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STABILIZATION FOR 1-D HYPERBOLIC DIFFERENTIAL EQUATIONS WITH BOUNDARY INPUT INCLUDING A NONLINEAR DISTURBANCE

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Abstract. We consider the stabilization for 1-D hyperbolic differential equations with boundary input including a nonlinear disturbance. The time varying extended state observer (ESO) is designed to estimate the disturbance. Based on the estimated disturbance, we obtain an explicit controller by applying the backstepping method. It is shown that the closed-loop system of the 1-D hyperbolic differential equation is asymptotically stable under this controller. This result is illustrated by simulation examples.

1. Introduction

Recently, stabilization problems of PDEs, such as a string, a beam, a chemical tubular reactor, have received a lot of attention [2, 5, 6, 8, 9, 10, 11, 12, 13]. For the first-order hyperbolic system, some stability problems were studied in [1, 3, 4, 15, 16, 17]. However, as far as we know, there are only a few papers that consider stability for the first-order hyperbolic system with boundary input matched with disturbance. It is well-known that when the small disturbance on boundary happens, the system can become instable, even has no solution.

In this article concerns the stabilization for 1-D hyperbolic differential equation with boundary input matched with nonlinear disturbance

$$\begin{gathered}
u_t(t,x) = u_x(t,x) + \int_0^x f(x,y)u(t,y)\,dy + g(x)u(t,0), \quad x \in (0,1),\ t > 0, \\
u(t,1) = U(t) + d(t), \quad t \ge 0, \\
u(0,x) = u_0(x),
\end{gathered} \tag{1.1}$$

where u is the state, U is the control input, the disturbance $d(t)$ is assumed to be bounded in the Euclidean norm. And $g \in C[0,1]$, $f \in C(\overline{\Omega})$, $\Omega = \{(x,y) : 0 < x, y < 1\}$.

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For (1.1), when $d(t)$ is absent, the system is null controllable and the backstepping controller can be chose as in [14, 15],

$$U(t) = \int_0^1 k(1,y)u(t,y)\,dy, \tag{1.2}$$

where $k(x,y)$ satisfies

$$\begin{gathered} k_x(x,y) + k_y(x,y) = \int_y^x k(x,\xi)f(\xi,y)\,d\xi - f(x,y), \quad x,y \in \overline{\Omega},\ t>0, \\ k(x,0) = \int_0^x k(x,y)g(y)\,dy - g(x), \quad x \in [0,1]. \end{gathered} \tag{1.3}$$

Note that (1.3) is well-posed, see [15, 16].

The objective of this article is to estimate the disturbance based on the time varying extended state observer designed, and redesign a continuous controller $U(t)$, to stabilize system (1.1) in the presence of a disturbance. We consider systems (1.1) in the state space $H = L^2(0,1)$.

2. Preliminary lemma

Following the ideas in [15, 16], we introduce a inverse transformation

$$V(t,x) = u(t,x) - \int_0^x k(x,y)u(t,y)\,dy. \tag{2.1}$$

This function transforms (1.1) into the system

$$\begin{gathered} V_t(t,x) = V_x(t,x), \quad x \in (0,1),\ t>0, \\ V(t,1) = U(t) - \int_0^1 k(1,y)u(t,y)\,dy + d(t), \quad t \ge 0, \\ V(0,x) = V_0(x). \end{gathered} \tag{2.2}$$

In what follows, we consider the stabilization of (2.2), and in the final step to go back to system (1.1), under the inverse transformation.

Introduce a new controller $U_0(t)$ such that

$$U(t) = U_0(t) + \int_0^1 k(1,y)u(t,y)\,dy. \tag{2.3}$$

Then system (2.2) becomes

$$\begin{gathered} V_t(t,x) = V_x(t,x), \quad x \in (0,1),\ t>0, \\ V(t,1) = U_0(t) + d(t), \quad t \ge 0, \\ V(0,x) = V_0(x). \end{gathered} \tag{2.4}$$

To write this system in operator form, we define the operator A and B as follows:

$$Af = f', \quad D(A) = \big\{ f \in H^1(0,1) \big| f(1) = 0 \big\}, \tag{2.5}$$

$$B = \delta(x-1). \tag{2.6}$$

Then we can write system (2.4) as an evolutionary equation in H:

$$\frac{d}{dt}V(t,x) = AV(t,x) + B(U_0(t) + d(t)). \tag{2.7}$$

Lemma 2.1. *Let A, B be defined in* (2.5) *and* (2.6)*. Then*

(i) *A generates a strongly continuous semigroup.*

(ii) *B is admissible to the semigroup e^{At}.*

Proof. It is well-known that A generates a strongly continuous semigroup $T(t)$, and $\sigma(A) = \emptyset$, $\omega_0(T(t)) = -\infty$ [18]. This shows that A generates an exponential stable C_0-semigroup e^{At} on H.

Now we show that B is admissible for e^{At}. Actually, a straightforward computation gives the adjoint of (2.5),

$$\begin{gathered} A^*g = -g', \\ D(A^*) = \big\{g \in H^1(0,1) | g(0) = 0\big\}. \end{gathered} \tag{2.8}$$

The dual system to (2.7) is

$$\begin{gathered} \frac{d}{dt}V^*(t,x) = A^*V^*(t,x), \\ y(t) = B^*V^*(t,x)\,. \end{gathered} \tag{2.9}$$

That is,

$$\begin{gathered} V_t^*(t,x) = -V_x^*(t,x), \\ V^*(t,0) = 0, \\ y(t) = V^*(t,1). \end{gathered} \tag{2.10}$$

On the one hand, for all $f \in H$,

$$(A^*)^{-1}f = -\int_0^x f(s)ds, \tag{2.11}$$

and

$$B^*(A^*)^{-1}f = -\int_0^1 f(s)ds, \tag{2.12}$$

which is bounded from H to C.

On the other hand, we define the energy function for (2.10) as

$$E(t) = \frac{1}{2}\int_0^1 (V^*)^2(t,x)dx. \tag{2.13}$$

Differentiate $E(t)$ with respect to t along the solution to (2.10) we obtain

$$\dot{E}(t) = -\frac{1}{2}(V^*)^2(t,1). \tag{2.14}$$

Choose the function

$$\rho(t) = \int_0^1 x(V^*)^2(t,x)dx. \tag{2.15}$$

Then, $|\rho(t)| \leqslant 2E(t)$. Differentiate $\rho(t)$ to give

$$\int_0^T (V^*)^2(t,1)dt \le 2(T+2)E(0), \tag{2.16}$$

This together with boundedness of $B^*(A^*)^{-1}$ shows that B is admissible to the semigroup generated by A [7]. $\square$

3. Estimate for the disturbance

The solution of (2.4) is understood in the sense that

$$\frac{d}{dt}\langle V(t,\cdot), f\rangle = \langle V(t,\cdot), A^*f\rangle + f(1)(U_0(t)+d(t)), \quad \forall f \in D(A^*). \tag{3.1}$$

Let $f(x) = 2x^2 + x \in D(A^*)$ in (3.1) to obtain

$$\dot{y}_1(t) = 3(U_0(t)+d(t)) - y_2(t), \tag{3.2}$$

where

$$y_1(t) = \int_0^1 (2x^2+x)V(t,x)dx,\ \ y_2(t) = \int_0^1 (4x+1)V(t,x)dx. \tag{3.3}$$

It is seen that (3.2) is an ODE with state $y_1(t)$ and control $U(t)$ with disturbance $d(t)$. We design a time varying high gain extended state observer to estimate disturbance $d(t)$ and $y_1(t)$ as follows:

$$\begin{gathered}\dot{\widehat{y}}(t) = 3(U_0(t)+\widehat{d}(t)) - y_2(t) + r(t)(y_1(t)-\widehat{y}(t)),\\ \dot{\widehat{d}}(t) = \frac{1}{3}r^2(t)(y_1(t)-\widehat{y}(t)),\end{gathered} \tag{3.4}$$

where $r(t)$ is time varying function satisfying

$$\dot{r}(t) > 0, \quad \lim_{t\to\infty} r(t) = \infty, \quad \frac{\dot{r}(t)}{r(t)} \le M, \quad \forall t \ge 0,\ M > 0. \tag{3.5}$$

Lemma 3.1. *Suppose that the disturbance $d(t)$ is bounded on $[0,\infty)$ and satisfies*

$$\lim_{t\to\infty} \frac{\dot{d}(t)}{r(t)} = 0. \tag{3.6}$$

Then, the solution of (3.2) *satisfies*

$$\lim_{t\to\infty} |y_1(t) - \hat{y}(t)| = \lim_{t\to\infty} |d(t) - \hat{d}(t)| = 0. \tag{3.7}$$

Proof. Let

$$\tilde{y}(t) = r(t)(y_1(t)-\hat{y}(t)),\ \ \tilde{d}(t) = (d(t)-\hat{d}(t)) \tag{3.8}$$

be the estimator errors. Then, by the system (3.2) and (3.4), the error $(\widetilde{y}, \widetilde{d})$ satisfies

$$\begin{gathered}\dot{\widetilde{y}}(t) = -r(t)\widetilde{y}(t) + 3r(t)\widetilde{d}(t) + \frac{\dot{r}(t)}{r(t)}\widetilde{y}(t),\\ \dot{\widetilde{d}}(t) = -\frac{1}{3}r(t)\widetilde{y}(t) + \dot{d}(t).\end{gathered} \tag{3.9}$$

For system (3.9), we construct the Lyapunov function

$$V\Big(\widetilde{y}(t), \widetilde{d}(t)\Big) = \tilde{y}^2(t) + \frac{21}{2}\tilde{d}^2(t) - \tilde{y}(t)\tilde{d}(t). \tag{3.10}$$

It follows that

$$\frac{1}{11}V\Big(\widetilde{y}(t), \widetilde{d}(t)\Big) \le \tilde{y}^2(t) + \tilde{d}^2(t) \le 2V\Big(\widetilde{y}(t), \widetilde{d}(t)\Big). \tag{3.11}$$

Along with (3.5), finding the derivative of V along the solution of (3.9), we obtain

$$\begin{aligned}\dot{V}(t) &= \Big(-\frac{5}{3}r(t)+2\frac{\dot{r}(t)}{r(t)}\Big)\tilde{y}^2(t)-3r(t)\tilde{d}^2(t)-\frac{\dot{r}(t)}{r(t)}\tilde{y}(t)\tilde{d}(t)+21\tilde{d}(t)\dot{d}(t)-\tilde{y}(t)\\ &\le \Big(-\frac{5}{3}r(t)+\frac{5}{2}\frac{\dot{r}(t)}{r(t)}\Big)\tilde{y}^2(t)-\Big(3r(t)-\frac{1}{2}\frac{\dot{r}(t)}{r(t)}\Big)\tilde{d}^2(t)+21\dot{d}(t)(|\tilde{d}|+|\tilde{y}|).\end{aligned} \tag{3.12}$$

$$K(t)=\min\Big\{\frac{5}{3}r(t)-\sup\frac{5}{2}\Big|\frac{\dot{r}(t)}{r(t)}\Big|,3r(t)-\sup\frac{1}{2}\Big|\frac{\dot{r}(t)}{r(t)}\Big|\Big\}. \tag{3.13}$$

By (3.5), we obtain

$$\lim_{t\to\infty}K(t)=\infty. \tag{3.14}$$

Noticing (3.11),

$$\dot{V}(t)\le -\frac{K(t)}{11}V(t)+42\sqrt{2}\sqrt{V(t)}|\dot{d}(t)|. \tag{3.15}$$

That is,

$$\frac{d\sqrt{V(t)}}{dt}\le -\frac{K(t)}{22}\sqrt{V(t)}+21\sqrt{2}\dot{d}(t). \tag{3.16}$$

Integrating (3.16), from 0 to t, yields

$$\sqrt{V(t)}\le 21\sqrt{2}\frac{\int_0^t|\dot{d}(s)|e^{\int_0^s\frac{1}{22}K(\tau)d\tau}ds}{e^{\int_0^t\frac{1}{22}K(\tau)d\tau}}. \tag{3.17}$$

We can apply the L'Hospital rule to the right side of (3.17) and the condition of Lemma 3.1 to obtain

$$\lim_{t\to\infty}\frac{\int_0^t|\dot{d}(s)|e^{\int_0^s\frac{1}{22}K(\tau)d\tau}ds}{e^{\int_0^t\frac{1}{22}K(\tau)d\tau}}=\lim_{t\to\infty}\frac{22|\dot{d}(t)|e^{\int_{t_0}^t\frac{1}{22}K(\tau)d\tau}}{e^{\int_0^t\frac{1}{22}K(\tau)d\tau}K(t)}=0. \tag{3.18}$$

By (3.17) and (3.18), we have

$$\lim_{t\to\infty}\sqrt{V(t)}=0. \tag{3.19}$$

Along with (3.11), this implies

$$\lim_{t\to\infty}\tilde{y}(t)=0,\quad \lim_{t\to\infty}\tilde{d}(t)=0. \tag{3.20}$$

Since $y_1(t)-\hat{y}(t)=\frac{\tilde{y}(t)}{r(t)}$, we finally obtain

$$\lim_{t\to\infty}|y_1(t)-\hat{y}(t)|=0. \tag{3.21}$$

Then, (3.7) follows from (3.20) and (3.21). $\square$

Remark 3.2. Note that in Lemma 3.1, the derivative of disturbance $d(t)$ is not bounded, and a time varying high gain extended state observer have been designed. However, when $r(t)$ has constant gain, a more strict condition the derivative of disturbance $d(t)$ is needed, to be bounded. In fact, after time varying high gain extended state observer reduce the peak value in initial state, the derivative of disturbance can become bounded. From the practice point of view, we begin to use the constant gain extended state observer to filter the noise. For example, choosing $r(t) = \frac{1}{\varepsilon}$, we design a constant high gain extended state observer for (3.2) to estimate $y_1(t)$ and $d(t)$ as follows:

$$\begin{aligned} \dot{\widehat{y}}(t) &= 3(U_0(t) + \widehat{d}(t)) - y_2(t) + \frac{1}{\varepsilon}(y_1(t) - \widehat{y}(t)), \\ \dot{\widehat{d}}(t) &= \frac{1}{3\varepsilon^2}(y_1(t) - \widehat{y}(t)), \end{aligned} \tag{3.22}$$

where ε is the tuning small parameter. Using the similar method, we can also prove $|\widehat{y}(t) - y_1(t)| + |\widehat{d}(t) - d(t)| \to 0$ as $t \to \infty, \varepsilon \to 0$. We omit the proof here.

4. Proof of main results

Choose $U_0(t) = -\hat{d}(t)$, the closed-loop is governed by

$$\begin{aligned} V_t(t,x) &= V_x(t,x), \quad x \in (0,1),\ t > 0, \\ V(t,1) &= U_0(t) + d(t), \quad t \geq 0, \\ \dot{y}_1(t) &= 3(U_0(t) + d(t)) - y_2(t), \\ \dot{\widehat{y}}(t) &= 3(U_0(t) + \widehat{d}(t)) - y_2(t) + r(t)(y_1(t) - \widehat{y}(t)), \\ \dot{\widehat{d}}(t) &= \frac{1}{3}r^2(t)(y_1(t) - \widehat{y}(t)), \end{aligned} \tag{4.1}$$

In the next section, we will prove that the closed-loop (4.1) is well-posed and stable.

Theorem 4.1. *Suppose that d is bounded measurable and satisfies* (3.6)*, $r(t)$ satisfies* (3.5)*. Then for any initial value $(V(0,x), y_1(0), \widehat{y}(0), \widehat{d}(0)) \in H \times \mathbb{R}^3$, the closed-loop system of* (4.1) *admits a unique solution $(V, y_1, \widehat{y}, \widehat{d}) \in C(0,\infty; H \times \mathbb{R}^3)$, and the solution V tends to zero as $t \to \infty$, $\widehat{y}(t), \widehat{d}(t)$ satisfy* (3.7).

Proof. Introduce error variables $\tilde{y}(t) = r(t)(y_1 - \hat{y}(t))$, $\tilde{d}(t) = (d(t) - \hat{d}(t))$, the system (4.1) is equivalent system (4.2)

$$\begin{aligned} V_t(t,x) &= V_x(t,x), \quad x \in (0,1),\ t > 0, \\ V(t,1) &= \widetilde{d}(t), \quad t \geq 0, \\ \dot{\widetilde{y}}(t) &= -r(t)\widetilde{y}(t) + 3r(t)\widetilde{d}(t) + \frac{\dot{r}(t)}{r(t)}\widetilde{y}(t), \\ \dot{\widetilde{d}}(t) &= -\frac{1}{3}r(t)\widetilde{y}(t) + \dot{d}(t), \end{aligned} \tag{4.2}$$

We can see the closed-loop system (4.2) is a "PDE" and "ODE" coupled system. By Lemma 3.1, the "ODE" section of system (4.2) is proved. We only need to prove the "PDE" section. The "PDE" section of the system (4.2) becomes

$$\begin{aligned} V_t(t,x) &= V_x(t,x), \quad x \in (0,1),\ t > 0, \\ V(t,1) &= \tilde{d}(t), \quad t \geq 0, \\ V(0,x) &= V_0(x). \end{aligned} \tag{4.3}$$

This system can be rewritten as an evolution equation in H,

$$\frac{d}{dt}V(t,x) = AV(t,x) + B\tilde{d}(t), \tag{4.4}$$

where A, B are the same as that in (2.5) and (2.6).

By Lemma 2.1, suppose that $\|e^{At}\| \le L_0 e^{-\omega t}$ for some $L_0, \omega > 0$. For any initial value $V(0,\cdot) \in H$, there exists a unique solution $V \in C(0,\infty; H)$ that can be written as

$$V(t,\cdot) = e^{At}V(0,\cdot) + \int_0^t e^{A(t-s)}B\widetilde{d}(s)ds. \tag{4.5}$$

By Lemma 3.1, for any given $\varepsilon_0 > 0$, there exist $t_1 > 0$ and $\varepsilon_1 > 0$ such that $|\widetilde{d}(t)| < \varepsilon_0$ for all $t > t_1$ and $0 < \varepsilon < \varepsilon_1$. We rewrite (4.5) as

$$V(t,\cdot) = e^{At}V(0,\cdot) + e^{A(t-t_1)}\int_0^{t_1} e^{A(t_1-s)}B\widetilde{d}(s)ds + \int_{t_1}^t e^{A(t-s)}B\widetilde{d}(s)ds. \tag{4.6}$$

The admissibility of B implies

$$\Big\| \int_0^t e^{A(t-s)}B\widetilde{d}(s)ds \Big\|_H^2 \le C_t \|\tilde{d}(t)\|^2_{L^2(0,t)} \le C_t t \|\tilde{d}(t)\|^2_{L^\infty(0,t)}. \tag{4.7}$$

for some constant C_t that is independent of $\tilde{d}(t)$ [7, Definition 6.6]. Because e^{At} is exponentially stable, it follows from [19, Proposition 2.5] that

$$\begin{aligned}\Big\| \int_{t_1}^t e^{A(t-s)}B\widetilde{d}(s)ds \Big\|_H &= \Big\| \int_0^t e^{A(t-s)}B(0 \diamond_{t_1} \tilde{d})(s)ds \Big\|_H \\ &\le L\|\tilde{d}(t)\|_{L^\infty(t_1,\infty)} \le L\varepsilon_0,\end{aligned} \tag{4.8}$$

where L is a constant that is independent of $\tilde{d}$, and

$$(d_1 \diamond_\tau d_2)(t) = \begin{cases} d_1(t), & 0 \le t \le \tau, \\ d_2(t) & t > \tau \end{cases} \tag{4.9}$$

where the left-hand side of (4.9) denotes the τ-concatenation of d_1 and d_2 [18].

By (4.6), (4.7) and (4.8), we have

$$\|V(t,\cdot)\| \le L_0 e^{-\omega t}\|V(0,\cdot)\| + L_0 C_{t_1} t_1 e^{-\omega(t-t_1)}\|\tilde{d}(t)\|_{L^\infty(0,t_1)} + L\varepsilon_0. \tag{4.10}$$

As $t \to \infty$, the first two terms of right hand side for (4.10) tend to zero. The result is then proved by the arbitrariness of ε_0. $\square$

Remark 4.2. Under the constant high gain extended estimated observer (3.22), the closed loop system is governed by

$$\begin{gathered} V_t(t,x) = V_x(t,x), \quad x \in (0,1),\ t > 0, \\ V(t,1) = U_0(t) + d(t), \quad t \ge 0, \\ \dot{y}_1(t) = 3(U_0(t) + d(t)) - y_2(t), \\ \dot{\widehat{y}}(t) = 3(U_0(t) + \widehat{d}(t)) - y_2(t) + \frac{1}{\varepsilon}(y_1(t) - \widehat{y}(t)), \\ \dot{\widehat{d}}(t) = \frac{1}{3\varepsilon^2}(y_1(t) - \widehat{y}(t)), \end{gathered} \tag{4.11}$$

This system is equivalent to the system

$$\begin{gathered} V_t(t,x) = V_x(t,x), \quad x \in (0,1),\ t > 0, \\ V(t,1) = \widetilde{d}(t), \quad t \ge 0, \\ \dot{\widetilde{y}}(t) = -\frac{1}{\varepsilon}\widetilde{y}(t) + \frac{3}{\varepsilon}\widetilde{d}(t), \\ \dot{\widetilde{d}}(t) = -\frac{1}{3\varepsilon}\widetilde{y}(t) + \dot{d}(t), \end{gathered} \tag{4.12}$$

The solution of system (4.12) also tends to zero as $t \to \infty$, $\varepsilon \to 0$.

Theorem 4.3. *Suppose that d is bounded measurable and satisfies (3.6), $r(t)$ satisfies (3.5). Choose the controller $U(t) = \int_0^1 k(1,y)u(t,y)\,dy - \hat{d}(t)$. Then for any initial value $(u(0,x), y_1(0), \widehat{y}(0), \widehat{d}(0)) \in H \times \mathbb{R}^3$, the closed-loop system of (1.1) following*

$$
\begin{gathered}
u_t(t,x) = u_x(t,x) + \int_0^x f(x,y)u(t,y)\,dy + g(x)u(t,0), \quad x \in (0,1),\ t > 0,\\
u(t,1) = \int_0^1 k(1,y)u(t,y)\,dy - \hat{d}(t) + d(t), \quad t \geq 0,\\
\dot{y}_1(t) = 3(-\hat{d}(t) + d(t)) - y_2(t),\\
\dot{\widehat{y}}(t) = 3(-\widehat{d}(t) + \widehat{d}(t)) - y_2(t) + r(t)(y_1(t) - \widehat{y}(t)),\\
\dot{\widehat{d}}(t) = \frac{1}{3}r^2(t)(y_1(t) - \widehat{y}(t)),
\end{gathered}
\tag{4.13}
$$

admits a unique solution $(u, y_1, \widehat{y}, \widehat{d}) \in C(0,\infty; H \times \mathbb{R}^3)$, and the solution $u(x,t)$ of system (4.13) tends to zero as $t \to \infty$. And $\widehat{y}(t), \widehat{d}(t)$ satisfies (3.7).

This theorem can be proved by the a inverse transformation of (2.1). We will omit the proofs.

Remark 4.4. Under the constant gain extended state observer (3.22), we choose the controller $U(t) = \int_0^1 k(1,y)u(t,y)\,dy - \hat{d}(t)$. Then for any initial value

$$(u(0,x), y_1(0), \widehat{y}(0), \widehat{d}(0)) \in H \times \mathbb{R}^3,$$

the closed-loop system of (1.1) following

$$
\begin{gathered}
u_t(t,x) = u_x(t,x) + \int_0^x f(x,y)u(t,y)\,dy + g(x)u(t,0), \quad x \in (0,1),\ t > 0,\\
u(t,1) = \int_0^1 k(1,y)u(t,y)\,dy - \hat{d}(t) + d(t), \quad t \geq 0,\\
\dot{y}_1(t) = 3(-\hat{d}(t) + d(t)) - y_2(t),\\
\dot{\widehat{y}}(t) = 3(-\hat{d}(t) + \widehat{d}(t)) - y_2(t) + \frac{1}{\varepsilon}(y_1(t) - \widehat{y}(t)),\\
\dot{\widehat{d}}(t) = \frac{1}{3\varepsilon^2}(y_1(t) - \widehat{y}(t)),
\end{gathered}
\tag{4.14}
$$

admits a unique solution $(u, y_1, \widehat{y}, \widehat{d}) \in C(0,\infty; H \times \mathbb{R}^3)$, and the solution $u(x,t)$ of system (4.13) tends to zero as $t \to \infty, \varepsilon \to 0$. Also $\widehat{y}(t), \widehat{d}(t)$ satisfies (3.7).

Corollary 4.5. *The special form of (1.1) is as follows:*

$$
\begin{gathered}
u_t(t,x) = u_x(t,x) + g(x)e^{bx}u(t,0), \quad x \in (0,1),\ t > 0,\\
u(t,1) = U(t) + d(t), \quad t \geq 0,\\
u(0,x) = u_0(x),
\end{gathered}
\tag{4.15}
$$

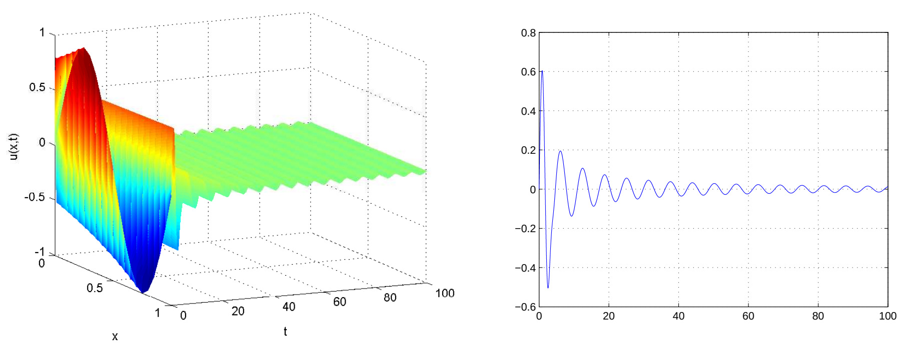
By Theorem 4.3, we can choose $U(t) = -\int_0^1 g(y)e^{(b+g)(1-y)}u(t,y)\,dy - \hat{d}(t)$, the closed-loop system (4.15) admits a unique solution $u \in C(0,\infty; H)$, and the solution of system (4.14) tends to zero as $t \to \infty$, and $\hat{d}(t)$ satisfies (3.6), $r(t)$ satisfies the condition of (3.5).

5. Numerical simulation

In this section, the finite difference method is applied to obtain computation of the displacement. We noticed the closed system (4.13) and (4.1) has the invertible transformation. And system (4.2) is equivalent to (4.1). The numerical simulation of system (4.2) is presented. The steps of space and time are taken as 0.05 and 0.001, respectively. The initial values are $V(0,x) = \sin(2\pi x), \tilde{y}(0) = 0, \tilde{d}(0) = 0$. From the practical view, we choose $d(t) = \sin t$,

$$r(t) = \begin{cases} 1 + \frac{2}{3}t, & t < 10.5, \\ 8, & t \geq 10.5. \end{cases}$$

Figure (1a) shows that system (4.2) is asymptotically stable under the time varying extended state observer. Figure (1b) shows that the time varying extended state observer is convergent.



(1a) (1b)

Figure 1. (1a) displacement of $V(x,t)$; (1b) the amplitude of error $\tilde{d}(t)$ (for interpretation of the estimation of disturbance)

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