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NONLOCAL SINGULAR PROBLEM WITH INTEGRAL CONDITION FOR A SECOND-ORDER PARABOLIC EQUATION

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Abstract. We prove the existence and uniqueness of a strong solution for a parabolic singular equation in which we combine Dirichlet with integral boundary conditions given only on parts of the boundary. The proof uses a priori estimate and the density of the range of the operator generated by the problem considered.

1. Introduction

In the rectangle $\Omega = [0,1] \times [0,T]$, we consider the equation

$$\pounds u = \frac{\partial u}{\partial t} - \frac{1}{x}\frac{\partial}{\partial x}\big(x\frac{\partial u}{\partial x}\big) = f(x,t), \tag{1.1}$$

with the initial condition

$$u(x,0) = \varphi(x), \quad x \in [0,1], \tag{1.2}$$

and the Dirichlet condition

$$u(1,t) = 0, \quad t \in [0,T], \tag{1.3}$$

and the nonlocal condition

$$\int_0^\alpha u(x,t)dx + \int_\beta^1 u(x,t)dx = 0, \quad 0 \le \alpha \le \beta \le 1,\ t \in [0,T]. \tag{1.4}$$

The functions $\varphi(x)$, $f(x,t)$ are given, and we assume that the matching conditions are satisfied

$$\varphi(1) = 0,$$
$$\int_0^\alpha \varphi(x)dx + \int_\beta^1 \varphi(x)dx = 0.$$

Over the previous few years, many physical phenomena were formulated by means of nonlocal mathematical models with integral boundary conditions. These integral boundary conditions appear when the data on the body can not be measured directly, but their average values are known. For instance, in some cases, describing the solution u (pressure, temperature, etc.) pointwise is not possible, because only

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the average value of the solution can be estimate along the boundary or along a part of it. These mathematical models are encountered in many engineering models such as heat conduction [7], plasma physics [14], thermoelasticity [15], electrochemistry , chemical diffusion [3] and underground water flow [7, 17]. The importance of this kind of problems have been also pointed out by Samarskii [14]. The first paper, devoted to second order partial differential equations with nonlocal integral conditions goes back to Cannon [4].This type of boundary value problems with combined Dirichlet or Newmann and integral condition, or with purely integral conditions has been investigated in [1, 2, 10, 19] for parabolic equations, for hyperbolic equations in [1, 13, 18], and in [6, 9] for mixed type equations. Problems for elliptic equations with operator nonlocal conditions were considered by Mikhailov and Gushin [8], A.L.Skubachevski, Steblov [16], Peneiah [12].

In this article we prove the existence and uniqueness of the strong solution of a class of non local mixed second-order singular parabolic problem in which we combine Dirichlet and integral conditions given only on parts of the boundary. Case of $\alpha = 0$, is treated in [5, 10]. This kind of problems for parabolic equations was considered in [11].

2. Preliminaries

In this article, we prove the existence and uniqueness of a strong solution of problem (1.1)-(1.4). For this, we consider the solution of problem (1.1)-(1.4) as a solution of operator equation $Lu = \mathcal{F} = (f, \varphi)$, where the operator L is considred from E to F, where E is the Banach space of the functions u, with the norm

$$\|u\|_E^2 = \int_\Omega x^2\Big(|\frac{\partial u}{\partial t}|^2 + |\frac{\partial^2 u}{\partial x^2}|^2\Big)\,dx\,dt + \sup_t \int_0^1 x^2\Big(|u|^2 + |\frac{\partial u}{\partial x}|^2\Big)dx,$$

F is the Hilbert space of vector valued functions $F = (f, \varphi)$ obtained by the completion of the space $L^2(\Omega) \times W_2^2(0,1)$, with respect to the norm

$$\|F\|_F^2 = \int_\Omega x^2|f|^2\,dx\,dt + \int_0^1 x^2\Big(|\varphi|^2 + |\frac{d\varphi}{dx}|^2\Big)dx,$$

with domain of definition $D(L)$ consisting of functions $u \in E$, such that $u, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x \partial t}$ belong to $L^2(\Omega)$ and u satisfies conditions (1.3)-(1.4). Then we establish an energy inequality

$$\|u\|_E \le C\|Lu\|_F, \quad \forall u \in D(L), \tag{2.1}$$

and we show that the operator L has a closure $\overline{L}$.

Definition 2.1. A solution of the operator equation $\overline{L}u = \mathcal{F}$ is called a strong solution of problem (1.1)-(1.4).

Inequality (2.1), can be extended to $u \in D(\overline{L})$, that is

$$\|u\|_E \le C\|\overline{L}u\|_F, \quad \forall u \in D(\overline{L}). \tag{2.2}$$

From this inequality, we obtain the uniqueness of a strong solution, if it exists, and the equality of the sets $R(\overline{L})$ and $\overline{R(L)}$. Thus, to prove the existence of the strong solution of the problem (1.1)-(1.4) for any $\mathcal{F} \in F$, it remains to prove that the set $R(L)$ is dense in F.

3. An energy inequality and its applications

Theorem 3.1. *There exists a positive constant C, such that, for any function $u \in D(L)$ we have*

$$\|u\|_E \le C\|Lu\|_F. \tag{3.1}$$

Proof. Let

$$Mu = x^2\frac{\partial u}{\partial t} - x\int_0^x \frac{\partial u}{\partial t}(\zeta,t)d\zeta + x\int_\alpha^x \frac{\partial u}{\partial t}(\zeta,t)d\zeta - x\int_\beta^x \frac{\partial u}{\partial t}(\zeta,t)d\zeta,$$

We consider the quadratic form obtained by multiplying (1.1) by $\exp(-ct)\overline{Mu}$, where $c > 0$ and integrating over $\Omega^s = [0,1]\times[0,s]$ with $0 \le s \le T$, and taking the real part, formally

$$\begin{aligned}
&\Phi(u,u)\\
&= \operatorname{Re}\int_{\Omega_s} \exp(-ct)\frac{\partial u}{\partial t}\overline{Mu}\,dx\,dt - \int_{\Omega_s} \exp(-ct)\frac{1}{x}\frac{\partial}{\partial x}(x\frac{\partial u}{\partial x})\overline{Mu}\,dx\,dt.
\end{aligned} \tag{3.2}$$

Integrating each term by parts in (3.2) with respect to x and using the condition (1.4), we obtain

$$\begin{aligned}
&\operatorname{Re}\int_{\Omega_s} \exp(-ct)\frac{\partial u}{\partial t}\overline{Mu}\,dx\,dt\\
&= \int_{\Omega_s} x^2\exp(-ct)|\frac{\partial u}{\partial t}|^2\,dx\,dt + \int_{\Omega_s}\exp(-ct)\frac{\partial u}{\partial t}\int_0^x \zeta\overline{\frac{\partial u}{\partial t}}d\zeta\\
&= \int_{\Omega_s} x^2\exp(-ct)|\frac{\partial u}{\partial t}|^2\,dx\,dt + \frac{1}{2}\int_0^s \exp(-ct)dt|\int_0^1 x\frac{\partial u}{\partial t}dx|^2\\
&\quad + \int_{\Omega_s}\exp(-ct)\frac{\int_0^x|\zeta\frac{\partial u}{\partial t}|^2d\zeta}{2x^2}\,dx\,dt\,.
\end{aligned} \tag{3.3}$$

Using conditions (1.3), (1.4), we obtain

$$-\int_{\Omega_s}\exp(-ct)\frac{1}{x}\frac{\partial}{\partial x}(x\frac{\partial u}{\partial x})\overline{Mu}\,dx\,dt = \int_{\Omega_s} x^2\frac{\partial u}{\partial x}\overline{\frac{\partial^2 u}{\partial x\partial t}}\,dx\,dt. \tag{3.4}$$

Integrating with respect to t, in the right hind side of (3.4), using (3.3), expression (3.2) becomes

$$\begin{aligned}
&\int_{\Omega_s} x^2|\frac{\partial u}{\partial t}|^2\,dx\,dt + \frac{1}{2}\int_0^s dt|\int_0^1 x\frac{\partial u}{\partial t}dx|^2 + \int_{\Omega_s}\frac{\int_0^x|\zeta\frac{\partial u}{\partial t}|^2d\zeta}{2x^2}\,dx\,dt\\
&+ c\int_{\Omega_s}\frac{x^2}{2}\exp(-ct)|\frac{\partial u}{\partial x}|^2\,dx\,dt + \int_0^1\frac{x^2}{2}\exp(-ct)|\frac{\partial u}{\partial x}|^2dx\Big|_{t=s}dx\\
&= \operatorname{Re}\int_{\Omega_s}\exp(-ct)\pounds u\overline{Mu}\,dx\,dt + \int_0^1\frac{x^2}{2}|\frac{d\varphi}{dx}|^2dx.
\end{aligned} \tag{3.5}$$

Substituting Mu by its expression in the first term in the right-hand side of (3.5), we obtain

$$\begin{aligned}
&\operatorname{Re}\int_{\Omega_s}\exp(-ct)\pounds u\overline{Mu}\,dx\,dt\\
&= \operatorname{Re}\int_{\Omega_s} x^2\exp(-ct)f\frac{\partial u}{\partial t}\,dx\,dt - \operatorname{Re}\int_{\Omega_s} xf\Big(\int_0^x\frac{\partial u}{\partial t}d\zeta - \int_\alpha^x\frac{\partial u}{\partial t}d\zeta
\end{aligned}$$

$$+\int_{\beta}^{x}\frac{\partial u}{\partial t}d\zeta\Big)\,dx\,dt.$$

By integrating with respect to x, using the condition (1.4), we obtain

$$-\operatorname{Re}\int_{\Omega_s} xf(\int_0^x \frac{\partial u}{\partial t}d\zeta-\int_{\alpha}^{x}\frac{\partial u}{\partial t}d\zeta+\int_{\beta}^{x}\frac{\partial u}{\partial t}d\zeta)\,dx\,dt=\int_{\Omega_s}\frac{\partial u}{\partial t}\int_0^x \zeta f d\zeta\,dx\,dt,$$

then by using ε-inequalities, we have

$$\begin{aligned}
&\operatorname{Re}\int_{\Omega_s} x^2\exp(-ct)f\frac{\partial u}{\partial t}\,dx\,dt\\
&\le \frac{\varepsilon_1}{2}\int_{\Omega_s} x^2\exp(-ct)|f|^2\,dx\,dt+\frac{1}{2\varepsilon_1}\int_{\Omega_s} x^2\exp(-ct)|\frac{\partial u}{\partial t}|^2\,dx\,dt,
\end{aligned}$$

$$\begin{aligned}
&\int_{\Omega_s}\frac{\partial u}{\partial t}\int_0^x \zeta f d\zeta\,dx\,dt\\
&\le \frac{1}{2\varepsilon_2}\int_{\Omega_s} x^2\exp(-ct)|\frac{\partial u}{\partial t}|^2\,dx\,dt+\frac{\varepsilon_2}{2}\int_{\Omega_s}\exp(-ct)\frac{|\int_0^x\zeta f d\zeta|^2}{x^2}\,dx\,dt
\end{aligned}$$

It is easy to show that

$$\int_{\Omega_s}\exp(-ct)\frac{|\int_0^x\zeta f(\zeta,t)d\zeta|^2}{x^2}\,dx\,dt\le 4\int_{\Omega_s} x^2\exp(-ct)|f|^2\,dx\,dt.$$

Then, from the previous inequalities, formula (3.5) becomes

$$\begin{aligned}
&\int_{\Omega_s}(1-\frac{1}{2\varepsilon_1}-\frac{1}{2\varepsilon_2})x^2\exp(-ct)|\frac{\partial u}{\partial t}|^2\,dx\,dt+\frac12\int_0^s dt|\int_0^1 x\frac{\partial u}{\partial t}dx|^2\\
&+\int_{\Omega_s}\frac{\int_0^x|\zeta\frac{\partial u}{\partial t}|^2d\zeta}{2x^2}\,dx\,dt+c\int_{\Omega_s}\frac{x^2}{2}\exp(-ct)|\frac{\partial u}{\partial x}|^2\,dx\,dt\\
&+\int_0^1\frac{x^2}{2}\exp(-ct)|\frac{\partial u}{\partial x}|^2dx\Big|_{t=s}dx\\
&\le(\frac{\varepsilon_1}{2}+2\varepsilon_2)\int_{\Omega_s} x^2\exp(-ct)|f|^2\,dx\,dt+\int_0^1\frac{x^2}{2}|\frac{d\varphi}{dx}|^2dx,
\end{aligned}$$

we choose $\varepsilon_1=\varepsilon_2=2$, then

$$\begin{aligned}
&\int_{\Omega_s} x^2\exp(-ct)|\frac{\partial u}{\partial t}|^2\,dx\,dt+\int_{\Omega_s} x^2\exp(-ct)|\frac{\partial u}{\partial x}|^2\,dx\,dt\\
&+\int_0^1 x^2\exp(-ct)|\frac{\partial u}{\partial x}|^2dx\Big|_{t=s}dx\\
&\le\frac{10}{\min(1,c)}\Big(\int_{\Omega_s} x^2\exp(-ct)|f|^2\,dx\,dt+\int_0^1 x^2|\frac{d\varphi}{dx}|^2dx\Big).
\end{aligned}\tag{3.6}$$

Hence from (1.1), (3.6) we deduce that

$$\begin{aligned}
&\int_{\Omega_s} x^2\exp(-ct)|\frac{\partial^2 u}{\partial x^2}|^2\,dx\,dt\\
&\le(\frac{80}{\min(1,c)}+2)\Big(\int_{\Omega_s} x^2\exp(-ct)|f|^2\,dx\,dt+\int_0^1 x^2|\frac{d\varphi}{dx}|^2dx\Big).
\end{aligned}$$

Integrating the term $x^2 \exp(-ct) u \frac{\partial u}{\partial t}$ with respect to t and using (3.6), we obtain

$$\begin{aligned}
&\int_0^1 x^2 \exp(-ct)|u|^2 dx\Big|_{t=s} dx \\
&\le (\frac{10}{c\min(1,c)}+1)\Big(\int_{\Omega_s} x^2 \exp(-ct)|f|^2\,dx\,dt + \int_0^1 x^2|\varphi|^2 dx\Big)\,.
\end{aligned}$$

Then from the previous inequalities we obtain

$$\begin{aligned}
&\int_{\Omega_s} x^2\Big(|\frac{\partial u}{\partial t}|^2 + |\frac{\partial^2 u}{\partial x^2}|^2\Big)\,dx\,dt + \int_0^1 x^2\Big(|\frac{\partial u}{\partial x}|^2 + |u|^2\Big)dx\Big|_{t=s} dx \\
&\le \beta\Big(\int_{\Omega} x^2|f|^2\,dx\,dt + \int_0^1 x^2(|\frac{d\varphi}{dx}|^2 + |\varphi|^2)dx\Big).
\end{aligned} \tag{3.7}$$

The left side of (3.7) is independent of t, then by taking the upper bound with respect to t from 0 to T, we obtain the desired inequality

$$\|u\|_E \le C\|Lu\|_F, \forall u \in D(L),$$

where

$$C^2 = \beta = \max\Big(\frac{90}{\min(1,c)} + 3 + \frac{10}{c\min(1,c)}\Big)e^{cT}.$$

□

Lemma 3.2. *The operator L from E to F admits a closure $\overline{L}$.*

The previous Theorem is valid for a strong solution, then we have the inequalities

$$\|u\|_E \le C\|\overline{L}u\|_F, \forall u \in D(\overline{L}).$$

Hence we obtain the following corollaries

Corollary 3.3. *A strong solution of problem* (1.1)-(1.4) *is unique if it exists, and depends continuously on $\mathcal{F}$.*

Corollary 3.4. *The range $R(\overline{L})$ of the operator $\overline{L}$ is closed in F, and $R(\overline{L}) = \overline{R(L)}$.*

4. Solvability of problem (1.1)-(1.4)

To prove the solvability of problem (1.1)-(1.4), it is sufficient to show that $R(L)$ is dense in F. The proof is based on the following lemma.

Lemma 4.1. *Let $D_0(L) = \{u \in D(L),\ u(x,0) = 0,\}$. If, for $u \in D_0(L)$ and for some function $w \in L^2(\Omega)$,*

$$\int_{\Omega} \phi(x)\pounds u\overline{w}\,dx\,dt = 0, \tag{4.1}$$

where

$$\phi(x) = \begin{cases} x^3, & x \in (0,\alpha)\cup(\alpha,\beta), \\ x(x-\beta)^2, & x \in (\beta,1), \end{cases}$$

then $w = 0$.

Proof. Equality (4.1) can be written as

$$\int_\Omega \frac{\partial u}{\partial t}\overline{Nv}\,dx\,dt = \int_\Omega A(t)u\overline{v}\,dx\,dt, \tag{4.2}$$

where

$$v = \begin{cases} xw - \int_x^\alpha w d\zeta, & x\in(0,\alpha),\\ x^2w, & x\in(\alpha,\beta),\\ (x-\beta)w - \int_x^1 wd\zeta, & x\in(\beta,1),\end{cases} \tag{4.3}$$

and

$$A(t)u = \frac{\partial}{\partial x}\big(x\rho(x)\frac{\partial u}{\partial x}\big),$$

where

$$\rho(x) = \begin{cases} x, & x\in(0,\alpha),\\ 1, & x\in(\alpha,\beta),\\ (x-\beta), & x\in(\beta,1),\end{cases}$$

and

$$Nv = \begin{cases} x^2v - x\int_0^x vd\zeta = x^3w, & x\in(0,\alpha),\\ xv = x^3w, & x\in(\alpha,\beta),\\ x(x-\beta)v - x\int_\beta^x vd\zeta = x(x-\beta)^2w, & x\in(\beta,1).\end{cases} \tag{4.4}$$

From (4.3), we conclude that $\int_0^\alpha vdx + \int_\beta^1 vdx = 0$.

We introduce the smoothing operators

$$J_\varepsilon^{-1} = \big(I+\varepsilon\frac{\partial}{\partial t}\big)^{-1},\quad \big(J_\varepsilon^{-1}\big)^* = \big(I-\varepsilon\frac{\partial}{\partial t}\big)^{-1},$$

with respect to t, then, these operators provide the solution of the problems:

$$u_\varepsilon(t) - \varepsilon\frac{\partial u_\varepsilon}{\partial t} = u(t)\quad u_\varepsilon(0)=0,$$
$$v_\varepsilon^*(t) + \varepsilon\frac{\partial v_\varepsilon^*}{\partial t} = v(t)\quad v_\varepsilon^*(T)=0\,.$$

We also have the following properties: for any $g\in L^2(0,T)$, the functions $J_\epsilon^{-1}g$, $(J_\epsilon^{-1})^*g\in W_2^1(0,T)$. If $g\in D(L)$, then $J_\epsilon^{-1}g\in D(L)$ and we have

$$\begin{gathered}\lim\|J_\epsilon^{-1}g-g\|_{L^2(0,T)}=0,\quad \text{for } \varepsilon\to 0,\\ \lim\|(J_\epsilon^{-1})^*g-g\|_{L^2(0,T)}=0,\quad \text{for } \varepsilon\to 0\,.\end{gathered} \tag{4.5}$$

Substituting u in (4.2) by the smoothing function u_ϵ and using the relation

$$A(t)u_\epsilon = J_\epsilon^{-1}Au,$$

we obtain

$$\int_\Omega uN\overline{\frac{\partial v_\varepsilon^*}{\partial t}}\,dx\,dt = -\int_\Omega A(t)u\overline{v_\varepsilon^*}\,dx\,dt. \tag{4.6}$$

The left-hand side of (4.6) is a continuous linear functional of u. Hence the function v_ε^* has the derivatives $x\rho(x)\frac{\partial v_\varepsilon^*}{\partial x}$, $\frac{\partial}{\partial x}(x\rho(x)\frac{\partial v_\varepsilon^*}{\partial x})\in L^2(\Omega)$ and the following conditions are satisfied:

$$\begin{gathered}v_\varepsilon^*\big|_{x=\alpha} = v_\varepsilon^*\big|_{x=\beta} = v_\varepsilon^*\big|_{x=1} = 0,\\ \frac{\partial v_\varepsilon^*}{\partial x}\Big|_{x=\alpha} = \frac{\partial v_\varepsilon^*}{\partial x}\Big|_{x=\beta} = x^2\frac{\partial v_\varepsilon^*}{\partial x}\Big|_{x=0} = \frac{\partial v_\varepsilon^*}{\partial x}\Big|_{x=1} = 0\,.\end{gathered} \tag{4.7}$$

Substituting $u=\int_0^t \exp(-c\tau)v_\varepsilon^* d\tau$ in (4.2), where the constant $c<0$, we obtain

$$\int_\Omega \exp(-ct)v_\varepsilon^*\overline{Nv}\,dx\,dt=\int_\Omega A(t)u\overline{v}\,dx\,dt. \tag{4.8}$$

Using the properties of the smoothing operators we have

$$\int_\Omega \exp(-ct)v_\varepsilon^*\overline{Nv}\,dx\,dt=\int_\Omega A(t)u\overline{v_\varepsilon^*}\,dx\,dt-\epsilon\int_\Omega A(t)u\frac{\partial\overline{v_\varepsilon^*}}{\partial t}\,dx\,dt. \tag{4.9}$$

Integrating with respect to x and t, using (4.7) we obtain

$$\begin{aligned}
&\operatorname{Re}\int_\Omega A(t)u\overline{v_\varepsilon^*}\,dx\,dt\\
&=-\int_\Omega x\rho(x)\exp(ct)\frac{\partial u}{\partial t}\overline{\frac{\partial^2 u}{\partial x\partial t}}\,dx\,dt\\
&=-\int_0^1\frac{x\rho(x)}{2}\exp(ct)|\frac{\partial u}{\partial x}|^2|_{t=T}+c\int_\Omega\frac{x\rho(x)}{2}\exp(ct)|\frac{\partial u}{\partial x}|^2\,dx\,dt\le 0.
\end{aligned}$$

Integrating by parts the second terms with respect to x and t in the right hand side of (4.9) we obtain

$$\begin{aligned}
-\epsilon\int_\Omega A(t)u\frac{\partial\overline{v_\varepsilon^*}}{\partial t}\,dx\,dt&=\epsilon\int_\Omega x\rho(x)\frac{\partial u}{\partial t}\overline{\frac{\partial^2 v_\varepsilon^*}{\partial x\partial t}}\,dx\,dt\\
&=-\epsilon\int_\Omega x\rho(x)\exp(ct)|\frac{\partial^2 u}{\partial x\partial t}|^2\,dx\,dt\le 0.
\end{aligned} \tag{4.10}$$

Substituting the expression of Nv in (4.8), we obtain

$$\int_\Omega \exp(-ct)v_\varepsilon^*\overline{Nv}\,dx\,dt=\int_\Omega \exp(-ct)(v_\varepsilon^*-v)\overline{Nv}\,dx\,dt+\int_\Omega \exp(-ct)v\overline{Nv}\,dx\,dt,$$

since

$$\begin{aligned}
&\int_\Omega \exp(-ct)v\overline{Nv}\,dx\,dt\\
&=\int_0^T\int_0^\alpha \exp(-ct)x^2|v|^2\,dx\,dt+\int_0^T\int_\alpha^\beta \exp(-ct)x|v|^2\,dx\,dt\\
&\quad+\int_0^T\int_\beta^1 \exp(-ct)x(x-\beta)|v|^2\,dx\,dt+\frac12\int_0^T\int_0^\alpha|\int_0^x v d\zeta|^2\,dx\,dt\\
&\quad+\frac12\int_0^T\int_\beta^1|\int_\beta^x v d\zeta|^2\,dx\,dt,
\end{aligned}$$

then by passing to the limit as $\varepsilon\longrightarrow 0$, we obtain $v=0$, so that $w=0$. □

Theorem 4.2. *The range $R(\overline{L})$ of $\overline{L}$ coincides with F.*

Proof. Since F is the Hilbert space, $R(\overline{L})=F$ if and only if the relation

$$\int_\Omega x^2\pounds u\overline{F}_1\,dx\,dt+\int_0^1 x^2\varphi\overline{\varphi_1}dx+\int_0^1 x^2\frac{d\varphi}{dx}\overline{\frac{d\varphi_1}{dx}}dx=0, \tag{4.11}$$

is satisfied for arbitrary $u\in D(L)$ and $F_1=(g,\varphi_1)\in F$ imply $F_1=0$. Taking $u\in D_0(L)$ in (4.11), we obtain that $\int_\Omega x^2\pounds u\overline{F}_1\,dx\,dt=0$ and using Lemma 4.1, we obtain that $\phi(x)w = x^2 g$, then $g = 0$. Consequently for $u \in D(L)$ we have

$$\int_0^1 x^2 \varphi \overline{\varphi_1} dx + \int_0^1 x^2 \frac{d\varphi}{dx} \overline{\frac{d\varphi_1}{dx}} dx = 0,$$

since the range of the operator trace is dense in the Hilbert space with the norm

$$\int_0^1 x^2 \Big(|\frac{d\varphi}{dx}|^2 + |\varphi|^2 \Big) dx.$$

then $\varphi = 0$. □

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