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# MULTIPLE POSITIVE SOLUTIONS FOR ELLIPTIC PROBLEM WITH CONCAVE AND CONVEX NONLINEARITIES

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Abstract. In this article, we consider the existence of multiple solutions to the elliptic problem

$$\begin{gathered} -\Delta u = \lambda u^q + u^s + \mu u^p \quad \text{in } \Omega, \\ u > 0 \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered}$$

where $\Omega \subseteq \mathbb{R}^N$ $(N \geq 3)$ is a bounded domain with smooth boundary $\partial\Omega$, $0 < q < 1 < s < 2^* - 1 \leq p$, $2^* := \frac{2N}{N-2}$, $\lambda$ and $\mu$ are nonnegative parameters. By using variational methods, truncation and Moser iteration techniques, we show that if the parameters $\lambda$ and $\mu$ are small enough, then the problem has at least two positive solutions.

## 1. Introduction and main results

In this article we study the existence of nontrivial solutions for the elliptic problem

$$\begin{gathered} -\Delta u = \lambda u^q + u^s + \mu u^p \quad \text{in } \Omega, \\ u > 0 \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N (N \geq 3)$ is a bounded smooth domain, $\lambda$ and $\mu$ are nonnegative parameters, $0 < q < 1$, $1 < s < 2^* - 1$, $p \geq 2^* - 1$, $2^* = \frac{2N}{N-2}$, i.e. the nonlinearity is a combination of a sublinear term, a subcritical term and a critical or supercritical term. From the perspective of the concavity and convexity of a function, problem (1.1) has one concave term, two convex terms.

We want to remark that if the subcritical term $u^s (1 < s < 2^* - 1)$ does not appear in our problem (1.1), i.e.

$$\begin{gathered} -\Delta u = \lambda u^q + \mu u^p \quad \text{in } \Omega, \\ u > 0 \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.2}$$

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by the linear transformation $v = \mu^{\frac{1}{p-1}} u$, problem (1.2) is equivalent to

$$\begin{gathered} -\Delta v = \widetilde{\lambda} v^q + v^p \quad \text{in } \Omega, \\ v > 0 \quad \text{in } \Omega, \\ v = 0 \quad \text{on } \partial\Omega \end{gathered} \tag{1.3}$$

with $\widetilde{\lambda} = \lambda \mu^{\frac{1-q}{p-1}}$. This concave-convex problem was first considered by Ambrosetti, Brezis and Cerami [2], they discover that there exists $\Lambda > 0$ such that for $0 < \widetilde{\lambda} < \Lambda$, problem (1.3) has a solution if $p > 1$, and has a second solution if $1 < p \leq (N+2)/(N-2)$. For supercritical case, i.e. $p > (N+2)/(N-2)$, the authors poses an open problem: When $\Omega$ is a ball in $\mathbb{R}^N$, does problem (1.3) have two solutions for $\widetilde{\lambda} > 0$ small enough? After this seminal work, many works have been devoted to problems with concave-convex nonlinearities, see for example [1, 4, 7, 11, 12, 14]. Especially in the literature [14], using a concept of radial singular solution, Zhao and Zhong prove that if $\widetilde{\lambda} > 0$ is small enough and $p > 2^* - 1$, then problem (1.3) has exactly one solution. In particular, this means that problem (1.2) cannot have a second solution if $p > (N+2)/(N-2)$, and gives a negative answer to that open problem. In other words, (1.2) has exactly one solution for $\lambda$ and $\mu$ small enough. Now, we are interested in what will happen with adding a subcritical term $u^s$ in (1.2). In this paper, we show that the appearance of the subcritical term $u^s$ in (1.2) destroys the uniqueness result. More precisely, we prove the following main results.

**Theorem 1.1.** *Let $\Omega \subset \mathbb{R}^N$ be a bounded smooth domain. Assume $0 < q < 1 < s < 2^* - 1 \leq p$, then* (1.1) *has at least two positive solutions if $\lambda$ and $\mu$ are sufficiently small.*

Our approach is variational, based on the critical point theory and we use truncation methods and Moser iteration technique to deal with the critical case and supercritical case in a unified approach.

Before we proceed, we recall that to use the Mountain Pass Theorem [3, 9, 10] the Palais-Smale (PS) condition is needed. A $C^1$ functional $J$ on a Banach space $X$ is said to satisfy the (PS) condition at $c \in \mathbb{R}$ if every sequence $u_n \subset X$ satisfying

$$J(u_n) \to c \text{ and } \|J'(u_n)\|_{X'} \to 0 \text{ as } n \to \infty$$

admits a strongly convergent subsequence. We say that $J$ satisfies the (PS) condition if $J$ satisfies the (PS) condition at any $c \in \mathbb{R}$. This compactness type condition, which compensates for the lack of local compactness in the underlying space $X$ being in general infinite dimensional, leads to the following well known Mountain Pass Theorem.

**Lemma 1.2.** *Let $X$ be a Banach space and $J \in C^1(X, \mathbb{R})$ satisfying the (PS) condition. Suppose $J(0) = 0$ and*

- (J1) *there are constants $\rho, \alpha > 0$ such that $J|_{\partial B_\rho} \geq \alpha$, and*
- (J2) *there is an $e \in X \setminus B_\rho$ such that $J(e) \leq 0$.*

*Then $J$ possesses a critical value $c \geq \alpha$. Moreover, $c$ can be characterized as*

$$c = \inf_{g \in \Gamma} \max_{u \in g([0,1])} J(u),$$

*where $\Gamma = \{g \in C([0,1], X) : g(0) = 0,\ g(1) = e\}$.*

In this article, the norm in $L^r(\Omega)$ $(1<r<\infty)$ is $\|u\|_r=\big(\int_\Omega |u|^r \mathrm{d}x\big)^{1/r}$, and the norm in $H_0^1(\Omega)$ is $\|u\|=\big(\int_\Omega |\nabla u|^2 \mathrm{d}x\big)^{1/2}$. Here $X'$ denotes the dual space of $X$. $S$ is the best Sobolev embedding constant

$$S=\inf_{u\in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2 \mathrm{d}x}{\big(\int_\Omega |u|^{2^*} \mathrm{d}x\big)^{2/2^*}}. \tag{1.4}$$

This article is organized as follows. In section 2, we consider a truncated problem (2.1) and obtain two solutions by using variational methods. In section 3, we finish the proof of Theorem 1.1 by demonstrating that solutions of (2.1) are actually solutions of the original problem (1.1), this reduces to an $L^\infty$ estimate.

## 2. Truncated problem

One of the main difficulty to prove the existence solutions of problem (1.1) by using variational methods is that $J(u)$ does not satisfying the (PS) condition for large energy level for $p=\frac{N+2}{N-2}$ and $J(u)$ is not well defined on $H_0^1(\Omega)$ for $p>\frac{N+2}{N-2}$. Following the idea in [6, 8, 9, 13], we first investigate the truncated problem

$$\begin{gathered} -\Delta u=\lambda u^q+u^s+\mu g_K(u) \quad \text{in } \Omega,\\ u>0 \quad \text{in } \Omega,\\ u=0 \quad \text{on } \partial\Omega, \end{gathered} \tag{2.1}$$

where $K>0$ is a real number, whose value will be fixed later, $g_K(u)$ is given by

$$g_K(u)=\begin{cases} u^p, & |u|\le K,\\ K^{p-r+1}u^{r-1}, & |u|\ge K, \end{cases} \tag{2.2}$$

where $p\ge 2^*-1$, $2<r:=s+1<2^*$, then

$$G_K(u):=\int_0^u g_K(t)\mathrm{d}t=\begin{cases} \frac{1}{p+1}u^{p+1}, & |u|\le K,\\ \big(\frac{1}{p+1}-\frac{1}{r}\big)K^{p+1}+\frac{1}{r}K^{p-r+1}u^r, & |u|\ge K, \end{cases} \tag{2.3}$$

and

$$|g_K(u)|\le K^{p-r+1}u^{r-1},\ |G_K(u)|\le \frac{1}{r}K^{p-r+1}u^r. \tag{2.4}$$

The associated functional in $H_0^1(\Omega)$ is

$$\begin{aligned} J_K(u)&=\frac{1}{2}\int_\Omega |\nabla u|^2 \mathrm{d}x\\ &\quad -\frac{\lambda}{q+1}\int_\Omega u^{q+1}\mathrm{d}x-\frac{1}{s+1}\int_\Omega u^{s+1}\mathrm{d}x-\mu\int_\Omega G_K(u)\mathrm{d}x. \end{aligned} \tag{2.5}$$

**Remark 2.1.** The original problem (1.1) is critical and supercritical, after truncation, it becomes subcritical and the functional $J_K(u)\in C^1$ is well defined, this fact allows us to use the usual minimax methods.

We have the following multiplicity theorem for problem (2.1).

**Theorem 2.2.** *There exist two positive constants $\lambda_0$ and $\mu_0$ such that for all $\lambda,\mu$ with $0<\lambda<\lambda_0$ and $0<\mu<\mu_0$, problem* (2.1) *has at least two positive solutions.*

*Proof.* Let $e$ denote the solution of

$$\begin{gathered}-\Delta e = 1 \quad \text{in } \Omega,\\ e = 0 \quad \text{on } \partial\Omega,\end{gathered}$$

then $e \in C_0^\infty(\Omega)$ is nonnegative, and $\|e\|_\infty \le C$ for some positive constant $C > 0$. Since $0 < q < 1 < s < 2^* - 1$, and $2 < r = s + 1 < 2^*$, we can find $\lambda_0 > 0$ and $\mu_0 > 0$ such that for all $0 < \lambda < \lambda_0$ and $0 < \mu < \mu_0$, there exits $M = M(\lambda, \mu) > 0$ satisfying

$$M \ge \lambda M^q \|e\|_\infty^q + M^s \|e\|_\infty^s + \mu K^{p-r+1} M^{r-1} \|e\|_\infty^{r-1}.$$

As a consequence, the function $Me$ satisfies

$$\begin{aligned}-\Delta(Me) = (-\Delta e)M = M &\ge \lambda M^q \|e\|_\infty^q + M^s \|e\|_\infty^s + \mu K^{p-r+1} M^{r-1} \|e\|_\infty^{r-1}\\ &\ge \lambda (Me)^q + (Me)^s + \mu g_K(Me),\end{aligned}$$

and hence it is a supersolution of (2.1). Moreover, any $\varepsilon\varphi_1$ is a subsolution of (2.1), provided

$$-\Delta(\varepsilon\varphi_1) = \lambda_1 \varepsilon\varphi_1 \le \lambda(\varepsilon\varphi_1)^q + (\varepsilon\varphi_1)^s + \mu g_K(\varepsilon\varphi_1),$$

which is satisfied for all $\varepsilon > 0$ small enough and all $\lambda > 0, \mu > 0$. Taking $\varepsilon$ possibly smaller, we also have

$$\varepsilon\varphi_1 < Me$$

If follows that (2.1) has a solution $\varepsilon\varphi_1 \le u_1 \le Me$ whenever $\lambda \le \lambda_0$ and $\mu \le \mu_0$. Actually, $u_1$ is a local minimum of $J_K$ in the $C^1$-topology, hence a local minimum for $J_K$ in the $H_0^1(\Omega)$-topology, see [2] for details.

Next, we look for a second solution of (2.1) by Mountain Pass Theorem, since $u_1$ is a local minimum in the $H_0^1(\Omega)$-topology, we only need to show that the $(PS)$ condition is satisfied and $J_K(tu) \to -\infty$, as $t \to +\infty$.

**Claim 1.** The functional $J_K(u)$ satisfies $(PS)_c$ for any $c \in \mathbb{R}$. To see this, take $c \in \mathbb{R}$ and assume that $\{u_n\}$ is a Palais-Smale sequence at level $c$, namely such that

$$J_K(u_n) \to c \text{ and } J_K'(u_n) \to 0 (\text{in } H_0^1(\Omega)'),$$

Consequently we obtain, by Sobolev embedding theorem, together with (2.2) and (2.3),

$$\begin{aligned}c(1 + \|u_n\|) &\ge J_K(u_n) - \frac{1}{s+1} J_K'(u_n)u_n\\ &= \Big(\frac12 - \frac{1}{s+1}\Big)\int_\Omega |\nabla u_n|^2 \mathrm{d}x - \Big(\frac{\lambda}{q+1} - \frac{\lambda}{s+1}\Big)\int_\Omega u_n^{q+1}\mathrm{d}x\\ &\quad + \mu\int_\Omega \Big[\frac{1}{s+1} g_K(u_n)u_n - G_K(u_n)\Big]\mathrm{d}x\\ &\ge \Big(\frac12 - \frac{1}{s+1}\Big)\int_\Omega |\nabla u_n|^2 \mathrm{d}x - \Big(\frac{\lambda}{q+1} - \frac{\lambda}{s+1}\Big)\int_\Omega u_n^{q+1}\mathrm{d}x\\ &\ge \Big(\frac12 - \frac{1}{s+1}\Big)\|u_n\|^2 - \Big(\frac{\lambda}{q+1} - \frac{\lambda}{s+1}\Big) S_b^{q+1}\|u_n\|^{q+1},\end{aligned} \tag{2.6}$$

where $S_b$ is the Sobolev constant, and we have also used the fact that (2.2) and (2.3) imply $g_K(t)t \ge (s+1)G_K(t)$ for all $t \in \mathbb{R}$. It follows from (2.6) (note $1 < q+1 < 2$), $\{u_n\}$ is bounded in $H_0^1(\Omega)$. Then we can assume that, up to a subsequence, there exists $u \in H_0^1(\Omega)$ such that

$$u_n \rightharpoonup u \quad \text{in } H_0^1(\Omega),$$

$$\begin{gathered} u_n(x) \to u(x) \quad \text{for almost every } x \in \Omega, \\ u_n \to u \quad \text{in } L^s(\Omega). \end{gathered}$$

As a consequence,

$$\begin{gathered} \int_\Omega (u_n^q - u^q)(u_n - u)\mathrm{d}x \to 0, \quad \int_\Omega (u_n^s - u^s)(u_n - u)\mathrm{d}x \to 0, \\ \int_\Omega [g_K(u_n) - g_K(u)](u_n - u)\mathrm{d}x \to 0 \quad \text{as } n \to \infty. \end{gathered}$$

We conclude by computing

$$\begin{aligned} o(1) &= (J_K'(u_n) - J_K'(u))(u_n - u) \\ &= \int_\Omega |\nabla(u_n - u)|^2\mathrm{d}x - \lambda\int_\Omega (u_n^q - u^q)(u_n - u)\mathrm{d}x \\ &\quad - \int_\Omega (u_n^s - u^s)(u_n - u)\mathrm{d}x - \mu\int_\Omega [g_K(u_n) - g_K(u)](u_n - u)\mathrm{d}x \\ &= \|u_n - u\|^2 + o(1), \end{aligned}$$

which shows that $u_n \to u$ in $H_0^1(\Omega)$. This proves Claim 1.

**Claim 2.** $J_K(tu) \to -\infty$, as $t \to +\infty$. For every $u \in H_0^1(\Omega) \setminus \{0\}$ and $t > 0$ we have

$$\begin{aligned} J_K(tu) &= \frac{t^2}{2}\|u\|^2 - \frac{\lambda t^{q+1}}{q+1}\int_\Omega u^{q+1}\mathrm{d}x - \frac{t^{s+1}}{s+1}\int_\Omega u^{s+1}\mathrm{d}x - \mu\int_\Omega G_K(tu)\mathrm{d}x \\ &= \frac{t^2}{2}\|u\|^2 - \frac{\lambda t^{q+1}}{q+1}\int_\Omega u^{q+1}\mathrm{d}x - \frac{t^{s+1}}{s+1}\int_\Omega u^{s+1}\mathrm{d}x \\ &\quad - \frac{\mu t^{p+1}}{p+1}\int_{\{|tu|\le K\}} u^{p+1}\mathrm{d}x - \frac{\mu t^r K^{p-r+1}}{r}\int_{\{|tu|\ge K\}} u^r\mathrm{d}x. \end{aligned}$$

Since

$$\int_{\{|tu|\le K\}} u^{p+1}\mathrm{d}x \to 0 \quad \text{as } t \to +\infty,$$

and $1 < q+1 < 2 < s+1 = r < 2^*$, it follows that $J(tu) \to -\infty$ as $t \to +\infty$. This proves Claim 2.

Since Claims 1 and 2 hold, by the mountain pass theorem there exists a $u_2 \in H_0^1(\Omega)$ such that $J_K(u_2) = c_M$, where

$$c_M = \inf_{\omega\in W}\max_{t\in[0,1]} J_K(\omega(t)) \quad \text{and} \quad W = \{\omega \in C([0,1]) : \omega(0) = u_1, J_K(\omega(1)) < 0\}.$$

We may assume that $u_2$ is positive. Indeed, we can extend the nonlinearity to zero if $u < 0$, with this extension, the maximum principle implies that every nontrivial solutions of (2.1) is positive. ☐

**Lemma 2.3.** *The solutions for problem* (2.1) *obtained by Theorem 2.2 are bounded in $H_0^1(\Omega)$, i.e.*

$$\|u_i\| \le \gamma, \quad i = 1, 2.$$

*where $\gamma > 0$ is independent of $\mu$.*

*Proof.* Let $c_M$ be the mountain pass level for $J_K$ obtained in previous section,

$$c_M \ge J_K(u_i) = J_K(u_i) - \frac{1}{s+1}J_K'(u_i)u_i$$

$$
\begin{aligned}
&= \Big(\frac{1}{2}-\frac{1}{s+1}\Big)\int_\Omega |\nabla u_i|^2 \mathrm{d}x - \Big(\frac{\lambda}{q+1}-\frac{\lambda}{s+1}\Big)\int_\Omega u_i^{q+1}\mathrm{d}x \\
&\quad + \mu\int_\Omega \Big[\frac{1}{s+1}g_K(u_i)u_i - G_K(u_i)\Big]\mathrm{d}x \\
&\geq \Big(\frac{1}{2}-\frac{1}{s+1}\Big)\int_\Omega |\nabla u_n|^2 \mathrm{d}x - \Big(\frac{\lambda}{q+1}-\frac{\lambda}{s+1}\Big)\int_\Omega u_i^{q+1}\mathrm{d}x \\
&\geq \Big(\frac{1}{2}-\frac{1}{s+1}\Big)\|u_i\|^2 - \Big(\frac{\lambda}{q+1}-\frac{\lambda}{s+1}\Big)S_b^{q+1}\|u_i\|^{q+1}.
\end{aligned}
$$

Since $1 < q+1 < 2$, we infer that $\|u_i\| \leq \gamma$ which is independent of $\mu$. ☐

**Remark 2.4.** Actually, $u_1$ and $u_2$ also solve problem (1.1), to show this, we only need to prove $\|u_i\|_{L^\infty(\Omega)} \leq K$, $i = 1, 2$. One should note that $c_M$ is decreasing with respect to $K$, so, $\gamma$ is also decreasing with respect to $K$, this fact is important in the following $L^\infty(\Omega)$ estimate (see inequality (3.14) in next section).

## 3. PROOF OF MAIN RESULT

To prove Theorem 1.1, we only need to show that solutions of (2.1) are actually bounded by some $K$. Our approach is a variant of Moser iteration technique inspired by [5, 6, 8, 13].

*Proof of Theorem 1.1.* For convenience, set $u := u_i$, $i = 1, 2$. Let $u$ be a weak solution of (2.1). Hence, for any $\varphi \in H_0^1(\Omega)$,

$$
\int_\Omega \nabla u \nabla \varphi \mathrm{d}x = \lambda \int_\Omega u^q \varphi \mathrm{d}x + \int_\Omega u^s \varphi \mathrm{d}x + \mu \int_\Omega g_K(u)\varphi \mathrm{d}x. \tag{3.1}
$$

For each $L > 0$, let us define the following functions

$$
u_L(x) = \begin{cases} u(x), & \text{if } u(x) \leq L, \\ L, & \text{if } u(x) > L, \end{cases}
$$

$z_L = u_L^{2(\beta-1)}u$ and $w_L = u_L^{\beta-1}u$, where $\beta > 1$ will be fixed later. Taking $z_L$ as a test function in (3.1), we obtain

$$
\int_\Omega \nabla u \nabla z_L \mathrm{d}x = \lambda \int_\Omega u^q z_L \mathrm{d}x + \int_\Omega u^s z_L \mathrm{d}x + \mu \int_\Omega g_K(u) z_L \mathrm{d}x. \tag{3.2}
$$

The left hand side of the above equality is

$$
\begin{aligned}
\int_\Omega \nabla u \nabla z_L \mathrm{d}x &= \int_\Omega \nabla u \nabla (u_L^{2(\beta-1)}u)\mathrm{d}x \\
&= \int_\Omega |\nabla u|^2 u_L^{2(\beta-1)}\mathrm{d}x + 2(\beta-1)\int_\Omega u u_L^{2(\beta-1)-1}\nabla u \nabla u_L \mathrm{d}x \\
&= \int_\Omega |\nabla u|^2 u_L^{2(\beta-1)}\mathrm{d}x + 2(\beta-1)\int_{\{0\leq u \leq L\}} |\nabla u|^2 u_L^{2(\beta-1)}\mathrm{d}x
\end{aligned}
$$

Since $2(\beta-1)\int_{\{0\le u\le L\}}|\nabla u|^2u_L^{2(\beta-1)}\mathrm{d}x\ge 0$, it follows that

$$\begin{aligned}
&\int_\Omega |\nabla u|^2u_L^{2(\beta-1)}\mathrm{d}x\\
&\le \int_\Omega \nabla u\nabla z_L\mathrm{d}x\\
&=\lambda\int_\Omega u^qz_L\mathrm{d}x+\int_\Omega u^sz_L\mathrm{d}x+\mu\int_\Omega g_K(u)z_L\mathrm{d}x\\
&=\lambda\int_\Omega u^qu_L^{2(\beta-1)}u\mathrm{d}x+\int_\Omega u^su_L^{2(\beta-1)}u\mathrm{d}x+\mu\int_\Omega g_K(u)u_L^{2(\beta-1)}u\mathrm{d}x\\
&\le\lambda\int_\Omega u^{q+1}u_L^{2(\beta-1)}\mathrm{d}x+\int_\Omega u^{s+1}u_L^{2(\beta-1)}\mathrm{d}x+\mu K^{p-r+1}\int_\Omega u^ru_L^{2(\beta-1)}\mathrm{d}x
\end{aligned}\tag{3.3}$$

where we have used (2.4), (3.1) and (3.2). By (1.4), we obtain

$$\begin{aligned}
&\Big(\int_\Omega |w_L|^{2^*}\Big)^{2/2^*}\mathrm{d}x\\
&\le S^{-1}\int_\Omega|\nabla w_L|^2\mathrm{d}x=S^{-1}\int_\Omega|\nabla(u_L^{\beta-1}u)|^2\mathrm{d}x\\
&=S^{-1}\int_\Omega|(\beta-1)uu_L^{\beta-2}\nabla u_L+u_L^{\beta-1}\nabla u|^2\mathrm{d}x\\
&\le 2S^{-1}\int_\Omega|(\beta-1)uu_L^{\beta-2}\nabla u_L|^2\mathrm{d}x+\int_\Omega|u_L^{\beta-1}\nabla u|^2\mathrm{d}x\\
&=2S^{-1}\int_{\{0\le u\le L\}}(\beta-1)^2u_L^{2(\beta-1)}|\nabla u|^2\mathrm{d}x+\int_\Omega u_L^{2(\beta-1)}|\nabla u|^2\mathrm{d}x\\
&\le 2S^{-1}[(\beta-1)^2+1]\int_\Omega u_L^{2(\beta-1)}|\nabla u|^2\mathrm{d}x\\
&=2S^{-1}\beta^2\big[\big(\frac{\beta-1}{\beta}\big)^2+\frac{1}{\beta^2}\big]\int_\Omega u_L^{2(\beta-1)}|\nabla u|^2\mathrm{d}x\\
&\le 4S^{-1}\beta^2\int_\Omega u_L^{2(\beta-1)}|\nabla u|^2\mathrm{d}x.
\end{aligned}\tag{3.4}$$

Since $u_L\le u, 0<q<1$, we can use (3.3) and (3.4) to obtain

$$\begin{aligned}
\Big(\int_\Omega |w_L|^{2^*}\Big)^{2/2^*}\mathrm{d}x&\le 4S^{-1}\beta^2\Big[\lambda\int_\Omega u^{q+1}u_L^{2(\beta-1)}\mathrm{d}x+\int_\Omega u^{s+1}u_L^{2(\beta-1)}\mathrm{d}x\\
&\quad+\mu K^{p-r+1}\int_\Omega u^ru_L^{2(\beta-1)}\mathrm{d}x\Big]\\
&\le 4S^{-1}\beta^2\Big[\lambda|\Omega|+\lambda\int_\Omega u^2u_L^{2(\beta-1)}\mathrm{d}x+\int_\Omega u^{s+1}u_L^{2(\beta-1)}\mathrm{d}x\\
&\quad+\mu K^{p-r+1}\int_\Omega u^ru_L^{2(\beta-1)}\mathrm{d}x\Big]
\end{aligned}\tag{3.5}$$

Considering the Sobolev embedding $H_0^1(\Omega))\hookrightarrow L^{2^*}(\Omega)$, and $\|u\|\le\gamma$ (see Lemma 2.3), we have

$$S^{1/2}\Big(\int_\Omega|u|^{2^*}\mathrm{d}x\Big)^{1/2^*}\le\Big(\int_\Omega|\nabla u|^2\mathrm{d}x\Big)^{1/2}\le\gamma,$$

then

$$\|u\|_{2^*}\le\gamma S^{-1/2}.\tag{3.6}$$

Let $\alpha^* = \frac{2^*\cdot 2}{2^*-r+2}$. Since

$$u^r u_L^{2(\beta-1)} = u^{r-2}w_L^2,\ u^{s+1}u_L^{2(\beta-1)} = u^{s-1}w_L^2$$

and $u^2u_L^{2(\beta-1)} = w_L^2$, we now use the Hölder inequality, (3.4), (3.5) and (3.6) to conclude that, whenever $w_L \in L^{\alpha^*}(\Omega)$, it holds

$$\begin{aligned}
\|w_L\|_{2^*}^2 &\le 4S^{-1}\beta^2\Big[\lambda|\Omega| + \lambda\int_\Omega w_L^2\mathrm{d}x + \int_\Omega u^{s-1}w_L^2\mathrm{d}x \\
&\quad + \mu K^{p-r+1}\int_\Omega u^{r-2}w_L^2\mathrm{d}x\Big] \\
&\le 4S^{-1}\beta^2\Big[\lambda|\Omega| + \lambda|\Omega|^{\frac{\alpha^*-2}{\alpha^*}}\|w_L\|_{\alpha^*}^2 + \|u\|_{2^*}^{s-1}\|w_L\|_{\alpha^*}^2 \\
&\quad + \mu K^{p-r+1}\|u\|_{2^*}^{2^*(1-\frac{2}{\alpha^*})}\|w_L\|_{\alpha^*}^2\Big] \\
&\le 4S^{-1}\beta^2\Big[\lambda|\Omega| + \Big(\lambda|\Omega|^{\frac{\alpha^*-2}{\alpha^*}} + (\gamma S^{-1/2})^{s-1} \\
&\quad + \mu K^{p-r+1}(\gamma S^{-1/2})^{2^*(1-\frac{2}{\alpha^*})}\Big)\|w_L\|_{\alpha^*}^2\Big] \\
&\le 4S^{-1}\beta^2\Big[2\lambda(1+|\Omega|) + \gamma^{s-1}S^{-\frac{s-1}{2}} \\
&\quad + \mu K^{p-r+1}(\gamma S^{-1/2}+1)^{2^*}\Big]\max\{1,\|w_L\|_{\alpha^*}^2\}
\end{aligned} \tag{3.7}$$

Set $\beta := 2^*/\alpha^*$, then $w_L \in L^{\alpha^*}(\Omega)$. From (3.7) we have

$$\|w_L\|_{2^*}^2 \le \beta^2 C_{\lambda,\mu,K}\max\{1,\|w_L\|_{\alpha^*}^2\} \tag{3.8}$$

where $C_{\lambda,\mu,K} = 4S^{-1}\Big[2\lambda(1+|\Omega|) + \gamma^{s-1}S^{-\frac{s-1}{2}} + \mu K^{p-r+1}(\gamma S^{-1/2}+1)^{2^*}\Big]$, which is independent of $u,\ \beta,\ \alpha^*$ and $L$. From (3.8) and the definition of $w_L$, we obtain

$$\begin{aligned}
\Big(\int_\Omega u_L^{(\beta-1)2^*}u^{2^*}\mathrm{d}x\Big)^{2/2^*} &\le \beta^2 C_{\lambda,\mu,K}\max\Big\{1,\Big(\int_\Omega u_L^{(\beta-1)\alpha^*}u^{\alpha^*}\mathrm{d}x\Big)^{2/\alpha^*}\Big\} \\
&\le \beta^2 C_{\lambda,\mu,K}\max\Big\{1,\Big(\int_\Omega u^{\beta\alpha^*}\mathrm{d}x\Big)^{2/\alpha^*}\Big\}
\end{aligned}$$

By Fatou's Lemma,

$$\Big(\int_\Omega u^{\beta 2^*}\mathrm{d}x\Big)^{2/2^*} \le \beta^2 C_{\lambda,\mu,K}\max\Big\{1,\Big(\int_\Omega u^{\beta\alpha^*}\mathrm{d}x\Big)^{2/\alpha^*}\Big\},$$

which is equivalent to

$$\|u\|_{\beta 2^*} \le \beta^{1/\beta}C_{\lambda,\mu,K}^{\frac{1}{2\beta}}\max\{1,\|u\|_{\beta\alpha^*}\} \tag{3.9}$$

Since $\beta = \frac{2^*}{\alpha^*} > 1$ and $u \in L^{2^*}(\Omega)$, the inequality (3.9) holds for this choice of $\beta$. Now, let us choose a sequence of positive numbers $\{\beta_m\}_m$ in the following way:

$$\beta_0 = \beta, \quad \beta_m = \beta^m. \tag{3.10}$$

Noting that $\beta^2\alpha^* = \beta 2^*$, we have

$$\beta_{m+1}\alpha^* = \beta^{m+1}\alpha^* = \beta^{m-1}\left(\beta^2\alpha^*\right) = \beta^{m-1}\cdot\beta 2^* = \beta^m 2^* = \beta_m 2^*. \tag{3.11}$$

In view of (3.10) and (3.11), we can restate (3.9) as

$$\|u\|_{\beta_m\alpha^*} \le \beta_{m-1}^{\frac{1}{\beta_{m-1}}}C_{\lambda,\mu,K}^{\frac{1}{2\beta_{m-1}}}\max\{1,\|u\|_{\beta_{m-1}\alpha^*}\}.$$

Define $b_m = \max\{1, \|u\|_{\beta_m \alpha^*}\}$, then

$$\begin{aligned}
\log b_m &\le \frac{1}{\beta_{m-1}} \log \beta_{m-1} + \frac{1}{2\beta_{m-1}} \log C_{\lambda,\mu,K} + \log b_{m-1} \\
&\le \sum_{i=1}^{m-1} \frac{\log \beta_i}{\beta_i} + \frac{\log C_{\lambda,\mu,K}}{2} \sum_{i=1}^{m-1} \frac{1}{\beta_i} + \log b_0 \\
&= \sum_{i=1}^{m-1} \frac{\log \beta^i}{\beta^i} + \frac{\log C_{\lambda,\mu,K}}{2} \sum_{i=1}^{m-1} \frac{1}{\beta^i} + \log \max\{1, \|u\|_{2^*}\}.
\end{aligned} \tag{3.12}$$

Notice that

$$\sum_{i=1}^{m-1} \frac{\log \beta^i}{\beta^i} + \frac{\log C_{\lambda,\mu,K}}{2} \sum_{i=1}^{m-1} \frac{1}{\beta^i} \to C_\beta + C'_\beta \log C_{\lambda,\mu,K} := C_0$$

as $m \to \infty$, with

$$C_\beta = \sum_{i=1}^{\infty} \frac{\log \beta^i}{\beta^i}, \quad 2C'_\beta = \sum_{i=1}^{\infty} \frac{1}{\beta^i}, \ \beta > 1.$$

Taking the limit as $m \to \infty$ in (3.12), and using (3.6), we deduce that

$$\|u\|_\infty \le e^{C_0} \max\{1, \|u\|_{2^*}\} \le e^{C_0} \max\{1, \gamma S^{-1/2}\}.$$

We should pay attention that $C_0$ depends on $\lambda, \mu, K, |\Omega|, S, \gamma$ and control the dependence of $C_0$ on $|\Omega|, S$ and $\gamma$. Now, to prove our theorem, we need choose suitable value of $\lambda, \mu, K$ carefully, such that

$$e^{C_0} \max\left\{1, \gamma S^{-1/2}\right\} = e^{C_\beta + C'_\beta \log C_{\lambda,\mu,K}} \max\{1, \gamma S^{-1/2}\} \le K. \tag{3.13}$$

this is equivalent to

$$C_{\lambda,\mu,K}^{C'_\beta} e^{C_\beta} \max\{1, \gamma S^{-1/2}\} \le K.$$

That is,

$$\begin{aligned}
&\Big[4S^{-1}(1 + |\Omega|)(2\lambda + \gamma^{s-1} S^{-\frac{s-1}{2}}) \\
&+ \mu K^{p-r+1} (\gamma S^{-1/2} + 1)^{2^*}\Big]^{C'_\beta} e^{C_\beta} \max\{1, \gamma S^{-1/2}\} \le K.
\end{aligned}$$

Choose $K > 0$ to satisfy the inequality (note that $\lambda \le \lambda_0$)

$$\Big(\frac{K}{e^{C_\beta} \max\{1, \gamma S^{-1/2}\}}\Big)^{1/C'_\beta} - 4S^{-1}(1 + |\Omega|)(2\lambda + \gamma^{s-1} S^{-\frac{s-1}{2}}) > 0, \tag{3.14}$$

and then fix $\mu_K$ such that

$$\begin{aligned}
\mu_K := \frac{1}{K^{p-r+1} (\gamma S^{-1/2} + 1)^{2^*}} \Big[ &\Big(\frac{K}{e^{C_\beta} \max\{1, \gamma S^{-1/2}\}}\Big)^{1/C'_\beta} \\
&- 4S^{-1}(1 + |\Omega|)(2\lambda + \gamma^{s-1} S^{-\frac{s-1}{2}})\Big].
\end{aligned}$$

Let $\mu^* := \min\{\mu_0, \mu_K\}$, we obtain (3.13) for $\mu \in [0, \mu^*]$ and some $K$ satisfying (3.14). This completes the proof. ☐

Since $u_i \in L^\infty(\Omega)$, $i = 1, 2$, using bootstrap technique, we obtain $u_i \in C^{2,\alpha}(\Omega)$, $i = 1, 2$ for some constant $0 < \alpha < 1$.

**Corollary 3.1.** *The solutions obtained in Theorem 2.2 are smooth; i.e., $u_i$ belongs to $C^{2,\alpha}(\overline{\Omega})$, $i = 1, 2$ for some constant $0 < \alpha < 1$.*

**Remark 3.2.** Our method could be generalized to obtain analogous results for equations with more general perturbation $h(x, u)$, i.e.

$$\begin{gathered} -\Delta u = \lambda u^q + u^s + \mu h(x,u) \quad \text{in } \Omega, \\ u > 0 \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{3.15}$$

where $0 < q < 1 < s < 2^* - 1$, $h(x, t) \geq 0$ for $t \geq 0$ and satisfies the growth condition $|h(x,t)| \leq C_0 \left(1 + |t|^{p-1}\right)$, $p \geq 2^*$ and $C_0 > 0$ is a constant.

We have the following result similar to Theorem 1.1.

**Theorem 3.3.** *Problem* (3.15) *has at least two positive solutions for $\lambda$ and $\mu$ small enough.*

*Proof.* In fact, the truncation of $h(x, t)$ can be given by

$$h_K(x,t) = \begin{cases} h(x,t), & |t| \leq K, \\ \min\{h(x,t), C_0(1 + K^{p-r}t^{r-1})\}, & |t| > K, \end{cases} \tag{3.16}$$

where $r \in (2, 2^*)$. Then $h_K$ satisfies

$$|h_K(x,t)| \leq C_0(1 + K^{p-r}|t|^{r-1}). \tag{3.17}$$

The truncated problem associated to problem (3.15) becomes

$$\begin{gathered} -\Delta u = \lambda u^q + u^s + \mu h_K(x,u) \quad \text{in } \Omega, \\ u > 0 \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{3.18}$$

By (3.16)–(3.18) and a technique similar to the one in Theorem 1.1, we can prove that the two solutions (one is a local minimum, the other is of Mountain Pass type) for truncated problem (3.18) satisfy $\|u_i\| \leq K$, $i = 1, 2$. In view of the definition of $h_K$, we know that $u_1$ and $u_2$ are also solutions of the original problem (3.15). ☐

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