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EXISTENCE OF SOLUTIONS TO SECOND-ORDER NONLINEAR COUPLED SYSTEMS WITH NONLINEAR COUPLED BOUNDARY CONDITIONS

IMRAN TALIB, NASEER AHMAD ASIF, CEMIL TUNC

Abstract. In this article, study the existence of solutions for the second-order nonlinear coupled system of ordinary differential equations

$$u''(t) = f(t, v(t)), \quad t \in [0, 1],$$
$$v''(t) = g(t, u(t)), \quad t \in [0, 1],$$

with nonlinear coupled boundary conditions

$$\phi(u(0), v(0), u(1), v(1), u'(0), v'(0)) = (0, 0),$$
$$\psi(u(0), v(0), u(1), v(1), u'(1), v'(1)) = (0, 0),$$

where $f, g : [0, 1] \times \mathbb{R} \to \mathbb{R}$ and $\phi, \psi : \mathbb{R}^6 \to \mathbb{R}^2$ are continuous functions. Our main tools are coupled lower and upper solutions, Arzela-Ascoli theorem, and Schauder's fixed point theorem. The results presented in this article extend those in [1, 3, 15].

1. Introduction

The applications of nonlinear differential equations can be seen in natural sciences including the treatment of problems in classical statistic, population dynamics, chemical kinetics, combustion theory, mechanics, optimal control, ecology, biotechnology, harvesting [9, 14]. Many physical problems can be modeled using nonlinear differential equations. For example, the dynamics of a pendulum under influence of gravity is discussed using second order dimensionless nonlinear differential equation.

The lower and upper solution technique has been widely investigated in studying boundary value problems (BVPs) of differential equations. The idea was firmly established by the work of Perron [10] on the Dirichlet problem for harmonic functions. In late 1960s, Jackson [6] established the theory for treating BVPs of second order nonlinear ordinary differential equations. Remarkable contributions were also made by Schmitt in [11]. In 1973, Gudkov [4] studied the applications of lower and upper solutions to nonlinear boundary conditions for a second order problems;

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Zhang [13] discussed some notable applications to characterize the existence of positive solutions to the Dirichlet problem of a type of sublinear differential equations.

Recently an increasing interest has been observed in investigating the existence of positive solutions for differential equations with nonlinear boundary conditions using coupled lower and upper solutions approach. The reader can see [3] and reference therein. In these articles, the monotonicity assumptions are imposed on the functions that defined nonlinear boundary conditions to generalize the classical results of linear boundary and initial conditions.

The study of system of BVPs has also attracted many authors. The reader can study [1, 5, 7, 12] and references therein; Zhou and Xu [15] established the existence and multiplicity of positive solutions of the following nonlinear coupled system of BVPs

$$\begin{gathered} -u''(t) = f(t, v(t)), \quad t \in (0,1), \\ -v''(t) = g(t, u(t)), \quad t \in (0,1), \\ u(0) = v(0) = 0, \\ \alpha u(\eta) = u(1), \quad \alpha v(\eta) = v(1), \quad \eta \in (0,1), \quad 0 < \alpha\eta < 1, \end{gathered} \tag{1.1}$$

by applying the fixed point index theory in cones. Asif and Khan [1] investigated the existence of a positive solution to the following four-point coupled boundary-value problem

$$\begin{gathered} -x''(t) = f(t, x(t), y(t)), \quad t \in (0,1), \\ -y''(t) = g(t, x(t), y(t)), \quad t \in (0,1), \\ x(0) = 0, \quad x(1) = \alpha y(\xi), \\ y(0) = 0, \quad y(1) = \beta x(\eta), \end{gathered} \tag{1.2}$$

where $\xi, \eta \in (0,1)$, $0 < \alpha\beta\xi\eta < 1$, $f, g : [0,1] \times [0,\infty) \times [0,\infty) \to [0,\infty)$ are continuous and allowed to be singular at $t = 0$ and $t = 1$.

Motivated by the works in [1, 3, 15], we consider the existence of the solution of the following nonlinear coupled system

$$\begin{gathered} u''(t) = f(t, v(t)), \quad t \in [0,1], \\ v''(t) = g(t, u(t)), \quad t \in [0,1], \end{gathered} \tag{1.3}$$

with nonlinear coupled boundary conditions (CBCs)

$$\begin{gathered} \phi(u(0), v(0), u(1), v(1), u'(0), v'(0)) = (0,0), \\ \psi(u(0), v(0), u(1), v(1), u'(1), v'(1)) = (0,0), \end{gathered} \tag{1.4}$$

where $f, g : [0,1] \times \mathbb{R} \to \mathbb{R}$ and $\phi, \psi : \mathbb{R}^6 \to \mathbb{R}^2$ are continuous functions.

Our proposed work is new and a productive addition in the existing literature of nonlinear CBCs. The results presented in [1, 3, 15] are extended in our work. In [15], the coupling is defined in the differential system only but not in the boundary conditions. But our problem (1.3)-(1.4) deals with the case where the coupling is not only in the differential system (1.3) but also in the boundary conditions (1.4). In [3], the idea of the coupled lower and upper solutions is discussed for a single differential equation but in our case it is extended for the system of differential equations (1.3). In [1], the coupling is discussed not only in the differential system but also in the boundary conditions. However, Asif and Khan [1] did not generalize the classical results. We mean to say that the existence results discussed in [1] dealt with specific type of boundary conditions. On the other hand, we took boundary conditions (1.4) in a generalized way. That is (1.4) generalizes most of the usual linear boundary conditions. For instance, if $\phi(j,k,l,m,n,o) = (S-j, W-k)$ and $\psi(j,k,l,m,n,o) = (X-l, Y-m)$ with $S,W,X,Y \in \mathbb{R}$, then (1.4) implies the Dirichlet boundary conditions. Similarly, if $\phi(j,k,l,m,n,o) = (n-S, o-W)$ and $\psi(j,k,l,m,n,o) = (X-n, Y-o)$, then (1.4) implies the Neumann boundary conditions.

Definitely in order to obtain a solution satisfying some boundary conditions and lying between a subsolution and a supersolution some more conditions are needed. For example for the Dirichlet case, it suffices that

$$\begin{aligned} \alpha_1(0) \le S \le \beta_1(0), \quad \alpha_2(0) \le W \le \beta_2(0), \\ \alpha_1(1) \le X \le \beta_1(1), \quad \alpha_2(1) \le Y \le \beta_2(1), \end{aligned} \tag{1.5}$$

and for the Neumann case, it suffices that

$$\begin{aligned} \beta_1'(0) \le S \le \alpha_1'(0), \quad \beta_2'(0) \le W \le \alpha_2'(0), \\ \alpha_1'(1) \le X \le \beta_1'(1), \quad \alpha_2'(1) \le Y \le \beta_2'(1). \end{aligned} \tag{1.6}$$

Definition 1.1. We say that a couple of functions $(\alpha_1, \alpha_2) \in C^2[0,1] \times C^2[0,1]$ is a subsolution of (1.3) if

$$\begin{aligned} \alpha_1''(t) \ge f(t, \alpha_2(t)), \quad t \in [0,1], \\ \alpha_2''(t) \ge g(t, \alpha_1(t)), \quad t \in [0,1]. \end{aligned} \tag{1.7}$$

In the same way, a supersolution is a couple of functions $(\beta_1, \beta_2) \in C^2[0,1] \times C^2[0,1]$ that satisfies the reversed inequalities in (1.7). In what follows we will write $(\alpha_1, \alpha_2) \preceq (\beta_1, \beta_2)$, if $\alpha_1(t) \le \beta_1(t)$ and $\alpha_2(t) \le \beta_2(t)$, for all $t \in [0,1]$.

For $u, v \in C^2[0,1]$, we define the set

$$[u,v] = \{w \in C^2[0,1] : u(t) \le w(t) \le v(t), t \in [0,1]\}.$$

This article is organized as follows. In Section 2, we study the second order nonlinear coupled system of BVPs along with definitions of lower and upper solutions that generalize the most of the usual linear boundary conditions. Moreover, we prove new existence results assuming the functions ϕ and ψ in (1.4) are monotone. In Section 3, examples are included to show the applicability of our results. In Section 4, the conclusion of the article is presented. Here

$$C_0^2[0,1] = \{w \in C^2[0,1] : w(0) = 0\}.$$

The following lemma is very much helpful in proving Theorem 2.2.

Lemma 1.2. *Let $L : C^1[0,1] \times C^1[0,1] \to C_0^2[0,1] \times C_0^2[0,1] \times \mathbb{R}^2 \times \mathbb{R}^2$ be defined by*

$$\begin{aligned} [L(u,v)](t) = \Big(& u'(t) - u'(0) - \lambda \int_0^t u(s) \mathrm{d}s, v'(t) - v'(0) - \lambda \int_0^t v(s)\, \mathrm{d}s, \\ & (au(0) + bu(1), cv(0) + dv(1)), (Eu(0) + Fu(1), Gv(0) + Hv(1)) \Big), \end{aligned} \tag{1.8}$$

where $\lambda, a, b, c, d, E, F, G$ and H are real constants with $\lambda > 0$, such that

$$(ad - bc)(EH - FG)\big(e^{-\sqrt{\lambda}} - e^{\sqrt{\lambda}}\big) \ne 0,$$

Then L^{-1} exists and is continuous and is defined by

$$
\begin{aligned}
&[L^{-1}(y,z,\gamma,\delta,\mu,\zeta)] \\
&= \Big(C_1 e^{\sqrt{\lambda}t} + C_2 e^{-\sqrt{\lambda}t} + \frac{1}{2}\int_0^t e^{\sqrt{\lambda}(t-s)} y(s)\,\mathrm{d}s + \frac{1}{2}\int_0^t e^{\sqrt{\lambda}(s-t)} y(s)\,\mathrm{d}s, \\
&C_3 e^{\sqrt{\lambda}t} + C_4 e^{-\sqrt{\lambda}t} + \frac{1}{2}\int_0^t e^{\sqrt{\lambda}(t-s)} z(s)\,\mathrm{d}s + \frac{1}{2}\int_0^t e^{\sqrt{\lambda}(s-t)} z(s)\,\mathrm{d}s\Big),
\end{aligned} \tag{1.9}
$$

with

$$
\begin{aligned}
C_1 = \frac{1}{(ad-bc)(e^{\sqrt{\lambda}} - e^{-\sqrt{\lambda}})} \Big(& 2\delta(a + be^{-\sqrt{\lambda}}) - d(a + be^{-\sqrt{\lambda}}) \\
& \times \int_0^1 e^{\sqrt{\lambda}(1-s)} y(s)ds + d(a + be^{-\sqrt{\lambda}}) \int_0^1 e^{\sqrt{\lambda}(s-1)} y(s)ds \\
& - 2\gamma(c + de^{-\sqrt{\lambda}}) + b(c + de^{-\sqrt{\lambda}}) \int_0^1 e^{\sqrt{\lambda}(1-s)} y(s)ds \\
& - b(c + de^{-\sqrt{\lambda}}) \int_0^1 e^{\sqrt{\lambda}(s-1)} y(s)ds \Big),
\end{aligned}
$$

$$
\begin{aligned}
C_2 = \frac{1}{(ad-bc)(e^{-\sqrt{\lambda}} - e^{\sqrt{\lambda}})} \Big(& 2\delta(a + be^{\sqrt{\lambda}}) - d(a + be^{\sqrt{\lambda}}) \\
& \times \int_0^1 e^{\sqrt{\lambda}(1-s)} y(s)ds + d(a + be^{\sqrt{\lambda}}) \int_0^1 e^{\sqrt{\lambda}(s-1)} y(s)ds \\
& - 2\gamma(c + de^{\sqrt{\lambda}}) + b(c + de^{\sqrt{\lambda}}) \int_0^1 e^{\sqrt{\lambda}(1-s)} y(s)ds \\
& - b(c + de^{\sqrt{\lambda}}) \int_0^1 e^{\sqrt{\lambda}(s-1)} y(s)ds \Big),
\end{aligned}
$$

$$
\begin{aligned}
C_3 = \frac{1}{(EH-FG)(e^{\sqrt{\lambda}} - e^{-\sqrt{\lambda}})} \Big(& 2\zeta(E + Fe^{-\sqrt{\lambda}}) - H(E + Fe^{-\sqrt{\lambda}}) \\
& \times \int_0^1 e^{\sqrt{\lambda}(1-s)} z(s)ds + F(E + Fe^{-\sqrt{\lambda}}) \int_0^1 e^{\sqrt{\lambda}(s-1)} z(s)ds \\
& - 2\mu(G + He^{-\sqrt{\lambda}}) + F(G + He^{-\sqrt{\lambda}}) \int_0^1 e^{\sqrt{\lambda}(1-s)} z(s)ds \\
& - F(G + He^{-\sqrt{\lambda}}) \int_0^1 e^{\sqrt{\lambda}(s-1)} z(s)ds \Big),
\end{aligned}
$$

and

$$
\begin{aligned}
C_4 = \frac{1}{(EH-FG)(e^{-\sqrt{\lambda}} - e^{\sqrt{\lambda}})} \Big(& 2\zeta(E + Fe^{\sqrt{\lambda}}) - H(E + Fe^{\sqrt{\lambda}}) \\
& \times \int_0^1 e^{\sqrt{\lambda}(1-s)} z(s)ds + F(E + Fe^{\sqrt{\lambda}}) \int_0^1 e^{\sqrt{\lambda}(s-1)} z(s)ds \\
& - 2\mu(G + He^{\sqrt{\lambda}}) + F(G + He^{\sqrt{\lambda}}) \int_0^1 e^{\sqrt{\lambda}(1-s)} z(s)ds \\
& - F(G + He^{\sqrt{\lambda}}) \int_0^1 e^{\sqrt{\lambda}(s-1)} z(s)ds \Big).
\end{aligned}
$$

Proof. Choose

$$y(t) = u'(t) - u'(0) - \lambda \int_0^t u(s)\,\mathrm{d}s, \tag{1.10}$$

$$z(t) = v'(t) - v'(0) - \lambda \int_0^t v(s)\,\mathrm{d}s, \tag{1.11}$$

$$\gamma = au(0) + bu(1), \tag{1.12}$$

$$\delta = cv(0) + dv(1), \tag{1.13}$$

$$\mu = Eu(0) + Fu(1), \tag{1.14}$$

$$\zeta = Gv(0) + Hv(1). \tag{1.15}$$

In light of (1.10)–(1.15), (1.8) can also be written as

$$[L(u, v)](t) = (y(t), z(t), (\gamma, \delta), (\mu, \zeta)). \tag{1.16}$$

Differentiating (1.10) with respect to t, we have

$$y'(t) = u''(t) - \lambda u(t), \quad \lambda > 0. \tag{1.17}$$

The general solution of (1.17) can be easily determined using variation of parameters technique along with integration by parts and taking limits of integration from 0 to t, we have

$$u(t) = C_1 e^{\sqrt{\lambda}t} + C_2 e^{-\sqrt{\lambda}t} + \frac{1}{2}\int_0^t e^{\sqrt{\lambda}(t-s)} y(s)\mathrm{d}s + \frac{1}{2}\int_0^t e^{\sqrt{\lambda}(s-t)} y(s)\mathrm{d}s. \tag{1.18}$$

C_1 and C_2 can be easily determined with the help of (1.12) and (1.13) as

$$\begin{aligned} \gamma &= (a + be^{\sqrt{\lambda}})C_1 + (a + be^{-\sqrt{\lambda}})C_2 + \frac{b}{2}\Big(\int_0^1 e^{\sqrt{\lambda}(1-s)} y(s)\mathrm{d}s + e^{\sqrt{\lambda}(s-1)} y(s)\mathrm{d}s\Big), \\ \delta &= (c + de^{\sqrt{\lambda}})C_1 + (c + de^{-\sqrt{\lambda}})C_2 + \frac{d}{2}\Big(\int_0^1 e^{\sqrt{\lambda}(1-s)} y(s)\mathrm{d}s + e^{\sqrt{\lambda}(s-1)} y(s)\mathrm{d}s\Big). \end{aligned} \tag{1.19}$$

Solving the system of equations (1.19), we have

$$\begin{aligned} C_1 = \frac{1}{(ad - bc)(e^{\sqrt{\lambda}} - e^{-\sqrt{\lambda}})} &\Big(2\delta(a + be^{-\sqrt{\lambda}}) - d(a + be^{-\sqrt{\lambda}}) \\ &\times \int_0^1 e^{\sqrt{\lambda}(1-s)} y(s)ds + d(a + be^{-\sqrt{\lambda}}) \int_0^1 e^{\sqrt{\lambda}(s-1)} y(s)ds \\ &- 2\gamma(c + de^{-\sqrt{\lambda}}) + b(c + de^{-\sqrt{\lambda}}) \int_0^1 e^{\sqrt{\lambda}(1-s)} y(s)ds \\ &- b(c + de^{-\sqrt{\lambda}}) \int_0^1 e^{\sqrt{\lambda}(s-1)} y(s)ds\Big), \end{aligned} \tag{1.20}$$

and

$$\begin{aligned} C_2 = \frac{1}{(ad-bc)(e^{-\sqrt{\lambda}}-e^{\sqrt{\lambda}})}\Big(& 2\delta(a+be^{\sqrt{\lambda}}) - d(a+be^{\sqrt{\lambda}}) \\ & \times \int_0^1 e^{\sqrt{\lambda}(1-s)}y(s)ds + d(a+be^{\sqrt{\lambda}})\int_0^1 e^{\sqrt{\lambda}(s-1)}y(s)ds \\ & - 2\gamma(c+de^{\sqrt{\lambda}}) + b(c+de^{\sqrt{\lambda}})\int_0^1 e^{\sqrt{\lambda}(1-s)}y(s)ds \\ & - b(c+de^{\sqrt{\lambda}})\int_0^1 e^{\sqrt{\lambda}(s-1)}y(s)ds\Big). \end{aligned} \tag{1.21}$$

Similarly on the same line, it can be easily shown that

$$v(t) = C_3e^{\sqrt{\lambda}t} + C_4e^{-\sqrt{\lambda}t} + \frac{1}{2}\int_0^t e^{\sqrt{\lambda}(t-s)}z(s)\mathrm{d}s + \frac{1}{2}\int_0^t e^{\sqrt{\lambda}(s-t)}z(s)\mathrm{d}s, \tag{1.22}$$

with

$$\begin{aligned} C_3 = \frac{1}{(EH-FG)(e^{\sqrt{\lambda}}-e^{-\sqrt{\lambda}})}\Big(& 2\zeta(E+Fe^{-\sqrt{\lambda}}) - H(E+Fe^{-\sqrt{\lambda}}) \\ & \times \int_0^1 e^{\sqrt{\lambda}(1-s)}z(s)ds + F(E+Fe^{-\sqrt{\lambda}})\int_0^1 e^{\sqrt{\lambda}(s-1)}z(s)ds \\ & - 2\mu(G+He^{-\sqrt{\lambda}}) + F(G+He^{-\sqrt{\lambda}})\int_0^1 e^{\sqrt{\lambda}(1-s)}z(s)ds \\ & - F(G+He^{-\sqrt{\lambda}})\int_0^1 e^{\sqrt{\lambda}(s-1)}z(s)ds\Big), \end{aligned} \tag{1.23}$$

and

$$\begin{aligned} C_4 = \frac{1}{(EH-FG)(e^{-\sqrt{\lambda}}-e^{\sqrt{\lambda}})}\Big(& 2\zeta(E+Fe^{\sqrt{\lambda}}) - H(E+Fe^{\sqrt{\lambda}}) \\ & \times \int_0^1 e^{\sqrt{\lambda}(1-s)}z(s)ds + F(E+Fe^{\sqrt{\lambda}})\int_0^1 e^{\sqrt{\lambda}(s-1)}z(s)ds \\ & - 2\mu(G+He^{\sqrt{\lambda}}) + F(G+He^{\sqrt{\lambda}})\int_0^1 e^{\sqrt{\lambda}(1-s)}z(s)ds \\ & - F(G+He^{\sqrt{\lambda}})\int_0^1 e^{\sqrt{\lambda}(s-1)}z(s)ds\Big). \end{aligned} \tag{1.24}$$

Equation (1.16) can also be written as

$$(u(t), v(t)) = [L^{-1}(y(t), z(t), (\gamma, \delta), (\mu, \zeta))]. \tag{1.25}$$

Hence (1.18)-(1.25) prove the required result. □

2. Coupled lower and upper solutions

The following definition is very helpful for constructing the statement of the main result Theorem 2.2; also it covers different possibilities for the nonlinear functions ϕ and ψ.

Definition 2.1. We say that (α_1, α_2), $(\beta_1, \beta_2) \in C^2[0,1]\times C^2[0,1]$ are coupled lower and upper solutions for the problem (1.3) and (1.4) if (α_1, α_2) is a subsolution and

(β_1, β_2) is a supersolution for the system (1.3) such that

$$\begin{aligned}
&\phi(\beta_1(0), \beta_2(0), \beta_1(1), \beta_2(1), \beta_1'(0), \beta_2'(0)) \\
&\preceq (0,0) \preceq \phi(\alpha_1(0), \alpha_2(0), \alpha_1(1), \alpha_2(1), \alpha_1'(0), \alpha_2'(0)) \\
&\phi(\beta_1(0), \beta_2(0), \alpha_1(1), \alpha_2(1), \beta_1'(0), \beta_2'(0)) \\
&\preceq (0,0) \preceq \phi(\alpha_1(0), \alpha_2(0), \beta_1(1), \beta_2(1), \alpha_1'(0), \alpha_2'(0)), \\
&\psi(\beta_1(0), \beta_2(0), \beta_1(1), \beta_2(1), \beta_1'(1), \beta_2'(1)) \\
&\preceq (0,0) \preceq \psi(\alpha_1(0), \alpha_2(0), \alpha_1(1), \alpha_2(1), \alpha_1'(1), \alpha_2'(1)) \\
&\psi(\alpha_1(0), \alpha_2(0), \beta_1(1), \beta_2(1), \beta_1'(1), \beta_2'(1)) \\
&\preceq (0,0) \preceq \psi(\beta_1(0), \beta_2(0), \alpha_1(1), \alpha_2(1), \alpha_1'(1), \alpha_2'(1)).
\end{aligned} \tag{2.1}$$

Theorem 2.2. *Assume that (α_1, α_2), (β_1, β_2) are coupled lower and upper solutions for the problem (1.3)-(1.4). Suppose that the functions ϕ and ψ are monotone nondecreasing and nonincreasing in the fifth and sixth arguments respectively. In addition, suppose that the functions*

$$\begin{aligned}
\phi_\alpha(x, y) &:= \phi(\alpha_1(0), \alpha_2(0), x, y, \alpha_1'(0), \alpha_2'(0)), \\
\phi_\beta(x, y) &:= \phi(\beta_1(0), \beta_2(0), x, y, \beta_1'(0), \beta_2'(0)),
\end{aligned}$$

are monotone on $[\alpha_1(1), \beta_1(1)] \times [\alpha_2(1), \beta_2(1)]$ and that the functions

$$\begin{aligned}
\psi_\alpha(x, y) &:= \psi(x, y, \alpha_1(1), \alpha_2(1), \alpha_1'(1), \alpha_2'(1)), \\
\psi_\beta(x, y) &:= \psi(x, y, \beta_1(1), \beta_2(1), \beta_1'(1), \beta_2'(1)),
\end{aligned}$$

are monotone on $[\alpha_1(0), \beta_1(0)] \times [\alpha_2(0), \beta_2(0)]$.

Then there exists at least one solution $(u, v) \in [\alpha_1, \beta_1] \times [\alpha_2, \beta_2]$ of problem (1.3)-(1.4).

Proof. Let $\lambda > 0$ and consider the modified system

$$\begin{aligned}
u''(t) - \lambda u(t) &= F^*(t, u(t), v(t)), \ t \in [0,1], \\
v''(t) - \lambda v(t) &= G^*(t, u(t), v(t)), \ t \in [0,1], \\
\phi^*(u(0), v(0), u(1), v(1), u'(0), v'(0)) &= (u(0), v(0)), \\
\psi^*(u(0), v(0), u(1), v(1), u'(1), v'(1)) &= (u(1), v(1)),
\end{aligned} \tag{2.2}$$

with

$$F^*(t, u(t), v(t)) = \begin{cases}
f(t, \beta_2(t)) - \lambda\beta_1(t) & \text{if } v(t) > \beta_2(t), u(t) > \beta_1(t), \\
f(t, v(t)) - \lambda\beta_1(t) & \text{if } \alpha_2(t) \le v(t) \le \beta_2(t), u(t) > \beta_1(t), \\
f(t, \alpha_2(t)) - \lambda\beta_1(t) & \text{if } v(t) < \alpha_2(t), u(t) > \beta_1(t), \\
f(t, \beta_2(t)) - \lambda u(t) & \text{if } v(t) > \beta_2(t), \alpha_1(t) \le u(t) \le \beta_1(t), \\
f(t, v(t)) - \lambda u(t) & \text{if } \alpha_2(t) \le v(t) \le \beta_2(t), \\
& \quad \alpha_1(t) \le u(t) \le \beta_1(t), \\
f(t, \alpha_2(t)) - \lambda u(t) & \text{if } v(t) < \alpha_2(t), \alpha_1(t) \le u(t) \le \beta_1(t), \\
f(t, \beta_2(t)) - \lambda\alpha_1(t) & \text{if } v(t) > \beta_2(t), u(t) < \alpha_1(t), \\
f(t, v(t)) - \lambda\alpha_1(t) & \text{if } \alpha_2(t) \le v(t) \le \beta_2(t), u(t) < \alpha_1(t), \\
f(t, \alpha_2(t)) - \lambda\alpha_1(t) & \text{if } v(t) < \alpha_2(t), u(t) < \alpha_1(t),
\end{cases}$$

and

$$G^*(t,u(t),v(t)) = \begin{cases} g(t,\beta_1(t)) - \lambda\beta_2(t) & \text{if } v(t) > \beta_2(t), u(t) > \beta_1(t), \\ g(t,u(t)) - \lambda\beta_2(t) & \text{if } \alpha_1(t) \le u(t) \le \beta_1(t), v(t) > \beta_2(t), \\ g(t,\alpha_1(t)) - \lambda\beta_2(t) & \text{if } u(t) < \alpha_1(t), v(t) > \beta_2(t), \\ g(t,\beta_1(t)) - \lambda v(t) & \text{if } u(t) > \beta_1(t), \alpha_2(t) \le v(t) \le \beta_2(t), \\ g(t,u(t)) - \lambda v(t) & \text{if } \alpha_1(t) \le u(t) \le \beta_1(t), \\ & \quad \alpha_2(t) \le v(t) \le \beta_2(t), \\ g(t,\alpha_1(t)) - \lambda v(t) & \text{if } u(t) < \alpha_1(t), \alpha_2(t) \le v(t) \le \beta_2(t), \\ g(t,\beta_1(t)) - \lambda\alpha_2(t) & \text{if } u(t) > \beta_1(t), v(t) < \alpha_2(t), \\ g(t,u(t)) - \lambda\alpha_2(t) & \text{if } \alpha_1(t) \le u(t) \le \beta_1(t), v(t) < \alpha_2(t), \\ g(t,\alpha_1(t)) - \lambda\alpha_2(t) & \text{if } u(t) < \alpha_1(t), v(t) < \alpha_2(t), \end{cases}$$

$$\phi^*(j,k,l,m,n,o) = p(0,(j,k) + \phi(j,k,l,m,n,o)),$$
$$\psi^*(j,k,l,m,n,o) = p(1,(l,m) + \psi(j,k,l,m,n,o)),$$

and

$$p(t,(x,y)) = \begin{cases} (\beta_1(t),\beta_2(t)) & \text{if } (x,y) \npreceq (\beta_1(t),\beta_2(t)), \\ (x,y) & \text{if } (\alpha_1(t),\alpha_2(t)) \preceq (x,y) \preceq (\beta_1(t),\beta_2(t)), \\ (\alpha_1(t),\alpha_2(t)) & \text{if } (x,y) \nsucceq (\alpha_1(t),\alpha_2(t)). \end{cases}$$

Note that if $(u,v) \in [\alpha_1,\beta_1] \times [\alpha_2,\beta_2]$ is a solution of (2.2), then (u,v) is a solution of (1.3)-(1.4). For the sake of simplicity we divide the proof into three steps:

Step 1: We define the mappings

$$L, N : C^1[0,1] \times C^1[0,1] \to C_0^2[0,1] \times C_0^2[0,1] \times \mathbb{R}^2 \times \mathbb{R}^2,$$

by

$$\begin{aligned}[L(u,v)](t) = \Big(& u'(t) - u'(0) - \lambda \int_0^t u(s)\,\mathrm{d}s, v'(t) - v'(0) - \lambda \int_0^t v(s)\,\mathrm{d}s, \\ &(u(0),v(0)),(u(1),v(1))\Big),\end{aligned}$$

and

$$\begin{aligned}[N(u,v)](t) = \Big(&\int_0^t F^*(s,u(s),v(s))ds, \int_0^t G^*(s,u(s),v(s))ds, \\ &\phi^*(u(0),v(0),u(1),v(1),u'(0),v'(0)), \\ &\psi^*(u(0),v(0),u(1),v(1),u'(1),v'(1))\Big).\end{aligned}$$

Since $F^*(s,u(s),v(s))$ and $G^*(s,u(s),v(s))$ are bounded on $[0,1] \times \mathbb{R}^2$ and integral is a continuous function on $[0,1]$. Further ϕ^* and ψ^* being constant functions are continuous. Therefore $[N(u,v)]$ is continuous on $[0,1]$. Further, the class $\{N(u,v) : u,v \in C^1[0,1]\}$ is uniformly bounded and equicontinuous. Therefore in view of Arzela-Ascoli theorem $\{N(u,v) : u,v \in C^1[0,1]\}$ is relatively compact. Consequently N is a compact map. Also from Lemma 1.2 with $a = 1, b = 0$, $c = 1, d = 0$ and $E = 0$, $F = 1$, $G = 0$, $H = 1$, L^{-1}, exists and is continuous.

On the other hand, solving (2.2) is equivalent to find a fixed point of

$$L^{-1}N : C^1[0,1] \times C^1[0,1] \to C^1[0,1] \times C^1[0,1].$$

Now, Schauder's fixed point theorem guarantees the existence of at least a fixed point since $L^{-1}N$ is continuous and compact.

Step 2: If (u,v) is a solution of (2.2), then $(u,v)\in[\alpha_1,\beta_1]\times[\alpha_2,\beta_2]$. We claim that $(u,v)\preceq(\beta_1,\beta_2)$. If $(u,v)\npreceq(\beta_1,\beta_2)$, then either $u\npreceq\beta_1$ and/or $v\npreceq\beta_2$. If $u\npreceq\beta_1$, then $u-\beta_1$ attains a positive maximum at some $t_0\in[0,1]$. Clearly, $u(t_0)-\beta_1(t_0)>0$. Thus $(u-\beta_1)'(t_0)=0$ and $(u-\beta_1)''(t_0)<0$. But,

$$\begin{aligned}(u-\beta_1)''(t_0)&>F^*(t_0,u(t_0),v(t_0))+\lambda u(t_0)-f(t_0,\beta_2(t_0))\\&=f(t_0,\beta_2(t_0))-\lambda\beta_1(t_0)+\lambda u(t_0)-f(t_0,\beta_2(t_0))\\&=\lambda(u(t_0)-\beta_1(t_0))>0,\end{aligned}$$

a contradiction. Similarly, one can show that $(\alpha_1,\alpha_2)\preceq(u,v)$. Hence $(u,v)\in[\alpha_1,\beta_1]\times[\alpha_2,\beta_2]$.

Step 3: If (u,v) is a solution of (2.2), then (u,v) satisfies (1.4). We claim that

$$(\alpha_1(0),\alpha_2(0))\preceq(u(0),v(0))+\phi(u(0),v(0),u(1),v(1),u'(0),v'(0))\preceq(\beta_1(0),\beta_2(0)). \tag{2.3}$$

If $(u(0),v(0))+\phi(u(0),v(0),u(1),v(1),u'(0),v'(0))\npreceq(\beta_1(0),\beta_2(0))$, then

$$\begin{aligned}(u(0),v(0))&=\phi^*(u(0),v(0),u(1),v(1),u'(0),v'(0))\\&=p(0,(u(0),v(0))+\phi(u(0),v(0),u(1),v(1),u'(0),v'(0)))\\&=(\beta_1(0),\beta_2(0)).\end{aligned}$$

From Step 2, we know that $(u,v)\preceq(\beta_1,\beta_2)$, and this together with $(u-\beta_1,v-\beta_2)\in C^2[0,1]\times C^2[0,1]$ and $(u(0),v(0))=(\beta_1(0),\beta_2(0))$ yields $u'(0)\le\beta_1'(0)$ and $v'(0)\le\beta_2'(0)$. If $\phi_\beta(x,y)$ is monotone nonincreasing, then

$$\begin{aligned}&(u(0),v(0))+\phi\big(u(0),v(0),u(1),v(1),u'(0),v'(0)\big)\\&=(\beta_1(0),\beta_2(0))+\phi\big(\beta_1(0),\beta_2(0),u(1),v(1),u'(0),v'(0)\big)\\&\preceq(\beta_1(0),\beta_2(0))+\phi\big(\beta_1(0),\beta_2(0),u(1),v(1),\beta_1'(0),\beta_2'(0)\big)\\&=(\beta_1(0),\beta_2(0))+\phi_\beta(u(1),v(1))\\&\preceq(\beta_1(0),\beta_2(0))+\phi_\beta(\alpha_1(1),\alpha_2(1))\\&=(\beta_1(0),\beta_2(0))+\phi\big(\beta_1(0),\beta_2(0),\alpha_1(1),\alpha_2(1),\beta_1'(0),\beta_2'(0)\big)\\&\preceq(\beta_1(0),\beta_2(0)),\end{aligned}\tag{2.4}$$

a contradiction. Similarly, if $\phi_\beta(x,y)$ is monotone nondecreasing, then we get same contradiction. Consequently, (2.3) holds. Similar reasoning shows the other boundary condition. Consequently, (u,v) satisfies (1.4). Hence the system of BVPs (1.3)-(1.4) has a solution $(u,v)\in[\alpha_1,\beta_1]\times[\alpha_2,\beta_2]$. □

3. Examples

Example 3.1. Consider the differential equations

$$\begin{aligned}u''(t)&=v(t)+\sin(t),\quad t\in[0,1],\\v''(t)&=u^3(t)+\cos(t),\quad t\in[0,1],\end{aligned}\tag{3.1}$$

with the nonlinear coupled boundary conditions

$$\begin{aligned}&\big(u^{\frac{1}{2n+1}}(1)-v(0),v^{\frac{1}{2n+1}}(1)-u(0)\big)=(0,0),\quad n\in\mathbb{N},\\&(u'(1)-v^{2n+1}(1),v'(1)-u^{2n+1}(1))=(0,0),\quad n\in\mathbb{N}.\end{aligned}\tag{3.2}$$

Let $\alpha_1(t) = \sin(t) - 5$, $\alpha_2(t) = \sin(t) - 6$ and $\beta_1(t) = \sin(t) + 5$, $\beta_2(t) = \sin(t) + 6$. It is easy to show that (α_1, α_2), (β_1, β_2) are subsolution and supersolution of the system (3.1), respectively. Further, (α_1, α_2), (β_1, β_2) satisfies the system (2.1). Hence by Theorem 2.2, the system of BVPs (3.1)-(3.2) has at least one solution $(u, v) \in [\alpha_1, \beta_1] \times [\alpha_2, \beta_2]$.

Example 3.2. Consider the differential equations

$$\begin{aligned} u''(t) &= v^3(t) - 10\sqrt{\sin(t)}, \quad && t \in [0, 1], \\ v''(t) &= u^3(t) - 8\sqrt{\cos(t-1)}, \quad && t \in [0, 1], \end{aligned} \tag{3.3}$$

with the nonlinear coupled boundary conditions

$$\begin{aligned} &\big(u(0)u(1) - v(0)v(1), u(0)u'(0) - v(0)v'(0)\big) = (0, 0), \\ &\big(u(0)u'(1) - v(0)v(1), u(1)u'(1) - v(1)v'(1)\big) = (0, 0). \end{aligned} \tag{3.4}$$

Let $\alpha_1(t) = -\frac{t^3}{2}$, $\alpha_2(t) = -\frac{t^5}{3}$ and $\beta_1(t) = t+2$, $\beta_2(t) = t+3$. It is easy to show that (α_1, α_2), (β_1, β_2) are subsolution and supersolution of system (3.3), respectively. Further, (α_1, α_2), (β_1, β_2) satisfies system (2.1). Hence by Theorem 2.2, system of BVPs (3.3)-(3.4) has at least one solution $(u, v) \in [\alpha_1, \beta_1] \times [\alpha_2, \beta_2]$.

Conclusion. In this article, the existence of the solution of the second order nonlinear coupled system with nonlinear (CBCs) is investigated using coupled lower and upper solutions approach. We extend the work presented in [1, 3, 15] and the boundary conditions (1.4) generalize most of the linear and nonlinear boundary conditions [2, 8]. Furthermore, the concept of coupled lower and upper solutions is defined in Section 2 that verifies the classical results (1.5)-(1.6). Some examples are taken to ensure the validity of the theoretical results.

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Imran Talib

Department of Mathematics, School of Science, University of Management and Technology, CII Johar Town, Lahore, Pakistan

E-mail address: imrantaalib@gmail.com

Naseer Ahmad Asif

Department of Mathematics, School of Science, University of Management and Technology, CII Johar Town, Lahore, Pakistan

E-mail address: naseerasif@yahoo.com

Cemil Tunc

Department of Mathematics Faculty of Sciences, Yuzuncu Yil University, Van - Turkey

E-mail address: cemtunc@yahoo.com