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BRANCHING ANALYSIS OF A COUNTABLE FAMILY OF GLOBAL SIMILARITY SOLUTIONS OF A FOURTH-ORDER THIN FILM EQUATION

PABLO ALVAREZ-CAUDEVILLA, VICTOR A. GALAKTIONOV

Abstract. The main goal in this article is to justify that source-type and other global-in-time similarity solutions of the Cauchy problem for the fourth-order thin film equation

$$u_t = -\nabla \cdot (|u|^n \nabla \Delta u) \quad \text{in } \mathbb{R}^N \times \mathbb{R}_+ \text{ where } n > 0,\ N \geq 1,$$

can be obtained by a continuous deformation (a homotopy path) as $n \to 0^+$. This is done by reducing to similarity solutions (given by eigenfunctions of a rescaled linear operator $\mathbf{B}$) of the classic *bi-harmonic equation*

$$u_t = -\Delta^2 u \quad \text{in } \mathbb{R}^N \times \mathbb{R}_+, \text{ where } \mathbf{B} = -\Delta^2 + \frac{1}{4} y \cdot \nabla + \frac{N}{4} I.$$

This approach leads to a countable family of various global similarity patterns of the thin film equation, and describes their oscillatory sign-changing behaviour by using the known asymptotic properties of the fundamental solution of bi-harmonic equation. The branching from $n = 0^+$ for thin film equation requires Hermitian spectral theory for a pair $\{\mathbf{B}, \mathbf{B}^*\}$ of non-self adjoint operators and leads to a number of difficult mathematical problems. These include, as a key part, the problem of multiplicity of solutions, which is under particular scrutiny.

1. Introduction: TFEs, connections with classic PDE theory, layout

1.1. Main models, applications, and preliminaries. We study the global-in-time behaviour of solutions of the fourth-order quasilinear evolution equation of parabolic type, called the *thin film equation* (TFE-4),

$$u_t = -\nabla \cdot (|u|^n \nabla \Delta u) \quad \text{in } \mathbb{R}^N \times \mathbb{R}_+\,, \tag{1.1}$$

where $\nabla = \text{grad}_x$, $\Delta = \nabla \cdot \nabla$ stands for the Laplace operator in $\mathbb{R}^N$, and $n > 0$ is a real parameter. The TFE-4 (1.1) is written for solutions of changing sign, which can occur in the *Cauchy problem* (CP) and also in some *free-boundary problems* (FBPs); see proper settings shortly.

Fourth- and sixth-order TFEs with a similar form to (1.1) such as (TFE-6)

$$u_t = \nabla \cdot (|u|^n \nabla \Delta^2 u)\,, \tag{1.2}$$

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as well as more complicated doubly nonlinear degenerate parabolic models (see [1,27] for other typical examples), have various applications in thin film, lubrication theory, and in several other hydrodynamic-type problems.

These equations model the dynamics of a thin film of viscous fluid, such as the spreading of a liquid film along a surface, where u stands for the height of the film (then clearly $u \geq 0$ that naturally leads to a FBP setting).

Specifically, when $n = 3$ we are dealing with a problem in the context of lubrication theory for thin viscous films that are driven by surface tension and when $n = 1$ with Hele–Shaw flows. We refer e.g. to [15, 17, 24, 25] for recent surveys and for extended lists of references concerning physical derivations of various models, key mathematical results and further applications. Moreover, since the 1980s such equations also play quite a special role in nonlinear PDE theory.

It is worth mentioning that *nonnegative solutions* with compact support of various FBPs are mostly physically relevant, and that the pioneering mathematical approaches by Bernis and Friedman in 1990 [7] were developed mainly for such solutions in the one dimensional case.

Furthermore, some very important extensions to the N-dimensional case were achieved by Beretta, Bertsch and Dal Passo in [5].

However, *solutions of changing sign* have already been under scrutiny for a few years (see [11, 16, 18]), which in particular, can have some biological motivations as stated in personnal communications with Professor J. R. King, to say nothing of general PDE theory.

It turned out that these classes of the so-called "oscillatory solutions of changing sign" of (1.1) were rather difficult to tackle rigorously by standard and classical methods. Principally, due to the additional fact that any kind of detailed analysis for higher-order equations is much more difficult than for second-order counterparts (such as the notorious classic *porous medium equation* $u_t = \Delta(|u|^{n-1}u)$) in view of the lack of maximum principle, comparison and order preserving semigroups and potential properties of the operators involved.

Thus, practically all the existing methods for monotone or variational operators are not applicable to the TFE-4 (1.1). Moreover, for TFEs such as (1.1) even their self-similar radial (i.e. ODE) representatives can lead to several surprises in trying to describe sign-changing features close to interfaces; see [16] for a collection of such hard properties.

On the other hand, it also turned out that, for better understanding of such singular oscillatory properties of solutions of the CP for (1.1), it is fruitful to consider the (homotopic) limit $n \to 0^+$ (see [2] for an extensive analysis of this homotopic approach) owing to Hermitian spectral theory developed in [14] for a pair $\{\mathbf{B}, \mathbf{B}^*\}$ of linear rescaled operators for $n = 0$, i.e. for the *bi-harmonic equation*

$$u_t = -\Delta^2 u \quad \text{in } \mathbb{R}^N \times \mathbb{R}_+ , \quad \mathbf{B} = -\Delta^2 + \frac{1}{4} y \cdot \nabla + \frac{N}{4} I, \quad \mathbf{B}^* = -\Delta^2 - \frac{1}{4} y \cdot \nabla, \tag{1.3}$$

which will be key for our further analysis and whose solutions are C^∞, have infinite speed of propagation and oscillates infinitely near the interfaces.

1.2. **Main results.** In this article, our goal is, using this continuity/homotopy deformation approach as "$n \to 0^+$", to reduce the nonlinear eigenvalue problem (described in detail later on)

$$-\nabla \cdot (|f|^n \nabla \Delta f) + \beta y \cdot \nabla f + \alpha f = 0 \quad \text{in } \mathbb{R}^N, \tag{1.4}$$

to the linear eigenvalue problem with the non-self-adjoint operator $\mathbf{B}$, after writing the solutions of equation (1.1) by

$$u(x,t) := t^{-\alpha} f(y), \quad \text{with } y := \frac{x}{t^{\beta}}, \quad \beta = \frac{1-n\alpha}{4}, \quad \text{for } t>0,$$

with f as the similarity profiles satisfying the nonlinear eigenvalue problem (1.4).

Thus, we shall focus our analysis on the Cauchy problem (1.1) for exponents $n>0$, which are assumed to be sufficiently small, studying large time behaviour of the solutions of (1.1), i.e. *source-type solutions* or *global asymptotic behaviour* (as $t \to \infty$) with conservation of mass, in the case on a non-zero mass, i.e.,

$$\int u(x,t)\,\mathrm{d}x \neq 0.$$

Thus, we perform a systematic analysis of the behaviour of the similarity solutions through a so-called *homotopic approach* (branching from $n=0$) via branching theory, by using the Lyapunov–Schmidt methods, obtaining relevant results and properties for the solutions of the self-similar equation associated with (1.1) and, hence, for the proper solutions of (1.1).

Loosely speaking, this approach is characterized as follows: good proper (similarity or not) solutions of the Cauchy problem for the TFE (1.1) are those that can be continuously deformed (via a homotopic path) as $n \to 0^+$ to the corresponding solutions of the bi-harmonic equation (1.3), which will play a crucial role in the subsequent analysis. Note that in [3] this analysis has been carried out directly on the parabolic TFE-4 (1.1).

Then, under the previous conditions we are able to show the following result.

Theorem 1.1. *For sufficiently small $n>0$*

$$\text{there exists a countable set of solutions } \{f_k, \alpha_k\}_{|\sigma|=k\geq 0}, \tag{1.5}$$

which depend directly on the dimension of the eigenfunctions of the linear operator $\mathbf{B}$, where σ is a multiindex in $\mathbb{R}^N$ to numerate the pairs.

Remark 1.2. Note that this continuity with respect to the parameter n was already observed by Bernis, Hulshof & Quirós in [9]. For the one dimensional case and assuming non-negative solutions for (1.1) they ascertained that the limit when $n \to 0^+$ cannot be the CP due to the oscillatory properties of (1.3). However, since we are supposing changing sign solutions for the the equation (1.1) our limiting problem might be the CP (1.3).

Furthermore, our homotopic-like approach is based upon the spectral properties known for the linear counterpart (1.3) of the TFE (1.1). Moreover, owing to the oscillatory character of the solutions of the bi-harmonic equation (1.3), being a "limit case" of the TFE (1.1), close to the interfaces this homotopy study exhibits a typical difficulty concerning the desired structure of the transversal zeros of solutions, at least for small $n>0$. The proof of such a transversality zero property is a difficult open problem, though qualitatively, this was rather well understood, [15].

Remark 1.3. It is worth mentioning that, unlike the FBPs, studied in hundreds of papers since the 1980s (see [24] and [16] for key references and alternative versions of uniqueness approaches), thin film theory for the Cauchy problem for (1.1) or (1.7) has recently led to a number of difficult open problems and is not still fully developed; see the above references as a guide to main difficulties and ideas.

In fact, the concept of proper solutions is still rather obscure for the Cauchy problem, since any classic or standard notions of weak-mild-generalized-... solutions fail in the CP setting. Therefore, the complete definition of the solutions of the CP for (1.1) is still unclear, although the results obtained in [3] provide us with a great improvement in that respect.

1.3. **Previous related results, further extensions and layout.** Observe that this approach certainly has great similarities to a previous one developed in the last two decades of the twentieth century for second-order operators. It is well understood that for any $n > 1$, (PME-2)

$$u_t = \Delta(|u|^{n-1}u) \tag{1.6}$$

has a family of exact self-similar compactly supported source-type solutions (the ZKB ones from 1950s), which describes the large time behaviour of compactly supported solutions with conservation of mass.

In addition, (1.6) also admits a countable family of other similarity solutions; see [22] for key references and most recent results.

The PME-2 (1.6) can be interpreted as a nonlinear degenerate version of the classic *heat equation* for $n = 0$,

$$u_t = \Delta u \quad \text{in } \mathbb{R}^N \times \mathbb{R}_+.$$

Note that passing to the limit $n \to 0^+$ in (1.6) for nonnegative solutions was considered a difficult mathematical problem in the 1970s-80s, which exhibited typical (but clearly simpler than in the TFE case) features of a "homotopy" transformation of PDEs. This study was initiated by Kalashnikov in 1978 [26] for the one-dimensional case.

Further detailed results in $\mathbb{R}^N$ were obtained in [4]; see also [12]. More recent estimates were obtained in [29, 30] for the 1D PME-2 (1.6) establishing the rate of convergence of solutions as $n \to 0^\pm$, such as $O(n)$ as $n \to 0^-$ (i.e. from $n < 0$, the fast diffusion range, where solutions are smoother) in $L^1(\mathbb{R})$ [29], and $O(n^2)$ as $n \to 0^+$ in $L^2(\mathbb{R} \times (0, T))$ [30].

However, most of such convergence results are obtained for *nonnegative* solutions of (1.6). For solutions of changing sign, even for this second-order equation (1.6), there are some open problems; see [22] for references and further details.

Thus, in the twenty-first century, higher-order TFEs such as (1.1), though looking like a natural counterpart/extension of the PME-2 (1.6), corresponding mathematical TFE theory is more complicated with several remaining problems still open.

Finally, to summarize let us also mention that higher-order semilinear and quasilinear parabolic equations occur in applications to thin film theory, nonlinear diffusion, lubrication theory, flame and wave propagation (the Kuramoto–Sivashinsky equation and the extended Fisher–Kolmogorov equation), phase transition at critical Lifshitz points and bi-stable systems (see Peletier-Troy [31] for further details, models and results).

Furthermore, the analysis carried out in this paper could be extended to more complicated models such as the *unstable fourth-order thin film equation* (the unstable TFE-4):

$$u_t = -\nabla \cdot (|u|^n \nabla \Delta u) - \Delta(|u|^{p-1}u)\,, \tag{1.7}$$

with the unstable homogeneous second-order diffusion term, where $p > 1$ is a fixed exponent; see [15] for physical motivations, references, and other basics. Here, (1.7) represents a fourth-order nonlinear parabolic equation with the backward (unstable) diffusion term in the second-order operator. Note, that the fourth-order term reflects surface tension effects and the second-order term can reflect gravity, van der Waals interactions, thermocapillarity effects, or geometry of the solid substrate.

Although, the analysis of equations such as (1.7) will be the ultimate goal, they are not within the scope of this paper. Thus, the layout of the paper is as follows:

(I) Problem setting and spectral theory for the linear equation (1.3). Sections 2 and 3

(II) Proof of the main results of the paper. Study of a countable family of global self-similar solutions of (1.1) via their branching from eigenspaces at $n = 0^+$, Section 4.

2. Problem setting and self-similar solutions

2.1. **The FBP and CP.** For both the FBP and the CP, the solutions are assumed to satisfy standard free-boundary conditions or boundary conditions at infinity:

$$\begin{aligned} u &= 0, \quad \text{zero-height},\\ \nabla u &= 0, \quad \text{zero contact angle},\\ -\mathbf{n}\cdot\nabla(|u|^n\Delta u) &= 0, \quad \text{conservation of mass (zero-flux)}, \end{aligned} \tag{2.1}$$

at the singularity surface (interface) $\Gamma_0[u]$, which is the lateral boundary of

$$\operatorname{supp} u \subset \mathbb{R}^N \times \mathbb{R}_+, \quad N \geq 1\,,$$

where $\mathbf{n}$ stands for the unit outward normal to $\Gamma_0[u]$. Note that, for sufficiently smooth interfaces, the condition on the flux can be read as

$$\lim_{\operatorname{dist}(x,\Gamma_0[u])\downarrow 0} -\mathbf{n}\cdot\nabla(|u|^n\Delta u) = 0.$$

For the FBP, dealing with *nonnegative* solutions, this setting is assumed to define a unique solution. However, this uniqueness result is known in 1D only; see [24], where the interface equation was included into the problem setting. We also refer to [16, § 6.2], where a "local" uniqueness is explained via *von Mises* transformation, which fixes the interface point. For more difficult, non-radial geometries in $\mathbb{R}^N$, there is no hope of getting any uniqueness for the FBP, in view of possible very complicated shapes of supports leading to various "self-focusing" singularities of interfaces at some points, which can dramatically change the required regularity of solutions.

For the CP, the assumption of non-negativity is got rid of, and solutions become oscillatory close to interfaces. It is then key, for the CP, that the solutions are expected to be "smoother" at the interface than those for the FBP, i.e. (2.1) are not sufficient to define their regularity. These *maximal regularity* issues for the CP, leading to oscillatory solutions, are under scrutiny in [16]. However, since as far as we know there is no knowledge of how the solutions for these problems should be, little more can be said about it.

Moreover, we denote by

$$M(t) := \int u(x,t)\,\mathrm{d}x,$$

the mass of the solution, where integration is performed over smooth support ($\mathbb{R}^N$ is allowed for the CP only). Then, differentiating $M(t)$ with respect to t and applying the divergence theorem (under natural regularity assumptions on solutions and free boundary) we have that

$$J(t) := \frac{\mathrm{d}M}{\mathrm{d}t} = -\int_{\Gamma_0 \cap \{t\}} \mathbf{n} \cdot \nabla(|u|^n \Delta u)\,.$$

The mass is conserved if $J(t) \equiv 0$, which is assured by the flux condition in (2.1).

The problem is completed with bounded, smooth, integrable, compactly supported initial data

$$u(x,0) = u_0(x) \quad \text{in } \Gamma_0[u] \cap \{t=0\}. \tag{2.2}$$

In the CP for (1.1) in $\mathbb{R}^N \times \mathbb{R}_+$, one needs to pose bounded compactly supported initial data (2.2) prescribed in $\mathbb{R}^N$. Then, under the same zero flux condition at finite interfaces (to be established separately), the mass is preserved.

2.2. **Global similarity solutions: towards a nonlinear eigenvalue problem.** We now begin to specify the self-similar solutions of the equation (1.1), which are admitted due to its natural scaling-invariant nature. In the case of the mass being conserved, we have global in time source-type solutions.

Using the following scaling in (1.1)

$$x := \mu \bar{x}, \quad t := \lambda \bar{t}, \quad u := \nu \bar{u}, \quad \text{with},$$

$$\frac{\partial u}{\partial t} = \frac{\nu}{\lambda}\frac{\partial \bar{u}}{\partial t}, \quad \frac{\partial u}{\partial x_i} = \frac{\nu}{\mu}\frac{\partial \bar{u}}{\partial x_i}, \quad \frac{\partial^2 u}{\partial x_i^2} = \frac{\nu}{\mu^2}\frac{\partial^2 \bar{u}}{\partial x_i^2},$$

and substituting those expressions in (1.1) yields

$$\frac{\nu}{\lambda}\frac{\partial \bar{u}}{\partial t} = -\frac{\nu^{n+1}}{\mu^4}\nabla \cdot (|\bar{u}|^n \nabla \Delta \bar{u})\,.$$

To keep this equation invariant, the following must be fulfilled:

$$\frac{\nu}{\lambda} = \frac{\nu^{n+1}}{\mu^4}, \tag{2.3}$$

so that

$$\mu := \lambda^\beta \Rightarrow \nu := \lambda^{\frac{4\beta-1}{n}} \quad \text{and} \quad u(x,t) := \lambda^{\frac{4\beta-1}{n}} \bar{u}(\bar{x},\bar{t}) = \lambda^{\frac{4\beta-1}{n}} \bar{u}(\frac{x}{\mu}, \frac{t}{\lambda}).$$

Consequently,

$$u(x,t) := t^{\frac{4\beta-1}{n}} f(\frac{x}{t^\beta}),$$

where $t = \lambda$ and $f(x/t^\beta) = \bar{u}(x/t^\beta, 1)$. Owing to (2.3), we obtain

$$n\alpha + 4\beta = 1,$$

which links the parameters α and β. Hence, substituting

$$u(x,t) := t^{-\alpha} f(y), \quad \text{with} \quad y := \frac{x}{t^\beta}, \quad \beta = \frac{1-n\alpha}{4}, \tag{2.4}$$

into (1.1) and rearranging terms, we find that the function f solves a quasilinear elliptic equation of the form

$$\nabla \cdot (|f|^n \nabla \Delta f) = \alpha f + \beta y \nabla \cdot f\,. \tag{2.5}$$

Finally, thanks to the above relation between α and β, we find a *nonlinear eigenvalue problem* of the form

$$-\nabla\cdot(|f|^n\nabla\Delta f)+\frac{1-\alpha n}{4}y\nabla\cdot f+\alpha f=0,\quad f\in C_0(\mathbb{R}^N)\,, \tag{2.6}$$

where we add to the equation (2.5) a natural assumption that f must be compactly supported (and, of course, sufficiently smooth at the interface, which is an accompanying question to be discussed as well).

Thus, for such degenerate elliptic equations, the functional setting in (2.6) assumes that we are looking for (weak) *compactly supported* solutions $f(y)$ as certain "nonlinear eigenfunctions" that hopefully occur for special values of nonlinear eigenvalues $\{\alpha_k\}_{k\ge 0}$. Our goal is to justify that, labelling the eigenfunctions via a multiindex σ,

Equation (2.6) possesses a countable set of eigenfunction/value pairs $\{f_k,\alpha_k\}_{|\sigma|=k\ge 0}$. (2.7)

Concerning the well-known properties of finite propagation for TFEs, we refer to papers [15]– [18], where a large amount of earlier references are available; see also [23] for more recent results and references in this elliptic area.

However, one should observe that there are still a few entirely rigorous results, especially those that are attributed to the Cauchy problem for TFEs; for example [3].

In the linear case $n=0$, the condition $f\in C_0(\mathbb{R}^N)$ is naturally replaced by the requirement that the eigenfunctions $\psi_\beta(y)$ exhibit typical exponential decay at infinity, a property that is reinforced by introducing appropriate weighted L^2-spaces. Actually, using the homotopy limit $n\to 0^+$, we will be obliged for small $n>0$, instead of C_0-setting in (2.6), to use the following weighted L^2-space:

$$f\in L^2_\rho(\mathbb{R}^N),\quad \text{where } \rho(y)=\mathrm{e}^{a|y|^{4/3}},\ a>0 \text{ small}\,. \tag{2.8}$$

Note that, in the case of the Cauchy problem with conservation of mass making use of the self-similar solutions (2.4), we have that

$$M(t):=\int_{\mathbb{R}^N}u(x,t)\,\mathrm{d}x=t^{-\alpha}\int_{\mathbb{R}^N}f\big(\frac{x}{t^\beta}\big)\,\mathrm{d}x=t^{-\alpha+\beta N}\int_{\mathbb{R}^N}f(y)\,\mathrm{d}y,$$

where the actual integration is performed over the support $\operatorname{supp} f$ of the nonlinear eigenfunction. Then, as is well known, if $\int f\neq 0$, the exponents are calculated giving the first explicit nonlinear eigenvalue:

$$-\alpha+\beta N=0\Rightarrow\alpha_0(n)=\frac{N}{4+Nn}\quad\text{and}\quad\beta_0(n)=\frac{1}{4+Nn}. \tag{2.9}$$

3. Hermitian spectral theory of the linear rescaled operators

In this section, we establish the spectrum $\sigma(\mathbf{B})$ of the linear operator $\mathbf{B}$ obtained from the rescaling of the linear counterpart of (1.1), i.e. the bi-harmonic equation (1.3), which will be essentially used in what follows.

3.1. How the operator B appears: a linear eigenvalue problem.

Let $u(x,t)$ be the unique solution of the CP for the linear parabolic bi-harmonic equation (1.3) with the initial data (the space as in (2.8) to be more properly introduced shortly)

$$u_0\in L^2_\rho(\mathbb{R}^N),$$

given by the convolution Poisson-type integral

$$u(x,t) = b(t) * u_0 \equiv t^{-N/4} \int_{\mathbb{R}^N} F((x-z)t^{-1/4}) u_0(z)\, \mathrm{d}z. \tag{3.1}$$

Here, by scaling invariance of the problem, in a similar way as was done in the previous section for (1.1), the unique fundamental solution of the operator $\frac{\partial}{\partial t} + \Delta^2$ has the self-similar structure

$$b(x,t) = t^{-N/4} F(y), \quad y := \frac{x}{t^{1/4}} \quad (x \in \mathbb{R}^N). \tag{3.2}$$

Substituting $b(x,t)$ into (1.3) yields that the rescaled fundamental kernel F in (3.2) solves the linear elliptic problem

$$\mathbf{B}F \equiv -\Delta_y^2 F + \frac{1}{4} y \cdot \nabla_y F + \frac{N}{4} F = 0 \quad \text{in} \quad \mathbb{R}^N, \quad \int_{\mathbb{R}^N} F(y)\, \mathrm{d}y = 1. \tag{3.3}$$

$\mathbf{B}$ is a non-symmetric linear operator, which is bounded from $H^4_\rho(\mathbb{R}^N)$ to $L^2_\rho(\mathbb{R}^N)$ with the exponential weight as in (2.8). Here, $a \in (0, 2d)$ is any positive constant, depending on the parameter $d > 0$, which characterises the exponential decay of the kernel $F(y)$:

$$|F(y)| \le D \mathrm{e}^{-d|y|^{4/3}} \quad \text{in} \quad \mathbb{R}^N \quad \big(D > 0, \quad d = 3 \cdot 2^{-11/3}\big). \tag{3.4}$$

By F we denote the oscillatory rescaled kernel as the only solution of (3.3), which has exponential decay, oscillates as $|y| \to \infty$, and satisfies the standard pointwise estimate (3.4).

Thus, we need to solve the corresponding *linear eigenvalue problem*:

$$\mathbf{B}\psi = \lambda \psi \quad \text{in } \mathbb{R}^N, \quad \psi \in L^2_\rho(\mathbb{R}^N). \tag{3.5}$$

One can see that the nonlinear one (2.6) formally reduces to (3.5) at $n = 0$ with the following shifting of the corresponding eigenvalues:

$$\lambda = -\alpha + \frac{N}{4}.$$

In fact, this is the main reason to calling (2.6) a nonlinear eigenvalue problem and, crucially, the discreteness of the real spectrum of the linear one (3.5) will be shown to be inherited by the nonlinear problem.

3.2. **Functional setting and semigroup expansion.** Thus, we solve (3.5) and calculate the spectrum of $\sigma(\mathbf{B})$ in the weighted space $L^2_\rho(\mathbb{R}^N)$. We then need the following Hilbert space:

$$H^4_\rho(\mathbb{R}^N) \subset L^2_\rho(\mathbb{R}^N) \subset L^2(\mathbb{R}^N).$$

The Hilbert space $H^4_\rho(\mathbb{R}^N)$ has the following inner product:

$$\langle v, w \rangle_\rho := \int_{\mathbb{R}^N} \rho(y) \sum_{k=0}^{4} D^k v(y) \overline{D^k w(y)}\, \mathrm{d}y,$$

where $D^k v$ stands for the vector $\{D^\beta v\,,\, |\beta| = k\}$, and the norm

$$\|v\|^2_\rho := \int_{\mathbb{R}^N} \rho(y) \sum_{k=0}^{4} |D^k v(y)|^2\, \mathrm{d}y.$$

Next, introducing the rescaled variables

$$u(x,t) = t^{-N/4} w(y,\tau), \quad y := \frac{x}{t^{1/4}}, \quad \tau = \ln t : \mathbb{R}_+ \to \mathbb{R}, \tag{3.6}$$

we find that the rescaled solution $w(y,\tau)$ satisfies the evolution equation

$$w_\tau = \mathbf{B} w\,, \tag{3.7}$$

since, substituting the representation of $u(x,t)$ (3.6) into (1.3) yields

$$-\Delta_y^2 w + \frac{1}{4} y \cdot \nabla_y w + \frac{N}{4} w = t \frac{\partial w}{\partial t} \frac{\partial \tau}{\partial t}.$$

Thus, to keep this invariant, the following should be satisfied:

$$t \frac{\partial \tau}{\partial t} = 1 \Rightarrow \tau = \ln t,$$

i.e., as defined in (3.6). Hence, $w(y,\tau)$ is the solution of the Cauchy problem for the equation (3.7) and with the following initial condition at $\tau = 0$, i.e. at $t = 1$:

$$w_0(y) = u(y,1) \equiv b(1) * u_0 = F * u_0\,. \tag{3.8}$$

Then, the linear operator $\frac{\partial}{\partial \tau} - \mathbf{B}$ is also a rescaled version of the standard parabolic one $\frac{\partial}{\partial t} + \Delta^2$. Therefore, the corresponding semigroup $e^{\mathbf{B}\tau}$ admits an explicit integral representation. This helps to establish some properties of the operator $\mathbf{B}$ and describes other evolution features of the linear flow.

Indeed, from (3.1) we find the following explicit representation of the semigroup:

$$w(y,\tau) = \int_{\mathbb{R}^N} F\big(y - z e^{-\frac{\tau}{4}}\big)\, u_0(z)\, dz \equiv e^{\mathbf{B}\tau} w_0,$$

where $x = t^{1/4} y$, $\tau = \ln t$. Subsequently, consider Taylor's power series of the analytic kernel[1]

$$F\big(y - z e^{-\frac{\tau}{4}}\big) = \sum_{(\beta)} e^{-\frac{|\beta|\tau}{4}} \frac{(-1)^{|\beta|}}{\beta!} D^\beta F(y) z^\beta \equiv \sum_{(\beta)} e^{-\frac{|\beta|\tau}{4}} \frac{1}{\sqrt{\beta!}} \psi_\beta(y) z^\beta, \tag{3.9}$$

for any $y \in \mathbb{R}^N$, where

$$z^\beta := z_1^{\beta_1} \dots z_N^{\beta_N}\,,$$

and ψ_β are the normalized eigenfunctions for the operator $\mathbf{B}$. The series in (3.9) converges uniformly on compact subsets in $z \in \mathbb{R}^N$. Indeed, denoting $|\beta| = l$ and estimating the coefficients

$$\Big| \sum_{|\beta|=l} \frac{(-1)^l}{\beta!} D^\beta F(y) z_1^{\beta_1} \dots z_N^{\beta_N} \Big| \le b_l |z|^l,$$

by Stirling's formula we have that, for $l \gg 1$,

$$b_l = \frac{N^l}{l!} \sup_{y \in \mathbb{R}^N, |\beta| = l} |D^\beta F(y)| \approx \frac{N^l}{l!} l^{-l/4} e^{l/4} \approx l^{-3l/4} c^l = e^{-l \ln 3l/4 + l \ln c}.$$

Note that, the series $\sum b_l |z|^l$, has radius of convergence $R = \infty$. Thus, we obtain the following representation of the solution:

$$w(y,\tau) = \sum_{(\beta)} e^{-\frac{|\beta|}{4}\tau} M_\beta(u_0) \psi_\beta(y), \quad \text{where } \lambda_\beta := -\frac{|\beta|}{4},$$

[1] We hope that returning here to the multiindex β instead of σ in (2.7) will not lead to a confusion with the exponent β in self-similar scaling (2.4).

and $\{\psi_\beta\}$ are the eigenvalues and eigenfunctions of the operator $\mathbf{B}$, respectively, and

$$M_\beta(u_0) := \frac{1}{\sqrt{\beta!}} \int_{\mathbb{R}^N} z_1^{\beta_1} \dots z_N^{\beta_N} u_0(z)\, \mathrm{d}z,$$

are the corresponding momenta of the initial datum w_0 defined by (3.8).

3.3. **Main spectral properties of the pair $\{\mathbf{B}, \mathbf{B}^*\}$.** Thus, the following holds [14]:

Theorem 3.1. (i) *The spectrum of $\mathbf{B}$ comprises real eigenvalues only with the form*

$$\sigma(\mathbf{B}) := \big\{\lambda_\beta = -\frac{|\beta|}{4},\ |\beta| = 0, 1, 2, \dots\big\}.$$

Eigenvalues λ_β have finite multiplicity with eigenfunctions,

$$\psi_\beta(y) := \frac{(-1)^{|\beta|}}{\sqrt{\beta!}} D^\beta F(y) \equiv \frac{(-1)^{|\beta|}}{\sqrt{\beta!}} \big(\frac{\partial}{\partial y_1}\big)^{\beta_1} \dots \big(\frac{\partial}{\partial y_N}\big)^{\beta_N} F(y). \tag{3.10}$$

(ii) *The subset of eigenfunctions $\Phi = \{\psi_\beta\}$ is complete in $L^2(\mathbb{R}^N)$ and in $L^2_\rho(\mathbb{R}^N)$.*
(iii) *For any $\lambda \notin \sigma(\mathbf{B})$, the resolvent $(\mathbf{B} - \lambda I)^{-1}$ is a compact operator in $L^2_\rho(\mathbb{R}^N)$.*

Subsequently, it was also shown in [14] that the adjoint (in the dual metric of $L^2(\mathbb{R}^N)$) operator of $\mathbf{B}$ given by

$$\mathbf{B}^* := -\Delta^2 - \frac{1}{4}\, y \cdot \nabla,$$

in the weighted space $L^2_{\rho^*}(\mathbb{R}^N)$, with the exponentially decaying weight function

$$\rho^*(y) \equiv \frac{1}{\rho(y)} = \mathrm{e}^{-a|y|^\alpha} > 0,$$

is a bounded linear operator, $\mathbf{B}^* : H^4_{\rho^*}(\mathbb{R}^N) \to L^2_{\rho^*}(\mathbb{R}^N)$, so

$$\langle \mathbf{B}v, w\rangle = \langle v, \mathbf{B}^* w\rangle, v \in H^4_\rho(\mathbb{R}^N),\ w \in H^4_{\rho^*}(\mathbb{R}^N).$$

Moreover, the following theorem establishes the spectral properties of the adjoint operator which will be very similar to those ones shown in Theorem 3.1 for the operator $\mathbf{B}$.

Theorem 3.2. (i) *The spectrum of $\mathbf{B}^*$ consists of eigenvalues of finite multiplicity,*

$$\sigma(\mathbf{B}^*) = \sigma(\mathbf{B}) := \big\{\lambda_\beta = -\frac{|\beta|}{4}\,,\ |\beta| = 0, 1, 2, \dots\big\},$$

and the eigenfunctions $\psi^*_\beta(y)$ are polynomials of order $|\beta|$.
(ii) *The subset of eigenfunctions $\Phi^* = \{\psi^*_\beta\}$ is complete in $L^2_{\rho^*}(\mathbb{R}^N)$.*
(iii) *For any $\lambda \notin \sigma(\mathbf{B}^*)$, the resolvent $(\mathbf{B}^* - \lambda I)^{-1}$ is a compact operator in $L^2_{\rho^*}(\mathbb{R}^N)$.*

It should be pointed out that, since $\psi_0 = F$ and $\psi^*_0 \equiv 1$, we have

$$\langle \psi_0, \psi^*_0\rangle = \int_{\mathbb{R}^N} \psi_0\, \mathrm{d}y = \int_{\mathbb{R}^N} F(y)\, \mathrm{d}y = 1.$$

However, thanks to (3.10), we have that

$$\int_{\mathbb{R}^N} \psi_\beta \equiv \langle \psi_\beta, \psi_0^* \rangle = 0 \quad \text{for } |\beta| \neq 0.$$

This expresses the orthogonality property to the adjoint eigenfunctions in terms of the dual inner product.

Note that as shown in [14], for the eigenfunctions $\{\psi_\beta\}$ of $\mathbf{B}$ denoted by (3.10), the corresponding adjoint eigenfunctions are *generalized Hermite polynomials* given by

$$\psi_\beta^*(y) := \frac{1}{\sqrt{\beta!}} \Big[y^\beta + \sum_{j=1}^{[|\beta|/4]} \frac{1}{j!}\, \Delta^{2j} y^\beta \Big]. \tag{3.11}$$

Hence, the orthonormality condition holds

$$\langle \psi_\beta, \psi_\gamma \rangle = \delta_{\beta\gamma} \quad \text{for any} \quad \beta,\ \gamma,$$

where $\langle \cdot, \cdot \rangle$ is the duality product in $L^2(\mathbb{R}^N)$ and $\delta_{\beta\gamma}$ is Kronecker's delta. Also, operators $\mathbf{B}$ and $\mathbf{B}^*$ have zero Morse index (no eigenvalues with positive real parts are available).

Moreover, the adjoint operator $\mathbf{B}^*$ was used in [2] to analyse blow-up solutions for the TFE-4 (1.1).

Some key spectral results can be extended as in [14] to $2m$th-order linear poly-harmonic flows

$$u_t = -(-\Delta)^m u \quad \text{in} \quad \mathbb{R}^N \times \mathbb{R}_+,$$

where the elliptic equation for the rescaled kernel $F(y)$ takes the form

$$\mathbf{B}F \equiv -(-\Delta_y)^m F + \frac{1}{2m} y \cdot \nabla_y F + \frac{N}{2m} F = 0 \quad \text{in} \quad \mathbb{R}^N, \quad \int_{\mathbb{R}^N} F(y)\, \mathrm{d}y = 1.$$

In particular, for $m = 1$, we find the *Hermite operator* and the *Gaussian kernel* (see [10] for further information)

$$\mathbf{B}F \equiv \Delta F + \frac{1}{2} y \cdot \nabla F + \frac{N}{2}\, F = 0 \;\Rightarrow\; F(y) = \frac{1}{(4\pi)^{N/2}}\, \mathrm{e}^{-|y|^2/4},$$

whose name is connected to fundamental works of Charles Hermite on orthogonal polynomials $\{H_\beta\}$ about 1870. These classic Hermite polynomials are obtained by differentiating the Gaussian: up to normalization constants,

$$D^\beta \mathrm{e}^{-\frac{|y|^2}{4}} = H_\beta(y)\, \mathrm{e}^{-\frac{|y|^2}{4}} \quad \text{for any } \beta. \tag{3.12}$$

Note that, for $N = 1$, such operators and polynomial eigenfunctions in 1D were studied earlier by Jacques C.F. Sturm in 1836; on this history and Sturm's main original calculations, see [19, Ch. 1].

The generating formula (3.12) for (generalized) Hermite polynomials is not available if $m \geq 2$, so that (3.11) are obtained via a different procedure, [14].

4. Similarity profiles for the Cauchy problem via n-branching

In general, construction of oscillatory similarity solutions of the Cauchy problem for the TFE-4 (1.1) is a difficult nonlinear problem, which is harder than for the corresponding FBP one.

On the other hand, for $n = 0$, such similarity profiles exist and are given by eigenfunctions $\{\psi_\beta\}$. In particular, the first mass-preserving profile is just the rescaled kernel $F(y)$, so it is unique, as was shown in Section 3.

Hence, somehow, a possibility to visualize such an oscillatory first "nonlinear eigenfunction" $f(y)$ of changing sign, which satisfies the *nonlinear eigenvalue problem* (2.6), at least, for sufficiently small $n > 0$ can be expected.

This suggests that, via an n-branching approach argument, it is possible to "connect" f with the rescaled fundamental profile F, satisfying the corresponding linear equation (3.3), with all the necessary properties of F presented in Section 3.

Thus, we plan to describe the behaviour of the similarity profiles $\{f_\beta\}$, as nonlinear eigenfunctions of (2.6) for the TFE performing a "homotopic" approach when $n \downarrow 0$.

Homotopic approaches are well-known in the theory of vector fields, degree and nonlinear operator theory (see [13, 28] for details). However, we shall be less precise in order to apply that approach and, here, a "homotopic path" just declares existence of a continuous connection (a curve) of solutions $f \in C_0$ that ends up at $n = 0^+$, at the linear eigenfunction $\psi_0(y) = F(y)$ or further eigenfunctions $\psi_\beta(y) \sim D^\beta F(y)$, as (3.10) claims.

Using classical branching theory in the case of finite regularity of nonlinear operators involved, we formally show that the necessary orthogonality condition holds deriving the corresponding *Lyapunov–Schmidt branching equation.* We will try to be as rigorous as possible in supporting the delivery of the nonlinear eigenvalues $\{\alpha_k\}$.

Further extensions of solutions for non-small $n > 0$ require a novel essentially non-local technique of such nonlinear analysis, which remains an open problem.

4.1. **Nonlinear eigenvalues $\{\alpha_k\}$ and transversality conditions for the nonlinear eigenfunctions f.** In this first part of the section we establish the conditions and terms necessary for the expansions of the parameter α and the nonlinear eigenfunctions, as well as the transversality oscillatory conditions for such nonlinear eigenfunctions.

This will allow us to obtain the desired countable number of solutions (1.5) for the similarity equation (1.4) via Lyapunov-Schmidt reduction through the subsequent analysis.

The nonlinear eigenvalues $\{\alpha_k\}$ are obtained according to non-self-adjoint spectral theory from Section 3. We then use the explicit expressions for the eigenvalues and eigenfunctions of the linear eigenvalue problem (3.5) given in Theorem 3.1, where we also need the main conclusions of the "adjoint" Theorem 3.2.

Thus, taking the corresponding linear equation from (2.6) with $n = 0$, we find, at least, formally, that

$$n = 0: \quad \mathcal{L}(\alpha) f := -\Delta^2 f + \frac{1}{4} y \cdot \nabla f + \alpha f = 0.$$

Moreover, from that equation, combined with the eigenvalues expressions obtained in the previous section, we ascertain the following critical values for the parameter $\alpha_k = \alpha_k(n)$,

$$n = 0: \quad \alpha_k(0) := -\lambda_k + \frac{N}{4} \equiv \frac{k+N}{4} \quad \text{for } k = 0, 1, 2, \dots, \tag{4.1}$$

where λ_k are the eigenvalues defined in Theorem 3.1, so that

$$\alpha_0(0) = \frac{N}{4}, \quad \alpha_1(0) = \frac{N+1}{4}, \quad \alpha_2(0) = \frac{N+2}{4}, \dots, \alpha_k(0) = \frac{k+N}{4} \dots .$$

In particular, when $k = 0$, we have that $\alpha_0(0) = \frac{N}{4}$ and the eigenfunction satisfies

$$\mathbf{B}F = 0, \quad \text{so that } \ker \mathcal{L}(\alpha_0) = \operatorname{span}\{\psi_0\} \quad (\psi_0 = F),$$

and, hence, since $\lambda_0 = 0$ is a simple eigenvalue for the operator $\mathcal{L}(\alpha_0) = \mathbf{B}$, its algebraic multiplicity is 1. In general, we find that

$$\ker \big(\mathbf{B} + \frac{k}{4}\, I\big) = \operatorname{span}\{\psi_\beta, |\beta| = k\}, \quad \text{for any } k = 0, 1, 2, 3, \dots ,$$

where the operator $\mathbf{B} + \frac{k}{4} I$ is Fredholm of index zero since it is a compact perturbation of the identity of linear type with respect to k. In other words, $R[\mathcal{L}(\alpha_k)]$ is a closed subspace of $L^2_\rho(\mathbb{R}^N)$ and, for each α_k,

$$\dim \ker(\mathcal{L}(\alpha_k)) < \infty \quad \text{and} \quad \operatorname{codim} R[\mathcal{L}(\alpha_k)] < \infty.$$

Then, for small $n > 0$ in (2.6), we can assume the following asymptotic expansions

$$\alpha_k(n) := \alpha_k + \mu_{1,k} n + o(n), \quad \text{and} \tag{4.2}$$

$$|f|^n \equiv \mathrm{e}^{n \ln |f|} := 1 + n \ln |f| + o(n). \tag{4.3}$$

As customary in bifurcation-branching theory [28, 32], existence of an expansion such as (4.2) will allow one to get further expansion coefficients in

$$\alpha_k(n) := \alpha_k + \mu_{1,k} n + \mu_{2,k} n^2 + \mu_{3,k} n^3 + \dots ,$$

as the regularity of nonlinearities allows and suggests, though the convergence of such an analytic series can be questionable and is not under scrutiny here.

Another principle question is that, for oscillatory sign changing profiles $f(y)$, the last expansion (4.3) cannot be understood in the pointwise sense. However, it can be naturally expected to be valid in other metrics such as weighted L^2 or Sobolev spaces, as in Section 3, that used to be appropriate for the functional setting of the equivalent integral equation and for that with $n = 0$.

Then, since (4.3) is obviously pointwise violated at the nodal set $\{f = 0\}$ of $f(y)$, this imposes some restrictions on the behaviour of corresponding eigenfunctions $\psi_\beta(y)$ $(n = 0)$ close to their zero sets. Using well-known asymptotic and other related properties of the *radial* analytic rescaled kernel $F(y)$ of the fundamental solutions (3.2), the generating formula of eigenfunctions (3.10) confirms that the nodal set of analytic eigenfunctions $\{\psi_\beta = 0\}$ consists of isolated zero surfaces, which are "transversal", at least in the a.e. sense, with the only accumulation point at $y = \infty$. Overall, under such conditions, this indicates that

Expansion (4.3) contains not more than "logarithmic" singularities a.e., (4.4)

which well suited the integral compact operators involved in the branching analysis, though we are far from claiming this as any rigorous issue.

Moreover, when $n > 0$ is not small enough, such an analogy and statements like (4.4) become unclear, and global extensions of continuous n-branches induced by some compact integral operators, i.e. nonexistence of turning (saddle-node) points in n, require, as usual, some unknown monotonicity-like results.

Then, to carry out our homotopic approach we assume the expansion (4.3) away from possible zero surfaces of $f(y)$, which, by transversality, can be localized in arbitrarily small neighbourhoods.

Indeed, it is clear that when

$$|f| > \delta > 0, \quad \text{for any} \quad \delta > 0,$$

there is no problem in approximating $|f|^n$ by (4.3), i.e.,

$$|f|^n = 1 + O(n) \quad \text{as } n \to 0^+.$$

However, when

$$|f| \le \delta, \quad \text{for any } \delta > 0,$$

sufficiently small, the proof of such an approximation in weak topology (as suffices for dealing with equivalent integral equations) is far from clear unless

the zeros of the f's are also transversal a.e.,

with a standard accumulating property at the only interface zero surface. The latter issues have been studied and described in [16] in the radial setting. Hence, we can suppose that such nonlinear eigenfunctions $f(y)$ are oscillatory and infinitely sign changing close to the interface surface.

Therefore, if we assume that their zero surface is transversal a.e. with a known geometric-like accumulation at the interface, we find that, for any n close to zero and any $\delta = \delta(n) > 0$ sufficiently small,

$$n|\ln |f|| \gg 1, \quad \text{if } |f| \le \delta(n),$$

and, hence, on such subsets, $f(y)$ must be exponentially small:

$$|\ln |f|| \gg \frac{1}{n} \;\Rightarrow\; \ln |f| \ll -\frac{1}{n} \;\Rightarrow\; |f| \ll \mathrm{e}^{-\frac{1}{n}}.$$

Recall that this happens in also exponentially small neighbourhoods of the transversal zero surfaces.

Overall, using the periodic structure of the oscillatory component at the interface [16] (we must admit that such delicate properties of oscillatory structures of solutions are known for the 1D and radial cases only, though we expect that these phenomena are generic), we can control the singular coefficients in (4.3), and, in particular, to see that

$$\ln |f| \in L^1_{\rm loc}(\mathbb{R}^N). \tag{4.5}$$

However, for most general geometric configurations of nonlinear eigenfunctions $f(y)$, we do not have a proper proof of (4.5) or similar estimates, so our further analysis is still essentially formal.

4.2. **Derivation of the branching equation.** Under the above-mentioned transversality conditions and assuming the expansions (4.2), for the nonlinear eigenvalues α_k, and (4.3), for the nonlinear eigenfunctions f, we are able to obtain the branching equation applying the classical Lyapunov-Schmidt method.

It is worth recalling again that our computations below are to be understood as those dealing with the equivalent integral equations and operators, so, in particular, we can use the powerful facts on compactness of the resolvent $(\mathbf{B} - \lambda I)^{-1}$ and of the adjoint one $(\mathbf{B}^* - \lambda I)^{-1}$ in the corresponding weighted L^2-spaces. Note that, in such an equivalent integral representation, the singular term in (4.3) satisfying (4.5) makes no principal difficulty, so the expansion (4.3) makes rather usual sense for applying standard nonlinear operator theory.

Thus, under natural assumptions, substituting (4.2) into (2.6), for any $k = 0, 1, 2, 3, \dots$, we find that, omitting $o(n)$ terms when necessary,

$$-\nabla \cdot [(1 + n \ln |f|)\nabla \Delta f] + \frac{1 - \alpha_k n - \mu_{1,k} n^2}{4} y \cdot \nabla f + (\alpha_k + \mu_{1,k} n) f = 0\,,$$

and, rearranging terms,

$$-\Delta^2 f - n\nabla \cdot (\ln |f| \nabla \Delta f) + \frac{1}{4} y \cdot \nabla f - \frac{\alpha_k n + \mu_{1,k} n^2}{4} y \cdot \nabla f + \alpha_k f + \mu_{1,k} n f = 0\,.$$

Hence, we finally have

$$\big(\mathbf{B} + \frac{k}{4}\, I\big) f + n\big[-\nabla \cdot (\ln |f| \nabla \Delta f) - \frac{\alpha_k}{4} y \cdot \nabla f + \mu_{1,k} f \big] + o(n) = 0\,,$$

which can be written in the form

$$\big(\mathbf{B} + \frac{k}{4}\, I\big) f + n \mathcal{N}_k(f) + o(n) = 0\,, \tag{4.6}$$

with the operator

$$\mathcal{N}_k(f) := -\nabla \cdot (\ln |f| \nabla \Delta f) - \frac{\alpha_k}{4} y \cdot \nabla f + \mu_{1,k} f\,.$$

Subsequently, as was shown in Section 3, we have that

$$\ker \big(\mathbf{B} + \frac{k}{4}\, I\big) = \operatorname{span}\{\psi_\beta\}_{|\beta| = k} \quad \text{for } k = 0, 1, 2, 3, \dots,$$

where the operator $\mathbf{B} + \frac{k}{4} I$ is Fredholm of index zero and

$$\dim \ker \big(\mathbf{B} + \frac{k}{4}\, I\big) = M_k \geq 1 \quad \text{for any } k = 0, 1, 2, 3, \dots,$$

where M_k stands for the length of the vector $\{D^\beta v,\, |\beta| = k\}$, so that $M_k > 1$ for $k \geq 1$.

Simple eigenvalue for $k = 0$**.** Since 0 is a simple eigenvalue of $\mathbf{B}$ when $k = 0$, i.e.,

$$\ker \mathbf{B} \oplus R[\mathbf{B}] = L^2_\rho(\mathbb{R}^N),$$

the study of the case $k = 0$ seems to be simpler than for other different k's because the dimension of the eigenspace is $M_0 = 1$.

Thus, we shall describe the behaviour of solutions for small $n > 0$ and apply the classical Lyapunov–Schmidt method to (4.6) (assuming, as usual, some extra necessary regularity hypothesis to be clarified later on), in order to accomplish the branching approach as $n \downarrow 0$, in two steps, when $k = 0$ and k is different from 0.

Thus, owing to Section 3, we already know that 0 is a simple eigenvalue of $\mathbf{B}$, i.e. $\ker \mathbf{B} = \operatorname{span}\{\psi_0\}$ is one-dimensional. Hence, denoting by Y_0 the complementary invariant subspace, orthogonal to ψ_0^*, we set

$$f = \psi_0 + V_0,$$

where $V_0 \in Y_0$.

Moreover, according to the spectral properties of the operator $\mathbf{B}$, we define P_0 and P_1 such that $P_0 + P_1 = I$, to be the projections onto $\ker \mathbf{B}$ and Y_0 respectively. Finally, setting

$$V_0 := n\Phi_{1,0} + o(n), \tag{4.7}$$

substituting the expression for f into (4.6) and passing to the limit as $n \to 0^+$ leads to a linear inhomogeneous equation for $\Phi_{1,0}$,

$$\mathbf{B}\Phi_{1,0} = -\mathcal{N}_0(\psi_0), \tag{4.8}$$

since $\mathbf{B}\psi_0 = 0$.

Furthermore, by Fredholm theory, $V_0 \in Y_0$ exists if and only if the right-hand side is orthogonal to the one dimensional kernel of the adjoint operator $\mathbf{B}^*$ with $\psi_0^* = 1$, because of (3.11). Hence, in the topology of the dual space L^2, this requires the standard orthogonality condition:

$$\langle \mathcal{N}_0(\psi_0), 1\rangle = 0. \tag{4.9}$$

Then, (4.8) has a unique solution $\Phi_{1,0} \in Y_0$ determining by (4.7) a bifurcation branch for small $n > 0$. In fact, the algebraic equation (4.9) yields the following explicit expression for the coefficient $\mu_{1,0}$ of the expansion (4.2) for the first eigenvalue $\alpha_0(n)$:

$$\mu_{1,0} := \frac{\langle \nabla\cdot(\ln|\psi_0|\nabla\Delta\psi_0) + \frac{N}{16}y\cdot\nabla\psi_0, \psi_0^*\rangle}{\langle \psi_0, \psi_0^*\rangle} = \langle \nabla\cdot(\ln|\psi_0|\nabla\Delta\psi_0) + \frac{N}{16}y\cdot\nabla\psi_0, \psi_0^*\rangle.$$

Consequently, in the particular case of having simple eigenvalues we just obtain one branch of solutions emanating at $n = 0$.

Multiple eigenvalues for $k \geq 1$. Next we ascertain the number of branches in the case when the eigenvalues of the operator $\mathbf{B}$ are semisimple. For any $k \geq 1$, we know that

$$\dim\ker\big(\mathbf{B} + \frac{k}{4}\,I\big) = M_k > 1.$$

Hence, in order to perform a similar analysis to the one done for simple eigenvalues we have to use the full eigenspace expansion

$$f = \sum_{|\beta|=k} c_\beta\hat{\psi}_\beta + V_k, \tag{4.10}$$

for every $k \geq 1$. Currently, for convenience, we denote

$$\{\hat{\psi}_\beta\}_{|\beta|=k} = \{\hat{\psi}_1, \dots, \hat{\psi}_{M_k}\},$$

the natural basis of the M_k-dimensional eigenspace $\ker\big(\mathbf{B} + \frac{k}{4}\,I\big)$ and set

$$\psi_k = \sum_{|\beta|=k} c_\beta\hat{\psi}_\beta.$$

Moreover,

$$V_k \in Y_k \quad\text{and}\quad V_k = \sum_{|\beta|>k} c_\beta\psi_\beta,$$

where Y_k is the complementary invariant subspace of $\ker\big(\mathbf{B} + \frac{k}{4}\,I\big)$.

Furthermore, in the same way, as we did for the case $k = 0$, we define the $P_{0,k}$ and $P_{1,k}$, for every $k \geq 1$, to be the projections of $\ker\big(\mathbf{B} + \frac{k}{4}\,I\big)$ and Y_k respectively. We also expand V_k as

$$V_k := n\Phi_{1,k} + o(n). \tag{4.11}$$

Subsequently, substituting (4.10) into (4.6) and passing to the limit as $n \downarrow 0^+$, we obtain the following equation:

$$\big(\mathbf{B} + \frac{k}{4}\,I\big)\Phi_{1,k} = -\mathcal{N}_k\big(\sum_{|\beta|=k} c_\beta\psi_\beta\big), \tag{4.12}$$

under the natural "normalizing" constraint

$$\sum_{|\beta|=k} c_\beta = 1 \quad (c_\beta \ge 0). \tag{4.13}$$

Therefore, applying the Fredholm alternative, $V_k \in Y_k$ exists if and only if the term on the right-hand side of (4.12) is orthogonal to ker $\big(\mathbf{B} + \frac{k}{4}\, I\big)$. Then, multiplying the right-hand side of (4.12) by ψ_β^*, for every $|\beta| = k$, in the topology of the dual space L^2, we obtain an algebraic system of $M_k + 1$ equations and the same number of unknowns, $\{c_\beta,\, |\beta| = k\}$ and $\mu_{1,k}$:

$$\langle \mathcal{N}_k\big(\sum_{|\beta|=k} c_\beta \psi_\beta\big), \psi_\beta^* \rangle = 0 \quad \text{for all } |\beta| = k, \tag{4.14}$$

which is indeed the Lyapunov–Schmidt branching equation [32]. In general, such algebraic systems are assumed to allow us to obtain the branching parameters and, hence, establish the number of different solutions induced on the given M_k-dimensional eigenspace as the kernel of the operator involved.

However, we must admit and urge that the algebraic system (4.14) is a truly difficult issue. One of the main features of it is as follows:

$$\text{Equation (4.14) is not variational.} \tag{4.15}$$

In other words, one cannot use for (4.14) the classic category-genus theory of calculus of variation [6, 28], to claim that the category of the kernel (equal to M_k) is the least number of different critical points and hence of different solutions.

To see (4.15), it suffices to note that, due to (3.10) and (3.11), the generalized Hermite polynomials ψ_β^* have nothing in common in the algebraic sense with the eigenfunctions ψ_β in the L^2-scalar products in (4.14).

4.3. **A digression to Hermite classic self-adjoint theory.** It is worth mentioning that for the classic second-order Hermite operator

$$\mathbf{B} = \Delta + \frac{1}{2} y \cdot \nabla + \frac{N}{2} I \quad (\text{then, in the } L^2\text{-metric, } \mathbf{B}^* = \Delta - \frac{1}{2} y \cdot \nabla), \tag{4.16}$$

statement (4.15) does not hold. Indeed, by classic theory [10, p. 48], these eigenfunctions are related to each other by

$$\psi_\beta(y) = D^\beta F(y) \equiv H_\beta(y) F(y), \quad \text{where } F(y) = (4\pi)^{-\frac{N}{2}}\, \mathrm{e}^{-\frac{|y|^2}{4}} \tag{4.17}$$

is the Gaussian kernel and $H_\beta(y)$ are standard Hermite polynomials, which also define the adjoint eigenfunctions:

$$\psi_\beta^*(y) = b_\beta H_\beta(y) \equiv \frac{b_\beta}{F(y)}\, \psi_\beta(y), \tag{4.18}$$

where b_β are normalization constants. One knows that this result comes from the symmetry of the operator (4.16) in the weighted metric of $L^2_\rho(\mathbb{R}^N)$, where

$$\rho(y) = \mathrm{e}^{\frac{|y|^2}{4}} \sim \frac{1}{F(y)} \ \Rightarrow\ \mathbf{B} = \frac{1}{\rho}\, \nabla \cdot (\rho \nabla) + \frac{N}{2}\, I, \quad \text{so } (\mathbf{B})^*_{L^2_\rho} = \mathbf{B}.$$

In view of the relations (4.17) and (4.18) of the bi-orthonormal bases $\{\psi_\beta\}$ and $\{\psi_\beta^*\}$, the corresponding algebraic systems such as (4.14) can be variational. Moreover, even the original nonlinear elliptic equation, similar to (2.6), where the 4th-order operator is replaced by a natural 2nd-order, one of the porous medium type:

$$-\nabla(|f|^n \nabla \Delta f) \mapsto \nabla(|f|^n \nabla f),$$

then, we are in a situation where it becomes variational.

Thus, in this case, both branching (local phenomena) and global extensions of n-bifurcation branches can be performed on the basis of the powerful Lusternik–Schnirel'man category variational theory from 1920s [28, § 56], so that existence and multiplicity (at least, not less than in the linear case $n = 0$) of solutions are guaranteed.

4.4. **Computations for branching of dipole solutions in 2D.** To avoid excessive computations and as a self-contained example, we now ascertain some expressions for those coefficients in the case when $|\beta| = 1$, $N = 2$, and $M_1 = 2$, so that, in our notations, $\{\psi_\beta\}_{|\beta|=1} = \{\hat\psi_1, \hat\psi_2\}$.

Consequently, in this case, we obtain the following algebraic system: the expansion coefficients of $\psi_1 = c_1\hat\psi_1 + c_2\hat\psi_2$ satisfy

$$\begin{gathered} c_1\langle \hat\psi_1^*, h_1\rangle - \frac{c_1\alpha_1}{4}\,\langle \hat\psi_1^*, y\cdot\nabla\hat\psi_1\rangle + c_1\mu_{1,1} + c_2\langle \hat\psi_1^*, h_2\rangle - \frac{c_2\alpha_1}{4}\,\langle \hat\psi_1^*, y\cdot\nabla\hat\psi_2\rangle = 0,\\ c_1\langle \hat\psi_2^*, h_1\rangle - \frac{c_1\alpha_1}{4}\,\langle \hat\psi_2^*, y\cdot\nabla\hat\psi_1\rangle + c_2\langle \hat\psi_2^*, h_2\rangle - \frac{c_2\alpha_1}{4}\,\langle \hat\psi_2^*, y\cdot\nabla\hat\psi_2\rangle + c_2\mu_{1,1} = 0,\\ c_1 + c_2 = 1, \end{gathered} \tag{4.19}$$

where

$$h_1 := -\nabla\cdot[\ln(c_1\hat\psi_1 + c_2\hat\psi_2)\nabla\Delta\hat\psi_1], \quad h_2 := -\nabla\cdot[\ln(c_1\hat\psi_1 + c_2\hat\psi_2)\nabla\Delta\hat\psi_2],$$

and, c_1, c_2, and $\mu_{1,1}$ are the coefficients that we want to calculate, α_1 is regarded as the value of the parameter α denoted by (4.1) and dependent on the eigenvalue λ_1, for which $\hat\psi_{1,2}$ are the associated eigenfunctions, and $\hat\psi_{1,2}^*$ the corresponding adjoint eigenfunctions. Hence, substituting the expression $c_2 = 1 - c_1$ from the third equation into the other two, we have the following nonlinear algebraic system

$$\begin{gathered} 0 = N_1(c_1, \mu_{1,1}) - c_1\frac{\alpha_1}{4}\left[\langle \hat\psi_1^*, y\cdot\nabla\hat\psi_1\rangle - \langle \hat\psi_1^*, y\cdot\nabla\hat\psi_2\rangle\right],\\ 0 = N_2(c_1, \mu_{1,1}) - c_1\frac{\alpha_1}{4}\left[\langle \hat\psi_2^*, y\cdot\nabla\hat\psi_1\rangle - \langle \hat\psi_2^*, y\cdot\nabla\hat\psi_2\rangle\right] + \mu_{1,1}, \end{gathered} \tag{4.20}$$

where

$$\begin{gathered} N_1(c_1, \mu_{1,1}) := c_1\langle \hat\psi_1^*, h_1\rangle + \langle \hat\psi_1^*, h_2\rangle - \frac{\alpha_1}{4}\,\langle \hat\psi_1^*, y\cdot\nabla\hat\psi_2\rangle - c_1\langle \hat\psi_1^*, h_2\rangle + c_1\mu_{1,1},\\ N_2(c_1, \mu_{1,1}) := c_1\langle \hat\psi_2^*, h_1\rangle + \langle \hat\psi_2^*, h_2\rangle - \frac{\alpha_1}{4}\,\langle \hat\psi_2^*, y\cdot\nabla\hat\psi_2\rangle - c_1\langle \hat\psi_2^*, h_2\rangle - c_1\mu_{1,1}, \end{gathered}$$

represent the nonlinear parts of the algebraic system, with h_0 and h_1 depending on c_1.

Subsequently, to guarantee existence of solutions of the system (4.19), we apply the Brouwer fixed point theorem to (4.20) by supposing that the values c_1 and $\mu_{1,1}$ are the unknowns, in a disc sufficiently big $D_R(\hat c_1, \hat\mu_{1,1})$ centered in a possible non-degenerate zero $(\hat c_1, \hat\mu_{1,1})$. Thus, we write the system (4.20) in the matrix form

$$\begin{pmatrix} 0\\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{\alpha_1}{4}\left[\langle \hat\psi_1^*, y\cdot\nabla\hat\psi_1\rangle - \langle \hat\psi_1^*, y\cdot\nabla\hat\psi_2\rangle\right] & 0\\ -\frac{\alpha_1}{4}\left[\langle \hat\psi_2^*, y\cdot\nabla\hat\psi_1\rangle - \langle \hat\psi_2^*, y\cdot\nabla\hat\psi_2\rangle\right] & 1 \end{pmatrix} \begin{pmatrix} c_1\\ \mu_{1,1} \end{pmatrix} + \begin{pmatrix} N_1(c_1, \mu_{1,1})\\ N_2(c_1, \mu_{1,1}) \end{pmatrix}.$$

Hence, we have that the zeros of the operator

$$\mathcal{F}(c_1,\mu_{1,1}) := \mathfrak{M}\begin{pmatrix} c_1 \\ \mu_{1,1} \end{pmatrix} + \begin{pmatrix} N_1(c_1,\mu_{1,1}) \\ N_2(c_1,\mu_{1,1}) \end{pmatrix},$$

are the possible solutions of (4.20), where $\mathfrak{M}$ is the matrix corresponding to the linear part of the system, while

$$(N_1(c_1,\mu_{1,1}), N_2(c_1,\mu_{1,1}))^T,$$

corresponds to the nonlinear part. The application $\mathcal{H} : \mathcal{A} \times [0,1] \to \mathbb{R}$, defined by

$$\mathcal{H}(c_1,\mu_{1,1},t) := \mathfrak{M}\begin{pmatrix} c_1 \\ \mu_{1,1} \end{pmatrix} + t\begin{pmatrix} N_1(c_1,\mu_{1,1}) \\ N_2(c_1,\mu_{1,1}) \end{pmatrix},$$

provides us with a homotopy transformation from $\mathcal{F}(c_1,\mu_{1,1}) = \mathcal{H}(c_1,\mu_{1,1},1)$ to its linearization

$$\mathcal{H}(c_1,\mu_{1,1},0) := \mathfrak{M}\begin{pmatrix} c_1 \\ \mu_{1,1} \end{pmatrix}. \tag{4.21}$$

Thus, the system (4.20) possesses a nontrivial solution if (4.21) has a non-degenerate zero, in other words, if the next condition is satisfied

$$\langle \hat{\psi}_1^*, y \cdot \nabla \hat{\psi}_1 \rangle - \langle \hat{\psi}_1^*, y \cdot \nabla \hat{\psi}_2 \rangle \neq 0. \tag{4.22}$$

Note that, if the substitution would have been $c_1 = 1 - c_2$, the condition might also be

$$\langle \hat{\psi}_2^*, y \cdot \nabla \hat{\psi}_2 \rangle - \langle \hat{\psi}_2^*, y \cdot \nabla \hat{\psi}_1 \rangle \neq 0.$$

Then, under condition (4.22), the system (4.20) can be written in the form

$$\begin{pmatrix} c_1 - \hat{c}_1 \\ \mu_{1,1} - \hat{\mu}_{1,1} \end{pmatrix} = -\mathcal{M}^{-1}\begin{pmatrix} N_1(c_1,\mu_{1,1}) - \hat{c}_1 \\ N_2(c_1,\mu_{1,1}) - \hat{\mu}_{1,1} \end{pmatrix},$$

which can be interpreted as a fixed point equation. Moreover, applying Brouwer's fixed point theorem, we have that

$$\begin{aligned} \operatorname{Ind}((\hat{c}_1,\hat{\mu}_{1,1}),\mathcal{H}(\cdot,\cdot,0)) &= \mathcal{Q}_{C_R(\hat{c}_1,\hat{\mu}_{1,1})}(\mathcal{H}(\cdot,\cdot,0)) \\ &= \deg(\mathcal{H}(\cdot,\cdot,0), D_R(\hat{c}_1,\hat{\mu}_{1,1})) \\ &= \deg(\mathcal{F}(c_1,\mu_{1,1}), D_R(\hat{c}_1,\hat{\mu}_{1,1})), \end{aligned}$$

where $\mathcal{Q}_{C_R(\hat{c}_1,\hat{\mu}_{1,1})}(\mathcal{H}(\cdot,\cdot,0))$ defines the number of rotations of the function $\mathcal{H}(\cdot,\cdot,0)$ around the curve $C_R(\hat{c}_1,\hat{\mu}_{1,1})$ and $\deg(\mathcal{H}(\cdot,\cdot,0), D_R(\hat{c}_1,\hat{\mu}_{1,1}))$ denotes the topological degree of $\mathcal{H}(\cdot,\cdot,0)$ in $D_R(\hat{c}_1,\hat{\mu}_{1,1})$. Owing to classical topological methods, both are equal.

Thus, once we have proved the existence of solutions, we achieve some expressions for the coefficients required:

$$\begin{aligned} \mu_{1,1} = c_2(\langle \hat{\psi}_1^* + \hat{\psi}_2^*, h_1 - h_2 \rangle - \frac{\alpha_1}{4}\langle \hat{\psi}_1^* + \psi_2^*, y \cdot \nabla \hat{\psi}_1 - y \cdot \nabla \hat{\psi}_2 \rangle) \\ - \langle \hat{\psi}_1^* + \hat{\psi}_2^*, h_1 \rangle + \frac{\alpha_1}{4}\langle \hat{\psi}_1^* + \hat{\psi}_2^*, y \cdot \nabla \hat{\psi}_1 \rangle, \\ c_1 = 1 - c_2. \end{aligned}$$

The expressions for the coefficients in a general case might be accomplished after some tedious calculations, otherwise similar to those performed above.

Note that, in general, those nonlinear finite-dimensional algebraic problems are rather complicated, and the problem of an optimal estimate of the number of different solutions remains open.

Moreover, reliable multiplicity results are very difficult to obtain. We expect that this number should be somehow related (and even sometimes coincides) with the dimension of the corresponding eigenspace of the linear operators $\mathbf{B}+\frac{k}{4}I$, for any $k=0,1,2,\dots$. This is a conjecture only, and may be too illusive; see further supportive analysis presented below.

However, we devote the remainder of this section to a possible answer to that conjecture, which is not totally complete though, since we are imposing some conditions.

Thus, in order to detect the number of solutions of the nonlinear algebraic system (4.19), we proceed to reduce this system to a single equation for one of the unknowns. As a first step, integrating by parts in the terms in which h_1 and h_2 are involved and rearranging terms in the first two equations of the system (4.19), we arrive at

$$\begin{aligned}
&\int_{\mathbb{R}^N} \nabla\psi_1^* \cdot \ln(c_1\hat\psi_1+c_2\hat\psi_2)\nabla\Delta(c_1\hat\psi_1+c_2\hat\psi_2)\\
&-c_1\frac{\alpha_1}{4}\int_{\mathbb{R}^N}\hat\psi_1^* y\cdot\nabla\hat\psi_1 + c_1\mu_{1,1} - c_2\frac{\alpha_1}{4}\int_{\mathbb{R}^N}\hat\psi_1^* y\cdot\nabla\hat\psi_2 = 0,\\
&\int_{\mathbb{R}^N} \nabla\hat\psi_2^* \cdot \ln(c_1\hat\psi_1+c_2\hat\psi_2)\nabla\Delta(c_2\hat\psi_1+c_2\hat\psi_2)\\
&-c_1\frac{\alpha_1}{4}\int_{\mathbb{R}^N}\hat\psi_2^* y\cdot\nabla\hat\psi_1 + c_2\mu_{1,1} - c_2\frac{\alpha_1}{4}\int_{\mathbb{R}^N}\hat\psi_2^* y\cdot\nabla\hat\psi_2 = 0.
\end{aligned}$$

By the third equation, we have that $c_1 = 1-c_2$, and hence, setting

$$c_1\hat\psi_1+c_2\hat\psi_2 = \hat\psi_1+(\hat\psi_2-\hat\psi_1)c_2,$$

and substituting these into those new expressions for the first two equations of the system, we find that

$$\begin{aligned}
&\int_{\mathbb{R}^N} \nabla\hat\psi_1^*\cdot\ln(\hat\psi_1+(\hat\psi_2-\hat\psi_1)c_2)\nabla\Delta(\hat\psi_1+(\hat\psi_2-\hat\psi_1)c_2)+\mu_{1,1}-c_2\mu_{1,1}\\
&-\frac{\alpha_1}{4}\int_{\mathbb{R}^N}\hat\psi_1^* y\cdot\nabla\hat\psi_1 + c_2\frac{\alpha_1}{4}\int_{\mathbb{R}^N}\hat\psi_1^* y\cdot(\nabla\hat\psi_1-\nabla\hat\psi_2)=0,\\
&\int_{\mathbb{R}^N} \nabla\hat\psi_2^*\cdot\ln(\hat\psi_1+(\hat\psi_2-\hat\psi_1)c_2)\nabla\Delta(\hat\psi_1+(\hat\psi_2-\hat\psi_1)c_2)+c_2\mu_{1,1}\\
&-\frac{\alpha_1}{4}\int_{\mathbb{R}^N}\hat\psi_2^* y\cdot\nabla\hat\psi_1 + c_2\frac{\alpha_1}{4}\int_{\mathbb{R}^N}\hat\psi_2^* y\cdot(\nabla\hat\psi_1-\nabla\hat\psi_2)=0.
\end{aligned}\tag{4.23}$$

Adding both equations, we have

$$\begin{aligned}
\mu_{1,1} = &-\int_{\mathbb{R}^N}(\nabla\hat\psi_1^*+\nabla\hat\psi_2^*)\cdot\ln(\hat\psi_1+(\hat\psi_2-\hat\psi_1)c_2)\nabla\Delta(\hat\psi_1+(\hat\psi_2-\hat\psi_1)c_2)\\
&+\frac{\alpha_1}{4}\int_{\mathbb{R}^N}(\psi_1^*+\psi_2^*)y\cdot\nabla\hat\psi_1 - c_2\frac{\alpha_1}{4}\int_{\mathbb{R}^N}(\hat\psi_1^*+\hat\psi_2^*)y\cdot(\nabla\hat\psi_2-\nabla\hat\psi_1).
\end{aligned}$$

Thus, substituting it into the second equation of (4.23), we obtain the following equation with the single unknown c_2:

$$\begin{aligned}
&-c_2^2\frac{\alpha_1}{4}\int_{\mathbb{R}^N}(\hat\psi_1^*+\hat\psi_2^*)y\cdot(\nabla\hat\psi_2-\nabla\hat\psi_1)\\
&+c_2\frac{\alpha_1}{4}\Big(\int_{\mathbb{R}^N}(\hat\psi_1^*+2\hat\psi_2^*)y\cdot\nabla\hat\psi_1-\int_{\mathbb{R}^N}\hat\psi_2^* y\cdot\nabla\hat\psi_2\Big)
\end{aligned}$$

$$
\begin{aligned}
&-\frac{\alpha_1}{4}\int_{\mathbb{R}^N}\hat{\psi}_2^* y\cdot\nabla\hat{\psi}_1+\int_{\mathbb{R}^N}\nabla\psi_2^*\cdot\ln(\hat{\psi}_1+(\hat{\psi}_2-\hat{\psi}_1)c_2)\nabla\Delta(\hat{\psi}_1+(\hat{\psi}_2-\hat{\psi}_1)c_2)\\
&-c_2\int_{\mathbb{R}^N}(\nabla\hat{\psi}_1^*+\nabla\hat{\psi}_2^*)\ln(\hat{\psi}_1+(\hat{\psi}_2-\hat{\psi}_1)c_2)\nabla\Delta(\hat{\psi}_1+(\hat{\psi}_2-\hat{\psi}_1)c_2)=0,
\end{aligned}
$$

which can be written as

$$
c_2^2A+c_2B+C+\omega(c_2)\equiv\mathfrak{F}(c_2)+\omega(c_2)=0.
$$

Here, $\omega(c_2)$ can be considered as a perturbation of the quadratic form $\mathfrak{F}(c_2)$ with the coefficients defined by

$$
\begin{gathered}
A:=-\frac{\alpha_1}{4}\int_{\mathbb{R}^N}(\hat{\psi}_1^*+\hat{\psi}_2^*)y\cdot(\nabla\hat{\psi}_2-\nabla\hat{\psi}_1),\\
B:=\frac{\alpha_1}{4}\Big(\int_{\mathbb{R}^N}(\hat{\psi}_1^*+2\hat{\psi}_2^*)y\cdot\nabla\hat{\psi}_1-\int_{\mathbb{R}^N}\hat{\psi}_2^* y\cdot\nabla\hat{\psi}_2\Big),\\
C:=-\frac{\alpha_1}{4}\int_{\mathbb{R}^N}\hat{\psi}_2^* y\cdot\nabla\hat{\psi}_1,
\end{gathered}
$$

$$
\begin{aligned}
\omega(c_2):=&\int_{\mathbb{R}^N}\nabla\hat{\psi}_2^*\cdot\ln(\hat{\psi}_1+(\hat{\psi}_2-\hat{\psi}_1)c_2)\nabla\Delta(\hat{\psi}_1+(\hat{\psi}_2-\hat{\psi}_1)c_2)\\
&-c_2\int_{\mathbb{R}^N}(\nabla\hat{\psi}_1^*+\nabla\hat{\psi}_2^*)\cdot\ln(\hat{\psi}_1+(\hat{\psi}_2-\hat{\psi}_1)c_2)\nabla\Delta(\hat{\psi}_1+(\hat{\psi}_2-\hat{\psi}_1)c_2).
\end{aligned}
$$

Since, by the normalizing constraint (4.13), $c_2\in[0,1]$, solving the quadratic equation $\mathfrak{F}(c_2)$ yields:

(i) $c_2=0\Rightarrow\mathfrak{F}(0)=C$;
(ii) $c_2=1\Rightarrow\mathfrak{F}(1)=A+B+C$; and
(iii) differentiating $\mathfrak{F}$ with respect to c_2, we obtain that $\mathfrak{F}'(c_2)=2c_2A+B$. Then, the critical point of the function $\mathfrak{F}$ is $c_2^*=-\frac{B}{2A}$ and its image is $\mathfrak{F}(c_2^*)=-\frac{B}{4A}+C$.

Consequently, the conditions to be imposed for having more than one solution (we already know the existence of at least one solution) are as follows:

(a) $C(A+B+C)>0$;
(b) $C\big(-\frac{B}{4A}+C\big)<0$; and
(c) $0<-\frac{B}{2A}<1$.

Note that, for $-\frac{B}{4A}+C=0$, we have just a single solution. Hence, considering the equation again in the form

$$
\mathfrak{F}(c_2)+\omega(c_2)=0,
$$

where $\omega(c_2)$ is a perturbation of the quadratic form $\mathfrak{F}(c_2)$, and bearing in mind that the objective is to detect the number of solutions of the system (4.19), we need to control somehow this perturbation.

Under conditions (a), (b), and (c), $\mathfrak{F}(c_2)$ possesses exactly two solutions. Therefore, controlling the possible oscillations of the perturbation $\omega(c_2)$ in such a way that

$$
\|\omega(c_2)\|_{L^\infty}\le\mathfrak{F}(c_2^*),
$$

we can assure that the number of solutions for (4.19) is exactly two. This is the dimension of the kernel of the operator $\mathbf{B}+\frac{1}{4}I$ (as we expected in our more general conjecture).

The above particular example shows how difficult the questions on existence and multiplicity of solutions for such non-variational branching problems are.

Recall that the actual values of the coefficients A, B, C, and others, for which the number of solutions crucially depends on, are very difficult to estimate, even numerically, in view of the complicated nature of the eigenfunctions (3.10) involved. To say nothing of the nonlinear perturbation $\omega(c_2)$.

4.5. **Branching computations for $|\beta| = 2$.** Overall, the above analysis provides us with some expressions for the solutions for the self-similar equation (2.6) depending on the value of k. Actually, we can achieve those expressions for every critical value α_k, but again the calculus gets rather difficult.

For the sake of completeness, we now analyze the case $|\beta| = 2$ and $M_2 = 3$, so that $\{\psi_\beta\}_{|\beta|=2} = \{\hat\psi_1, \hat\psi_2, \hat\psi_3\}$ stands for a basis of the eigenspace $\ker\big(\mathbf{B} + \frac{1}{2} I\big)$, with $k = 2$ ($\lambda_k = -k/4$).

Thus, in this case, performing in a similar way as was done for (4.19) with

$$\psi_2 = c_1\hat\psi_1 + c_2\hat\psi_2 + c_3\hat\psi_3,$$

we arrive at the following algebraic system:

$$\begin{aligned}
&c_1\langle\hat\psi_1^*, h_1\rangle + c_2\langle\hat\psi_1^*, h_2\rangle + c_3\langle\hat\psi_1^*, h_3\rangle - \frac{c_1\alpha_2}{4}\langle\hat\psi_1^*, y\cdot\nabla\hat\psi_1\rangle \\
&- \frac{c_2\alpha_2}{4}\langle\hat\psi_1^*, y\cdot\nabla\hat\psi_2\rangle - \frac{c_3\alpha_2}{4}\langle\hat\psi_1^*, y\cdot\nabla\hat\psi_3\rangle + c_1\mu_{1,2} = 0, \\
&c_1\langle\hat\psi_2^*, h_1\rangle + c_2\langle\hat\psi_2^*, h_2\rangle + c_2\langle\hat\psi_2^*, h_3\rangle - \frac{c_1\alpha_2}{4}\langle\hat\psi_2^*, y\cdot\nabla\hat\psi_1\rangle \\
&- \frac{c_2\alpha_2}{4}\langle\hat\psi_2^*, y\cdot\nabla\hat\psi_2\rangle - \frac{c_3\alpha_2}{4}\langle\hat\psi_2^*, y\cdot\nabla\hat\psi_3\rangle + c_2\mu_{1,2} = 0, \\
&c_1\langle\hat\psi_3^*, h_1\rangle + c_2\langle\hat\psi_3^*, h_2\rangle + c_2\langle\hat\psi_3^*, h_3\rangle - \frac{c_1\alpha_2}{4}\langle\hat\psi_3^*, y\cdot\nabla\hat\psi_1\rangle \\
&- \frac{c_2\alpha_2}{4}\langle\hat\psi_3^*, y\cdot\nabla\hat\psi_2\rangle - \frac{c_3\alpha_2}{4}\langle\hat\psi_3^*, y\cdot\nabla\hat\psi_3\rangle + c_3\mu_{1,2} = 0, \\
&c_1 + c_2 + c_3 = 1,
\end{aligned} \tag{4.24}$$

where

$$\begin{aligned}
h_1 &:= -\nabla\cdot[\ln(c_1\hat\psi_1 + c_2\hat\psi_2 + c_3\hat\psi_3)\nabla\Delta\hat\psi_1], \\
h_2 &:= -\nabla\cdot[\ln(c_1\hat\psi_1 + c_2\hat\psi_2 + c_3\hat\psi_3)\nabla\Delta\hat\psi_2], \\
h_3 &:= -\nabla\cdot[\ln(c_1\hat\psi_1 + c_2\hat\psi_2 + c_3\hat\psi_3)\nabla\Delta\hat\psi_3],
\end{aligned}$$

and c_1, c_2, c_3, and $\mu_{1,2}$ are the unknowns to be evaluated. Moreover, α_2 is regarded as the value of the parameter α denoted by (4.1) and is dependent on the eigenvalue λ_2 with $\hat\psi_1, \hat\psi_2, \hat\psi_3$ representing the associated eigenfunctions and $\hat\psi_1^*, \hat\psi_2^*, \hat\psi_3^*$ the corresponding adjoint eigenfunctions.

Subsequently, substituting $c_3 = 1 - c_1 - c_2$ into the first three equations and performing an argument based upon the Brouwer fixed point theorem and the topological degree as the one done above for the case $|\beta| = 1$, we ascertain the existence of a non-degenerate solution of the algebraic system if the following condition is satisfied:

$$\langle\hat\psi_1^*, y\cdot\nabla(\hat\psi_3 - \hat\psi_1)\rangle\langle\hat\psi_2^*, y\cdot\nabla(\hat\psi_3 - \hat\psi_2)\rangle - \langle\hat\psi_1^*, y\cdot\nabla(\hat\psi_3 - \hat\psi_2)\rangle\langle\hat\psi_2^*, y\cdot\nabla(\hat\psi_3 - \hat\psi_1)\rangle \neq 0.$$

Note that, by similar substitutions, other conditions might be obtained.

Furthermore, once we know the existence of at least one solution, we proceed now with a possible way of computing the number of solutions of the nonlinear algebraic system (4.24). Obviously, since the dimension of the eigenspace is bigger than that in the case $|\beta| = 1$, the difficulty in obtaining multiplicity results increases.

Firstly, integrating by parts in the nonlinear terms, in which h_1, h_2 and h_3 are involved, and rearranging terms in the first three equations gives

$$\begin{aligned}
&\int_{\mathbb{R}^N} \nabla \psi_1^* \cdot \ln(c_1\hat{\psi}_1 + c_2\hat{\psi}_2 + c_3\hat{\psi}_3)\nabla\Delta(c_1\hat{\psi}_1 + c_2\hat{\psi}_2 + c_3\hat{\psi}_3)\\
&- c_1\frac{\alpha_2}{4}\int_{\mathbb{R}^N} \hat{\psi}_1^* y\cdot\nabla\hat{\psi}_1 + c_1\mu_{1,2} - c_2\frac{\alpha_2}{4}\int_{\mathbb{R}^N} \hat{\psi}_1^* y\cdot\nabla\hat{\psi}_2 - c_3\frac{\alpha_2}{4}\int_{\mathbb{R}^N} \hat{\psi}_1^* y\cdot\nabla\hat{\psi}_3 = 0,\\
&\int_{\mathbb{R}^N} \nabla \hat{\psi}_2^* \cdot \ln(c_1\hat{\psi}_1 + c_2\hat{\psi}_2 + c_3\hat{\psi}_3)\nabla\Delta(c_2\hat{\psi}_1 + c_2\hat{\psi}_2 + c_3\hat{\psi}_3)\\
&- c_1\frac{\alpha_2}{4}\int_{\mathbb{R}^N} \hat{\psi}_2^* y\cdot\nabla\hat{\psi}_1 + c_2\mu_{1,2} - c_2\frac{\alpha_2}{4}\int_{\mathbb{R}^N} \hat{\psi}_2^* y\cdot\nabla\hat{\psi}_2 - c_3\frac{\alpha_2}{4}\int_{\mathbb{R}^N} \hat{\psi}_2^* y\cdot\nabla\hat{\psi}_3 = 0,\\
&\int_{\mathbb{R}^N} \nabla \hat{\psi}_3^* \cdot \ln(c_1\hat{\psi}_1 + c_2\hat{\psi}_2 + c_3\hat{\psi}_3)\nabla\Delta(c_2\hat{\psi}_1 + c_2\hat{\psi}_2 + c_3\hat{\psi}_3)\\
&- c_1\frac{\alpha_2}{4}\int_{\mathbb{R}^N} \hat{\psi}_3^* y\cdot\nabla\hat{\psi}_1 + c_3\mu_{1,2} - c_2\frac{\alpha_2}{4}\int_{\mathbb{R}^N} \hat{\psi}_3^* y\cdot\nabla\hat{\psi}_2 - c_3\frac{\alpha_2}{4}\int_{\mathbb{R}^N} \hat{\psi}_3^* y\cdot\nabla\hat{\psi}_3 = 0.
\end{aligned}$$

According to the fourth equation, we have that $c_1 = 1 - c_2 - c_3$. Then, setting

$$c_1\hat{\psi}_1 + c_2\hat{\psi}_2 + c_3\hat{\psi}_3 = \hat{\psi}_1 + c_2(\hat{\psi}_2 - \hat{\psi}_1) + c_3(\hat{\psi}_3 - \hat{\psi}_1),$$

and substituting it into the expressions obtained above for the first three equations of the system yield

$$\begin{aligned}
&\int_{\mathbb{R}^N} \nabla\hat{\psi}_1^* \cdot \ln(\hat{\psi}_1 + (\hat{\psi}_2 - \hat{\psi}_1)c_2 + (\hat{\psi}_3 - \hat{\psi}_1)c_3)\nabla\Delta\Big(\hat{\psi}_1 + (\hat{\psi}_2 - \hat{\psi}_1)c_2\\
&+ (\hat{\psi}_3 - \hat{\psi}_1)c_3\Big) + \mu_{1,2} - c_2\mu_{1,2} - c_3\mu_{1,2} - \frac{\alpha_2}{4}\int_{\mathbb{R}^N} \hat{\psi}_1^* y\cdot\nabla\hat{\psi}_1\\
&+ \frac{\alpha_2}{4}\int_{\mathbb{R}^N} \hat{\psi}_1^* y\cdot((\nabla\hat{\psi}_1 - \nabla\hat{\psi}_2)c_2 + (\nabla\hat{\psi}_1 - \nabla\hat{\psi}_3)c_3) = 0,\\
&\int_{\mathbb{R}^N} \nabla\hat{\psi}_2^* \cdot \ln(\hat{\psi}_1 + (\hat{\psi}_2 - \hat{\psi}_1)c_2 + (\hat{\psi}_3 - \hat{\psi}_1)c_3)\nabla\Delta\Big(\hat{\psi}_1\\
&+ (\hat{\psi}_2 - \hat{\psi}_1)c_2 + (\hat{\psi}_3 - \hat{\psi}_1)c_3\Big) + c_2\mu_{1,2} - \frac{\alpha_2}{4}\int_{\mathbb{R}^N} \hat{\psi}_2^* y\cdot\nabla\hat{\psi}_1\\
&+ \frac{\alpha_2}{4}\int_{\mathbb{R}^N} \hat{\psi}_2^* y\cdot((\nabla\hat{\psi}_1 - \nabla\hat{\psi}_2)c_2 + (\nabla\hat{\psi}_1 - \nabla\hat{\psi}_3)c_3) = 0,\\
&\int_{\mathbb{R}^N} \nabla\hat{\psi}_3^* \cdot \ln(\hat{\psi}_1 + (\hat{\psi}_2 - \hat{\psi}_1)c_2 + (\hat{\psi}_3 - \hat{\psi}_1)c_3)\nabla\Delta\Big(\hat{\psi}_1\\
&+ (\hat{\psi}_2 - \hat{\psi}_1)c_2 + (\hat{\psi}_3 - \hat{\psi}_1)c_3\Big) + c_3\mu_{1,2} - \frac{\alpha_2}{4}\int_{\mathbb{R}^N} \hat{\psi}_3^* y\cdot\nabla\hat{\psi}_1\\
&+ \frac{\alpha_2}{4}\int_{\mathbb{R}^N} \hat{\psi}_3^* y\cdot((\nabla\hat{\psi}_1 - \nabla\hat{\psi}_2)c_2 + (\nabla\hat{\psi}_1 - \nabla\hat{\psi}_3)c_3) = 0.
\end{aligned} \tag{4.25}$$

Now, adding the first equation of (4.25) to the other two, we have

$$\int_{\mathbb{R}^N} (\nabla\hat{\psi}_1^* + \nabla\hat{\psi}_2^*) \cdot \ln(\hat{\psi}_1 + (\hat{\psi}_2 - \hat{\psi}_1)c_2 + (\hat{\psi}_3 - \hat{\psi}_1)c_3)\nabla\Delta\Big(\hat{\psi}_1$$

$$
\begin{aligned}
&+ (\hat\psi_2-\hat\psi_1)c_2+(\hat\psi_3-\hat\psi_1)c_3\Big)+\mu_{1,2}-c_3\mu_{1,2}-\frac{\alpha_2}{4}\int_{\mathbb{R}^N}(\hat\psi_1^*+\hat\psi_2^*)y\cdot\nabla\hat\psi_1\\
&+\frac{\alpha_2}{4}\int_{\mathbb{R}^N}(\hat\psi_1^*+\hat\psi_2^*)y\cdot((\nabla\hat\psi_1-\nabla\hat\psi_2)c_2+(\nabla\hat\psi_1-\nabla\hat\psi_3)c_3)=0,
\end{aligned}
$$

$$
\begin{aligned}
&\int_{\mathbb{R}^N}(\nabla\hat\psi_1^*+\nabla\hat\psi_3^*)\cdot\ln(\hat\psi_1+(\hat\psi_2-\hat\psi_1)c_2+(\hat\psi_3-\hat\psi_1)c_3)\nabla\Delta\Big(\hat\psi_1\\
&+(\hat\psi_2-\hat\psi_1)c_2+(\hat\psi_3-\hat\psi_1)c_3\Big)+\mu_{1,2}-c_2\mu_{1,2}-\frac{\alpha_2}{4}\int_{\mathbb{R}^N}(\hat\psi_1^*+\hat\psi_3^*)y\cdot\nabla\hat\psi_1\\
&+\frac{\alpha_2}{4}\int_{\mathbb{R}^N}(\hat\psi_1^*+\hat\psi_3^*)y\cdot((\nabla\hat\psi_1-\nabla\hat\psi_2)c_2+(\nabla\hat\psi_1-\nabla\hat\psi_3)c_3)=0.
\end{aligned}
$$

Subsequently, subtracting those equations yields

$$
\begin{aligned}
\mu_{1,2}=\frac{1}{c_3-c_2}\Big[&\int_{\mathbb{R}^N}(\nabla\hat\psi_2^*-\nabla\hat\psi_3^*)\ln\Psi\nabla\Delta\Psi-\frac{\alpha_2}{4}\int_{\mathbb{R}^N}(\hat\psi_2^*-\hat\psi_3^*)y\cdot\nabla\hat\psi_1\\
&+\frac{\alpha_2}{4}\int_{\mathbb{R}^N}(\hat\psi_2^*-\hat\psi_3^*)y\cdot((\nabla\hat\psi_1-\nabla\hat\psi_2)c_2+(\nabla\hat\psi_1-\nabla\hat\psi_3)c_3)\Big],
\end{aligned}
$$

where $\Psi=\hat\psi_1+(\hat\psi_2-\hat\psi_1)c_2+(\hat\psi_3-\hat\psi_1)c_3$. Thus, substituting it into (4.25) (note that, from the substitution into one of the last two equations, we obtain the same equation), we arrive at the following system, with c_2 and c_3 as the unknowns:

$$
\begin{aligned}
&c_3\int_{\mathbb{R}^N}(\nabla\hat\psi_1^*-\nabla\hat\psi_2^*+\nabla\hat\psi_3^*)\ln\Psi\nabla\Delta\Psi-c_2\int_{\mathbb{R}^N}(\nabla\hat\psi_1^*+\nabla\hat\psi_2^*-\nabla\hat\psi_3^*)\ln\Psi\nabla\Delta\Psi\\
&+\int_{\mathbb{R}^N}(\nabla\hat\psi_2^*-\nabla\hat\psi_3^*)\ln\Psi\nabla\Delta\Psi-\frac{\alpha_2}{4}\int_{\mathbb{R}^N}(\hat\psi_2^*-\hat\psi_3^*)y\cdot\nabla\hat\psi_1\\
&+c_2\frac{\alpha_2}{4}\Big[\int_{\mathbb{R}^N}(\hat\psi_2^*-\hat\psi_3^*)y\cdot\nabla(2\hat\psi_1-\hat\psi_2)+\int_{\mathbb{R}^N}\hat\psi_1^*y\cdot\nabla\hat\psi_1\Big]\\
&+c_3\frac{\alpha_2}{4}\Big[\int_{\mathbb{R}^N}(\hat\psi_2^*-\hat\psi_3^*)y\cdot\nabla(2\hat\psi_1-\hat\psi_3)-\int_{\mathbb{R}^N}\hat\psi_1^*y\cdot\nabla\hat\psi_1\Big]\\
&+c_2c_3\frac{\alpha_2}{4}\Big[\int_{\mathbb{R}^N}\hat\psi_1^*y\cdot(\nabla\hat\psi_3-\nabla\hat\psi_2)-\int_{\mathbb{R}^N}(\hat\psi_2^*-\hat\psi_3^*)y\cdot(2\nabla\hat\psi_1-\nabla\hat\psi_2-\nabla\hat\psi_3)\Big]\\
&+c_3^2\frac{\alpha_2}{4}\int_{\mathbb{R}^N}(\hat\psi_1^*-\hat\psi_2^*+\hat\psi_3^*)y\cdot(\nabla\hat\psi_1-\nabla\hat\psi_3)\\
&-c_2^2\frac{\alpha_2}{4}\int_{\mathbb{R}^N}(\hat\psi_1^*+\hat\psi_2^*-\hat\psi_3^*)y\cdot(\nabla\hat\psi_1-\nabla\hat\psi_2)=0,
\end{aligned}
$$

$$
\begin{aligned}
&c_3\int_{\mathbb{R}^N}\nabla\hat\psi_2^*\ln\Psi\nabla\Delta\Psi-c_2\int_{\mathbb{R}^N}\nabla\hat\psi_3^*\ln\Psi\nabla\Delta\Psi-c_3\frac{\alpha_2}{4}\int_{\mathbb{R}^N}\hat\psi_2^*y\cdot\nabla\hat\psi_1\\
&+c_2\frac{\alpha_2}{4}\int_{\mathbb{R}^N}\hat\psi_3^*y\cdot\nabla\hat\psi_1+c_3\frac{\alpha_2}{4}\int_{\mathbb{R}^N}\hat\psi_2^*y\cdot((\nabla\hat\psi_1-\nabla\hat\psi_2)c_2+(\nabla\hat\psi_1-\nabla\hat\psi_3)c_3)\\
&-c_2\frac{\alpha_2}{4}\int_{\mathbb{R}^N}\hat\psi_3^*y\cdot((\nabla\hat\psi_1-\nabla\hat\psi_2)c_2+(\nabla\hat\psi_1-\nabla\hat\psi_3)c_3)=0.
\end{aligned}
$$

These can be re-written in the form

$$
\begin{gathered}
A_1c_2^2+B_1c_3^2+C_1c_2+D_1c_3+E_1c_2c_3+\omega_1(c_2,c_3)=0,\\
A_2c_2^2+B_2c_3^2+C_2c_2+D_2c_3+E_2c_2c_3+\omega_2(c_2,c_3)=0,
\end{gathered}
\tag{4.26}
$$

where

$$\begin{aligned}\omega_1(c_2,c_3) :=\ & c_3\int_{\mathbb{R}^N}(\nabla\hat\psi_1^*-\nabla\hat\psi_2^*+\nabla\hat\psi_3^*)\ln\Psi\nabla\Delta\Psi\\ &- c_2\int_{\mathbb{R}^N}(\nabla\hat\psi_1^*+\nabla\hat\psi_2^*-\nabla\hat\psi_3^*)\ln\Psi\nabla\Delta\Psi\\ &+\int_{\mathbb{R}^N}(\nabla\hat\psi_2^*-\nabla\hat\psi_3^*)\cdot\ln\Psi\nabla\Delta\Psi-\frac{\alpha_2}{4}\int_{\mathbb{R}^N}(\hat\psi_2^*-\hat\psi_3^*)y\cdot\nabla\hat\psi_1,\end{aligned}$$

and

$$\omega_2(c_2,c_3) := c_3\int_{\mathbb{R}^N}\nabla\hat\psi_2^*\cdot\ln\Psi\nabla\Delta\Psi - c_2\int_{\mathbb{R}^N}\nabla\hat\psi_3^*\cdot\ln\Psi\nabla\Delta\Psi,$$

are the perturbations of the quadratic polynomials

$$\mathfrak{F}_i(c_2,c_3) := A_ic_2^2+B_ic_3^2+C_ic_2+D_ic_3+E_ic_2c_3,$$

with $i=1,2$. The coefficients of those quadratic expressions are

$$\begin{gathered}A_1 := -\frac{\alpha_2}{4}\int_{\mathbb{R}^N}(\hat\psi_1^*+\hat\psi_2^*-\hat\psi_3^*)y\cdot(\nabla\hat\psi_1-\nabla\hat\psi_2),\\ B_1 := \frac{\alpha_2}{4}\int_{\mathbb{R}^N}(\hat\psi_1^*-\hat\psi_2^*+\hat\psi_3^*)y\cdot(\nabla\hat\psi_1-\nabla\hat\psi_3),\\ C_1 := \frac{\alpha_2}{4}[\int_{\mathbb{R}^N}(\hat\psi_2^*-\hat\psi_3^*)y\cdot\nabla(2\hat\psi_1-\hat\psi_2)+\int_{\mathbb{R}^N}\hat\psi_1y\cdot\nabla\hat\psi_1],\\ D_1 := \frac{\alpha_2}{4}[\int_{\mathbb{R}^N}(\hat\psi_2^*-\hat\psi_3^*)y\cdot\nabla(2\hat\psi_1-\hat\psi_3)-\int_{\mathbb{R}^N}\hat\psi_1y\cdot\nabla\hat\psi_1],\\ E_1 := \frac{\alpha_2}{4}[\int_{\mathbb{R}^N}\hat\psi_1^*y\cdot(\nabla\hat\psi_3-\nabla\hat\psi_2)-\int_{\mathbb{R}^N}(\hat\psi_2^*-\hat\psi_3^*)y\cdot(2\nabla\hat\psi_1-\nabla\hat\psi_2-\nabla\hat\psi_3)],\\ A_2 := -\frac{\alpha_2}{4}\int_{\mathbb{R}^N}\hat\psi_3^*y\cdot(\nabla\hat\psi_1-\nabla\hat\psi_2),\\ B_2 := \frac{\alpha_2}{4}\int_{\mathbb{R}^N}\hat\psi_2^*y\cdot((\nabla\hat\psi_1-\nabla\hat\psi_3),\\ C_2 := \frac{\alpha_2}{4}\int_{\mathbb{R}^N}\hat\psi_3^*y\cdot\nabla\hat\psi_1,\quad D_2 := -\frac{\alpha_2}{4}\int_{\mathbb{R}^N}\hat\psi_2^*y\cdot\nabla\hat\psi_1,\\ E_2 := \frac{\alpha_2}{4}\int_{\mathbb{R}^N}\hat\psi_2^*y\cdot(\nabla\hat\psi_1-\nabla\hat\psi_2)-\hat\psi_3^*y\cdot(\nabla\hat\psi_1-\nabla\hat\psi_3).\end{gathered}$$

Therefore, using the conic classification to solve (4.26), we have the number of solutions through the intersection of two conics. Then, depending on the type of conic, we shall always obtain one to four possible solutions for our system. Hence, somehow, the number of solutions depends on the coefficients we have for the system and, at the same time, on the eigenfunctions that generate the subspace $\ker\big(\mathbf{B}+\frac{k}{4}\big)$.

Thus, we have the following conditions, which will provide us with the conic section of each equation of the system (4.26):

(i) If $B_i^2-4A_iE_i<0$, the equation represents an *ellipse*, unless the conic is degenerate, for example $c_2^2+c_3^2+k=0$ for some positive constant k. So, if $A_i=B_j$ and $E_i=0$, the equation represents a *circle*;
(ii) If $B_i^2-4A_iE_i=0$, the equation represents a *parabola*;
(iii) If $B_i^2-4A_iE_i>0$, the equation represents a *hyperbola*. If we also have $A_i+E_i=0$ the equation represents a hyperbola (a rectangular hyperbola).

Consequently, the zeros of the system (4.26) and, hence, of the system (4.24), adding the "normalizing" constraint (4.13), are ascertained by the intersection of those two conics in (4.26) providing us with the number of possible n-branches between one and four. Note that in case those conics are two circles we only have two intersection points at most. Moreover, due to the dimension of the eigenspaces it looks like in this case that we have four possible intersection points two of them will coincide. However, the justification for this is far from clear.

Moreover, as it was done for the previous case when $|\beta| = 1$, we need to control the oscillations of the perturbation functions in order to maintain the number of solutions. Therefore, imposing that

$$\|\omega_i(c_2, c_3)\|_{L^\infty} \leq \mathfrak{F}_i(c_2^*, c_3^*), \quad \text{with } i = 1, 2,$$

we ascertain that the number of solutions must be between one and four. This again gives us an idea of the difficulty of more general multiplicity results.

4.6. **Further comments on mathematical justification of existence.** We return to the self-similar nonlinear eigenvalue problem (2.6), associated with (1.1), which can be written in the form

$$\mathcal{L}(\alpha, n)f + \mathcal{N}(n, f) = 0, \quad \text{where } \mathcal{N}(n, f) := \nabla \cdot ((1 - |f|^n)\nabla\Delta f)\,.$$

As we have seen, the main difficulty in justifying the n-branching behaviour concerns the distribution and "transversal topology" of zero surfaces of solutions close to finite interface hyper-surfaces.

Recall that, as in classic nonlinear operator theory [13,28,32], our analysis above always assumed that we actually dealt with and performed computations for the integral equation:

$$f = -\mathcal{L}^{-1}(\alpha, n)\mathcal{N}(n, f) \equiv \mathcal{G}(n, f), \quad \mathcal{L}(\alpha, n) := -\Delta^2 + \frac{1 - \alpha n}{4} y \cdot \nabla + \alpha I, \tag{4.27}$$

where $\mathcal{L}(\alpha, n)$ is invertible in L^2_ρ (this is directly checked via Section 3) and, hence compact, for a fixed α, and $f \in C_0(\mathbb{R}^N)$ for small $n > 0$. This confirms that the zeros of the function $\mathcal{F}(n, f)$ are fixed points of the map $\mathcal{G}(n, f)$.

Note again that (4.27) is an eigenvalue problem, where admissible real values of α are supposed to be defined together with its solvability. This makes existence/multiplicity questions for (4.27) extremely difficult.

There are two cases of this problem. The first and simpler one occurs when the eigenvalue α is determined *a priori*, e.g. in the case $k = 0$, where $\alpha_0(0) = N/4$ denoted as $\alpha_0(0) = \alpha_0$, and where, for $n > 0$, the first nonlinear eigenvalue is given explicitly (see (2.9)):

$$\alpha_0(n) = \frac{N}{4 + Nn}.$$

Then (4.27) with $\alpha = \alpha_0(n)$ for $n > 0$ becomes a standard nonlinear integral equation with, however, a quite curious and hard-to-detect functional setting. Indeed, the right-hand side in (4.27), where the nonlinearity is not in a fully divergent form, assumes the extra regularity at least such as

$$f \in H^3_\rho.$$

In view of the known good properties of the compact resolvent $(\mathcal{L} - \lambda I)^{-1}$, it is clear that the action of the inverse one $\mathcal{L}^{-1}$ is sufficient to restore the regularity, since locally in $\mathbb{R}^N$ this acts like Δ^{-2}. Therefore, it is plausible that

$$\mathcal{G} : H^3_\rho \to H^3_\rho,$$

and it is not difficult to get an *a priori* bounds at least for small enough f's. The accompanying analysis as $y \to \infty$ (due to the unbounded domain) assumes no novelties or special difficulties and is standard for such weighted L^2 and Sobolev spaces.

Therefore, application of Schauder's fixed point theorem (see e.g. [6, p. 90]) to (4.27) is the most powerful tool to imply existence of a solution, and moreover a continuous curve of fixed points $\Gamma_n = \{f : n > 0 \text{ small}\}$.

By scaling invariance of the similarity equation, we are obliged to impose the normalization condition, say,

$$f(0) = \delta_0 > 0 \quad \text{sufficiently small.}$$

On the other hand, uniqueness remains a completely open problem (apart from partial results such as [3] when n is sufficiently close to zero). However, studying the behaviour of the solution curve Γ_n as $n \to 0$ and applying (under suitable hypothesis) the branching techniques developed above, we may conclude that any such continuous curve must be originated at a properly scaled eigenfunction $\psi_0 = F$, so that such a curve is unique due to well-posedness of all the asymptotic expansions.

A possibility of extension of Γ_n for larger values of $n > 0$ represents an essentially more difficult nonlocal open problem. Indeed, via compactness of linear operators involved in (4.27), one can expect that such a curve can end up at a bifurcation point only (unless it blows up). However, nonexistence of turning saddle-node points at some $n_* > 0$ (meaning that the n-branch is nonexistent for some $n > n_*$) is not that easy to rule out. Moreover, such turning points with thin film operators involved are actually possible, [21].

After establishing the existence of such solutions for small $n > 0$, we face the next problem on their asymptotic properties including the fact that these are compactly supported. On a qualitative level, these questions were discussed in [15].

In the case of higher-order nonlinear eigenfunctions of (4.27) for $k \geq 1$ including the dipole case $k = 1$, the parameter α becomes an eigenvalue that is essentially involved into the problem setting. This assumes the consideration of the equation (4.27) in the extended space

$$(f, \alpha) \in X = H^3_\rho \times \{\alpha \in \mathbb{R}\} \quad \text{and } \mathcal{G} : X \to X, \tag{4.28}$$

where proving the latter mapping for some compact subsets becomes a hard open problem. Note that here even the necessary convexity issue for applying Schauder's Theorem can be difficult. We still do not know whether representations such as (4.28) may lead to any rigorous treatment of the nonlinear eigenvalue problem (4.27) for $k \geq 1$.

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Pablo Alvarez-Caudevilla
Universidad Carlos III de Madrid, Av. Universidad 30, 28911-Leganés, Spain
E-mail address: pacaudev@math.uc3m.es Phone +34-916249099

Victor A. Galaktionov
Department of Mathematical Sciences, University of Bath, Bath BA2 7AY, UK
E-mail address: vag@maths.bath.ac.uk Phone +44-1225826988