent to the unconstrained optimization problem of functional (3.4) minimization.

Let us put $\Psi(t) = \lambda w(t)\psi$, where the vector-function $w(t)$ is an element of the set $W_1$. Since by Lemma 4.3 if $z \in \Omega$, the functional $\varphi(z)$ is subdifferentiable and its subdifferential is represented in (4.9), and the functional $I(z)$ is Gateaux differentiable and its gradient is represented in (4.5), then from necessary minimum condition [7]

$$0_n \in \partial\Phi(z^*).$$

Considering (4.10) we have that at the minimum point for all $t \in [0,T]$,

$$\int_t^T \frac{\partial f_0(x^*,t)}{\partial x} d\tau + \Psi(t) + \int_t^T -\frac{\partial c(F(x^*,t),\Psi(t))}{\partial x} d\tau = 0_n, \tag{5.5}$$

where $0_n$ is a zero element of the space $P_n[0,T]$. Differentiating (5.5) on the interval $[0,T]$, one obtains a system of differential equations

$$\dot{\Psi}(t) = -\frac{\partial c(F(x^*,t),\Psi(t))}{\partial x} + \frac{\partial f_0(x^*,t)}{\partial x}$$

with the terminal condition $\Psi(T) = \mathbf{0}$, hence we obtain the relations (5.2), (5.4).

If $t \in T_0$, from the formula of the functional $l(z,t)$ one gets $(z,\Psi) = c(F,\Psi)$, if $t \in T_-$, then $w(t) = 0$ and relation (5.3) still takes place. Thus (5.3) holds for all $t \in [0,T]$. The proof is complete. $\square$

**Remark 5.3.** Theorem 5.2 has been formulated for the problem with the free right end. It is not difficult to show that relations (5.2), (5.3) will also hold for the problem with the fixed right end, but the terminal value $\Psi(T)$ for this problem will not be equal to zero in the general case, i. e. relation (5.4) will not hold in this case.

Using the known minimum conditions in terms of the derivative in directions from Lemma 4.2 one gets the following lemma.

**Lemma 5.4.** *Let the conditions of Theorem 5.1 be satisfied. For the point* $x^*$ *to satisfy inclusion* (2.1) *and condition* (2.2) *and to minimize functional* (2.3)*, it is necessary to have*

$$\begin{aligned} &\max_{\|w\|\le 1} \Big[ \int_0^T \Big( \int_t^T \frac{\partial f_0(x^*,\tau)}{\partial x} d\tau + \lambda w(t)\psi, v(t) \Big) dt \\ &+ \min_{f\in F} \int_0^T \Big( \int_t^T -\Big(\frac{\partial f(x^*,\tau)}{\partial x}\Big)' \lambda w(\tau)\psi \, d\tau, v(t) \Big) dt \Big] \ge 0 \quad \forall v \in P_n[0,T]. \end{aligned} \tag{5.6}$$

Let $\overline{F} \subset F$ be a set of such $f \in F$ that (5.6) holds. It can be shown that the following lemma holds.

**Lemma 5.5.** *Relation* (5.6) *is equivalent to the condition: for every fixed* $\overline{f} \in \overline{F}$ *there exists such a vector-function* $\overline{w} \in W_1$ *that the following relation holds*

$$\int_t^T \frac{\partial f_0(x^*,\tau)}{\partial x} d\tau + \lambda \overline{w}(t)\psi + \int_t^T -\Big(\frac{\partial \overline{f}(x^*,\tau)}{\partial x}\Big)' \lambda \overline{w}(\tau)\psi \, d\tau = 0_n \tag{5.7}$$

*for all* $t \in [0,T]$.

**Theorem 5.6.** *Let the conditions of Theorem 5.1 be satisfied. For the point* $x^*$ *to satisfy inclusion* (2.1) *and condition* (2.2) *and to minimize functional* (2.3)*, the existence of such a vector-function* $\overline{f} \in F$ *and such a vector-function* $\overline{\Psi}(t)$*, for which for all* $t \in [0,T]$ *the following relations hold*

$$\dot{\overline{\Psi}}(t) = -\Big(\frac{\partial \overline{f}(x^*,t)}{\partial x}\Big)' \overline{\Psi}(t) + \frac{\partial f_0(x^*,t)}{\partial x}, \tag{5.8}$$

$$\big(\dot{x}^*, \overline{\Psi}(t)\big) - c\big(F(x^*,t), \overline{\Psi}(t)\big) = 0, \tag{5.9}$$

$$\overline{\Psi}(T) = \mathbf{0}, \tag{5.10}$$

*is necessary.*

*Proof.* In view of Lemma 5.4 it is sufficient to show that (5.6) is equivalent to (5.8), (5.10) for some $\overline{f} \in F$ and $\overline{\Psi}(t)$.

In view of Lemma 5.5 relation (5.6) is equivalent to (5.7) for every fixed $\overline{f} \in \overline{F}$. Differentiating (5.7) on the interval $[0,T]$ and denoting

$$\overline{\Psi}(t) = \lambda \overline{w}(t)\psi,$$

one gets a system of differential equations (5.8) with terminal condition (5.10). Relation (5.9) can be proved in the same way as in Theorem 5.2. □

**Remark 5.7.** Theorem 5.6 has been formulated for the problem with the free right end. It is not difficult to show that relations (5.8), (5.9) will also hold for the problem with the fixed right end, but the terminal value $\Psi(T)$ for this problem will not be equal to zero in the general case, i. e. relation (5.10) will not hold in this case.

## 6. Example

Consider the system of differential equations

$$\dot{x}_1 = u_1, \quad \dot{x}_2 = x_1,$$

where the restriction on control is given by the set

$$U = \{u \in R^2 : |u_1| \le 1,\, u_2 = 0\}.$$

Let the initial condition $x_0 = (0,0)$ and the terminal state $x(1) = (-1/2, -1/3)$ of the system be given. It is required to find such control $u^* \in U$, which minimizes the functional

$$I(x) = \int_0^1 x_2(t) dt.$$

This system can be rewritten in the form of the inclusion $\dot{x} \in F(x)$, where

$$F(x) = \begin{pmatrix} [-1,1] \\ x_1 \end{pmatrix}.$$

Since the support function $c(A, b)$ of the segment $A = \{a \in R : a \in [-1, 1]\}$ is $|b|$, then in this case the support function of the multivalued mapping $F(x)$ is expressed by the formula

$$c(F, \psi) = |\psi_1| + x_1\psi_2.$$

One can see that the function $c(F, \psi)$ is continuously differentiable in the phase variables and its gradient may be written as follows

$$\frac{\partial c}{\partial x} = (\psi_2, 0).$$

Further, we have

$$\frac{\partial f_0}{\partial x} = (0, 1).$$

From Theorem 5.2 and Remark 5.7 it follows that the vector-function $\psi(t)$ must satisfy the system of differential equations

$$\dot{\psi}_1 = -\psi_2, \quad \dot{\psi}_2 = 1. \tag{6.1}$$

From Theorem 5.2 and Remark 5.7 one also gets that for $\psi(t)$ and for all $t$ the following relations hold

$$(\dot{x}, \psi(t)) = u_1\psi_1 + x_1\psi_2 = c(F, \psi) = |\psi_1| + x_1\psi_2,$$

therefore for all $t$,

$$u_1(t)\psi_1(t) = |\psi_1(t)|. \tag{6.2}$$

From (6.1), (6.2) it is not difficult to obtain the optimal control

$$\begin{gathered} u_1^*(t) = -1, \quad t \in [0, \tau_1), \\ u_1^*(t) = 1, \quad t \in [\tau_1, \tau_2), \\ u_1^*(t) = -1, \quad t \in [\tau_2, 1], \end{gathered} \tag{6.3}$$

and the corresponding optimal trajectory

$$\begin{gathered} x_1(t) = -t, \quad x_2(t) = -t^2/2, \quad t \in [0, \tau_1), \\ x_1(t) = t + S_1, \quad x_2(t) = t^2/2 + S_1 t + S_2, \quad t \in [\tau_1, \tau_2), \\ x_1(t) = -t + 1/2, \quad x_2(t) = -t^2/2 + 1/2t - 1/2, \quad t \in [\tau_2, 1], \end{gathered} \tag{6.4}$$

where

$$\tau_1 = 13/24, \quad \tau_2 = 19/24, \quad S_1 = -13/12, S_2 = 169/576.$$

The values $\tau_1$, $\tau_2$, $S_1$, $S_2$ in (6.3), (6.4) are found using the boundary conditions and the condition of trajectory continuity.

One can easily check that conditions (6.1), (6.2) may be obtained directly from the Pontryagin's maximum principle. Here a different approach has been demonstrated, when we transit from the original system to the corresponding differential inclusion, for which we apply the conditions of optimality (Theorem 5.2) to find the optimal process $(x^*(t), u^*(t))$.

**Conclusion.** Thus, in this paper application of the theory of exacts penalty functions to the problem of optimal control of differential inclusion is demonstrated. The apparatus of support functions gives opportunity to reduce the original problem to the optimization problem under constraints. With the help of exact penalties this problem is reduced to minimization of the nonsmooth functional $\Phi(z)$ on the whole space. Provided that the support function $c(F(x,t),\psi)$ is continuously differentiable in the vector of the phase coordinates, this functional appears to be subdifferentiable, which allows to write out the necessary minimum conditions in terms of a subdifferential, which coincide with some classical results for this problem. In the case of nondifferentiability of $c(F(x,t),\psi)$ in the phase variable $D$-derivative in directions of the functional $\Phi(z)$ is found, which allows to formulate necessary minimum conditions. The example of theoretical results application is given.

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