EXACT CONTROLLABILITY PROBLEM OF A WAVE EQUATION IN NON-CYLINDRICAL DOMAINS

HUA WANG, YIJUN HE, SHENGJIA LI

Abstract. Let \( \alpha : [0, \infty) \rightarrow (0, \infty) \) be a twice continuous differentiable function which satisfies that \( \alpha(0) = 1 \), \( \alpha' \) is monotone and \( 0 < c_1 \leq \alpha'(t) \leq c_2 < 1 \) for some constants \( c_1, c_2 \). The exact controllability of a one-dimensional wave equation in a non-cylindrical domain is proved. This equation characterizes small vibrations of a string with one of its endpoint fixed and the other moving with speed \( \alpha'(t) \). By using the Hilbert Uniqueness Method, we obtain the exact controllability results of this equation with Dirichlet boundary control on one endpoint. We also give an estimate on the controllability time that depends only on \( c_1 \) and \( c_2 \).

1. Introduction and main results

Suppose \( \alpha : [0, \infty) \rightarrow (0, \infty) \) is a twice continuous differentiable function satisfying the following assumptions:

(A1) \( 0 < c_1 \leq \alpha'(t) \leq c_2 < 1 \) for all \( 0 \leq t < \infty \);

(A2) \( \alpha' \) is monotone;

(A3) \( \alpha(0) = 1 \).

Let \( T > 0 \). We define the non-cylindrical domain \( \tilde{Q}_T^\alpha \) by

\[
\tilde{Q}_T^\alpha = \{(y, t) \in \mathbb{R}^2 : 0 < y < \alpha(t), t \in (0, T)\}.
\]

This article concerns the exact controllability of the one-dimensional wave equation

\[
\begin{align*}
&u_{tt}(y, t) - u_{yy}(y, t) = 0, \quad (y, t) \in \tilde{Q}_T^\alpha, \\
&u(0, t) = 0, \quad u(\alpha(t), t) = v(t), \quad t \in (0, T), \\
&u(y, 0) = u^0(y), \quad u_t(y, 0) = u^1(y), \quad y \in (0, 1),
\end{align*}
\]

(1.1)

where \( v \in L^2(0, T) \) and \( (u^0, u^1) \in L^2(0, 1) \times H^{-1}(0, 1) \). Since \( \sup_{t \in (0, T)} |\alpha'(t)| < 1 \), by [9], the system of (1.1) admits a unique solution in the sense of transposition. Here, as in [10], \( u \in L^\infty(0, T; L^2(0, \alpha(t))) \) is called a solution by transposition of

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problem (1.1) if \( u \) verifies
\[
\int_0^T \int_0^T u(y,t) \hat{h}(y,t) \, dy \, dt = \int_0^1 [u^1(y) \theta(y,0) - u^0(y) \theta_0(y,0)] \, dy - \int_0^T v(t) \theta_y(\alpha(t),t) \, dt,
\]
for all \( \hat{h} \in L^1(0,T; L^2(0, \alpha(t))) \), where \( \theta \) is the weak solution of the problem
\[
\begin{align*}
\theta_{tt}(y,t) - \theta_{yy}(y,t) &= \hat{h}, \quad (y,t) \in \hat{Q}_T^\alpha, \\
\theta(0,t) &= \theta(\alpha(t)), \quad t \in (0,T), \\
\theta(T) &= \theta'(T) = 0, \quad x \in (0,1).
\end{align*}
\]

The exact controllability problem of system (1.1) is stated as follows.

**Definition 1.1.** We say system (1.1) is exactly controllable at time \( T \), if for any \((u^0, u^1) \in L^2(0,1) \times H^{-1}(0,1), \) \((u^d, u^d_t) \in L^2(0, \alpha(T)) \times H^{-1}(0, \alpha(T))\), there exists \( v \in L^2(0,T) \) such that the solution by transposition of (1.1) satisfies \( u(T) = u^d \) and \( u_t(T) = u^d_t \).

For a function \( \alpha \) satisfying conditions (A1)–(A3), we define
\[
T^* = \frac{1}{c_2} \{ \exp \left( \frac{2 \alpha^2 (1 - c_1)(1 + c_2)}{c_1 (1 - c_2)^2} \right) - 1 \},
\]
\[
T^*_1 = \frac{1}{c_2} \{ \exp \left( \frac{2 \alpha^2 (1 - c_1)}{c_1 (1 - c_2)^3} \right) - 1 \}.
\]

One of the main results of this article as follows.

**Theorem 1.2.** For any given \( T > T^* \), (1.1) is exactly controllable at time \( T \).

Similarly, for the exact controllability problem, when the control is acting on the fixed endpoint,
\[
\begin{align*}
w_{tt}(y,t) - w_{yy}(y,t) &= 0, \quad (y,t) \in \hat{Q}_T^\alpha, \\
w(0,t) &= v(t), \quad w(\alpha(t),t) = 0, \quad t \in (0,T), \\
w(y,0) &= u^0(y), \quad w_t(y,0) = u^1(y), \quad y \in (0,1),
\end{align*}
\]
we have the following result.

**Theorem 1.3.** For any given \( T > T^*_1 \), (1.6) is exactly controllable at time \( T \).

**Remark 1.4.** When \( \alpha(t) = 1 + kt \) for some constant \( k \in (0,1) \), \( T^* \) is reduced to \( T^*_k \) defined in [4], and Theorem 1.2 is reduced to [4] Theorem 1.1.

**Remark 1.5.** Theorem 1.3 extends the results in [5] and [6]. In fact, when \( \alpha(t) = 1 + kt \), an exact controllability result of system (1.6) has been proved for \( 0 < k < 1 - \frac{1}{\sqrt{2}} \) in [5] and for \( 0 < k < 1 - \frac{2}{\sqrt{2} + \sqrt{2}^2} \) in [6]. We also note that the controllability time \( T^*_k \) given here is better than the constants \( T^*_k \) in [5] and [6] in this case.

**Remark 1.6.** We note that there are many functions \( \alpha(t) \) satisfying conditions (A1)–(A3) but are not the form \( 1 + kt \), for example \( \alpha(t) = 1 + (t + \arctan t)/c \) where \( c \) is any constant that is greater than 2.
Recently, several works on the controllability problems of wave equations in non-
cylindrical domains have been published. The existence of solutions of the initial
boundary value problem for the nonlinear wave equation in non-cylindrical domains
has been studied in [3] [8]. The controllability problem for a multi-dimensional wave
equation in a non-cylindrical domain has been investigated in [2] [3] [10]. About the
one-dimensional cases, there have been extensive study of the controllability problem
in a non-cylindrical domain. We refer the reader to [1] [4] [5] [6].

When \( \alpha(t) = 1 + kt \) for some constant \( 0 < k < 1 \), in [4], the exact controllability
of the system (1.1) has been acquired. When \( \alpha(t) = 1 + kt \), Cui and Song obtained
that the system (1.6) is exactly controllable for \( 0 < k < 1 - \frac{1}{\sqrt{e}} \) in [5] and is exactly
controllable for \( 0 < k < 1 - \frac{2}{1 + e^2} \) in [6].

There are also other results on the exact controllability problem for wave equa-
tions of variable coefficients in cylindrical domains, see [7] [10] [11] [12] and the refer-
ces therein. So, our first aim is to transform (1.1) and (1.6) into wave equations
with variable coefficients in a cylindrical domain.

Let \( x = \frac{y}{\alpha(t)} \) and \( w(x, t) = u(y, t) = u(\alpha(t)x, t) \) for \( (y, t) \in \hat{Q}^2_T \). Then, it
is straightforward to show that \( (x, t) \) varies in \( Q_T := (0, 1) \times (0, T) \) and (1.1)
is transformed into the wave equation with variable coefficients,

\[
\begin{align*}
\frac{\partial^2 w}{\partial t^2} - \left[ \frac{\beta(x, t)}{\alpha(t)} w_x \right]_x + \frac{\gamma(x, t)}{\alpha(t)} w_{tx} + \frac{\tau(x, t)}{\alpha(t)} w_x &= 0, \quad \text{in } Q_T, \\
w(0, t) &= 0, \quad w(1, t) = v(t) \quad t \in (0, T), \\
w(x, 0) &= w^0(x), \quad w_t(x, 0) = w^1(x), \quad x \in (0, 1),
\end{align*}
\]

where \( \beta(x, t) = \frac{1 - \alpha''(t)x^2}{\alpha(t)} \), \( \gamma(x, t) = -2\alpha'(t)x \), \( \tau(x, t) = -\alpha''(t)x \), \( w^0 = u^0 \), \( w^1 = u^1 + \alpha'(0)xu^0 \).

From [10], we know that for \( (u^0, u^1) \in L^2(0, 1) \times H^{-1}(0, 1) \) and \( v \in L^2(0, T) \),
(1.7) admits a unique solution \( w \in \mathcal{C}([0, T]; L^2(0, 1)) \cap \mathcal{C}([0, T]; H^{-1}(0, 1)) \) in the
sense of transposition, where \( w \) is called a solution by transposition of problem (1.7)
if

\[
\begin{align*}
\int_0^T \int_0^1 w(t, x) h(x) \, dx \, dt &= \int_0^1 [\alpha'(0)w^0(x)z(x, 0) + w'(x)z(x, 0)] \, dx \\
- \int_0^T \beta(1, t)z_x(1, t)v(t) \, dt + \int_0^1 [\gamma(x, 0)w^0(x)z(x, 0) + \gamma(x, 0)w_x^0(x)z(x, 0)] \, dx,
\end{align*}
\]

for every \( h \in L^1(0, T; L^2(0, 1)) \) and \( z \) is the weak solution of the problem

\[
L^*z = h, \quad \text{in } Q_T, \\
z(0, t) = z(1, t) = 0, \quad t \in (0, T), \\
z(x, T) = z_t(x, 0) = 0, \quad x \in (0, 1),
\]

where the formal adjoint \( L^* \) of \( L \) is defined by

\[
L^*z = \alpha(t)z_{tt} - [\beta(x, t)z_x]_x + \gamma(x, t)z_{tx} + \tau(x, t)z_x.
\]

Thus, Theorem 1.2 can be restated as the following exact controllability result for
equation (1.7).
Theorem 1.7. For any $T > T^*$ where $T^*$ is given by (1.4), any $(w_0, w_1) \in L^2(0,1) \times H^{-1}(0,1)$ and $(w_0^w, w_2^w) \in L^2(0,1) \times H^{-1}(0,1)$, we can always find a control $v \in L^2(0, T)$ such that the corresponding solution by transposition $w$ of (1.7) satisfies $w(T) = w_0^w$, $w_i(T) = w_1^w$.

Similarly, (1.6) can be transformed into the wave equation with variable coefficients,
\[
\begin{align*}
&w_{tt} - \left[\frac{\beta(x,t)}{\alpha(t)} w_x\right]_x + \frac{\gamma(x,t)}{\alpha(t)} w_{tx} + \frac{\tau(x,t)}{\alpha(t)} w_x = 0, \quad \text{in } Q_T, \\
&w(0,t) = v(t), \quad w(1,t) = 0, \quad t \in (0,T), \\
&w(x,0) = w^0(x), \quad w_i(x,0) = w^1(x), \quad x \in (0,1),
\end{align*}
\]
(1.10)

and Theorem 1.8 can be restated as the following exact controllability result for equation (1.10).

Theorem 1.8. For any $T > T^*_1$ where $T^*_1$ is given by (1.5), any $(w_0^w, w_1^w) \in L^2(0,1) \times H^{-1}(0,1)$ and $(w_0^w, w_2^w) \in L^2(0,1) \times H^{-1}(0,1)$, we can always find a control $v \in L^2(0,T)$ such that the corresponding solution by transposition $w$ of (1.10) satisfies $w(T) = w_0^w$, $w_i(T) = w_1^w$.

2. Description of the Hilbert Uniqueness Method

In this section, we describe the Hilbert uniqueness method which is used in the proof of Theorems 1.7 and 1.8. Next, we consider Theorem 1.7 in detail.

Firstly, for any $(w_0^w, w_1^w) \in L^2(0,1) \times H^{-1}(0,1)$, the system
\[
\begin{align*}
\alpha(t)\xi_{tt} - \left[\beta(x,t)\xi_x\right]_x + \gamma(x,t)\xi_{tx} + \tau(x,t)\xi_x &= 0, \quad \text{in } Q_T, \\
\xi(0,t) &= 0, \quad \xi(1,t) = 0, \quad t \in (0,T), \\
\xi(x,T) &= w_0^w(x), \quad \xi_i(x,T) = w_1^w(x), \quad x \in (0,1)
\end{align*}
\]
(2.1)

has a unique solution $\xi \in C([0,T]; L^2(0,1)) \cap C^1([0,T]; H^{-1}(0,1))$ in the sense of transportation.

Secondly, for any $(z_0^0, z_1^0) \in H^1_0(0,1) \times L^2(0,1)$, we solve
\[
\begin{align*}
\alpha(t)z_{tt} - \left[\beta(x,t)z_x\right]_x + \gamma(x,t)z_{tx} + \tau(x,t)z_x &= 0, \quad \text{in } Q_T, \\
z(0,t) &= z(1,t) = 0, \quad t \in (0,T), \\
z(x,0) &= z_0^0(x), \quad z_i(x,0) = z_1^0(x), \quad x \in (0,1),
\end{align*}
\]
(2.2)

and
\[
\begin{align*}
\alpha(t)\eta_{tt} - \left[\beta(x,t)\eta_x\right]_x + \gamma(x,t)\eta_{tx} + \tau(x,t)\eta_x &= 0, \quad \text{in } Q_T, \\
\eta(0,t) &= 0, \quad \eta(1,t) = z_x(1,t), \quad t \in (0,T), \\
\eta(x,T) &= 0, \quad \eta_i(x,T) = 0, \quad x \in (0,1).
\end{align*}
\]
(2.3)

Then we define a linear operator $\Lambda : H^1_0(0,1) \times L^2(0,1) \rightarrow H^{-1}(0,1) \times L^2(0,1)$, by
\[
(z_0^0, z_1^0) \mapsto \left(\eta_\gamma(\cdot, 0) + \gamma(\cdot, 0)\eta_\tau(\cdot, 0) - \alpha(0)\eta(\cdot, 0), -\eta(\cdot, 0)\right),
\]
Lastly, the problem is reduced to prove the existence of some $(z_0^0, z_1^0) \in H^1_0(0,1) \times L^2(0,1)$ such that
\[
\Lambda(z_0^0, z_1^0) = \left([w^1 - \xi_t(0)] - \alpha'(0)[w^0 - \xi(0)] + \gamma(0)[w^0_x - \xi_x(0)] - [w^0 - \xi(0)]\right).
\]
(2.4)
To solve (2.4), we observe that
\[
\int_0^1 \beta(t, 1)|z_x(1, t)|^2 dt = \langle \Lambda(z^0, z^1), (z^0, z^1) \rangle_{H^{-1}(0, 1) \times L^2(0, 1)}.
\]
(2.5)

In section 3, we prove the following observability inequality for system (2.2): there exists a constant \( C > 0 \) such that
\[
\int_0^T \beta(t, 1)|z_x(1, t)|^2 dt \geq C (\|z^0\|_{H^1(0, 1)}^2 + \|z^1\|_{L^2(0, 1)}^2).
\]
(2.6)

Also, we prove that \( \Lambda \) is a bounded linear operator; i.e., there exists a constant \( C > 0 \) such that
\[
\int_0^T \beta(t, 1)|z_x(1, t)|^2 dt \leq C (\|z^0\|_{H^1(0, 1)}^2 + \|z^1\|_{L^2(0, 1)}^2).
\]
(2.7)

Combining (2.6), (2.7) and the Lax-Milgram Theorem, we can show that \( \Lambda \) is an isomorphism.

Then, the equation (2.4) has a unique solution \((z^0, z^1) \in H^1(0, 1) \times L^2(0, 1)\), and the function \( z_x(1, t) \) is the desired control such that the solution \( w \) of (1.7) satisfies \( w(T) = 0 \), \( w_1(T) = w^0_1 \).

For the proof of Theorem 1.8, the steps are similar to those of Theorem 1.7. In this case, instead of (2.3), we consider the following homogeneous wave equation
\[
\alpha(t) \eta_t - [\beta(t, x) \eta_x]_x + \gamma(t, x) \eta_x + \tau(t, x) \eta_x = 0, \quad \text{in} \quad Q_T,
\]
\[
\eta(0, t) = z_x(0, t), \quad \eta(1, t) = 0, \quad t \in (0, T),
\]
\[
\eta(x, T) = 0, \quad \eta_x(x, T) = 0, \quad x \in (0, 1),
\]
(2.8)

and define a linear operator \( \Lambda \) just same as (2.4), then we observe that
\[
\int_0^1 \beta(0, t)|z_x(0, t)|^2 dt = -\langle \Lambda(z^0, z^1), (z^0, z^1) \rangle_{H^{-1}(0, 1) \times L^2(0, 1)}.
\]
(2.9)

We omit the details of the proof here.

3. Observability estimates

The main purpose of this section is to prove the observability inequalities for system (2.2). To prove those estimates, we need some technical lemmas.

From [10], we know that: for any \((z^0, z^1) \in H^1(0, 1) \times L^2(0, 1)\), the equation (2.2) has a unique weak solution \( z \in C([0, T]; H^1(0, 1)) \cap C^1([0, T]; L^2(0, 1)) \) in the sense of transportation.

The energy for (2.2) is defined as
\[
E(t) = \frac{1}{2} \int_0^1 [\alpha(t)|z_t(x, t)|^2 + \beta(x, t)|z_x(x, t)|^2] dx, \quad \text{for} \quad t \geq 0,
\]
(3.1)

where \( z \) is the solution of (2.2). Since \( \alpha(0) = 1 \), we have
\[
E_0 := E(0) = \frac{1}{2} \int_0^1 [|z^1(x)|^2 + \beta(x, 0)|z^0_0(x)|^2] dx.
\]
(3.2)

First, we prove a lemma which is related to the decay rate of the energy \( E(t) \).
Lemma 3.1. If $\alpha(0) = 1$, $0 < c_1 \leq \alpha'(t) \leq c_2 < 1$ and $\alpha'$ is monotone, then
\[
\frac{c_3 E_0}{\alpha(t)} \leq E(t) \leq \frac{c_4 E_0}{\alpha(t)},
\]
where
\[
(c_3, c_4) = \begin{cases} 
\left(\frac{c_2}{1 - c_1}, \frac{c_2}{c_1}\right), & \text{if } \alpha' \text{ is increasing}, \\
\left(\frac{c_1}{c_2}, \frac{c_1}{1 - c_2}\right), & \text{if } \alpha' \text{ is decreasing}.
\end{cases}
\]

Proof. For any $0 < t \leq T$, through multiplying the first equation of (2.2) by $z_t$ and integrating the result on $(0, 1) \times (0, t)$, we conclude that
\[
0 = \int_0^t \int_0^1 \left\{ \alpha(s) z_{tt}(x, s) z_t(x, s) - \left[ \beta(x, s) z_x(x, s) \right]_x z_t(x, s) \\
+ \gamma(x, s) z_{tx}(x, s) z_t(x, s) + \tau(x, s) z_x(x, s) z_t(x, s) \right\} dx \, ds
\]
:= I_1 + I_2 + I_3 + I_4,
where
\[
I_1 = \frac{1}{2} \int_0^t \int_0^1 \alpha(s) |z_t(x, s)|^2 dx ds - \frac{1}{2} \int_0^t \int_0^1 \alpha'(s) |z_t(x, s)|^2 dx ds,
\]
\[
I_2 = \frac{1}{2} \int_0^t \int_0^1 \beta(x, s) |z_x(x, s)|^2 dx ds - \frac{1}{2} \int_0^t \int_0^1 \beta_t(x, s) |z_x(x, s)|^2 dx ds
\]
\[
= \frac{1}{2} \int_0^t \int_0^1 \beta(x, s) |z_x(x, s)|^2 dx ds + \frac{1}{2} \int_0^t \int_0^1 \frac{\alpha'(s)}{\alpha(s)} \beta(x, s) |z_x(x, s)|^2 dx ds
\]
\[
+ \int_0^t \int_0^1 \frac{\alpha'(s) \alpha''(s)}{\alpha(s)} x^2 |z_x(x, s)|^2 dx ds,
\]
\[
I_3 = \int_0^t \int_0^1 \alpha'(s) |z_t(x, s)|^2 dx ds,
\]
\[
I_4 = - \int_0^t \int_0^1 \alpha''(s) x z_x(x, s) z_t(x, s) dx ds.
\]

We thereby obtain:
\[
E(t) = E_0 - \int_0^t \frac{\alpha'(s)}{\alpha(s)} E(s) ds - \int_0^t \int_0^1 \frac{\alpha'(s) \alpha''(s)}{\alpha(s)} x^2 |z_x(x, s)|^2 dx ds
\]
\[
+ \int_0^t \int_0^1 \alpha''(s) x z_x(x, s) z_t(x, s) dx ds.
\]
\[
E'(t) = - \frac{\alpha'(t)}{\alpha(t)} E(t) - \int_0^t \frac{\alpha'(t) \alpha''(t)}{\alpha(t)} x^2 |z_x(x, t)|^2 dx + \int_0^t \frac{\alpha''(t)}{\alpha(t)} x z_x(x, t) z_t(x, t) dx.
\]

We subdivide the proof into two cases:

(1) $\alpha'$ is increasing; that is, $\alpha''(t) \geq 0$. By using the inequalities
\[
- \frac{\alpha'(t) \alpha''(t)}{2\alpha(t) \alpha(t)} x^2 |z_x(x, t)|^2 - \frac{\alpha(t) \alpha''(t)}{2\alpha'(t)} |z_t(x, t)|^2
\]
\[
\leq \alpha''(t) x z_x(x, t) z_t(x, t)
\]
\[
\leq \frac{\alpha'(t) \alpha''(t)}{2\alpha(t)} x^2 |z_x(x, t)|^2 + \frac{\alpha(t) \alpha''(t)}{2\alpha'(t)} |z_t(x, t)|^2,
\]
where \( \epsilon(t) = \frac{\alpha'(t)}{1 - \alpha'(t)} \), we easily obtain
\[
-\left( \frac{\alpha'(t)}{\alpha(t)} + \frac{\alpha''(t)}{1 - \alpha'(t)} \right) E(t) \leq E'(t) \leq -\left( \frac{\alpha'(t)}{\alpha(t)} - \frac{\alpha''(t)}{1 - \alpha'(t)} \right) E(t),
\]
so
\[
\frac{(1 - \alpha'(t))E_0}{(1 - \alpha'(0))\alpha(t)} \leq E(t) \leq \frac{\alpha'(t)E_0}{\alpha'(0)\alpha(t)}. \tag{3.7}
\]
Using \( 0 < c_1 \leq \alpha'(t) \leq c_2 < 1 \), we conclude that
\[
\frac{c_3E_0}{\alpha(t)} \leq E(t) \leq \frac{c_4E_0}{\alpha(t)}, \tag{3.8}
\]
where \( c_3 = \frac{1 - c_2}{1 - c_1} \), \( c_4 = \frac{c_2}{\alpha_1} \).

(2) \( \alpha' \) is decreasing; that is, \( \alpha''(t) \leq 0 \). By using the inequalities
\[
\begin{align*}
\frac{\alpha'(t)\alpha''(t)}{2\alpha(t)} x^2 |z_x(x,t)|^2 + \frac{\alpha(t)\alpha''(t)}{2\alpha'(t)} |z_t(x,t)|^2 \\
\leq \alpha''(t)x z_x(x,t) z_t(x,t) \\
\leq -\frac{\alpha'(t)\alpha''(t)}{2\alpha(t)} x^2 |z_x(x,t)|^2 - \epsilon(t)\alpha(t)\alpha''(t)|z_t(x,t)|^2,
\end{align*}
\]
where \( \epsilon(t) = \frac{\alpha'(t)}{1 - \alpha'(t)} \), we easily get
\[
-\left( \frac{\alpha'(t)}{\alpha(t)} - \frac{\alpha''(t)}{\alpha'(t)} \right) E(t) \leq E'(t) \leq -\left( \frac{\alpha'(t)}{\alpha(t)} + \frac{\alpha''(t)}{1 - \alpha'(t)} \right) E(t), \tag{3.9}
\]
so
\[
\frac{\alpha'(t)E_0}{\alpha'(0)\alpha(t)} \leq E(t) \leq \frac{(1 - \alpha'(t))E_0}{(1 - \alpha'(0))\alpha(t)}. \tag{3.10}
\]
Using \( 0 < c_1 \leq \alpha'(t) \leq c_2 < 1 \), we conclude that
\[
\frac{c_3E_0}{\alpha(t)} \leq E(t) \leq \frac{c_4E_0}{\alpha(t)}, \tag{3.11}
\]
where \( c_3 = \frac{c_2}{\alpha_1} \), \( c_4 = \frac{1 - c_2}{1 - c_1} \).

\[\square\]

**Remark 3.2.** When \( \alpha'(t) \equiv 0 \), that is, \( \alpha(t) = 1 + kt \) for some constant \( k \in (0,1) \), then \( c_3 = c_4 = 1 \), Lemma 3.1 is reduced to Lemma 3.1 in [4].

Next, similar to the proof of [4] Lemma 3.2, we can get the following estimate for each weak solution \( z \) of (2.2) by the multiplier method.

**Lemma 3.3.** For any function \( q \in C^1([0,1]) \), the solution \( z \) of (2.2) satisfies the estimate
\[
\frac{1}{2} \int_0^T \beta(x,t)q(x)|z_x(x,t)|^2 dt \bigg|_0^1 \\
= \frac{1}{2} \int_0^T \int_0^1 q'(x)|z_t(x,t)|^2 + \beta(x,t)|z_x(x,t)|^2 \, dx \, dt \\
- \int_0^T \int_0^1 \alpha'(t)q(x)z_x(x,t)z_t(x,t) \, dx \, dt - \frac{1}{2} \int_0^T \int_0^1 \beta_z(x,t)q(x)|z_x(x,t)|^2 \, dx \, dt \\
+ \int_0^1 [\alpha(t)q(x)z_x(x,t)z_t(x,t) - x\alpha'(t)q(x)|z_x(x,t)|^2] \, dx \bigg|_0^T. \tag{3.12}
\]
Finally, we derive the continuity estimate.

**Theorem 3.4.** Assume $T > 0$, for any $(z^0, z^1) \in H^1_0(0, 1) \times L^2(0, 1)$, there exists a constant $C > 0$ such that the solution of (2.2) satisfies the following two estimates:

\[
\int_0^T \beta(1, t)|z_x(1, t)|^2 dt \leq C(\|z^0\|^2_{H^1_0(0, 1)} + \|z^1\|^2_{L^2(0, 1)}), 
\tag{3.13}
\]

\[
\int_0^T \beta(0, t)|z_x(0, t)|^2 dt \leq C(\|z^0\|^2_{H^1_0(0, 1)} + \|z^1\|^2_{L^2(0, 1)}); 
\tag{3.14}
\]

so $z_x(0, \cdot) \in L^2(0, T)$ and $z_x(1, \cdot) \in L^2(0, T)$.

**Proof.** First, we prove inequality (3.13). Let $q(x) = x$ for $x \in [0, 1]$ in (3.12) and noticing that $\beta_x(x, t) = -\frac{2\alpha'(t)x}{\alpha(t)}$, $\gamma(x, t) = -2\alpha'(t)x$, it follows that

\[
\frac{1}{2} \int_0^T \beta(1, t)|z_x(1, t)|^2 dt 
= \int_0^T E(t) dt - \int_0^T \int_0^1 \alpha'(t)x z_t(x, t) z_x(x, t) dx dt 
+ \int_0^T \int_0^1 \frac{\alpha'(t)^2}{\alpha(t)} x^2 |z_x(x, t)|^2 dx dt 
+ \int_0^1 \left[ |\alpha(t) x z_t(x, t) z_x(x, t)| - \alpha'(t)x^2 |z_x(x, t)|^2 \right] dx \bigg|_0^T. 
\tag{3.15}
\]

We estimate every terms on the right side of (3.15). By the assumption for $\alpha$, we have $1 \leq \alpha(t) \leq 1 + c_2 T$ and $0 < \frac{1 - c_2^2}{c_2^2 T} \leq \beta(x, t) \leq 1$ for any $(x, t) \in Q_T$, these inequalities together with (3.3) and the boundedness of $\alpha'(t)$ imply

\[
\int_0^T E(t) dt - \int_0^T \int_0^1 \alpha'(t)x z_t(x, t) z_x(x, t) dx dt 
+ \int_0^T \int_0^1 \frac{\alpha'(t)^2}{\alpha(t)} x^2 |z_x(x, t)|^2 dx dt 
\leq \int_0^T E(t) dt + C \int_0^T \int_0^1 \|z_t(x, t)\|^2 + |z_x(x, t)|^2 \] dx dt 
\leq \int_0^T E(t) dt + C \int_0^T \int_0^1 [\alpha(t)|z_t(x, t)|^2 + \beta(x, t)|z_x(x, t)|^2] dx dt 
\leq C E_0. 
\tag{3.16}
\]

For each $t \in [0, T]$ and $\epsilon(t) > 0$, it holds that

\[
\left| \int_0^1 [\alpha(t) x z_t(x, t) z_x(x, t) - \alpha'(t)x^2 |z_x(x, t)|^2] dx \right| 
\leq \int_0^1 [\alpha(t)|z_t(x, t)||z_x(x, t)| + \alpha'(t)|z_x(x, t)|^2] dx 
\leq \frac{1}{2\epsilon(t)} \int_0^1 \alpha^2(t)|z_t(x, t)|^2 dx + \frac{\epsilon(t)}{2} \int_0^1 |z_t(x, t)|^2 dx + \int_0^1 \alpha'(t)|z_x(x, t)|^2 dx 
\leq \frac{\alpha(t)}{2\epsilon(t)} \int_0^1 \alpha(t)|z_t(x, t)|^2 dx + \frac{\epsilon(t)}{2} + \alpha'(t) \left[ \frac{\alpha(t)}{1 - \alpha'^2(t)} \right] \int_0^1 \beta(x, t)|z_x(x, t)|^2 dx.
\]
Choosing $\epsilon(t) = 1 - \alpha(t)$, then it is easy to see

$$
\epsilon(t) > 0 \text{ and } \frac{\alpha(t)}{\epsilon} = \frac{\cfrac{\epsilon}{2} + \alpha'(t)}{1 - \alpha'^2(t)} = \frac{\alpha(t)}{1 - \alpha'(t)}.
$$

This implies that

$$
\left| \int_0^1 \left[ \alpha(t)x_z(x,t)z_x(x,t) - \alpha'(t)x^2|z_x(x,t)|^2 \right]dx \right| \leq \frac{\alpha(t)}{1 - \alpha'(t)} E(t) \leq \frac{\alpha(t)}{c_2} E(t).
$$

Then, using (3.3), it follows that

$$
\left| \int_0^T \left[ \alpha(t)(x)(x,t)z_x(x,t) - \alpha'(t)x^2|z_x(x,t)|^2 \right]dx \right| \leq c_5 E_0, \quad (3.17)
$$

where $c_5 = \frac{2c_1}{1-c_2}$. Therefore, combining (3.15), (3.16) and (3.17), it follows that

$$
\int_0^T \beta(1,t)|z_x(1,t)|^2 dt \leq CE_0 \leq C(\|z^0\|^2_{H^0(0,1)} + \|z^1\|^2_{L^2(0,1)}).
$$

Next, we prove the inequality (3.14). Let $q(x) = x - 1$ for $x \in [0,1]$ in (3.12) and noticing that $\beta_x(x) = -\frac{2\alpha'(t)x}{\alpha(t)}$, $\gamma(x,t) = -2\alpha'(t)x$, it follows that

$$
\frac{1}{2} \int_0^T \beta(0,t)|z_x(0,t)|^2 dt
$$

$$
= \int_0^T E(t) dt - \int_0^T \int_0^1 \alpha'(t)(x-1)z_x(x,t)z_x(x,t) dx dt
$$

$$
+ \int_0^T \int_0^1 \frac{\alpha'(t)}{\alpha(t)} x(x-1)|z_x(x,t)|^2 dx dt
$$

$$
+ \int_0^T \int_0^1 \alpha(t)(x-1)z_x(x,t)z_x(x,t) - \alpha'(t)x(x-1)|z_x(x,t)|^2 dx dt \Bigg|_0^T.
$$

Through estimating every terms on the right side of (3.18), similar to the derive of (3.16), it follows that

$$
\int_0^T E(t) dt - \int_0^T \int_0^1 \alpha'(t)(x-1)z_x(x,t)z_x(x,t) dx dt
$$

$$
+ \int_0^T \int_0^1 \frac{\alpha'(t)}{\alpha(t)} x(x-1)|z_x(x,t)|^2 dx dt \leq CE_0. \quad (3.19)
$$

Since

$$
\left| \int_0^1 \left[ \alpha(t)(x-1)z_x(x,t)z_x(x,t) - \alpha'(t)x(x-1)|z_x(x,t)|^2 \right]d_x \right|
$$

$$
\leq \int_0^1 \left[ \alpha(t)|z_x(x,t)||z_x(x,t)| + \alpha'(t)|z_x(x,t)| \right]dx,
$$

similar to the derive of (3.17), it follows that

$$
\left| \int_0^T \left[ \alpha(t)(x-1)z_x(x,t)z_x(x,t) - \alpha'(t)x(x-1)|z_x(x,t)|^2 \right]dx \right|_0^T \leq c_5 E(0), \quad (3.20)
$$

where $c_5 = \frac{2c_4}{1-c_2}$. 

From (3.18), (3.19) and (3.20), it follows that
\[
\int_0^T \beta(0, t)|z_x(0, t)|^2 dt \leq CE_0 \leq (\|z^0\|^2_{H^1_0(0, 1)} + \|z^1\|^2_{L^2(0, 1)}).
\]

□

Now, we give the proof of the observability inequalities.

**Theorem 3.5.** For \( T > T^* \) where \( T^* \) satisfies (1.4) and any \((z^0, z^1) \in H^1_0(0, 1) \times L^2(0, 1)\), there exists a constant \( C > 0 \) such that the corresponding solution of (2.2) satisfies
\[
\int_0^T \beta(1, t)|z_x(1, t)|^2 dt \geq C(\|z^0\|^2_{H^1_0(0, 1)} + \|z^1\|^2_{L^2(0, 1)}).
\]

**Proof.** If we choose \( \epsilon(t) = \frac{\alpha'(t)}{1 + \alpha'(t)} \), then it is obvious that
\[
0 < \epsilon(t) < \frac{1}{2}, \quad \text{and} \quad 1 - \epsilon(t) = 1 + \left(2 - \frac{1}{\epsilon(t)}\right) \frac{\alpha'(t)}{1 - \alpha'(t)} = \frac{1}{1 + \alpha'(t)}.
\]
Thus, using
\[
x^2 = \frac{\alpha(t)x^2}{1 - \alpha'(t)x^2} \beta(x, t) \leq \frac{\alpha(t)}{1 - \alpha'(t)} \beta(x, t)
\]
and (3.3), it follows that
\[
\begin{align*}
\int_0^T E(t) dt & = \int_0^T \int_0^1 \alpha'(t)xz_x(t)z_x(t) \, dx \, dt \\
& + \int_0^T \int_0^1 \frac{\alpha'(t)}{\alpha(t)} x^2 |z_x(t)|^2 \, dx \, dt \\
& \geq \int_0^T \int_0^1 \frac{1 - \epsilon(t)}{2} \alpha(t)|z_t(t)|^2 \, dx \, dt \\
& + \int_0^T \int_0^1 \left(1 + \frac{1}{2\epsilon(t)} \frac{\alpha'(t)}{\alpha(t)} x^2\right) |z_x(t)|^2 \, dx \, dt \\
& \geq \int_0^T \int_0^1 \frac{1 - \epsilon(t)}{2} \alpha(t)|z_t(t)|^2 \, dx \, dt \\
& + \int_0^T \int_0^1 \left[1 + \frac{(2 - \frac{1}{\epsilon(t)})\alpha'(t)}{1 - \alpha'(t)} \frac{1}{2} \beta(x, t)|z_x(t)|^2 \, dx \, dt \\
& = \int_0^T \frac{1}{1 + \alpha'(t)} E(t) dt \\
& \geq c_6 E_0 \int_0^T \frac{1}{\alpha(t)} dt,
\end{align*}
\]
where \( c_6 = c_3/(1 + c_2) \). By (3.15), (3.17) and (3.22), we obtain
\[
\frac{1}{2} \int_0^T \beta(1, t)|z_x(1, t)|^2 dt \geq c_6 E_0 \int_0^T \frac{1}{1 + c_2 t} dt - c_5 E_0 \\
= \left(\frac{c_6}{c_2}\right) \log(1 + c_2 T) - c_5 E_0.
\]
If we choose $T^* = \frac{1}{c_2} \left\{ \exp \left( \frac{c_2 c_5}{c_6} \right) - 1 \right\}$, then it is easy to see that

$$T^* = \frac{1}{c_2} \left\{ \exp \left( \frac{c_2 c_5}{c_6} \right) - 1 \right\},$$

(3.23)

so for $T > T^*$,

$$\int_0^T \beta(1, t)|z_x(1, t)|^2 dt \geq C \left( \|z^0\|_{H^1_0(0, 1)}^2 + \|z^1\|_{L^2(0, 1)}^2 \right),$$

holds for $C = \frac{2\alpha}{c_2} \log(1 + c_2 T) - 2c_5 > 0$. □

**Theorem 3.6.** For $T > T_1^*$ where $T_1^*$ satisfies (1.5) and any $(z^0, z^1) \in H^1_0(0, 1) \times L^2(0, 1)$, there exists a constant $C > 0$ such that the corresponding solution of (2.2) satisfies

$$\int_0^T \beta(0, t)|z_x(0, t)|^2 dt \geq C \left( \|z^0\|_{H^1_0(0, 1)}^2 + \|z^1\|_{L^2(0, 1)}^2 \right).$$

(3.24)

**Proof.** Choosing $\epsilon(x, t) = \frac{\alpha'(t)(1 - \alpha'(t))}{\alpha(t)}$, it is easy to see that

$$\alpha'(t)z_x(x, t)z_t(x, t)$$

$$\leq \frac{\alpha^2(t)}{2\epsilon(x, t)}|z_t(x, t)|^2 + \frac{\epsilon(x, t)}{2} |z_x(x, t)|^2$$

$$= \frac{\alpha'(t)}{1 - \alpha'(t)x} \frac{\alpha(t)}{2} |z_t(x, t)|^2 + \frac{\alpha'(t)}{1 + \alpha'(t)x} \frac{\beta(x, t)}{2} |z_x(x, t)|^2.$$

Since $x - 1 \leq 0$ for $x \in [0, 1]$, we have

$$\int_0^T E(t) dt - \int_0^T \int_0^1 \alpha'(t)(x - 1)z_t(x, t)z_x(x, t) \, dx \, dt$$

$$+ \int_0^T \int_0^1 \frac{\alpha'^2(t)}{\alpha(t)} x(x - 1)|z_x(x, t)|^2 \, dx \, dt$$

$$\geq \int_0^T E(t) dt + \int_0^T \int_0^1 \frac{\alpha'(t)(x - 1)}{1 - \alpha'(t)x} \frac{\alpha(t)}{2} |z_t(x, t)|^2 \, dx \, dt$$

$$+ \frac{\alpha'(t)}{1 + \alpha'(t)x} \frac{\beta(x, t)}{2} |z_x(x, t)|^2 \, dx \, dt$$

(3.25)

$$+ \int_0^T \int_0^1 \frac{2\alpha'^2(t)x(x - 1)}{1 - \alpha'^2(t)x^2} \frac{\beta(x, t)}{2} |z_x(x, t)|^2 \, dx \, dt$$

$$= \frac{1}{2} \int_0^T \int_0^1 \frac{1 - \alpha'(t)}{1 - \alpha'(t)x} |\alpha(t)| z_t(x, t)|^2 + \beta(x, t) |z_x(x, t)|^2 \, dx \, dt$$

$$\geq \int_0^T (1 - \alpha'(t))E(t) dt$$

$$\geq c_6 E_0 \int_0^T \frac{1}{\alpha(t)} dt,$$

where $c_6 = (1 - c_2)c_3$.

From (3.18), (3.20) and (3.25), we arrive at

$$\frac{1}{2} \int_0^T \beta(0, t)|z_x(0, t)|^2 dt \geq c_6 E_0 \int_0^T \frac{1}{\alpha(t)} dt - c_5 E_0.$$
\[
\geq c_6^*E_0 \int_0^T \frac{1}{1 + c_2 t} dt - c_5 E_0 = \left( \frac{c_6^*}{c_2} \log(1 + c_2 T) - c_5 \right) E_0.
\]

If we choose \( T_1^∗ \) as in (1.5), then it is easy to see that

\[
T_1^* = \frac{1}{c_2} \left\{ \exp \left( \frac{c_2 c_5}{c_6^*} \right) - 1 \right\};
\]

thus, when \( T > T_1^* \),

\[
\int_0^T \beta(0, t)|z_x(0, t)|^2 dt \geq C \left( \|z^0\|_{H^1_{0, 1}}^2 + \|z^1\|^2_{L^2_{0, 1}} \right).
\]

holds for \( C = \frac{2c_6^*}{c_2^2} \log(1 + c_2 T) − 2c_5 > 0 \). □

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