Electronic Journal of Differential Equations, Vol. 2015 (2015), No. 73, pp. 1–19.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

EXISTENCE AND MULTIPLICITY OF SOLUTIONS TO OPERATOR EQUATIONS INVOLVING DUALITY MAPPINGS ON SOBOLEV SPACES WITH VARIABLE EXPONENTS

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Abstract. The aim of this article is to study the existence and multiplicity of solutions to operator equations involving duality mappings on Sobolev spaces with variable exponents. Our main tools are the well known Mountain Pass Theorem and its $\mathbb{Z}_2$-symmetric version.

1. Introduction

Our starting point for this article is the references [13, 12], where the existence of the weak solution for Dirichlet's problem with p-Laplacian (when p is a constant $1 < p < \infty$) was obtained using (among other methods) the Mountain Pass Theorem. It is well known that the p-Laplacian is in fact the duality mapping on $W_0^{1,p}(\Omega)$ corresponding to the gauge function $\varphi(t) = t^{p-1}$. In [7] some results from [12] are generalized considering operator equations with an arbitrary duality mapping on a real reflexive and smooth Banach space, compactly imbedded in $L^q(\Omega)$, where $1 < q < \infty$ and $\Omega \in \mathbb{R}^N$, $N \geq 2$, is a bounded domain with smooth boundary. In [6] the authors consider more general elliptic equations than those with p-Laplacian and prove the existence of nontrivial weak solutions of mountain type in an Orlicz-Sobolev space. Later, by using variational and topological methods, operator equations involving duality mappings on Orlicz-Sobolev spaces are studied in [16]. In [15] the multiplicity of solutions of operator equations involving duality mappings on a real reflexive and smooth Banach space, having the Kadeč-Klee property, compactly imbedded in a real Banach space has studied by using the $\mathbb{Z}_2$-symmetric version of the Mountain Pass Theorem. Equations of this type in Orlicz-Sobolev spaces are considered as applications.

In recent years there has been a great interest in the field of operator equations involving various forms of the $p(\cdot)$-Laplacian. The $p(\cdot)$-Laplacian is the operator $-\Delta_{p(\cdot)} : W_0^{1,p(\cdot)}(\Omega) \to (W_0^{1,p(\cdot)}(\Omega))^*$, $\Delta_{p(\cdot)}u := \operatorname{div}(|\nabla u|^{p(\cdot)-2}\nabla u)$ for $u \in W_0^{1,p(\cdot)}(\Omega)$. Many properties of the classical p-Laplacian may be recuperated except that of being a duality mapping on $W_0^{1,p(\cdot)}(\Omega)$. So, in this article, we will use a natural version of the $p(\cdot)$-Laplacian which is appropriate from the standpoint of

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2000 *Mathematics Subject Classification.* 35J60, 35B38, 47J30, 46E30.
Key words and phrases. Mountain Pass Theorem; duality mapping; critical point; Sobolev space with variable exponent.

Submitted September 18, 2014. Published March 24, 2015.

duality mappings (see [17] or [21, Section 9.3]): if φ is a gauge function, the $(\varphi, p(\cdot))$-Laplacian is the operator $-\Delta_{(\varphi,p(\cdot))} : W_0^{1,p(\cdot)}(\Omega) \to (W_0^{1,p(\cdot)}(\Omega))^*$, $-\Delta_{(\varphi,p(\cdot))}u := J_\varphi u$ for $u \in W_0^{1,p(\cdot)}(\Omega)$, where J_φ is the duality mapping on $W_0^{1,p(\cdot)}(\Omega)$, corresponding to the gauge function φ.

In particular, if $p(x)$ is constant and $\varphi(t) := t^{p-1}$, $t \geq 0$, then $\Delta_{(\varphi,p(\cdot))}$ coincides with Δ_p (see Remark 4.1 below).

The plan of this article is as follows. The main abstract result obtained in Section 2 is concerned with the existence of critical points of functional (2.1) defined on a real reflexive and smooth Banach space. The Mountain Pass Theorem and its $\mathbb{Z}_2$-symmetric version (see, e.g. Rabinowitz [23]) are the basic ingredients which are used.

Section 3 gathers various definitions and basic properties related to Lebesgue and Sobolev spaces with variable exponents, needed through the paper. The standard reference for the basic properties of variable exponent spaces is [19]. Additionally, the reader may also consult [8, 18]. Note that these spaces occur naturally in connection with various applications such as the modelling of electrorheological fluids [24].

Let Ω be a domain in $\mathbb{R}^N$, i.e. a bounded and connected open subset of $\mathbb{R}^N$ whose boundary $\partial\Omega$ is Lipschitz-continuous, the set Ω being locally on the same side of $\partial\Omega$. Consider the space

$$U_{\Gamma_0} = \{u \in W^{1,p(\cdot)}(\Omega) : u = 0 \text{ on } \Gamma_0 \subset \Gamma = \partial\Omega\},$$

where $\mathrm{d}\Gamma - \operatorname{meas}\Gamma_0 > 0$, with $p(\cdot) \in \mathcal{C}(\overline{\Omega})$ and $p(x) > 1$ for all $x \in \overline{\Omega}$. For details see [4, Section 2].

The main result of this article given in Section 4 and concerns the existence and multiplicity results for operator equation

$$J_\varphi u = N_g u, \tag{1.1}$$

where J_φ is a duality mapping on U_{Γ_0} corresponding to the gauge function φ. N_g is the Nemytskij operator generated by a Carathéodory function g satisfying an appropriate growth condition ensuring that N_g may be viewed as acting from U_{Γ_0} into its dual. In [10], the author used a topological method to prove the existence of the weak solution in $W_0^{1,p(\cdot)}(\Omega)$ for the problem $J_\varphi u = N_g u$. In [5], the existence of suitable solutions in U_{Γ_0} to equation (1.1) is proven by three different methods based, respectively, on reflexivity and smoothness of the space U_{Γ_0}, the Schauder fixed point theorem, and the Leray-Schauder degree.

All vector and function spaces considered in this paper are real. Given a normed vector space X, the notation X^* denotes its dual space and $\langle \cdot, \cdot \rangle_{X,X^*}$ designates the associated duality pairing. Often, we shall omit the spaces in duality and, simply write $\langle \cdot, \cdot \rangle$. Strong and weak convergence are denoted by $\to$ and $\rightharpoonup$, respectively.

2. An abstract result

The main result of this article is obtained via the following theorem.

Theorem 2.1. *Let X be a real reflexive and smooth Banach space, compactly imbedded in the real Banach space V with the compact injection $X \overset{i}{\hookrightarrow} V$. Let $H \in \mathcal{C}^1(X, \mathbb{R})$ be a functional given by*

$$H(u) := \Psi(u) - G(iu), \quad u \in X, \tag{2.1}$$

where:
(i) $\Psi : X \to \mathbb{R}$ *satisfies:*
(i.1) *at any* $u \in X$,
$$\Psi(u) := \Phi(\|u\|_X), \tag{2.2}$$
with
$$\Phi(t) := \int_0^t \varphi(\tau)\mathrm{d}\tau \quad \textit{for any } t \geq 0, \tag{2.3}$$
$\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *being a gauge function which satisfies*
$$\varphi^* := \sup_{t>0} \frac{t\varphi(t)}{\Phi(t)} < \infty. \tag{2.4}$$
(i.2) $\Psi' = J_\varphi$ *satisfies condition* $(S)_2$ *(see (2.8));*
(ii) $G : V \to \mathbb{R}$ *satisfies:*
(ii.0) $G(0_V) = 0$;
(ii.1) $G \in \mathcal{C}^1(V, \mathbb{R})$;
(ii.2) *there is a constant* $\theta > \varphi^*$ *such that, for any* $u \in V$,
$$\langle G'(u), u\rangle_{V,V^*} - \theta G(u) \geq C = const.; \tag{2.5}$$
(iii) *there exists* $c_0 > 0$ *such that for any* $u \in X$, *with* $\|u\|_X < c_0$, *one has*
$$H(u) > c_1\|u\|_X^p - c_2\|i(u)\|_V^q, \tag{2.6}$$
where i *stands for the compact injection of* X *in* V *while* $0 < p < q$ *and* $c_1 > 0$, $c_2 > 0$;
(iv) *for any finite dimensional subspace* $X_1 \subset X$, *there exist real constants* $d_0 > 0$, d_1, $d_2 > 0$, d_3, $s > 0$ *and* $r < s$ *(generally depending on* X_1*) such that*
$$H(u) \leq d_1\|u\|_X^r - d_2\|u\|_X^s + d_3, \tag{2.7}$$
for any $u \in X_1$ *with* $\|u\|_X > d_0$.
Then, the functional H *possesses a critical value. Moreover, if the functional* H *is even, then* H *has un unbounded sequence of critical values.*

Before proving of Theorem 2.1, we list some of the results to be used.

A function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be a *gauge* function if φ is continuous, strictly increasing, $\varphi(0) = 0$ and $\varphi(t) \to \infty$ as $t \to \infty$.

Firstly, we recall that a real Banach space X is said to be *smooth* if it has the following property: for any $x \in X$, $x \neq 0$, there exists a unique $u^*(x) \in X^*$ such that $\langle u^*(x), x\rangle = \|x\|_X$ and $\|u^*(x)\|_{X^*} = 1$. It is well known (see, for instance, Diestel [9], Zeidler [25]) that the smoothness of X is equivalent to the Gâteaux differentiability of the norm. Consequently, if $(X, \|\cdot\|_X)$ is smooth, then, for any $x \in X$, $x \neq 0$, the only element $u^*(x) \in X^*$ with the properties $\langle u^*(x), x\rangle = \|x\|_X$ and $\|u^*(x)\|_{X^*} = 1$ is $u^*(x) = \|\cdot\|'_X(x)$ (where $\|\cdot\|'_X(x)$ denotes the Gâteaux gradient of the $\|\cdot\|_X$-norm at x).

Secondly, if X is a real Banach space, the operator $T : X \to X^*$ is said to satisfy *condition* $(S)_2$ if
$$(S)_2 : \quad x_n \rightharpoonup x, \text{ and } Tx_n \to Tx \text{ imply } x_n \to x \text{ as } n \to \infty. \tag{2.8}$$
An operator T is said to satisfy *condition* $(S)_+$ if
$$(S)_+ : \quad x_n \rightharpoonup x \text{ and } \limsup_{n\to\infty}\langle Tx_n, x_n - x\rangle \leq 0 \text{ imply } x_n \to x \text{ as } n \to \infty.$$

It is known that if T satisfies condition $(S)_+$, then T satisfies condition $(S)_2$ (see Zeidler [25, p. 583]).

Let X be a real Banach space and let $H \in \mathcal{C}^1(X, \mathbb{R})$ be a functional. We say that H satisfies the *Palais-Smale condition* on X ((PS)-condition, for short) if any sequence $(u_n) \subset X$ with $(H(u_n))$ bounded and $H'(u_n) \to 0$ as $n \to \infty$, possesses a convergent subsequence. By (PS)-*sequence* for H we understand a sequence $(u_n) \subset X$ which satisfies $(H(u_n))$ is bounded and $H'(u_n) \to 0$ as $n \to \infty$.

The main tools used in proving Theorem 2.1 are the well known Mountain Pass Theorem and its $\mathbb{Z}_2$-symmetric version.

Theorem 2.2 ([23, Theorem 2.2]). *Let X be a real Banach space and let H belong to $\mathcal{C}^1(X, \mathbb{R})$ satisfying the (PS)-condition. Suppose that $H(0) = 0$ and that the following conditions hold:*

- (G1) *There exist $\rho > 0$ and $r > 0$ such that $H(u) \geq r$ for $\|u\| = \rho$;*
- (G2) *There exists $e \in X$ with $\|e\| > \rho$ such that $H(e) \leq 0$.*

Let

$$\Gamma = \{\gamma \in \mathcal{C}([0,1]; X) : \gamma(0) = 0, \gamma(1) = e\},$$
$$c = \inf_{\gamma \in \Gamma} \max_{0 \leq t \leq 1} H(\gamma(t)). \tag{2.9}$$

Then, H possesses a critical value $c > r$.

Theorem 2.3 ([23, Theorem 9.12]). *Let X be an infinite dimensional real Banach space and let $H \in \mathcal{C}^1(X, \mathbb{R})$ be even, satisfying the (PS)-condition, and $H(0) = 0$. Assume* (G1) *and*

- (G2') *for each finite dimensional subspace X_1 of X the set $\{u \in X_1 \mid H(u) \geq 0\}$ is bounded.*

Then H possesses an unbounded sequence of critical values.

Now, we show that under the assumptions of Theorem 2.1, the functional H has a mountain pass geometry. More precisely:

Proposition 2.4. *Let X be a real Banach space, imbedded in the real Banach space V, with the injection $X \overset{i}{\hookrightarrow} V$. Let $H \in \mathcal{C}^1(X, \mathbb{R})$ be given with $H(0) = 0$. Suppose that H satisfies the hypotheses* (iii) *and* (iv) *in Theorem 2.1. Then, the functional H satisfies the conditions* (G1), (G2), *and* (G2') *in Theorems 2.2 and 2.3.*

Proof. Indeed, let C be such that$\|i(u)\|_V \leq C\|u\|_X$, for any $u \in X$. According to [15, Theorem 1, p. 422], from (2.6) it follows that (G1) is satisfied with

$$0 < \rho < \min\Big(c_0, \big(\frac{c_1}{2C^q c_2}\big)^{1/(q-p)}\Big) \tag{2.10}$$

and $r = c_1 \rho^p / 2$.

Next we show that (G2) is also satisfied. Let X_1 be a finite dimensional subspace of X and let $e_0 \in X_1$ with $\|e_0\|_X > d_0$. Since for any $\lambda > 1$, one has $\|\lambda e_0\|_X > d_0$, it follows from (2.7) that,

$$H(\lambda e_0) \leq d_1 \lambda^r \|e_0\|_X^r - d_2 \lambda^s \|e_0\|_X^s + d_3. \tag{2.11}$$

Since, in general $s > r$, from (2.11) we deduce that $H(\lambda e_0) \to -\infty$ as $\lambda \to \infty$. Consequently, there exists a λ_0 such that, for $\lambda \geq \lambda_0$, $H(\lambda e_0) < 0$. Let $e := \lambda e_0$ with $\lambda > \max(1, \lambda_0, \rho/\|e_0\|_X)$, ρ being given by (2.10). Clearly with such a choice one has $\|e\|_X > \rho$ and $H(e) < 0$.

Finally, according to [15, Theorem 1, p. 422], from (2.7) it follows that (G2') is fulfilled. The proof is complete. □

To prove that the functional H satisfies the (PS)-condition, the following result will be useful.

Proposition 2.5 ([14, Corollary 1])**.** *Let X be a real reflexive Banach space, compactly imbedded in the real Banach space V and $H \in \mathcal{C}^1(X,\mathbb{R})$ be such that*

$$H'(u) = Su - Nu,$$

where $S : X \to X^*$ is monotone, hemicontinuous, satisfies condition $(S)_2$ and $N : V \to V^*$ is demicontinuous. Assume that any Palais-Smale sequence for H is bounded. Then H satisfies the (PS)-condition.

To apply Proposition 2.5, we recall that, if X is a real smooth Banach space and $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a gauge function, the duality mapping on X corresponding to φ is the mapping $J_\varphi : X \to X^*$ defined by

$$J_\varphi 0 := 0, \quad J_\varphi x := \varphi(\|x\|_X)\|\cdot\|'_X(x), \quad \text{if } x \neq 0.$$

The following result is standard in the theory of monotone operators (see, e.g. Browder [3], Zeidler [25]).

Proposition 2.6. *Let X be a real reflexive and smooth Banach space. Then, any duality mapping $J_\varphi : X \to X^*$ is:*

(a) *monotone ($\langle J_\varphi u - J_\varphi v, u - v\rangle \geq 0$, $u, v \in X$);*
(b) *demicontinuous ($x_n \to x \Rightarrow J_\varphi x_n \rightharpoonup J_\varphi x$).*

Since, generally, demicontinuity implies hemicontinuity, it follows that any duality mapping $J_\varphi : X \to X^*$ is hemicontinuous ($\langle J_\varphi(u+\lambda v), w\rangle \to \langle J_\varphi u, w\rangle$ as $\lambda \searrow 0$ for all $u, v, w \in X$). Consequently, from Proposition 2.5, we obtain the following result.

Corollary 2.7. *Let X be a real reflexive Banach space, compactly imbedded in the real Banach space V and $H \in \mathcal{C}^1(X,\mathbb{R})$ such that*

$$H'(u) = J_\varphi u - Nu,$$

where J_φ is a duality mapping corresponding to the gauge function φ, satisfying condition $(S)_2$ and $N : V \to V^*$ is demicontinuous. Assume that any Palais-Smale sequence for H is bounded. Then H satisfies the (PS)-condition.

Taking into account [14, Corollary 2, p. 897], we obtain

Corollary 2.8. *Let X be a real reflexive and smooth Banach space, compactly imbedded in the real Banach space V with the compact injection $X \overset{i}{\hookrightarrow} V$. Let $H \in \mathcal{C}^1(X,\mathbb{R})$ be a functional given by*

$$H(u) = \Psi(u) - G(iu), \quad u \in X,$$

where:

(i.1) *at any $u \in X$, $\Psi(u) = \Phi(\|u\|_X)$ with Φ given by (2.3), where $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a gauge function which satisfies (2.4);*
(i.2) *Ψ' satisfies condition $(\mathrm{S})_2$;*

(ii) $G : V \to \mathbb{R}$ *is* $\mathcal{C}^1$ *on* V *and satisfies: there is a constant* $\theta > \varphi^*$ *such that, at any* $u \in V$,

$$\langle G'(u), u\rangle_{V,V^*} - \theta G(u) \geq C = const.;$$

Then, the functional H *satisfies the* (PS)*-condition.*

Proof. The hypotheses of Corollary 2.7 are fulfilled with $N = G'$. Indeed, by Asplund's Theorem [2], $\Psi' = J_\varphi$ and, by hypothesis (i.2) J_φ satisfies condition $(S)_2$. The demicontinuity of G' is assumed by (ii.2). According to [14, Corollary 2, p. 897] we obtain that any (PS) sequence for H is bounded. □

Proof of Theorem 2.1. The assumptions of Theorem 2.1 entail the fulfillment of those of Corollary 2.8, therefore the functional H satisfies the (PS)-condition. According to Proposition 2.4, the functional H satisfies the conditions (G1), (G2), and (G2') from Theorems 2.2 and 2.3. Applying these theorems, the conclusions of Theorem 2.1 follow. □

3. Lebesgue and Sobolev spaces with variable exponent

The Lebesgue measure in $\mathbb{R}^N$ is denoted $\mathrm{d}x$. No distinction will be made between $\mathrm{d}x$-measurable functions and their equivalence classes modulo the relation of $\mathrm{d}x$-almost everywhere equality. The notation $\mathcal{D}(\Omega)$ denotes the space of functions that are infinitely differentiable in Ω and whose support is a compact subset of Ω.

The usual Lebesgue and Sobolev spaces, i.e., *with constant exponent* $p \geq 1$, are denoted $L^p(\Omega)$ and $W^{1,p}(\Omega)$.

Given a function $p(\cdot) \in L^\infty(\Omega)$ that satisfies

$$1 \leq p^- := \operatorname{ess\,inf}_{x\in\Omega} p(x) \leq p^+ := \operatorname{ess\,sup}_{x\in\Omega} p(x),$$

the Lebesgue space $L^{p(\cdot)}(\Omega)$ with variable exponent $p(\cdot)$ is defined as

$$L^{p(\cdot)}(\Omega) := \{v : \Omega \to \mathbb{R}; v \text{ is } \mathrm{d}x\text{-measurable and } \rho_{0,p(\cdot)}(v) := \int_\Omega |v(x)|^{p(x)}\mathrm{d}x < \infty\},$$

where $\rho_{0,p(\cdot)}(v)$ is called the *convex modular* of v.

Theorem 3.1. *Let* Ω *be a domain in* $\mathbb{R}^N$.
(a) *Let* $p(\cdot) \in L^\infty(\Omega)$ *be such that* $p^- \geq 1$. *Equipped with the norm*

$$v \in L^{p(\cdot)}(\Omega) \to \|v\|_{0,p(\cdot)} := \inf\{\lambda > 0;\ \int_\Omega |\frac{v(x)}{\lambda}|^{p(x)}\mathrm{d}x \leq 1\},$$

the space $L^{p(\cdot)}(\Omega)$ *is a separable Banach space. If* $p^- > 1$, *the space* $L^{p(\cdot)}(\Omega)$ *is uniformly convex, hence reflexive.*
(b) *Let* $p_1(\cdot) \in L^\infty(\Omega)$ *and* $p_2(\cdot) \in L^\infty(\Omega)$ *be such that* $p_1^- \geq 1$ *and* $p_2^- \geq 1$. *Then*

$$L^{p_2(\cdot)}(\Omega) \hookrightarrow L^{p_1(\cdot)}(\Omega)$$

if and only if

$$p_1(x) \leq p_2(x) \quad \text{for almost all } x \in \Omega.$$

(c) *For any* $u \in L^{p(\cdot)}(\Omega)$ *with* $p(\cdot) \in L^\infty(\Omega)$ *satisfying* $p^- > 1$ *and* $v \in L^{p'(\cdot)}(\Omega)$,

$$\int_\Omega |u(x)v(x)|dx \leq \Big(\frac{1}{p^-} + \frac{1}{(p')^-}\Big)\|u\|_{0,p(\cdot)}\|v\|_{0,p'(\cdot)}\,. \tag{3.1}$$

Remark 3.2 ([18, p. 430]). *If* $p(x)$ *is constant, then the space* $L^{p(\cdot)}(\Omega)$ *coincides with the classical Lebesgue space* $L^p(\Omega)$ *and the norms on these spaces are equal.*

The next theorem sums up the relations between the norm $\|\cdot\|_{0,p(\cdot)}$ and the convex modular $\rho_{0,p(\cdot)}$. Its proof can be found in [18].

Theorem 3.3. *Let $p(\cdot) \in L^\infty(\Omega)$ be such that $p^- \geq 1$ and let $u \in L^{p(\cdot)}(\Omega)$. The following properties hold:*

(a) *If $u \neq 0$, then $\|u\|_{0,p(\cdot)} = a$ if and only if $\rho_{0,p(\cdot)}(a^{-1}u) = 1$.*
(b) *$\|u\|_{0,p(\cdot)} < 1$ (resp. $= 1$ or > 1) if and only if $\rho_{0,p(\cdot)}(u) < 1$ (resp. $= 1$, or > 1).*
(c) *$\|u\|_{0,p(\cdot)} > 1$ implies $\|u\|_{0,p(\cdot)}^{p^-} \leq \rho_{0,p(\cdot)}(u) \leq \|u\|_{0,p(\cdot)}^{P^+}$.*
(d) *$\|u\|_{0,p(\cdot)} < 1$ implies $\|u\|_{0,p(\cdot)}^{p^+} \leq \rho_{0,p(\cdot)}(u) \leq \|u\|_{0,p(\cdot)}^{p^-}$.*

The Sobolev space $W^{1,p(\cdot)}(\Omega)$ with variable exponent $p(\cdot)$ is defined as

$$W^{1,p(\cdot)}(\Omega) := \{v \in L^{p(\cdot)}(\Omega) : \partial_i v \in L^{p(\cdot)}(\Omega), 1 \leq i \leq N\},$$

where, for each $1 \leq i \leq N$, ∂_i denotes the distributional derivative operator with respect to the i-th variable.

Theorem 3.4. *Let Ω be a domain in $\mathbb{R}^N$.*
(a) *Let $p(\cdot) \in L^\infty(\Omega)$ be such that $p^- \geq 1$. Equipped with the norm*

$$v \in W^{1,p(\cdot)}(\Omega) \to \|v\|_{1,p(\cdot)} := \|v\|_{0,p(\cdot)} + \textstyle\sum_{i=1}^N \|\partial_i v\|_{0,p(\cdot)},$$

the space $W^{1,p(\cdot)}(\Omega)$ is a separable Banach space. If $p^- > 1$, the space $W^{1,p(\cdot)}(\Omega)$ is reflexive.
(b) *Let $p_1(\cdot) \in L^\infty(\Omega)$ with $p_1^- \geq 1$ and $p_2(\cdot) \in L^\infty(\Omega)$ with $p_2^- \geq 1$ be such that*

$$p_1(x) \leq p_2(x) \text{ for almost all } x \in \Omega.$$

Then

$$W^{1,p_2(\cdot)}(\Omega) \hookrightarrow W^{1,p_1(\cdot)}(\Omega).$$

(c) *Let $p(\cdot) \in \mathcal{C}(\overline{\Omega})$ be such that $p^- \geq 1$. Given any $x \in \overline{\Omega}$, let*

$$p^*(x) := \frac{Np(x)}{N - p(x)} \text{ if } p(x) < N, \quad \text{and} \quad p^*(x) := \infty \text{ if } p(x) \geq N, \tag{3.2}$$

and let $q(\cdot) \in \mathcal{C}(\overline{\Omega})$ be a function that satisfies

$$1 \leq q(x) < p^*(x) \quad \text{for each } x \in \overline{\Omega}. \tag{3.3}$$

Then the following compact injection holds:

$$W^{1,p(\cdot)}(\Omega) \Subset L^{q(\cdot)}(\Omega),$$

so that, in particular, $W^{1,p(\cdot)}(\Omega) \Subset L^{p(\cdot)}(\Omega)$.
(d) *The function defined by*

$$v \in W^{1,p(\cdot)}(\Omega) \to \|v\|_{1,p(\cdot),\nabla} := \|v\|_{0,p(\cdot)} + \||\nabla v|\|_{0,p(\cdot)},$$

is a norm on $W^{1,p(\cdot)}(\Omega)$, equivalent with the norm $\|\cdot\|_{1,p(\cdot)}$.

The following theorem concerns the definition of the space U_{Γ_0} ([4, Theorem 6]).

Theorem 3.5. *Let Ω be a domain in $\mathbb{R}^N$, $N \geq 2$, let Γ_0 be a $d\Gamma$-measurable subset of $\Gamma = \partial\Omega$ that satisfies $d\Gamma - \operatorname{meas} \Gamma_0 > 0$, let $p(\cdot) \in \mathcal{C}(\overline{\Omega})$ be such that $p(x) > 1$ for all $x \in \overline{\Omega}$ and let*

$$U_{\Gamma_0} := \{u \in (W^{1,p(\cdot)}(\Omega), \|\cdot\|_{1,p(\cdot),\nabla}) : \operatorname{tr} u = 0 \text{ on } \Gamma_0\}.$$

Then:
(a) *The space* U_{Γ_0} *is closed in* $(W^{1,p(\cdot)}(\Omega), \|\cdot\|_{1,p(\cdot),\nabla})$*; hence* $(U_{\Gamma_0}, \|\cdot\|_{1,p(\cdot),\nabla})$ *is a separable reflexive Banach space.*
(b) *The map*

$$u \in U_{\Gamma_0} \to \|u\|_{0,p(\cdot),\nabla} := \||\nabla u|\|_{0,p(\cdot)} \tag{3.4}$$

is a norm on U_{Γ_0} *equivalent with the norm* $\|\cdot\|_{1,p(\cdot),\nabla}$.
(c) *The norm* $\|u\|_{0,p(\cdot),\nabla}$ *is Fréchet-differentiable at any nonzero* $u \in U_{\Gamma_0}$ *and the Fréchet-differential of this norm at any nonzero* $u \in U_{\Gamma_0}$ *is given for any* $h \in U_{\Gamma_0}$ *by*

$$\langle \|\cdot\|'_{0,p(\cdot),\nabla}(u), h\rangle = \frac{\int_{\Omega\setminus\Omega_{0,u}} p(x)\frac{|\nabla u(x)|^{p(x)-2}\,\langle \nabla u(x), \nabla h(x)\rangle}{\|u\|_{0,p(\cdot),\nabla}^{p(x)-1}}\mathrm{d}x}{\int_\Omega p(x)\frac{|\nabla u(x)|^{p(x)}}{\|u\|_{0,p(\cdot),\nabla}^{p(x)}}\mathrm{d}x},$$

where $\Omega_{0,u} := \{x \in \Omega; |\nabla u(x)| = 0\}$.

By Theorem 3.4 (c) and Theorem 3.5 (a)–(b) we derive the following result.

Lemma 3.6. *Let* $p(\cdot) \in \mathcal{C}(\overline{\Omega})$ *be such that* $p^- \geq 1$. *Given any* $x \in \overline{\Omega}$, *let* p^* *be given by* (3.2) *and let* $q(\cdot) \in \mathcal{C}(\overline{\Omega})$ *be a function that satisfies* (3.3). *Then the following compact inclusion holds:*

$$\big(U_{\Gamma_0}, \|\cdot\|_{1,p(\cdot),\nabla}\big) \Subset \big(L^{q(\cdot)}(\Omega), \|\cdot\|_{0,q(\cdot)}\big).$$

Remark 3.7. If $\varphi^* < q^-$, then $L^{q(\cdot)}(\Omega) \hookrightarrow L^{\varphi^*}(\Omega)$, therefore U_{Γ_0} is compactly imbedded in $L^{\varphi^*}(\Omega)$.

The above remark will be useful in the upcoming section.

Proposition 3.8 ([11, Proposition 4])**.** *Let* X *be a real reflexive Banach space, compactly embedded in the real Banach space* Z. *Denote by* i *the compact injection of* X *into* Z *and, for any* $r \in [1,\infty)$, *define*

$$\lambda_{1,r} = \inf\{\frac{\|u\|_X^r}{\|i(u)\|_Z^r} \mid u \in X\setminus\{0_X\}\}.$$

Then, $\lambda_{1,r}$ *is attained and* $\lambda_{1,r}^{-1/r}$ *is the best constant* c_Z *in the writing of the imbedding of* X *into* Z*:*

$$\|i(u)\|_Z \leq c_Z\|u\|_X, \quad \text{for all } u \in X.$$

Taking into account Remark 3.7, we obtain the following result.

Corollary 3.9. *Let* Ω *be a domain in* $\mathbb{R}^N$ $(N \geq 2)$, *let* $p \in \mathcal{C}(\overline{\Omega})$ *and* $q \in \mathcal{C}(\overline{\Omega})$ *be two functions such that* $p^- > 1$, $q^- > 1$ *and* (3.3) *holds. For* $\varphi^* < q^-$ *define*

$$\lambda_{1,\varphi^*} := \inf\Big\{\frac{\|u\|_{0,p(\cdot),\nabla}^{\varphi^*}}{\|u\|_{L^{\varphi^*}(\Omega)}^{\varphi^*}} : u \in U_{\Gamma_0}\setminus\{0\}\Big\}, \tag{3.5}$$

where i *is the compact injection* $i : U_{\Gamma_0} \to L^{\varphi^*}(\Omega)$. *Then* λ_{1,φ^*} *is attained and* $\lambda_{1,\varphi^*}^{-1/\varphi^*}$ *is the best constant* c *in the imbedding of* U_{Γ_0} *in* $L^{\varphi^*}(\Omega)$, *namely,*

$$\|i(u)\|_{L^{\varphi^*}(\Omega)} \leq c\|u\|_{0,p(\cdot),\nabla} \quad \text{for all } u \in U_{\Gamma_0}.$$

4. Main result

In this section we study the existence and multiplicity of weak solutions for the boundary value problem

$$J_\varphi u = g(x,u) \quad \text{in } \Omega, \tag{4.1}$$

$$u = 0 \quad \text{on } \Gamma_0 \subset \partial\Omega, \tag{4.2}$$

in the following framework:

• $J_\varphi : \big(U_{\Gamma_0}, \|\cdot\|_{0,p(\cdot),\nabla}\big) \to \big(U_{\Gamma_0}, \|\cdot\|_{0,p(\cdot),\nabla}\big)^*$ is the duality mapping on $(U_{\Gamma_0}, \|\cdot\|_{0,p(\cdot),\nabla})$ subordinated to the gauge function φ: such that $J_\varphi 0 = 0$, and

$$\langle J_\varphi u, h\rangle = \varphi(\|u\|_{0,p(\cdot),\nabla}) \frac{\int_{\Omega\setminus\Omega_{0,u}} p(x) \frac{|\nabla u(x)|^{p(x)-2}\langle \nabla u(x), \nabla h(x)\rangle}{\|u\|_{0,p(\cdot),\nabla}^{p(x)-1}} \mathrm{d}x}{\int_\Omega p(x) \frac{|\nabla u(x)|^{p(x)}}{\|u\|_{0,p(\cdot),\nabla}^{p(x)}} \mathrm{d}x},$$

at any nonzero $u \in U_{\Gamma_0}$, for any $h \in U_{\Gamma_0}$ (here $\Omega_{0,u} := \{x \in \Omega : |\nabla u(x)| = 0\}$).

• $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function.

Remark 4.1. By Remark 3.2, if $p(x)$ is constant on Ω, then $\|u\|_{0,p(\cdot)} = \|u\|_{L^p(\Omega)}$, and

$$\int_\Omega p(x) \frac{|\nabla u(x)|^{p(x)}}{\|u\|_{0,p(\cdot),\nabla}^{p(x)}} \mathrm{d}x = p;$$

therefore,

$$\langle J_\varphi u, h\rangle = \varphi(\|u\|_{0,p(\cdot),\nabla}) \frac{\int_{\Omega\setminus\Omega_{0,u}} |\nabla u(x)|^{p-2} \ \langle \nabla u(x), \nabla h(x)\rangle \mathrm{d}x}{\|u\|_{L^p(\Omega)}^{p-1}}.$$

Moreover, if $\varphi(t) = t^{p-1}$, $t \geq 0$, we obtain that

$$\langle J_\varphi u, h\rangle = \int_{\Omega\setminus\Omega_{0,u}} |\nabla u(x)|^{p-2}\langle \nabla u(x), \nabla h(x)\rangle \mathrm{d}x;$$

that is,

$$\langle J_\varphi u, h\rangle = \langle -\Delta_p u, h\rangle.$$

Consequently, in this case equation (4.1) can be rewritten as

$$-\Delta_p u = g(x,u) \quad \text{in } \Omega.$$

By a (weak) solution to the problem (4.1), (4.2) we understand a solution to the equation

$$J_\varphi u = N_g u, \tag{4.3}$$

N_g being the Nemytskij operator generated by g.

Our goal is to prove the main result of this paper.

Theorem 4.2. *Let Ω be a domain in $\mathbb{R}^N$ $(N \geq 2)$, let $p \in \mathcal{C}(\overline{\Omega})$ be a function such that $p^- > 1$, and let $p^*(\cdot)$ be given by (3.2). Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a gauge function which satisfies (2.4), where Φ is given by (2.3). Let there be given a Carathéodory function $g : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfying the hypotheses:*

(H1) *there exists a function $q(\cdot) \in \mathcal{C}(\overline{\Omega})$ that satisfies (3.3) such that*

$$|g(x,s)| \leq C_1 |s|^{q(x)/q'(x)} + a(x) \quad \textit{for almost all } x \in \Omega \textit{ and all } s \in \mathbb{R}, \tag{4.4}$$

where $\frac{1}{q(x)} + \frac{1}{q'(x)} = 1$, a is a bounded function, $a(x) \geq 0$ for almost all $x \in \Omega$, and C_1 is a constant, $C_1 > 0$;

(H2) *there exist* $s_0 > 0$ *and* $\theta > \varphi^* := \sup_{t>0} \frac{t\varphi(t)}{\Phi(t)}$ *such that*

$$0 < \theta G(x,s) \le s g(x,s), \tag{4.5}$$

for almost every $x \in \Omega$ *and all* s *with* $|s| \ge s_0$*, where*

$$G(x,s) := \textstyle\int_0^s g(x,\tau)\mathrm{d}\tau. \tag{4.6}$$

Also assume that

(H3)

$$\limsup_{s\to 0} \frac{g(x,s)}{|s|^{\varphi^*-2}s} < \frac{\varphi^*\Phi(1)}{2}\lambda_{1,\varphi^*} \tag{4.7}$$

uniformly with respect to almost all $x \in \Omega$*, where* λ_{1,φ^*} *is given by* (3.5)*.*

(H4) $\varphi^* < q^-$*.*

Let $N_g : L^{q(\cdot)}(\Omega) \to L^{q'(\cdot)}(\Omega)$*, with* $(N_g u)(x) = g(x,u(x))$ *for almost all* $x \in \Omega$*, denote the Nemytskij operator generated by* g*.*

Then under these assumptions, problem (4.1)*,* (4.2) *has a weak non-trivial solution in the space* U_{Γ_0} *(endowed with the norm* (3.4)*). Moreover, if* g *is odd in the second argument:* $g(x,-s) = -g(x,s)$*,* $s \in \mathbb{R}$*, then the problem* (4.1)*,* (4.2) *has a sequence of weak solutions.*

To prove this theorem, we apply Theorem 2.1 to the functional $H : U_{\Gamma_0} \to \mathbb{R}$,

$$H(u) := \Phi(\|u\|_{0,p(\cdot),\nabla}) - \mathcal{G}(u), \tag{4.8}$$

where

$$\mathcal{G}(u) := \int_\Omega G(x,u(x))\mathrm{d}x. \tag{4.9}$$

Proposition 4.3. *Under the hypotheses of Theorem 4.2, the functional* H *given by* (4.8)*, is well-defined and* $\mathcal{C}^1$ *on* U_{Γ_0}*, with*

$$H'(u) = J_\varphi(u) - g(x,u)$$

Proof. The well-definedness of functional H is reduced to proving that for any $u \in U_{\Gamma_0}$, $\int_\Omega G(x,u(x))\mathrm{d}x$ makes sense. Indeed, by using (4.4) it follows that

$$|G(x,s)| \le \frac{C_1}{q^-}|s|^{q(x)} + a(x)|s|. \tag{4.10}$$

Thus

$$\int_\Omega G(x,u(x))\mathrm{d}x \le \frac{C_1}{q^-}\int_\Omega |u(x)|^{q(x)}\mathrm{d}x + \int_\Omega a(x)|u(x)|\mathrm{d}x.$$

Since, for any $u \in U_{\Gamma_0}$, we have $u \in L^{q(\cdot)}(\Omega)$ and $a \in L^{q'(\cdot)}(\Omega)$, it follows that $\int_\Omega a(x)|u(x)|\mathrm{d}x$ makes sense. Consequently $\int_\Omega G(x,u(x))\mathrm{d}x < \infty$.

Now, we show that $H \in \mathcal{C}^1$ over U_{Γ_0}. First, we will prove that $\Psi : U_{\Gamma_0} \to \mathbb{R}$, $\Psi(u) := \Phi(\|u\|_{1,p(\cdot),\nabla})$, is $\mathcal{C}^1$ over U_{Γ_0}. Indeed, according to [4, Theorem 6], Ψ is continuously Fréchet differentiable at any nonzero $u \in U_{\Gamma_0}$ and, for any $h \in U_{\Gamma_0}$ one has

$$\langle \Psi'(u), h\rangle = \varphi(\|u\|_{0,p(\cdot),\nabla})\frac{\int_{\Omega\setminus\Omega_{0,u}} p(x)\frac{|\nabla u(x)|^{p(x)-2}\langle \nabla u(x), \nabla h(x)\rangle}{\|u\|_{0,p(\cdot),\nabla}^{p(x)-1}}\mathrm{d}x}{\int_\Omega p(x)\frac{|\nabla u(x)|^{p(x)}}{\|u\|_{0,p(\cdot),\nabla}^{p(x)}}\mathrm{d}x},$$

where $\Omega_{0,u} := \{x \in \Omega; |\nabla u(x)| = 0\}$.

If $u = 0$, then a direct calculus shows that Ψ is Gâteaux differentiable at zero and

$$\langle \Psi'(0), h\rangle = \lim_{t\to 0} t^{-1}\Phi(|t|\|h\|_{0,p(\cdot),\nabla}) = \lim_{t\to 0} \varphi(|t|\|h\|_{0,p(\cdot),\nabla}) sgn\ t\|h\|_{0,p(\cdot),\nabla} = 0.$$

Moreover, $u \to \Psi'(u)$ is continuous at zero. Indeed, from Theorem 3.3 (b), we obtain

$$\int_\Omega p(x)\frac{|\nabla u(x)|^{p(x)}}{\|u\|_{0,p(\cdot),\nabla}^{p(x)}}\mathrm{d}x \geq p^-\rho_{p(\cdot)}\Big(\frac{|\nabla u|}{\|u\|_{0,p(\cdot),\nabla}}\Big) = p^-. \tag{4.11}$$

On the other hand, by using Schwarz's inequality for nonnegative bilinear symmetric forms and inequality (3.1), it follows that

$$\begin{aligned}
&\Big|\int_{\Omega\setminus\Omega_{0,u}} p(x)\frac{|\nabla u(x)|^{p(x)-2}\langle\nabla u(x),\nabla h(x)\rangle}{\|u\|_{0,p(\cdot),\nabla}^{p(x)-1}}\mathrm{d}x\Big| \\
&\leq p^+\int_\Omega\Big(\frac{|\nabla u(x)|}{\|u\|_{0,p(\cdot),\nabla}}\Big)^{p(x)-1}|\nabla h(x)|\mathrm{d}x \\
&\leq M\||\nabla h|\|_{0,p(\cdot)}\Big\|\Big(\frac{|\nabla u|}{\|u\|_{0,p(\cdot),\nabla}}\Big)^{p(\cdot)-1}\Big\|_{0,p'(\cdot)} \\
&= M\|h\|_{0,p(\cdot),\nabla}\Big\|\Big(\frac{|\nabla u|}{\|u\|_{0,p(\cdot),\nabla}}\Big)^{p(\cdot)-1}\Big\|_{0,p'(\cdot)},
\end{aligned} \tag{4.12}$$

where $M = p^+\cdot(\frac{1}{p^-}+\frac{1}{p'^-})$. Since

$$\rho_{p'(\cdot)}\Big(\Big(\frac{|\nabla u|}{\|u\|_{0,p(\cdot),\nabla}}\Big)^{p(\cdot)-1}\Big) = \rho_{p(\cdot)}\Big(\frac{|\nabla u|}{\|u\|_{0,p(\cdot),\nabla}}\Big) = 1,$$

by Theorem 3.3 (b) we have

$$\Big\|\Big(\frac{|\nabla u|}{\|u\|_{0,p(\cdot),\nabla}}\Big)^{p(\cdot)-1}\Big\|_{0,p'(\cdot)} = 1\,,$$

therefore, from (4.12) we obtain

$$\Big|\int_{\Omega\setminus\Omega_{0,u}} p(x)\frac{|\nabla u(x)|^{p(x)-2}\ \nabla u(x)\cdot\nabla h(x)}{\|u\|_{0,p(\cdot),\nabla}^{p(x)-1}}\mathrm{d}x\Big| \leq M\|h\|_{0,p(\cdot),\nabla}\,.$$

From (4.11) and (4.12) we infer that

$$|\langle\Psi'(u),h\rangle| \leq \frac{M}{p^-}\cdot\varphi(\|u\|_{0,p(\cdot),\nabla})\cdot\|h\|_{0,p(\cdot),\nabla}\,,$$

for any nonzero $u\in U_{\Gamma_0}$ and for any $h\in U_{\Gamma_0}$. Thus

$$\|\Psi'(u)\| \leq \frac{M}{p^-}\varphi(\|u\|_{0,p(\cdot),\nabla}) \to 0 \quad \text{as } \|u\|_{0,p(\cdot),\nabla}\to 0\,;$$

therefore Ψ is $\mathcal{C}^1$. To conclude that H is $\mathcal{C}^1$, the $\mathcal{C}^1$-property of the functional $\mathcal{G}$ given by (4.9), has to be proven.

As far as the $\mathcal{C}^1$-regularity of $\mathcal{G}$ is concerned, for a later use, we shall prove more: $\mathcal{G}$ is $\mathcal{C}^1$ on $L^{q(\cdot)}(\Omega)$ and

$$\langle\mathcal{G}'(u),h\rangle = \textstyle\int_\Omega g(x,u(x))h(x)\mathrm{d}x, u,h\in L^{q(\cdot)}(\Omega). \tag{4.13}$$

Indeed, let $u, h \in L^{q(\cdot)}(\Omega)$. According to [20, p. 178] and by using Hölder's type inequality (3.1),

$$\begin{aligned}
&|\mathcal{G}(u+h) - \mathcal{G}(u) - \langle \mathcal{G}'(u), h\rangle| \\
&= \Big| \int_\Omega [g(x, u(x) + \theta(x)h(x))h(x) - g(x, u(x))h(x)]\mathrm{d}x\Big| \\
&\leq M\|g(x, u(x) + \theta(x)h(x)) - g(x, u(x))\|_{0,q'(\cdot)}\|h\|_{0,q(\cdot)},
\end{aligned}$$

where $0 \leq \theta(x) \leq 1$. Consequently,

$$\frac{|\mathcal{G}(u+h) - \mathcal{G}(u) - \langle \mathcal{G}'(u), h\rangle|}{\|h\|_{0,q(\cdot)}} \leq M\|g(x, u(x) + \theta(x)h(x)) - g(x, u(x))\|_{0,q'(\cdot)}.$$

Suppose $\|h\|_{0,q(\cdot)} \to 0$. Taking into account the continuity of Nemytskij operators [18, Theorem 1.16], it follows that $\mathcal{G}$ is Fréchet differentiable on $L^{q(\cdot)}(\Omega)$ and $\mathcal{G}'$ is given by (4.13).

Moreover, the operator $\mathcal{G}' : L^{q(\cdot)}(\Omega) \to (L^{q(\cdot)}(\Omega))^*$ given by (4.13) is continuous [18, Theorem 1.16].

Now, since U_{Γ_0} is continuously imbedded in $L^{q(\cdot)}(\Omega)$ and $\mathcal{G}$ is $\mathcal{C}^1$ on $L^{q(\cdot)}(\Omega)$, it follows that $\mathcal{G}$ is $\mathcal{C}^1$ on U_{Γ_0}. □

Proposition 4.4. *Let $q \in \mathcal{C}_+(\overline{\Omega})$ and $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function which satisfies the growth condition* (4.4) *and the hypothesis* (H2) *modified as follows: there exist $s_0 > 0$ and $\theta > 0$ such that* (4.5) *holds for almost all $x \in \Omega$ and all s with $|s| \geq s_0$, where $\mathcal{G}$ is given by* (4.9)*. Then, the functional $\mathcal{G} : L^{q(\cdot)}(\Omega) \to \mathbb{R}$ given by* (4.9) *satisfies the inequality* (2.5)*.*

Proof. One has

$$\langle \mathcal{G}'(u), u\rangle - \theta\mathcal{G}(u) = \int_\Omega [g(x, u(x))u(x) - \theta G(x, u(x))]\mathrm{d}x.$$

Now, we shall give an estimation for the right term of this equality. Define $\overline{\Omega} = \{x \in \Omega : |u(x)| > s_0\}$. Taking into account (4.5), one has

$$\int_{\overline{\Omega}} [g(x, u(x))u(x) - \theta G(x, u(x))]\mathrm{d}x \geq 0. \tag{4.14}$$

Also, considering (4.10), one has

$$\begin{aligned}
\Big|\int_{\Omega\setminus\overline{\Omega}} G(x, u(x))\mathrm{d}x\Big| &\leq \int_{\Omega\setminus\overline{\Omega}} [c|u(x)|^{q(x)} + |u(x)|a(x)]\mathrm{d}x \\
&\leq cs_0^{q^+}\operatorname{vol}(\Omega) + s_0\int_\Omega a(x)\mathrm{d}x = K,
\end{aligned}$$

where $c := C_1/q^-$.

On the other hand, from (4.4), it follows that

$$\begin{aligned}
\Big|\int_{\Omega\setminus\overline{\Omega}} g(x, u(x))u(x)\mathrm{d}x\Big| &\leq \int_{\Omega\setminus\overline{\Omega}} [c|u(x)|^{q(x)} + |u(x)|a(x)]\mathrm{d}x \\
&\leq cs_0^{q^+}\operatorname{vol}(\Omega) + s_0\int_\Omega a(x)\mathrm{d}x = K.
\end{aligned}$$

Thus

$$\Big|\int_{\Omega\setminus\overline{\Omega}} [g(x, u(x))u(x) - \theta G(x, u(x))]\mathrm{d}x\Big| \leq C, \tag{4.15}$$

with $C := K(1+\theta)$. From (4.14) and (4.15), we infer that

$$\int_\Omega [g(x,u(x))u(x) - \theta G(x,u(x))]\mathrm{d}x \geq -C,$$

that is (2.5). $\square$

Using the same arguments as in [16, Remark 7.2, p. 26], we obtain the following result.

Lemma 4.5. *Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a gauge function which satisfies* (2.4)*, where Φ is given by* (2.3)*. Then, for all $u \in U_{\Gamma_0}$ with $\|u\|_{0,p(\cdot),\nabla} < 1$ one has*

$$\Phi(\|u\|_{0,p(\cdot),\nabla}) \geq \Phi(1)\|u\|_{0,p(\cdot),\nabla}^{\varphi^*}. \tag{4.16}$$

Also for all $u \in U_{\Gamma_0}$ with $\|u\|_{0,p(\cdot),\nabla} > 1$ one has

$$\Phi(\|u\|_{0,p(\cdot),\nabla}) \leq \Phi(1)\|u\|_{0,p(\cdot),\nabla}^{\varphi^*}.$$

Proof of Theorem 4.2. We use Theorem 2.1 with $X = U_{\Gamma_0}$ and $V = L^{q(\cdot)}(\Omega)$. Indeed, X is reflexive (Theorem 3.5, (a)) and smooth (Theorem 3.5 (c)). Also, by Theorem 3.5 (a) and Theorem 3.4, (c) $(U_{\Gamma_0}, \|\cdot\|_{0,p(\cdot),\nabla})$ is compactly embedded in $(L^{q(\cdot)}(\Omega), \|\cdot\|_{0,q(\cdot)})$. According to [5, Theorem 4.6 a)], Ψ' satisfies condition $(S)_2$.

Obviously $\mathcal{G}(0) = 0$ and taking into account Propositions 4.3 and 4.4, it follows that $\mathcal{G}$ is $\mathcal{C}^1$ and that the hypothesis (ii) of Theorem 2.1 is fulfilled.

Let us prove that hypothesis (iii) of Theorem 2.1 is fulfilled. For the first term in (4.8), we have (4.16) for all $u \in U_{\Gamma_0}$ with $\|u\|_{0,p(\cdot),\nabla} < 1$.

Arguing as in [12, p. 239], from (H3) we deduce that there exists

$$0 < \mu < (\varphi^*\Phi(1)/2)\lambda_{1,\varphi^*} \tag{4.17}$$

and $\underline{s} > 0$ such that

$$G(x,s) < (\mu/\varphi^*)|s|^{\varphi^*}, \quad \text{for } x \in \Omega, 0 < |s| < \underline{s}. \tag{4.18}$$

Now, let us consider $|s| \in [\underline{s}, \infty)$. The function $|s|^{q(x)-1}$ being increasing as function of $|s|$, we have

$$|s| \leq \frac{1}{\underline{s}^{q(x)-1}}|s|^{q(x)}.$$

Since the function a in (4.4) is assumed to be bounded, it follows from (4.10) that

$$|G(x,s)| \leq c_3 \cdot s^{q(x)}, \quad \text{for } |s| \geq \underline{s}\,,$$

where $c_3 := C_1/q^- + \|a\|_\infty/\underline{s}^{q^- -1}$.

Now, we denote $\underline{\Omega} = \{x \in \Omega : |u(x)| \geq \underline{s}\}$. Then, for every $u \in L^{q(\cdot)}(\Omega)$, we have

$$\int_{\underline{\Omega}} G(x,u(x))\mathrm{d}x \leq c_3 \int_\Omega |u(x)|^{q(x)}\mathrm{d}x. \tag{4.19}$$

But U_{Γ_0} is continuously imbedded in $L^{q(\cdot)}(\Omega)$ (Lemma 3.6), therefore there exists a positive constant $\underline{c}$ such that

$$\|u\|_{0,q(\cdot)} \leq \underline{c}\|u\|_{0,p(\cdot),\nabla} \quad \text{for all } u \in U_{\Gamma_0}.$$

Consequently, for all $u \in U_{\Gamma_0}$ with $\|u\|_{0,p(\cdot),\nabla} < 1/\underline{c}$ it follows that $\|u\|_{0,q(\cdot)} < 1$. Therefore, taking into account (4.19) and Theorem 3.3 (d), we obtain

$$\int_{\underline{\Omega}} G(x,u(x))\mathrm{d}x \leq c_3\|u\|_{0,q(\cdot)}^{q^-}, \tag{4.20}$$

for all $u \in U_{\Gamma_0}$ with $\|u\|_{0,p(\cdot),\nabla} < 1/\underline{c}$.

On the other hand, from (4.18), for $u \in U_{\Gamma_0}$, we deduce

$$\int_{\Omega\setminus\underline{\Omega}} G(x,u(x))\mathrm{d}x \le \frac{\mu}{\varphi^*}\int_{\Omega}|u(x)|^{\varphi^*}\mathrm{d}x = \frac{\mu}{\varphi^*}\|u\|^{\varphi^*}_{L^{\varphi^*}(\Omega)}. \tag{4.21}$$

Since $\varphi^* < q^-$, then U_{Γ_0} is compactly imbedded in $L^{\varphi^*}(\Omega)$ (Remark 3.7). Taking into account (4.21), (4.17), and the definition (3.5) of λ_{1,φ^*}, for $u \in U_{\Gamma_0}$, we obtain

$$\int_{\Omega\setminus\underline{\Omega}} G(x,u(x))\mathrm{d}x \le \frac{\mu}{\varphi^*\lambda_{1,\varphi^*}}\|u\|^{\varphi^*}_{0,p(\cdot),\nabla} \le \frac{\Phi(1)}{2}\|u\|^{\varphi^*}_{0,p(\cdot),\nabla}. \tag{4.22}$$

Then, from Lemma 4.5, (4.22), (4.20), we obtain

$$\begin{aligned} H(u) &> \Phi(1)\|u\|^{\varphi^*}_{0,p(\cdot),\nabla} - \frac{\Phi(1)}{2}\|u\|^{\varphi^*}_{0,p(\cdot),\nabla} - c_3\|u\|^{q^-}_{0,q(\cdot)} \\ &= \frac{\Phi(1)}{2}\|u\|^{\varphi^*}_{0,p(\cdot),\nabla} - c_3\|u\|^{q^-}_{0,q(\cdot)}, \end{aligned}$$

for all $u \in U_{\Gamma_0}$ with $\|u\|_{0,p(\cdot),\nabla} < \min(1, 1/\underline{c})$. Therefore, the hypothesis (iii) of Theorem 2.1 is fulfilled.

Now, we shall verify the hypothesis (iv) of Theorem 2.1. Let θ and s_0 be as in (H2). We shall deduce that one has

$$G(x,s) \ge \gamma(x)|s|^\theta, \quad \text{for almost all } x \in \Omega \text{ and } |s| \ge s_0, \tag{4.23}$$

where the function γ will be specified below. Indeed, it follows from [12, p. 236] that

$$G(x,s) \ge (G(x,s_0)/s_0^\theta)s^\theta, \quad \text{for almost all } x \in \Omega \text{ and } s \ge s_0. \tag{4.24}$$

On the other hand, for almost all $x \in \Omega$ and $\tau \le -s_0$, from (4.5), we have $G(x,s) > 0$ for almost all $x \in \Omega$ and $|s| \ge s_0$, and

$$\frac{\theta}{\tau} \ge \frac{g(x,\tau)}{G(x,\tau)}.$$

By integrating from $s \le -s_0$ to $-s_0$, it follows that

$$\frac{s_0^\theta}{|s|^\theta} \ge \frac{G(x,-s_0)}{G(x,s)},$$

which implies

$$G(x,s) \ge (G(x,-s_0)/s_0^\theta)|s|^\theta, \quad \text{for almost all } x \in \Omega \text{ and } s \le -s_0. \tag{4.25}$$

Setting

$$\gamma(x) = \begin{cases} (G(x,s_0)/s_0^\theta), & \text{if } s \ge s_0 \\ (G(x,-s_0)/s_0^\theta), & \text{if } s \le -s_0, \end{cases}$$

from (4.24) and (4.25), we obtain (4.23).

For $v \in U_{\Gamma_0}$, we define

$$\Omega_{\ge} := \{x \in \Omega : |v(x)| \ge s_0\}, \Omega_{<} := \Omega\setminus\Omega_{\ge}.$$

From (4.23) it follows that

$$\begin{aligned} \int_\Omega G(x,v(x))\mathrm{d}x &\ge \int_{\Omega_\ge}\gamma(x)|v(x)|^\theta\mathrm{d}x + \int_{\Omega_<}G(x,v(x))\mathrm{d}x \\ &= \int_\Omega \gamma(x)|v(x)|^\theta\mathrm{d}x + \int_{\Omega_<}G(x,v(x))\mathrm{d}x - \textstyle\int_{\Omega_<}\gamma(x)|v(x)|^\theta\mathrm{d}x \end{aligned}$$

Since
$$\int_{\Omega_<} \gamma(x)|v(x)|^\theta \mathrm{d}x \le \|\gamma\|_\infty s_0^\theta \operatorname{vol}(\Omega),$$
we have
$$\int_\Omega G(x,v(x))\mathrm{d}x \ge \int_\Omega \gamma(x)|v(x)|^\theta \mathrm{d}x + \int_{\Omega_<} G(x,v(x))\mathrm{d}x - k,$$
where $k := \|\gamma\|_\infty s_0^\theta \operatorname{vol}(\Omega)$. On the other hand, it follows from (4.10) that
$$\int_{\Omega_<} G(x,v(x))\mathrm{d}x \le \|a\|_\infty s_0 + c_4 \max(s_0^{q^+}, s_0^{q^-}) \operatorname{vol}(\Omega),$$
where $c_4 = c_1/q^-$. Therefore
$$\int_\Omega G(x,v(x))\mathrm{d}x \ge \int_\Omega \gamma(x)|v(x)|^\theta \mathrm{d}x - K,$$
where $K := k + \|a\|_\infty s_0 + c_4 \max(s_0^{q^+}, s_0^{q^-}) \operatorname{vol}(\Omega)$. Consequently,
$$H(v) \le \Phi(\|v\|_{0,p(\cdot),\nabla}) - \int_\Omega \gamma(x)|v(x)|^\theta \mathrm{d}x + K,$$
where K is a positive constant and θ is given by $(\mathrm{H})_2$. Taking into account Lemma 4.5, for $\|v\|_{0,p(\cdot),\nabla} > 1$ we have
$$H(v) \le \Phi(1)\|v\|_{0,p(\cdot),\nabla}^{\varphi^*} - \int_\Omega \gamma(x)|v(x)|^\theta \mathrm{d}x + K. \tag{4.26}$$
Now, the functional $\|\cdot\|_\gamma : U_{\Gamma_0} \to \mathbb{R}$ defined by
$$\|v\|_\gamma = \Big(\int_\Omega \gamma(x)|v(x)|^\theta \mathrm{d}x\Big)^{1/\theta}$$
is a norm on U_{Γ_0}. Let X_1 be a finite dimensional subspace of U_{Γ_0}. Since the tow norms $\|\cdot\|_{0,p(\cdot),\nabla}$ and $\|\cdot\|_\gamma$ are equivalent on the finite dimensional subspace X_1, there is a constant $\delta = \delta(X_1) > 0$ such that
$$\|v\|_{0,p(\cdot),\nabla} \le \delta \|v\|_\gamma.$$
Therefore, from (4.26) it follows that
$$H(v) \le \Phi(1)\|v\|_{0,p(\cdot),\nabla}^{\varphi^*} - \frac{1}{\delta^\theta}\|v\|_{0,p(\cdot),\nabla}^\theta + K,$$
if $v \in X_1$, $\|v\|_{0,p(\cdot),\nabla} > 1$, that is the hypothesis (iv) is fulfilled.

Taking into account Theorem 2.1, it follows that the functional F possesses a sequence of critical positive values. By Proposition 4.3, equation
$$J_\varphi u = g(x,u)$$
has a sequence of solutions in U_{Γ_0} or, equivalently, the problem (4.1), (4.2) possesses a sequence of weak solutions in U_{Γ_0}. □

Taking into account Remark 4.1, if $p(x) = p$ =const. and $\varphi(t) = t^{r-1}$, $r > 1$, from Theorem 4.2 it follows:

Corollary 4.6. *Let Ω be a domain in $\mathbb{R}^N$ $(N \ge 2)$, $p \in (1,\infty)$, and let p^* be given by*
$$p^* := \frac{Np}{N-p} \text{ if } p < N \quad \text{and} \quad p^* := \infty \text{ if } p \ge N,$$
Let there be given a Carathéodory function $g : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfying the hypotheses:

(1) *there exists a function* $q(\cdot) \in \mathcal{C}(\overline{\Omega})$ *that satisfies*

$$1 \leq q(x) < p^* \quad \text{for each } x \in \overline{\Omega}$$

such that

$$|g(x,s)| \leq C_1 |s|^{q(x)/q'(x)} + a(x), \quad \text{for almost all } x \in \Omega \text{ and all } s \in \mathbb{R},$$

where $\frac{1}{q(x)} + \frac{1}{q'(x)} = 1$, *a is a bounded function,* $a(x) \geq 0$ *for almost all* $x \in \Omega$, *and* C_1 *is a constant,* $C_1 > 0$;

(2) *there exist* $s_0 > 0$ *and* $\theta > r$ *such that* (4.5) *holds for almost every* $x \in \Omega$ *and all s with* $|s| \geq s_0$, *where G is given by* (4.6).

Also assume that

(3)

$$\limsup_{s \to 0} \frac{g(x,s)}{|s|^{r-2}s} < \frac{\lambda_{1,r}}{2}$$

uniformly with respect to almost all $x \in \Omega$, *where* $\lambda_{1,r}$ *is given by* (3.5).

(4) $r < q^-$.

Let $N_g : L^{q(\cdot)}(\Omega) \to L^{q'(\cdot)}(\Omega)$, *with* $(N_g u)(x) = g(x,u(x))$ *for almost all* $x \in \Omega$, *denote the Nemytskij operator generated by g. Under these assumptions, the problem*

$$-\operatorname{div}\big(\||\nabla u|\|_{L^p(\Omega)}^{r-p} |\nabla u|^{p-2} \nabla u\big) = g(x,u) \quad \text{in } \Omega, \tag{4.27}$$

$$u = 0 \quad \text{on } \Gamma_0 \subset \partial\Omega, \tag{4.28}$$

has a weak non-trivial solution in the space U_{Γ_0}. *Moreover, if g is odd in the second argument:* $g(x,-s) = -g(x,s)$, $s \in \mathbb{R}$, *then problem* (4.27), (4.28) *has a sequence of weak solutions.*

In particular, if $r = p$ and $q(x) = q$ =const., we obtain a result similar to [12, Theorem 18, p. 370]:

Corollary 4.7. *Let* Ω *be a domain in* $\mathbb{R}^N$ $(N \geq 2)$, *let* $p \in \mathbb{R}$ *be such that* $p > 1$, *and let* p^* *be given by*

$$p^* := \frac{Np}{N-p} \text{ if } p < N, \quad \text{and} \quad p^* := \infty \text{ if } p \geq N,$$

Let there be given a Carathéodory function $g : \Omega \times \mathbb{R} \to \mathbb{R}$ *satisfying the hypotheses:*

(1) *there exists* $q \in (1, p^*)$ *such that*

$$|g(x,s)| \leq C_1 |s|^{q-1} + a(x), \quad \text{for almost all } x \in \Omega \text{ and all } s \in \mathbb{R},$$

where $\frac{1}{q} + \frac{1}{q'} = 1$, *a is a bounded function,* $a(x) \geq 0$ *for almost all* $x \in \Omega$, *and* C_1 *is a constant,* $C_1 > 0$;

(2) *there exist* $s_0 > 0$ *and* $\theta > p$ *such that* (4.5) *holds for almost every* $x \in \Omega$ *and all s with* $|s| \geq s_0$, *where G is given by* (4.6).

Also assume that

(3)

$$\limsup_{s \to 0} \frac{g(x,s)}{|s|^{p-2}s} < \frac{\lambda_{1,p}}{2}$$

uniformly with respect to almost all $x \in \Omega$, *where* $\lambda_{1,p}$ *is given by* (3.5).

(4) $p < q$.

Let $N_g : L^q(\Omega) \to L^{q'}(\Omega)$, with $(N_g u)(x) = g(x, u(x))$ for almost all $x \in \Omega$, denote the Nemytskij operator generated by g. Under these assumptions, the problem

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u) = g(x,u) \quad \text{in } \Omega, \tag{4.29}$$

$$u = 0 \quad \text{on } \Gamma_0 \subset \partial\Omega, \tag{4.30}$$

has a weak non-trivial solution in the space U_{Γ_0}. Moreover, if g is odd in the second argument, then problem (4.29), (4.30) has a sequence of weak solutions.

Now, let us consider the gauge function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$, $\varphi(t) = t^{r-1}\ln(1+t)$, $r > 1$. From (2.3) we have

$$\Phi(t) = \frac{t^r}{r}\ln(1+t) - \frac{1}{r}\int_0^t \frac{\tau^r}{1+\tau}\mathrm{d}\tau, t > 0.$$

According to [6, p. 54], $\varphi^* = r+1$. We shall apply Theorem 4.2 with $\varphi^* = r+1$. From definition of φ^* it follows that

$$\varphi^*\Phi(1) \geq \varphi(1) = \ln 2\,.$$

From Theorem 4.2 we have the following result.

Theorem 4.8. *Let Ω be a domain in $\mathbb{R}^N$ $(N \geq 2)$, let $p \in \mathcal{C}(\overline{\Omega})$ be a function such that $p^- > 1$, and let $p^*(\cdot)$ be given by (3.2). Let us consider the function*

$$\varphi : \mathbb{R}_+ \to \mathbb{R}_+, \quad \varphi(t) = t^{r-1}\ln(1+t), r > 1. \tag{4.31}$$

Let there be given a Carathéodory function $g : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfying the hypotheses:

(1) *there exists a function $q(\cdot) \in \mathcal{C}(\overline{\Omega})$ that satisfies (3.3) such that*
$$|g(x,s)| \leq C_1|s|^{q(x)/q'(x)} + a(x), \quad \text{for almost all } x \in \Omega \text{ and all } s \in \mathbb{R},$$
where $\frac{1}{q(x)} + \frac{1}{q'(x)} = 1$, a is a bounded function, $a(x) \geq 0$ for almost all $x \in \Omega$, and C_1 is a constant, $C_1 > 0$;

(2) *there exist $s_0 > 0$ and $\theta > r+1$ such that*
$$0 < \theta G(x,s) \leq sg(x,s),$$
for almost every $x \in \Omega$ and all s with $|s| \geq s_0$, where
$$G(x,s) := \textstyle\int_0^s g(x,\tau)\mathrm{d}\tau.$$

Also assume that

(3)
$$\limsup_{s\to 0}\frac{g(x,s)}{|s|^{r-1}s} < \frac{\ln 2}{2}\lambda_{1,r+1}$$
uniformly with respect to almost all $x \in \Omega$, where $\lambda_{1,r+1}$ is given by (4.21).

(4) $r + 1 < q^-$.

Let $N_g : L^{q(\cdot)}(\Omega) \to L^{q'(\cdot)}(\Omega)$, with $(N_g u)(x) = g(x,u(x))$ for almost all $x \in \Omega$, denote the Nemytskij operator generated by g. Under these assumptions, problem (4.1), (4.2), where φ is given by (4.31), has a weak non-trivial solution in the space U_{Γ_0} (endowed with the norm (3.4). Moreover, if g is odd in the second argument, then problem (4.1), (4.2) has a sequence of weak solutions.

Acknowledgements. The author thanks the anonymous referee for his/her valuable suggestions which contributed to improving the final form of this article.

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