DYNAMICS OF THE $p$-LAPLACIAN EQUATIONS WITH NONLINEAR DYNAMIC BOUNDARY CONDITIONS

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Abstract. In this article, we study the long-time behavior of the $p$-Laplacian equation with nonlinear dynamic boundary conditions for both autonomous and non-autonomous cases. For the autonomous case, some asymptotic regularity of solutions is proved. For the non-autonomous case, we obtain the existence and structure of a compact uniform attractor in $L^{r_1}(\Omega) \times L^{r}(\Gamma)$ ($r = \min(r_1, r_2)$).

1. Introduction

In this article, we consider the asymptotic behavior of solutions of the following $p$-Laplacian equations with nonlinear dynamic boundary conditions:

$$
\begin{align*}
    u_t - \Delta_p u + f(u) &= h(x,t), & \text{in } \Omega, \\
    u_t + |\nabla u|^{p-2}\partial_n u + g(u) &= 0, & \text{on } \Gamma,
\end{align*}
$$

where $\Omega$ is a bounded domain in $\mathbb{R}^N$ ($N \geq 3$) with a smooth boundary $\Gamma$, $\Delta_p$ denotes the $p$-Laplacian operator, which is defined as $\Delta_p u = \text{div}(|\nabla u|^{p-2}\nabla u)$, $p \geq 2$, and about the external forcing $h(x,t)$, we consider two cases: the autonomous case $h(x,t) = h(x) \in L^{r_1}(\Omega)$, where $r_1'$ is conjugate to $r_1$, and the non-autonomous case $h(x,t)$, which will be given later in Sections 3 and 4 respectively. The functions $f$ and $g \in C^1(\mathbb{R}, \mathbb{R})$, satisfy the following conditions:

$$
\begin{align*}
    C_1|s|^{r_1} - k_1 \leq f(s)s &\leq C_2|s|^{r_1} + k_2, & r_1 &\geq p, \\
    C_3|s|^{r_2} - k_3 \leq g(s)s &\leq C_4|s|^{r_2} + k_4, & r_2 &\geq 2, \\
    f'(s) &\geq -l, & g'(s) &\geq -m,
\end{align*}
$$

here $l, m > 0$, $C_i, k_i > 0$, $i = 1, 2, 3, 4$.

In the case $p = 2$, the problem (1.1) is a general reaction-diffusion equation, the dynamical behavior have been studied in [3, 4, 8, 22, 25, 26, 27, 31] for the Dirichlet boundary conditions and [10, 11, 14, 15, 28, 29] for the dynamic boundary conditions.

The long-time behavior of the solutions of (1.1) has been considered by many researchers, e.g., see [3, 4, 8, 27] and the references therein.
For the autonomous systems, i.e., \( h(x, t) = h(x) \), in the Dirichlet boundary case, the nonlinear eigenvalue problem for the \( p \)-Laplacian operator was considered in [18] by using the Ljusternik-Schnirelman principle. In [3], Babin & Vishik established the existence of a \((L^2(\Omega), (W^{1,p}_0(\Omega) \cap L^q(\Omega)))\)-global attractor. In [27], a special case of \( f = ku \) was discussed by Temam. In [5], Carvalho, Cholewa and Dlotko considered the existence of global attractors for problems with monotone operators, and as an application, they proved the existence of \((L^2(\Omega), L^2(\Omega))\)-global attractor for \( p \)-Laplacian equation, see also Cholewa & Dlotko [8]. In [6], Carvalho & Gentile obtained that the corresponding semigroup has a \((L^2(\Omega), W^{1,p}_0(\Omega))\)-global attractor under some additional conditions. In [30], Yang, Sun and Zhong obtained the existence of a \((L^2(\Omega), (W^{1,p}_0(\Omega) \cap L^q(\Omega)))\)-global attractor, which holds only under the assumptions (1.2) and (1.4). Some asymptotic regularity of the solutions was proved by Liu, Yang and Zhong in [20]. In the dynamic boundary case, recently, Gal et al [16] presented firstly the general result for the problem (1.1), the well-posedness and the asymptotic behavior of the solutions were studied.

Inspired by the ideas of [20, 26, 29], we obtain the asymptotic regularity of the solutions of equation (1.1), where we only assume the external forcing \( h \) holds only under the assumptions (1.2)–(1.4), and no any restrictions on \( p, r \). The main results of this article are Theorem 3.4 (asymptotic regularity), Theorem 3.5 (global attractor) and Theorem 4.5 (uniform attractor and its structure).

Hereafter, we assume that

\[ 2 < p < N. \]

For the case \( p = 2 \), system (1.1) is a reaction-diffusion equation and we refer the reader to [15, 28]; and if \( p \geq N \), then embeddings \( W^{1,p}(\Omega) \hookrightarrow L^{s_1}(\Omega) \) and \( W^{1,p}(\Omega) \hookrightarrow L^{s_2}(\Gamma) \) hold for any \( s_1, s_2 \in [1, \infty) \), which make the nonlinear terms \( f(\cdot) \) and \( g(\cdot) \) to be trivial terms.

For convenience, hereafter \( \| \cdot \| \) and \( \| \cdot \|_\Gamma \) stand for the norm in \( L^2(\Omega) \) and \( L^2(\Gamma) \), \( \langle \cdot, \cdot \rangle \) and \( \langle \cdot, \cdot \rangle_\Gamma \) stand for the inner product in \( L^2(\Omega) \) and \( L^2(\Gamma) \), respectively. \( C_i \) denote general positive constants, \( i = 1, \ldots, \), which will be different in different estimates.

This article is organized as follows: in Section 2, we introduce some preliminary results; in Section 3, for the autonomous cases, i.e., \( h(x, t) = h(x) \), we prove some asymptotic regularity of the solution; in Section 4, for the non-autonomous cases, the existence and structure of a uniform attractor in \( L^{r_1}(\Omega) \times L^r(\Gamma) \) \((r = \min(r_1, r_2))\) is obtained.
2. Preliminaries

In this section, we give some auxiliary results which will be used later. We first introduce the spaces of time-dependent external forcing \( h(x,t) \) to be considered in this article (see [4]).

**Definition 2.1** ([4]). A function \( \varphi \) is said to be translation bounded in \( L^2_{\text{loc}}(\mathbb{R}; X) \), if
\[
\| \varphi \|_2^2 = \sup_{t \in \mathbb{R}} \int_t^{t+1} \| \varphi \|_X^2 \, ds < +\infty.
\]
Denote by \( L^2_{\text{tb}}(\mathbb{R}; X) \) the set of all translation bounded functions in \( L^2_{\text{loc}}(\mathbb{R}; X) \).

We now introduce a class of functions that was defined first in [21].

**Definition 2.2** ([21]). A function \( \varphi \in L^2_{\text{loc}}(\mathbb{R}; X) \) is said to be normal if for any \( \varepsilon > 0 \), there exists \( \eta > 0 \) such that
\[
\sup_{t \in \mathbb{R}} \int_t^{t+\eta} \| \varphi \|_X^2 \, ds \leq \varepsilon.
\]
Denote by \( L^2_{\text{n}}(\mathbb{R}; X) \) the set of all normal functions in \( L^2_{\text{loc}}(\mathbb{R}; X) \).

**Lemma 2.3** ([21]). If \( \varphi_0 \in L^2_{\text{n}}(\mathbb{R}; X) \), then for any \( \tau \in \mathbb{R} \),
\[
\lim_{\gamma \to \infty} \sup_{t \geq \tau} \int_t^{t+\eta} e^{-\gamma(t-s)} \| \varphi(s) \|_X^2 \, ds = 0,
\]
uniformly (with respect to \( \varphi \in H(\varphi_0) \)), where \( H(\varphi_0) = \{ \varphi_0(t+h) \mid h \in \mathbb{R} \} \).

The next result is an estimate of the \( p \)-Laplacian operator; see [9] for the proof.

**Lemma 2.4.** Let \( p \geq 2 \). Then there exists constant \( K > 0 \) such that for any \( a, b \in \mathbb{R}^N \),
\[
\langle |a|^{p-2}a - |b|^{p-2}b, a - b \rangle \geq K|a - b|^p,
\]
where \( K \) depends only on \( p \) and \( N \); \( \langle \cdot, \cdot \rangle \) denotes the inner product of \( \mathbb{R}^N \).

3. Autonomous cases: \( h(x,t) = h(x) \)

In this section, we consider the autonomous case of (1.1); that is,
\[
\begin{align*}
&u_t - \Delta_p u + f(u) = h(x), \quad \text{in } \Omega, \\
&u_t + |\nabla u|^{p-2} \partial_n u + g(u) = 0, \quad \text{on } \Gamma, \\
u(x,0) = u_0(x),
\end{align*}
\]
where \( h(x) \in L^{r'_1}(\Omega), r'_1 \) is conjugate to \( r_1 \).

3.1. Mathematical setting. At first, following [17], it is more convenient to introduce the unknown function \( v(t) := u(t)|\Gamma \), defined on the boundary \( \Gamma \), so we think of our problem as a coupled system of two parabolic equations, one in the bulk \( \Omega \) and the other on the boundary \( \Gamma \). Thus, the function \( u(t) \) tracks diffusion in the bulk, while \( v(t) \) records the information on the boundary. Throughout the remainder of this section, we formulate the problem as following:
**Problem (P).** Let $\Omega \subset \mathbb{R}^N (N \geq 3)$ be a bounded domain with a smooth boundary $\Gamma := \partial \Omega$ (e.g., of class $C^2$). The nonlinearities $f$ and $g$ satisfy (1.2)–(1.4). For any given pair of initial data $(u_0, v_0) \in L^2(\Omega) \times L^2(\Gamma)$, find $(u(t), v(t))$ with

$$
(\mathbf{1.2}) \quad (u, v) \in C([0, +\infty); L^2(\Omega) \times L^2(\Gamma)) \cap L^\infty((0, +\infty); W^{1,p}(\Omega) \times W^{1-1/p,p}(\Gamma)),
$$

$$
(\mathbf{1.3}) \quad (u, v) \in W^{1,2}_0((0, \infty); L^2(\Omega) \times L^2(\Gamma)),
$$

$$
(\mathbf{1.4}) \quad u \in L^p_{\text{loc}}([0, +\infty); W^{1,p}(\Omega)),
$$

$$
(\mathbf{1.5}) \quad v \in L^p_{\text{loc}}([0, +\infty); W^{1-1/p,p}(\Gamma))
$$

such that $(u(0), v(0)) = (u_0, v_0)$, and for almost all $t \geq 0$, $(u(t), v(t))$ satisfies $u(t)|\Gamma = v(t)$ a.e. for $t \in (0, \infty)$, and the following partial differential equations:

$$
\partial_t u - \text{div}(|\nabla u|^{p-2}\nabla u) + f(u) = h(x), \quad \text{in } \Omega \times (0, +\infty),
$$

$$
\partial_t v + |\nabla u|^{p-2}\partial_n u + g(v) = 0, \quad \text{on } \Gamma \times (0, +\infty).
$$

Secondly, we give the following existence and uniqueness results, where we use the definition of weak solution as in [17, Definition 2.3]. For more details we refer the reader to [17].

**Theorem 3.1 ([17]).** Let $\Omega$ be a bounded smooth domain in $\mathbb{R}^N (N \geq 3)$, $f$ and $g$ satisfy (1.2)–(1.4), $h(x) \in L^{r_1}(\Omega)$. Then for any initial data $(u_0, v_0) \in L^2(\Omega) \times L^2(\Gamma)$ and any $T > 0$, the problem (P) has a unique weak solution $(u(t), v(t)) \in C([0, T]; L^2(\Omega) \times L^2(\Gamma))$. In addition to the regularity stated in (3.3), we also have that

$$
u(t) \in L^{r_1}(0, T; L^{r_1}(\Omega)), \quad v(t) \in L^{r_2}(0, T; L^{r_2}(\Gamma)).$$

Furthermore, $(u_0, v_0) \mapsto (u(t), v(t))$ is continuous on $L^2(\Omega) \times L^2(\Gamma)$.

By Theorem 2.3, we can define the operator semigroup $\{S(t)\}_{t \geq 0}$ on the phase space $L^2(\Omega) \times L^2(\Gamma)$ as follows:

$$
S(t) : L^2(\Omega) \times L^2(\Gamma) \to L^2(\Omega) \times L^2(\Gamma), \quad S(t)(u_0, v_0) = (u(t), v(t)),
$$

which is continuous in $L^2(\Omega) \times L^2(\Gamma)$.

Next, exactly as in [17], we have the following dissipative results.

**Lemma 3.2 ([17]).** Under the assumption of Theorem 2.3, $\{S(t)\}_{t \geq 0}$ has a positively invariant $(L^2(\Omega) \times L^2(\Gamma), W^{1,p}(\Omega) \cap L^{r_1}(\Omega) \times W^{1-1/p,p}(\Gamma) \cap L^{r_2}(\Gamma))$-bounded absorbing set; that is, there is a positive constant $M$, such that for any bounded subset $B \subset L^2(\Omega) \times L^2(\Gamma)$, there exists a positive constant $T$ which depends only on the $L^2(\Omega) \times L^2(\Gamma)$-norm of $B$ such that

$$
\int_{\Omega} |\nabla u(t)|^p \, dx + \int_{\Omega} |u(t)|^{r_1} \, dx + \int_{\Gamma} |v(t)|^{r_2} \, dS \leq M \quad \text{for all } t \geq T \text{ and } (u_0, v_0) \in B.
$$

**Lemma 3.3 ([17]).** Under the assumption of Theorem 2.3, for any bounded subset $B \subset L^2(\Omega) \times L^2(\Gamma)$, there exists a positive constant $T_1$ which depends only on the $L^2(\Omega) \times L^2(\Gamma)$-norm of $B$ such that

$$
\int_{\Omega} |u(s)|^2 \, dx + \int_{\Gamma} |v(s)|^2 \, dS \leq M' \quad \text{for all } s \geq T_1 \text{ and } (u_0, v_0) \in B,
$$

where $M'$ is a positive constant which depends on $M$. 


Hereafter, from Lemma 3.2 we denote one of the positively invariant absorbing set by \( B_0 \) with
\[
B_0 \subset \{(u(t), v(t)) : \|u(t)\|_{W^{1,p}(\Omega)} \cap L^{r_1}(\Omega) + \|v(t)\|_{W^{1-1/p,p}(\Gamma) \cap L^{r_2}(\Gamma)} \leq M\},
\]

note that here the positive invariance means \( S(t)B_0 \subset B_0 \) for any \( t \geq 0 \).

3.2. Asymptotic regularity. In this subsection, we consider the asymptotic regularity of solutions of systems (3.1), which excel the regularity allowed by the corresponding elliptic equation.

At first, we consider the elliptic equation
\[
- \text{div}(|\nabla \phi|^{p-2} \nabla \phi) + f(\phi) = h(x) \quad \text{in } \Omega,
\]
\[
|\nabla \phi|^{p-2} \partial_n \phi + g(\phi) = 0 \quad \text{on } \Gamma.
\]
Due to the assumptions (1.2)–(1.4), from the classical results about elliptic equations, we know that (3.6) at least has one solution \( \phi(x) \) with
\[
\phi(x) \in W^{1,p}(\Omega) \cap L^{r_1}(\Omega).
\]

For the rest of this article, we assume that \( \phi(x) \) denotes a fixed solution of (3.6). Then, for the solution \( (u(x,t), v(x,t)) \) of (3.1), we decompose \( (u(x,t), v(x,t)) \) as follows
\[
(u(x,t), v(x,t)) = (\phi(x) + w(x,t), \phi(x) + \tilde{w}(x,t))
\]
with \( u_0(x) = \phi(x) + w(x,0), v_0(x) = \phi(x) + \tilde{w}(x,0) \), where \( (w(x,t), \tilde{w}(x,t)) \) solves the equation
\[
w_t - \text{div}(|\nabla u|^{p-2} \nabla u) + \text{div}(|\nabla \phi|^{p-2} \nabla \phi) + f(u) - f(\phi) = 0 \quad \text{in } \Omega,
\]
\[
\tilde{w}_t + |\nabla u|^{p-2} \partial_n u - |\nabla \phi|^{p-2} \partial_n \phi + g(v) - g(\phi) = 0, \quad \text{on } \Gamma,
\]
\[
w(x,0) = u_0(x) - \phi(x),
\]
\[
\tilde{w}(x,0) = v_0(x) - \phi(x).
\]

It is easy to see that this equation is also globally well posed. Moreover, thanks to Lemma 3.2 without loss of generality, hereafter we assume \((u_0, v_0) \in B_0\) and so \((w(x,0), \tilde{w}(x,0)) \in (W^{1,p}(\Omega) \cap L^{r_1}(\Omega)) \times (W^{1-1/p,p}(\Gamma) \cap L^{r_2}(\Gamma))\).

At the same time, from the positive invariance of \( B_0 \) and (3.7) we have that
\[
\|w(x,t)\|_{W^{1,p}(\Omega) \cap L^{r_1}(\Omega)} + \|\tilde{w}(x,t)\|_{W^{1-1/p,p}(\Gamma) \cap L^{r_2}(\Gamma)} \leq M_1
\]
for all \( t \geq 0 \), with some positive constant \( M_1 \).

The main result of this section reads as follows.

**Theorem 3.4.** Let \( \Omega \) be a bounded smooth domain in \( \mathbb{R}^N \) (\( N \geq 3 \)), \( f \) and \( g \) satisfy (1.2)–(1.4), \( h(x) \in L^{r_1}(\Omega) \), and suppose that \( \{S(t)\}_{t \geq 0} \) is the semigroup generated by the solutions of equation (3.1) with initial data \((u_0, v_0) \in L^2(\Omega) \times L^2(\Gamma)\). Then, for any \( \delta, \gamma \in [0, \infty) \), there exists a bounded subset \( B_{\delta,\gamma} \) satisfying the following properties:
\[
B_{\delta,\gamma} = \{ (w, \tilde{w}) : \|w\|_{W^{1,p}(\Omega) \cap L^{r_1+\delta}(\Omega)} + \|\tilde{w}\|_{W^{1-1/p,p}(\Gamma) \cap L^{r_2+\gamma}(\Gamma)} \leq \Lambda_{p,r_1,r_2,N,\delta,\gamma} < \infty \},
\]
and for any bounded subset \( B \subset L^2(\Omega) \times L^2(\Gamma) \), there exists a
\[
T = T(\|B\|_{L^2(\Omega)}, \|B\|_{L^2(\Gamma)}, \delta, \gamma)
\]
such that
\[ S(t)B \subset \phi(x) + B_{\delta,\gamma} \quad \text{for all } t \geq T, \tag{3.11} \]
where \( \phi(x) \) is a fixed solution of \( (3.6) \), \((w(x,t),\tilde{w}(x,t))\) satisfies \( (3.9) \); the constant \( \Lambda_{p,r_1,r_2,N,\delta,\gamma} \) depends only on \( p,r_1,r_2,N,\delta,\gamma \).

**Proof.** We use the Moser-Alikakos iteration technique [2] to prove the following

\[ \text{Proof.} \]

\[ \text{into two steps.} \]

Step 1: We first claim that

For each \( k = 0,1,2,\ldots \), there exist two positive constants \( T_k \) and \( M_k \), which depend only on \( k,p,r_1,r_2,N \) and \( \| B_0 \|_{W^{1,p}(\Omega)\cap L^1(\Omega)\times W^{1-1/p,r}(\Gamma)\cap L^2(\Gamma)} \), such that for any \( (u_0,v_0) \in B_0 \) and \( t \geq T_k \), we have

\[ \int_\Omega |w(t)|^{\sigma_k} \, dx + \int_\Gamma |\tilde{w}(t)|^{\sigma_k} \, dS \leq M_k, \tag{A_k} \]

and

\[ \int_t^{t+1} \left( \int_\Omega |w(s)|^{\sigma_{k+1}} \, dx \right)^{\frac{N-p}{N-1}} \, ds + \int_t^{t+1} \left( \int_\Gamma |\tilde{w}(s)|^{\sigma_{k+1}} \, dS \right)^{\frac{N-p}{N-1}} \, ds \leq M_k. \tag{B_k} \]

where \((w(t),\tilde{w}(t))\) is the solution of equation \( (3.9) \), and

\[ \sigma_k = 2\left(\frac{N-1}{N-p}\right)^k + (p-2)\left[ \sum_{i=0}^k \left( \frac{N-1}{N-p} \right)^i - 1 \right], \quad k = 0,1,2,\ldots \tag{3.12} \]

(i) Initialization of the induction \((k = 0)\). From \( (3.10) \), we can deduce \((A_0)\) immediately. To prove \((B_0)\), we multiply \( (3.9) \) by \( w \) and \( \tilde{w} \), and integrate over \( \Omega \), then we obtain

\[ \frac{1}{2} \frac{d}{dt} \int_\Omega |w|^2 \, dx + \frac{1}{2} \frac{d}{dt} \int_\Gamma |\tilde{w}|^2 \, dS + \frac{\int_\Omega (|\nabla u|^{p-2}\nabla u - |\nabla \phi|^{p-2} \nabla \phi, \nabla w) \, dx}{p} + \int_\Omega (f(u) - f(\phi)) w \, dx + \int_\Gamma (g(v) - g(\phi)) \tilde{w} \, dS = 0. \tag{3.13} \]

By \( (1.4) \), we have

\[ \int_\Omega (f(u) - f(\phi)) w \, dx \geq -l \int_\Omega |w|^2 \, dx, \tag{3.14} \]

\[ \int_\Omega (g(v) - g(\phi)) \tilde{w} \, dS \geq -m \int_\Gamma |\tilde{w}|^2 \, dS. \tag{3.15} \]

Then applying Lemma \( 2.4 \) we have

\[ \int_\Omega (|\nabla u|^{p-2}\nabla u - |\nabla \phi|^{p-2} \nabla \phi, \nabla w) \, dx \geq K \int_\Omega |\nabla w|^p \, dx. \tag{3.16} \]

Inserting \( (3.14) \), \( (3.16) \) into \( (3.13) \), we obtain

\[ \frac{1}{2} \frac{d}{dt} \int_\Omega |w|^2 \, dx + \frac{1}{2} \frac{d}{dt} \int_\Gamma |\tilde{w}|^2 \, dS + K \int_\Omega |\nabla w|^p \, dx \leq l \int_\Omega |w|^2 \, dx + m \int_\Gamma |\tilde{w}|^2 \, dS \tag{3.17} \]

\[ \leq C \left( \int_\Omega |w|^2 \, dx + \int_\Gamma |\tilde{w}|^2 \, dS \right). \]
Then, for any $t \geq 0$, integrating the above inequality over $[t, t+1]$ and using (3.10), we deduce that
\[
\int_t^{t+1} \int_{\Omega} |\nabla w(x,s)|^p \, dx \, ds \leq C_{K,M,M}, \quad \text{for all} \quad t \geq 0. \tag{3.18}
\]

By the Sobolev embeddings (e.g., see Adams and Fourier [1])
\[
W^{1,p}(\Omega) \hookrightarrow L^{\frac{p(N-1)}{N-p}}(\Omega), \quad W^{1,p}(\Omega) \hookrightarrow L^{\frac{p(N-1)}{N-p}}(\Gamma),
\]
from (3.18), for all $t \geq 0$, we have
\[
\int_t^{t+1} \left( \int_{\Omega} \left| w(x,s) \right|^\frac{p(N-1)}{N-p} \, dx \right)^{\frac{N-p}{N}} \, ds \leq C_1 \int_t^{t+1} \int_{\Omega} |\nabla w(x,s)|^p \, dx \, ds \leq C_{K,M,M_1,N}, \tag{3.19}
\]
\[
\int_t^{t+1} \left( \int_{\Gamma} \left| \tilde{w}(x,s) \right|^\frac{p(N-1)}{N-p} \, dS \right)^{\frac{N-p}{N}} \, ds \leq C_2 \int_t^{t+1} \int_{\Omega} |\nabla w(x,s)|^p \, dx \, ds \leq C_{K,M,M_1,N}, \tag{3.20}
\]
where $C_1, C_2$ are constants of embeddings $W^{1,p}(\Omega) \hookrightarrow L^{\frac{p(N-1)}{N-p}}(\Omega)$ and $W^{1,p}(\Omega) \hookrightarrow L^{\frac{p(N-1)}{N-p}}(\Gamma)$, note that here $C_1, C_2$ depend only on $N$. This implies $(B_0)$ holds.

(ii) The induction argument. We now assume that $(A_k)$ and $(B_k)$ hold for $k \geq 1$, and we need only to prove that $(A_{k+1})$ and $(B_{k+1})$ hold. Multiplying (3.9) by $|w|^\sigma_k w$ and $|\tilde{w}|^\sigma_k \tilde{w}$, and integrating over $\Omega$, we obtain
\[
\frac{1}{\sigma_{k+1}} \frac{d}{dt} \left( \int_{\Omega} |w|^{\sigma_{k+1}} \, dx + \int_{\Gamma} |\tilde{w}|^{\sigma_{k+1}} \, dS \right)
+ (\sigma_{k+1} - 1) \int_{\Omega} (|\nabla w| |\phi| - |\nabla \phi| |w|^{\sigma_k - 1} - 2) \, dx \tag{3.21}
\]
\[
+ \int_{\Omega} (f(u) - f(\phi)) |w|^{\sigma_k - 1} w \, dx + \int_{\Gamma} (g(v) - g(\phi)) |\tilde{w}|^{\sigma_k - 1} \tilde{w} \, dS = 0.
\]

Similar to (3.14)–(3.16), we have
\[
\int_{\Omega} (f(u) - f(\phi)) |w|^{\sigma_k - 1} w \, dx \geq -l \int_{\Omega} |w|^{\sigma_k + 1} \, dx, \tag{3.22}
\]
\[
\int_{\Gamma} (g(v) - g(\phi)) |\tilde{w}|^{\sigma_k - 1} \tilde{w} \, dS \geq -m \int_{\Gamma} |\tilde{w}|^{\sigma_k + 1} \, dS, \tag{3.23}
\]
\[
(\sigma_{k+1} - 1) \int_{\Omega} (|\nabla w| |\phi| - |\nabla \phi| |w|^{\sigma_k - 1} - 2) \, dx \geq K(\sigma_{k+1} - 1) \int_{\Omega} |\nabla w|^{\sigma_k + 1} \, dx, \tag{3.24}
\]
so we have
\[
\frac{1}{\sigma_{k+1}} \frac{d}{dt} \left( \int_{\Omega} |w|^{\sigma_{k+1}} \, dx + \int_{\Gamma} |\tilde{w}|^{\sigma_{k+1}} \, dS \right) + K(\sigma_{k+1} - 1) \int_{\Omega} |\nabla w|^{\sigma_k + 1} \, dx
\]
\[
\leq l \int_{\Omega} |w|^{\sigma_{k+1}} \, dx + m \int_{\Gamma} |\tilde{w}|^{\sigma_{k+1}} \, dS \leq C \left( \int_{\Omega} |w|^{\sigma_{k+1}} \, dx + \int_{\Gamma} |\tilde{w}|^{\sigma_{k+1}} \, dS \right), \tag{3.25}
\]
Then, combining with \((B_k)\) and application of the uniform Gronwall lemma to \((3.25)\) we can get \((A_{k+1})\) immediately. For \((B_{k+1})\), we integrate the above inequality over \([t, t+1]\) and use \((A_{k+1})\), we have

\[
\int_{t}^{t+1} \int_{\Omega} |\nabla w|^p |w|^{|\sigma_{k+1} - 2|} \, dx \, ds \leq M_{k+1} \quad \text{for all } t \geq 0,
\]

(3.26)

where \(M_{k+1}\) depends on \(k, p, r_1, r_2, N, M, M_1\). By the embeddings \(W^{1,p}(\Omega) \hookrightarrow L^{\frac{p(N-1)}{N-p}}(\Omega)\) and \(W^{1,p}(\Gamma) \hookrightarrow L^{\frac{p(N-1)}{N-p}}(\Gamma)\) again, we have

\[
\left( \int_{\Omega} |w|^{(\sigma_{k+1} - 2 + p) \frac{N-1}{N-p}} \, dx \right)^{\frac{N-p}{N-1}} \\
\leq C_1 \cdot \left( \frac{p}{\sigma_{k+1} - 2 + p} \right)^{p} \int_{\Omega} |w|^{\sigma_{k+1} - 2} |\nabla w|^p \, dx,
\]

(3.27)

\[
\left( \int_{\Gamma} |\tilde{w}|^{(\sigma_{k+1} - 2 + p) \frac{N-1}{N-p}} \, ds \right)^{\frac{N-p}{N-1}} \\
\leq C_2 \cdot \left( \frac{p}{\sigma_{k+1} - 2 + p} \right)^{p} \int_{\Omega} |w|^{\sigma_{k+1} - 2} |\nabla w|^p \, dx,
\]

(3.28)

and from the definition of \(\sigma_k\), we have

\[
(\sigma_{k+1} - 2 + p) \frac{N-1}{N-p} = \sigma_{k+2}.
\]

(3.29)

Combining \(3.26-3.29\), we deduce \((B_{k+1})\) immediately.

**Step 2**: Based on Step 1, since \(N \geq 3\), from the definition of \(\sigma_k\) given in \((3.12)\), it is easy to see that \(\sigma_k \to \infty\) as \(k \to \infty\).

Hence, for any \(\delta, \gamma \in [0, \infty)\), we can take \(k\) so large that \(r_1 + \delta \leq \sigma_k, r_2 + \gamma \leq \sigma_k\). Consequently, we can define \(B_{k, \gamma}\) as

\[
B_{k, \gamma} := \left\{ (z, \tilde{z}) : \|z + \phi\|_{W^{1,p}(\Omega)} + \|z\|_{L^{r_1+\delta}(\Omega)}^{r_1+\delta} + \|\tilde{z} + \phi\|_{W^{1,1,p}(\Gamma)} + \|\tilde{z}\|_{L^{r_2+\gamma}(\Gamma)}^{r_2+\gamma} \leq M + M_k \right\},
\]

where \(z(t)|_{\Gamma} = \tilde{z}(t)\), and recall that \(\phi(x)\) is a fixed solution of \((3.6)\).

Hence, from Theorem 3.4 using the interpolation inequality, we can obtain immediately the following results.

**Theorem 3.5.** Under the assumptions of Theorem 3.4, the semigroup \(\{S(t)\}_{t \geq 0}\) has a \((L^2(\Omega) \times L^2(\Gamma), W^{1,p}(\Omega) \cap L^{r_1}(\Omega) \times W^{1,1/p,p}(\Gamma) \cap L^{r_2}(\Gamma))\)-global attractor \(\mathcal{A}\). Moreover, \(\mathcal{A}\) attracts every \(L^2(\Omega) \times L^2(\Gamma)\)-bounded subset with \((W^{1,p}(\Omega) \cap L^{r_1+\delta}(\Omega)) \times (W^{1,1/p,p}(\Gamma) \cap L^{r_2+\gamma}(\Gamma))\)-norm for any \(\delta, \gamma \in [0, \infty)\); and \(\mathcal{A}\) allows the decomposition \(\mathcal{A} = \mathcal{A}_0 + \mathcal{A}_0\) with \(\mathcal{A}_0\) is bounded in \((W^{1,p}(\Omega) \cap L^{r_1+\delta}(\Omega)) \times (W^{1,1/p,p}(\Gamma) \cap L^{r_2+\gamma}(\Gamma))\) for any \(\delta, \gamma \in (0, \infty)\), and \(\phi(x)\) is a fixed solution of \((3.6)\).

**Proof.** From Theorem 3.4 combining with the \((L^2(\Omega) \times L^2(\Gamma), L^2(\Omega) \times L^2(\Gamma))\)-asymptotic compactness (obtained in \((12)\)) and the interpolation inequality, it is easily to verify that \(\{S(t)\}_{t \geq 0}\) is asymptotically compact in \(L^{r_1}(\Omega) \times L^{r_2}(\Gamma)\), then it is sufficient to verify that \(\{S(t)\}_{t \geq 0}\) is asymptotically compact in \(W^{1,p}(\Omega) \times W^{1,1/p,p}(\Gamma)\).
Let $B_0$ be a $(W^{1,p}(\Omega) \cap L^{r_1}(\Omega)) \times (W^{1-1/p,p}(\Gamma) \cap L^{r_2}(\Gamma))$-bounded absorbing set obtained in Lemma 3.2, then we need only to show that for any $\{(u_{0n},v_{0n})\} \subset B_0$ and $t_n \to \infty$, \(\{(u_n(t_n),v_n(t_n))\}_{n=1}^{\infty}\) is precompact in $W^{1,p}(\Omega) \times W^{1-1/p,p}(\Gamma)$, where $u_n(t_n) = S(t_n)u_{0n}, v_n(t_n) = S(t_n)v_{0n}$.

In fact, we know that $\{(u_n(t_n),v_n(t_n))\}_{n=1}^{\infty}$ is precompact in $L^2(\Omega) \times L^2(\Gamma)$ and in $L^{r_1}(\Omega) \times L^{r_2}(\Gamma)$.

Without loss of generality, we assume that $\{(u_{nk}(t_{nk}),v_{nk}(t_{nk}))\}_{n=1}^{\infty}$ is a Cauchy sequence in $L^2(\Omega) \times L^2(\Gamma)$ and $L^{r_1}(\Omega) \times L^{r_2}(\Gamma)$.

Now, we prove that $\{(u_{nk}(t_{nk}),v_{nk}(t_{nk}))\}_{n=1}^{\infty}$ is a Cauchy sequence in $W^{1,p}(\Omega) \times W^{1-1/p,p}(\Gamma)$. From Lemma 2.4 and after standard transformations, we know that there exists a constant $K > 0$, such that

\[
K \|\nabla(u_{nk}(t_{nk}) - u_{nj}(t_{nj}))\|_{L^p(\Omega)}^p \\
\leq \left\langle -\frac{d}{dt}u_{nj}(t_{nj}) - f(u_{nj}(t_{nj})), u_{nk}(t_{nk}) - u_{nj}(t_{nj})\right\rangle + \left\langle -\frac{d}{dt}v_{nj}(t_{nj}) - g(v_{nj}(t_{nj})), v_{nk}(t_{nk}) - v_{nj}(t_{nj})\right\rangle_{\Gamma} \\
\leq \int_\Omega \left|\frac{d}{dt}u_{nk}(t_{nk}) - \frac{d}{dt}u_{nj}(t_{nj})\right| \|u_{nk}(t_{nk}) - u_{nj}(t_{nj})\| \\
+ \int_\Omega \left|f(u_{nk}(t_{nk})) - f(u_{nj}(t_{nj}))\right| \|u_{nk}(t_{nk}) - u_{nj}(t_{nj})\| \\
+ \int_\Gamma \left|\frac{d}{dt}v_{nk}(t_{nk}) - \frac{d}{dt}v_{nj}(t_{nj})\right| \|v_{nk}(t_{nk}) - v_{nj}(t_{nj})\| \\
+ \int_\Gamma \left|g(v_{nk}(t_{nk})) - g(v_{nj}(t_{nj}))\right| \|v_{nk}(t_{nk}) - v_{nj}(t_{nj})\|,
\]

so we have

\[
K \|\nabla(u_{nk}(t_{nk}) - u_{nj}(t_{nj}))\|_{L^p(\Omega)}^p \\
\leq \left\|\frac{d}{dt}u_{nk}(t_{nk}) - \frac{d}{dt}u_{nj}(t_{nj})\right\| \|u_{nk}(t_{nk}) - u_{nj}(t_{nj})\| \\
+ \left\|\frac{d}{dt}v_{nk}(t_{nk}) - \frac{d}{dt}v_{nj}(t_{nj})\right\| \|v_{nk}(t_{nk}) - v_{nj}(t_{nj})\|_{\Gamma} \\
+ C(1 + \|u_{nk}(t_{nk})\|_{L^{r_1}(\Omega)}^{r_1} + \|u_{nj}(t_{nj})\|_{L^{r_1}(\Omega)}^{r_1}) \|u_{nk}(t_{nk}) - u_{nj}(t_{nj})\|_{L^{r_1}(\Omega)} \\
+ \tilde{C}(1 + \|v_{nk}(t_{nk})\|_{L^{r_2}(\Gamma)}^{r_2} + \|v_{nj}(t_{nj})\|_{L^{r_2}(\Gamma)}^{r_2}) \|v_{nk}(t_{nk}) - v_{nj}(t_{nj})\|_{L^{r_2}(\Gamma)}.
\]

Combining Lemma 3.2, Lemma 3.3 and the compactness of $L^{r_1}(\Omega) \times L^{r_2}(\Gamma)$, and since $W^{1,p}(\Omega) \hookrightarrow W^{1-1/p,p}(\Gamma)$, we know that the norms on $W^{1,p}(\Omega) \times W^{1-1/p,p}(\Gamma)$ and $W^{1,p}(\Omega)$ are equivalent, \((3.31)\) yields \((3.30)\) immediately.

\[4.\text{ Non-autonomous case}\]

In this section, we discuss the non-autonomous case of \((1.1)\): that is,

\[
\begin{aligned}
&u_t - \Delta_p u + f(u) = h(x,t), \quad \text{in } \Omega, \\
&u_t + |\nabla u|^{p-2}\partial_n u + g(u) = 0, \quad \text{on } \Gamma, \\
u(x,\tau) = u_\tau(x), \quad \text{in } \Omega,
\end{aligned}
\]
where $h(x, t) \in L^2_\nu(\mathbb{R}; L^2(\Omega))$.

4.1. **Mathematical setting.** Similar to the autonomous cases (e.g., Problem (p) and Theorem 2.3), for each $h \in \Sigma$, we can also easily obtain the following well-posedness result and the time-dependent terms make no essential complications.

**Theorem 4.1** ([17]). Let $\Omega$ be a bounded smooth domain in $\mathbb{R}^N$ ($N \geq 3$), $f$ and $g$ satisfy ([1.2], [1.4]), $h(x, t) \in L^2_\nu(\mathbb{R}; L^2(\Omega))$. Then for any initial data $(u_\tau, v_\tau) \in L^2(\Omega) \times L^2(\Gamma)$, and any $\tau, T \in \mathbb{R}$, $T > \tau$, the solution $(u(t), v(t))$ of problem (4.1) is globally defined and satisfies

$$
u(t) \in C([\tau, T]; L^2(\Omega)) \cap L^p_{loc}(\tau, T; W^{1,p}(\Omega)) \cap L^{r_1}(\tau, T; L^{r_1}(\Omega)),
\nu(t) \in C([\tau, T]; L^2(\Gamma)) \cap L^p_{loc}(\tau, T; W^{1-1/p,p}(\Gamma)) \cap L^{r_2}(\tau, T; L^{r_2}(\Gamma)).$$

where $\nu(t) := u(t)|_\Gamma$. Furthermore, $(u_\tau, v_\tau) \mapsto (u(t), v(t))$ is continuous on $L^2(\Omega) \times L^2(\Gamma)$.

We now define the symbol space $\Sigma$ for (4.1). Taking a fixed symbol $\sigma_0(s) = h_0(s), h_0(s) \in L^2(\mathbb{R}; L^2(\Omega))$. We denote by $L^w_{loc}(\mathbb{R}; L^2(\Omega))$ the space $L^2_{loc}(\mathbb{R}; L^2(\Omega))$ endowed with local weak convergence topology. Set

$$\Sigma_0 = \{ h_0(s + h) | h \in \mathbb{R} \},$$

and let

$$\Sigma = \text{the closure of } \Sigma_0 \text{ in } L^w_{loc}(\mathbb{R}; L^2(\Omega)).$$

Systems (4.1) can be rewritten in the operator form

$$\partial_t y = A_\tau(y), \quad y|_{t=\tau} = y_\tau,$$

where $\sigma(t) = h(t)$ is the symbol of equation (4.4). Thus, from Theorem 1.1 we know that problem (4.1) is well posed for all $\sigma(s) \in \Sigma$ and generates a family of processes $\{ U_\sigma(t, \tau) \}, \sigma \in \Sigma$ given by the formula $U_\sigma(t, \tau)y_\tau = y(t)$, and the $y(t)$ is the solution of (4.1).

4.2. **Existence of a bounded uniformly (w. r. t. $\sigma \in \Sigma$) absorbing set in $(W^{1,p}(\Omega) \cap L^{r_1}(\Omega)) \times (W^{1-1/p,p}(\Gamma) \cap L^{r_2}(\Gamma))$.** In this subsection, $(W^{1,p}(\Omega) \cap L^{r_1}(\Omega) \times W^{1-1/p,p}(\Gamma) \cap L^{r_2}(\Gamma))$-bounded uniformly (with respect to $\sigma \in \Sigma$) absorbing set is obtained. The proof is similar to [17] (autonomous case).

**Theorem 4.2.** Let $\Omega$ be a bounded smooth domain in $\mathbb{R}^N$ ($N \geq 3$), $f$ and $g$ satisfy ([1.2], [1.4]), $h(x, t) \in L^2_\nu(\mathbb{R}; L^2(\Omega))$. Then the family of processes $\{ U_\sigma(t, \tau) \}, \sigma \in \Sigma$ corresponding to (4.1) has a bounded uniformly (with respect to $\sigma \in \Sigma$) absorbing set $B_0$ in $(W^{1,p}(\Omega) \cap L^{r_1}(\Omega)) \times (W^{1-1/p,p}(\Gamma) \cap L^{r_2}(\Gamma))$, that is, there is a positive constant $M$, such that for any $\tau \in \mathbb{R}$ and any bounded subset $B$, there exists a positive constant $T = T(B, \tau) \geq \tau$ such that

$$\int_{\Omega} |\nabla u(t)|^p dx + \int_{\Omega} |u(t)|^{r_1} dx + \int_{\Gamma} |v(t)|^{r_2} dS \leq M$$

for all $t \geq T$, $(u_\tau, v_\tau) \in B, \sigma \in \Sigma$.

**Proof.** Multiplying (4.1) by $u$ and $v$, and integrating by parts, we obtain

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} |u|^2 dx + \frac{1}{2} \frac{d}{dt} \int_{\Gamma} |v|^2 dS + \int_{\Omega} |\nabla u|^p dx + \int_{\Omega} f(u) u dx + \int_{\Gamma} g(v) v dS$$

$$= \int_{\Omega} h_0(t) u dx.$$
combining with assumptions (1.2)-(1.4), Young’s inequality and Poincaré inequality, we obtain
\[
\frac{d}{dt} \int_{\Omega} |u|^2 \, dx + \frac{d}{dt} \int_{\Gamma} |v|^2 \, dS + C \left( \int_{\Omega} |u|^2 \, dx + \int_{\Gamma} |v|^2 \, dS \right) \
\leq C_{[\Omega],S(\Gamma)} + C \|h_0\|^2. \tag{4.6}
\]
Applying the suitable version of Gronwall’s inequality to (4.6), we can find \(T_0 > 0\) and \(\rho_0 > 0\), such that
\[
\|u(t)\|^2 + \|v(t)\|^2_\Gamma \leq \rho_0^2, \quad \text{for any } t \geq T_0. \tag{4.7}
\]
Let \(F(s) = \int_0^s f(\tau) \, d\tau\), \(G(s) = \int_0^s g(\tau) \, d\tau\), by assumptions (1.2)-(1.3) again, from (4.5), we obtain
\[
\begin{align*}
\frac{d}{dt} \int_{\Omega} |u|^2 \, dx + & \frac{d}{dt} \int_{\Gamma} |v|^2 \, dS + \int_{\Omega} |\nabla u|^p \, dx + C_1 \int_{\Omega} F(u) \, dx + C_2 \int_{\Gamma} G(v) \, dS \\
\leq & C_{[\Omega],S(\Gamma)} + C \|h_0\|^2.
\end{align*}
\]
Integrating this inequality above from \(t\) to \(t+1\), and combining (4.7), it follows that for any \(t \geq T_0\),
\[
\int_t^{t+1} \left( \int_{\Omega} |\nabla u|^p \, dx + C_1 \int_{\Omega} F(u) \, dx + C_2 \int_{\Gamma} G(v) \, dS \right) \, ds
\leq C_{[\Omega],S(\Gamma),\rho_0} + C \int_t^{t+1} \|h_0\|^2 \, ds \tag{4.8}
\leq C_{[\Omega],S(\Gamma),\rho_0,\|h_0\|^2}.
\]
On the other hand, multiplying (1.1) by \(u_t\) and \(v_t\), we have
\[
\begin{align*}
\int_{\Omega} |u_t|^2 \, dx + & \int_{\Gamma} |v_t|^2 \, dS + \frac{1}{p} \frac{d}{dt} \int_{\Omega} |\nabla u|^2 \, dx + \frac{d}{dt} \left( \int_{\Omega} F(u) \, dx + \int_{\Gamma} G(v) \, dS \right) \\
\leq & \frac{1}{2} \int_{\Omega} |h_0|^2 \, dx + \frac{1}{2} \int_{\Omega} |u_t|^2 \, dx,
\end{align*}
\]
so we obtain
\[
\frac{d}{dt} \left( \int_{\Omega} |\nabla u|^p \, dx + p \int_{\Omega} F(u) \, dx + p \int_{\Gamma} G(v) \, dS \right) \leq C \|h_0\|^2. \tag{4.10}
\]
Combining (4.8) and (4.10), by the uniformly Gronwall lemma, we have that for any \(t \geq T_0 + 1, \sigma \in \Sigma\),
\[
\int_{\Omega} |\nabla u|^p \, dx + \int_{\Omega} F(u) \, dx + \int_{\Gamma} G(v) \, dS \leq C_{[\Omega],S(\Gamma),\rho_0,\|h_0\|^2}, \tag{4.11}
\]
which implies that for any \(t \geq T_0 + 1, \sigma \in \Sigma\),
\[
\int_{\Omega} |\nabla u|^p \, dx + \int_{\Omega} |u|^r \, dx + \int_{\Gamma} |v|^2 \, dS \leq M, \tag{4.12}
\]
where \(M\) depends on \(|\Omega|, S(\Gamma), \rho_0, \|h_0\|^2\).
\[\square\]
As a direct result of Theorem 4.2 we have the existence of a uniform attractor in \(L^2(\Omega) \times L^2(\Gamma)\):
Corollary 4.3. Under the assumptions of Theorem 4.2, the family of processes \( \{ U_\sigma(t, \tau) \} \), \( \sigma \in \Sigma \) corresponding to \( \{ 4.1 \} \) has a uniform attractor \( A_\Sigma \) in \( L^2(\Omega) \times L^2(\Gamma) \), which is compact in \( L^2(\Omega) \times L^2(\Gamma) \) and attracts every \( L^2(\Omega) \times L^2(\Gamma) \)-bounded subset with \( L^2(\Omega) \times L^2(\Gamma) \)-norm. Moreover,

\[
A_\Sigma = \omega_{0, \Sigma}(B_0) = \cup_{\sigma \in \Sigma} K_\sigma(s), \quad \forall s \in \mathbb{R},
\]

where \( K_\sigma(s) \) is the section at \( t = s \) of the kernel \( K_\sigma \) of the process \( \{ U_\sigma(t, \tau) \} \) with symbol \( \sigma \).

Proof. Theorem 4.2 and the Sobolev compactness imbedding theorem imply the existence of a uniform attractor \( A_\Sigma \) in \( L^2(\Omega) \times L^2(\Gamma) \) immediately. \( \square \)

4.3. Existence of a uniform attractor in \( L^{r_1}(\Omega) \times L^r(\Gamma) \) \( (r = \min(r_1, r_2)) \).

First, we give some a priori estimates for the solution of \( \{ 4.1 \} \) to verify the uniformly asymptotic compactness in \( L^{r_1}(\Omega) \times L^r(\Gamma) \). The idea of the proof comes from [31].

Theorem 4.4. Assume that \( h(t) \) is normal in \( L^2_{\text{loc}}(\mathbb{R}; L^2(\Omega)) \), \( f \) and \( g \) satisfy \( \{ 1.2 \} - \{ 1.3 \} \). Then for any \( \varepsilon > 0 \), \( \tau \in \mathbb{R} \) and any bounded subset \( B \subset L^2(\Omega) \times L^2(\Gamma) \), there exist two positive constants \( T = T(B, \varepsilon, \tau) \) and \( M = M(\varepsilon) \), such that

\[
\int_{\Omega(U_\sigma(t, \tau) u) \geq M} |U_\sigma(t, \tau) u|^{r_1} + \int_{\Gamma(U_\sigma(t, \tau) v) \geq M} |U_\sigma(t, \tau) v|^{r_1} \leq \varepsilon,
\]

for all \( t \geq T \), \( (u_\tau, v_\tau) \in B \), \( \sigma \in \Sigma \).

Proof. We multiply \( \{ 4.1 \} \) by \( (u - M)^{r_1-1}_+ \) and \( (v - M)^{r_1-1}_+ \), and integrate over \( \Omega \), then we have

\[
\frac{1}{r_1} \int_{\Omega(u \geq M)} |u - M|^{r_1} dx + \frac{1}{r_1} \int_{\Gamma(v \geq M)} |v - M|^{r_1} dS \\
+ (r_1 - 1) \int_{\Omega(u \geq M)} (u - M)^{r_1-2} |\nabla u|^2 dx + \int_{\Omega(u \geq M)} f(u)(u - M)^{r_1-1} dx \\
+ \int_{\Gamma(v \geq M)} g(v)(v - M)^{r_1-1} dS \\
= \int_{\Omega(u \geq M)} h_0(t)(u - M)^{r_1-1} dx,
\]

where \( (u - M)_+ \) denotes the positive part of \( (u - M) \); that is,

\[
(u - M)_+ = \begin{cases} 
    u - M, & u \geq M, \\
    0, & u \leq M.
\end{cases}
\]

From conditions \( \{ 1.2 \} - \{ 1.3 \} \), we can take \( M \) large enough such that

\[
C_3|v|^{r_2-1}_+ \leq g(v), \quad \text{in } \Gamma(v(t) \geq M),
\]

\[
C_4|u|^{r_1-1}_+ \leq f(u), \quad \text{in } \Omega(u(t) \geq M).
\]
Let $\Omega_1 = \Omega(u(t) \geq M)$, $\Gamma_1 = \Gamma(v(t) \geq M)$, using Young’s inequality and the inequalities above, we obtain
\[
\frac{1}{r_1} \int_{\Omega_1} |u - M|^{r_1} \, dx + \frac{1}{r_1} \int_{\Gamma_1} |v - M|^{r_1} \, dS \\
+ (r_1 - 1) \int_{\Omega_1} (u - M)^{r_1 - 2} |\nabla u|^p \, dx \\
+ C_4 \int_{\Omega_1} |u|^{r_1 - 1} (u - M)^{r_1 - 1} \, dx + C_5 \int_{\Gamma_1} |v|^{r_1 - 1} (v - M)^{r_1 - 1} \, dS
\]
\[
\leq \frac{C_4}{2} \int_{\Omega_1} |u - M|^{2r_1 - 2} \, dx + \frac{1}{2C_3} \int_{\Omega_1} |h_0(t)|^2 \, dx,
\]
so we have
\[
\frac{1}{r_1} \int_{\Omega_1} |u - M|^{r_1} \, dx + \frac{1}{r_1} \int_{\Gamma_1} |v - M|^{r_1} \, dS \\
+ (r_1 - 1) \int_{\Omega_1} (u - M)^{r_1 - 2} |\nabla u|^p \, dx \\
+ \frac{C_4 M^{r_1 - 2}}{2} \int_{\Omega_1} |u - M| \, dx + C_4 M^{r_2 - 2} \int_{\Gamma_1} |v - M|^{r_1} \, dS \\
\leq \frac{1}{2C_4} \int_{\Omega_1} |h_0(t)|^2 \, dx.
\]
By using the Gronwall lemma and together with the Lemma 2.3 we can choose $M$ large enough, such that
\[
\int_{\Omega_1} |u - M|^{r_1} \, dx + \int_{\Gamma_1} |v - M|^{r_1} \, dS \leq \varepsilon. \tag{4.15}
\]
Noting that
\[
\frac{1}{2r_1} \int_{\Omega(u \geq 2M)} |u|^{r_1} \, dx \leq \int_{\Omega(u \geq M)} |u - M|^{r_1} \, dx, \tag{4.16}
\]
\[
\frac{1}{2r_1} \int_{\Gamma(v \geq 2M)} |v|^{r_1} \, dS \leq \int_{\Gamma(v \geq M)} |v - M|^{r_1} \, dS, \tag{4.17}
\]
combining (4.15)–(4.17), we obtain
\[
\int_{\Omega(u \geq 2M)} |u(t)|^{r_1} \, dx + \int_{\Gamma(v \geq 2M)} |v(t)|^{r_1} \, dS \leq 2^{r_1} \varepsilon. \tag{4.18}
\]
Repeating the same steps above, just taking $(u + M)^{r_1 - 1}$ instead of $(u - M)^{r_1 - 1}$, $(v + M)^{r_1 - 1}$ instead of $(v - M)^{r_1 - 1}$, we deduce that
\[
\int_{\Omega(u \leq -2M)} |u(t)|^{r_1} \, dx + \int_{\Gamma(v \leq -2M)} |v(t)|^{r_1} \, dS \leq 2^{r_1} \varepsilon. \tag{4.19}
\]
Combining (4.18)–(4.19), we obtain
\[
\int_{\Omega(|u(t)| \geq 2M)} |u(t)|^{r_1} \, dx + \int_{\Gamma(|v(t)| \geq 2M)} |v(t)|^{r_1} \, dS \leq 2^{r_1} \varepsilon. \tag{4.20}
\]

Now we state the existence and structure of a uniform attractor in $L^{r_1}(\Omega) \times L^r(\Gamma)$ ($r = \min(r_1, r_2)$).
Theorem 4.5. Assume that $h(t)$ is normal in $L^2_{\text{loc}}(\mathbb{R}; L^2(\Omega))$, $f$ and $g$ satisfy \((1.2)-(1.4)\). Then the family of processes $\{U_\sigma(t, \tau)\}, \sigma \in \Sigma$ corresponding to \((4.1)\) has a compact uniform (with respect to $\sigma \in \Sigma$) attractor $\mathcal{A}_{\Sigma}$ in $L^{r_1}(\Omega) \times L^{r_2}(\Gamma)$ ($r = \min(r_1, r_2)$) and $\mathcal{A}_{\Sigma}$ satisfies

$$\mathcal{A}_{\Sigma} = \omega_{0, \Sigma}(B_0) = \bigcup_{\sigma \in \Sigma} \mathcal{K}_{\sigma}(s), \quad \forall s \in \mathbb{R},$$

where $\mathcal{K}_{\sigma}(s)$ is the section at $t = s$ of the kernel $\mathcal{K}_{\sigma}$ of the process $\{U_\sigma(t, \tau)\}$ with symbol $\sigma$.

Proof. From Corollary 4.3 and Theorem 4.4, it is easy to verify that $\{U_\sigma(t, \tau)\}, \sigma \in \Sigma$ has uniformly asymptotic compactness in $L^{r_1}(\Omega) \times L^{r_2}(\Gamma)$, which combining with Theorem 4.2, we can obtain the existence of a compactly uniform attractor in $L^{r_1}(\Omega) \times L^{r_2}(\Gamma)$ ($r = \min(r_1, r_2)$). Then, similar to \[24, 28\], we can obtain the structure of $\mathcal{A}_{\Sigma}$, see more details in \[24, 28\]. □

Acknowledgments. This work is partly supported by the NNSF of China (11101404, 11201204, 11471148) and by the State Scholarship Fund of China Scholarship Council (201308620021).

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