BOUNDEDNESS AND LARGE-TIME BEHAVIOR OF SOLUTIONS FOR A GIERER-MEINHARDT SYSTEM OF THREE EQUATIONS

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ABSTRACT. The aim of this work is to prove the uniform boundedness and the existence of global solutions for Gierer-Meinhardt model of three substance described by reaction-diffusion equations with Neumann boundary conditions. Based on a Lyapunov functional we establish the asymptotic behaviour of the solutions.

1. INTRODUCTION

In this article, we consider the Gierer-Meinhardt type system of three equations

\[
\begin{align*}
\frac{\partial u}{\partial t} - a_1 \Delta u &= -b_1 u + f(u, v, w), \quad \text{in } \mathbb{R}^+ \times \Omega, \\
\frac{\partial v}{\partial t} - a_2 \Delta v &= -b_2 v + g(u, v, w), \quad \text{in } \mathbb{R}^+ \times \Omega, \\
\frac{\partial w}{\partial t} - a_3 \Delta w &= -b_3 w + h(u, v, w), \quad \text{in } \mathbb{R}^+ \times \Omega,
\end{align*}
\]

(1.1)

where

\[
\begin{align*}
f(u, v, w) &= \rho_1(x, u, v, w) \frac{u^{p_1}}{v^{q_1}(w^{r_1} + \epsilon)} + \sigma_1(x), \\
g(u, v, w) &= \rho_2(x, u, v, w) \frac{u^{p_2}}{v^{q_2}w^{r_2}} + \sigma_2(x), \\
h(u, v, w) &= \rho_3(x, u, v, w) \frac{u^{p_3}}{v^{q_3}w^{r_3}} + \sigma_3(x),
\end{align*}
\]

with homogeneous Neumann boundary conditions

\[
\begin{align*}
\frac{\partial u}{\partial \eta} &= \frac{\partial v}{\partial \eta} = \frac{\partial w}{\partial \eta} = 0 \quad \text{on } \mathbb{R}^+ \times \partial \Omega,
\end{align*}
\]

(1.2)

and initial data

\[
\begin{align*}
u(0, x) &= \varphi_1(x), \quad v(0, x) = \varphi_2(x), \quad w(0, x) = \varphi_3(x), \quad \text{in } \Omega.
\end{align*}
\]

(1.3)
Here $\Omega$ is an open bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial \Omega$ and outer normal $\eta(x)$. The constants $c, p_i, q_i, r_i, a_i$ and $b_i$, $i = 1, 2, 3$ are real numbers such that
\[ c, p_i, q_i, r_i \geq 0, \quad \text{and} \quad a_i, b_i > 0, \]
and
\[ 0 < p_1 - 1 < \min \left\{ p_2 \min \left( \frac{q_1}{q_2 + 1}, \frac{r_1}{r_2} \right), p_3 \min \left( \frac{r_1}{r_3 + 1}, \frac{q_1}{q_3} \right) \right\}. \tag{1.4} \]
The initial data are assumed to be positive and continuous functions on $\overline{\Omega}$. For $i = 1, 2, 3$, we assume that $\sigma_i$ are positive functions in $C(\overline{\Omega})$, and $\rho_i$ are positive bounded functions in $C^1(\overline{\Omega} \times \mathbb{R}^3_+)$. In 1972, following the ingenious idea of Turing [15], Gierer and Meinhardt [2] proposed a mathematical model for pattern formations of spatial tissue structure of hydra in morphogenesis, a biological phenomenon discovered by Trembley in 1744 [14]. It can be expressed in the following system
\[
\frac{\partial u}{\partial t} = a_1 \Delta u - \mu_1 u + u^p v^q + \sigma, \quad \text{in } \mathbb{R}^+ \times \Omega, \\
\frac{\partial v}{\partial t} = a_2 \Delta v - \mu_2 v + u^r v^s, \quad \text{in } \mathbb{R}^+ \times \Omega, \tag{1.5}
\]
on a bounded $\Omega \subset \mathbb{R}^N$, with the homogeneous Neumann boundary conditions and positive initial data: $a_1, a_2, \mu_1, \mu_2$ and $\sigma$ are positive constants, and $p, q, r, s$ are non negative constants satisfying the relation
\[ \frac{p - 1}{r} < \frac{q}{s + 1}. \]
The existence of global solutions to the system [1.5] is proved by Rothe [11] with special cases $N = 3, p = 2, q = 1, r = 2$ and $s = 0$. The Rothe’s method can not be applied (at least directly) to general $p, q, r, s$. Wu and Li [16] obtained the same results for the problem (1.5) so long as $u, v^{-1}$ and $\sigma$ are suitably small. Li, Chen and Qin [7] showed that the solutions of this problem are bounded all the time for each pair of initial values in $L^\infty(\Omega)$ if
\[ \frac{p - 1}{r} < \min \left\{ 1, \frac{q}{s + 1} \right\}. \tag{1.6} \]
Masuda and Takahashi [8] considered the generalized Gierer-Meinhardt system
\[
\frac{\partial u_i}{\partial t} = a_i \Delta u_i - \mu_i u_i + g_i(x, u_1, u_2), \quad \text{in } \mathbb{R}^+ \times \Omega (i = 1, 2), \tag{1.7}
\]
where $a_i, \mu_i, i = 1, 2$ are positive constants, and
\[
g_1(x, u_1, u_2) = \rho_1(x, u_1, u_2) \frac{u_1^p}{u_2} + \sigma_1(x), \\
g_2(x, u_1, u_2) = \rho_2(x, u_1, u_2) \frac{u_1^r}{u_2} + \sigma_2(x),
\]
with $\sigma_1(\cdot)$ (resp. $\sigma_2(\cdot)$) is a positive (resp. non-negative) $C^1$ function on $\Omega$, and $\rho_1$ (resp. $\rho_2$) is a non negative (resp. positive) bounded and $C^1$ function on $\Omega \times \mathbb{R}^2_+$. They extended the result of global existence of solutions for (1.7) of Li, Chen and Qin [7] to
\[ \frac{p - 1}{r} < \frac{2}{N + 2}, \tag{1.8} \]
and

\[ \varphi_1, \varphi_2 \in W^{2,1}(\Omega), \quad \gamma > \max\{N, 2\}, \]
\[ \frac{\partial \varphi_1}{\partial \eta} = \frac{\partial \varphi_2}{\partial \eta} = 0 \quad \text{on} \ \partial \Omega \quad \text{and} \quad \varphi_1 \geq 0, \varphi_2 > 0 \quad \text{in} \ \bar{\Omega}. \]  

(1.9)

Jiang [6] obtained the same results as Masuda and Takahashi [8] by another method such that (1.6) and (1.9) are satisfied.

Abdelmalek, et al [1] considered the following Gierer-Meinhardt system of three equations

\[ \frac{\partial u}{\partial t} - a_1 \Delta u = -b_1 u + \frac{u^{p_1}}{v_1^{(r_1 + e)}} + \sigma, \quad \text{in} \ \mathbb{R}^+ \times \Omega, \]
\[ \frac{\partial v}{\partial t} - a_2 \Delta v = -b_2 v + \frac{u^{p_2}}{v_2^{r_2}}, \quad \text{in} \ \mathbb{R}^+ \times \Omega, \]  

(1.10)
\[ \frac{\partial w}{\partial t} - a_3 \Delta w = -b_3 w + \frac{u^{p_3}}{v_3^{r_3}}, \quad \text{in} \ \mathbb{R}^+ \times \Omega, \]

with homogeneous Neumann boundary conditions

\[ \frac{\partial u}{\partial \eta} = \frac{\partial v}{\partial \eta} = \frac{\partial w}{\partial \eta} = 0 \quad \text{on} \ \mathbb{R}^+ \times \partial \Omega, \]  

(1.11)

and the initial data

\[ u(0, x) = \varphi_1(x) > 0, \]
\[ v(0, x) = \varphi_2(x) > 0, \]  

(1.12)
\[ w(0, x) = \varphi_3(x) > 0 \]
in \( \Omega \), and \( \varphi_i \in C(\bar{\Omega}) \) for all \( i = 1, 2, 3 \). Under the condition (1.4) and by using a suitable Lyapunov functional, they studied the global existence of solutions for the system (1.10), (1.12). Their method gave only the existence of global solutions, and they did not obtain results about the uniform boundedness of solutions on \( (0, +\infty) \).

For the asymptotic behavior of the solutions, Wu and Li [16] considered the system

\[ \frac{\partial u_1}{\partial t} = a_1 \Delta u_1 - u_1 + \frac{u_1^p}{u_2^q} + \sigma_1(x), \quad \text{in} \ \mathbb{R}^+ \times \Omega, \]
\[ \tau \frac{\partial u_2}{\partial t} = a_2 \Delta u_2 - u_2 + \frac{u_1^p}{u_2^q} + \sigma_2(x), \quad \text{in} \ \mathbb{R}^+ \times \Omega, \]  

(1.13)

with the constant of relaxation time \( \tau > 0 \), and they proved that if \( \sigma_1 \equiv \sigma_2 \equiv 0 \) and \( \tau > \frac{2}{p+q} \), then \( (u(t, x), v(t, x)) \to (0, 0) \) uniformly on \( \bar{\Omega} \) as \( t \to +\infty \).

Under suitable conditions on \( \tau \) and on the initial data, Suzuki and Takagi [12, 13] also studied the behavior of the solutions for (1.13) with the constant of relaxation time \( \tau \).

We first treat the uniform boundedness of the solutions for Gierer-Meinhardt system of three equations by proving that the Lyapunov function argument proposed in [1] can be adapted to our situation. Interestingly, we show that the same Lyapunov function satisfies a differential inequality from which the uniform boundedness of the solutions is deduced for any positive time. Then under reasonable conditions on the coefficients \( b_1, b_2 \) and \( b_3 \), and by using the uniform boundedness of the solutions and the Lyapunov function which is non-increasing function, we deal with the long-time behavior of solutions as the time goes to +\( \infty \). In particular we are concerned with \( \sigma_1 \equiv 0, \sigma_2 \) and \( \sigma_3 \) are non-negative constants to assure that

\[ \lim_{t \to +\infty} \|u(t, .)\|_\infty = \lim_{t \to +\infty} \|v(t, .) - \frac{\sigma_2}{b_2}\|_\infty = \lim_{t \to +\infty} \|w(t, .) - \frac{\sigma_3}{b_3}\|_\infty = 0. \]
2. Notation and preliminary results

2.1. Existence of local solutions. For $i = 1, 2, 3$ we set

$$
\varphi_i = \min_{x \in \Omega} \varphi_i(x), \quad \bar{\varphi}_i = \max_{x \in \Omega} \varphi_i(x),
$$

$$
\rho_i = \min_{x \in \Omega, \xi \in \mathbb{R}^3_+} \rho_i(x, \xi), \quad \bar{\rho}_i = \max_{x \in \Omega, \xi \in \mathbb{R}^3_+} \rho_i(x, \xi),
$$

$$
\sigma_i = \min_{x \in \Omega} \sigma_i(x), \quad \bar{\sigma}_i = \max_{x \in \Omega} \sigma_i(x).
$$

The basic existence theory for abstract semi linear differential equations directly leads to a local existence result to system (1.1)–(1.3) (see, Henry [5]). All solutions are classical on $(0, T_{\text{max}}) \times \Omega$, where $T_{\text{max}}(\|u_0\|_{\infty}, \|w_0\|_{\infty})$ denotes the eventual blowing-up time in $L^\infty(\Omega)$.

2.2. Positivity of solutions.

**Lemma 2.1.** If $(u, v, w)$ is a solution of the problem (1.1)–(1.3), then for all $(t, x) \in (0, T_{\text{max}}) \times \Omega$, we have

(1) \[
\begin{align*}
    u(t, x) &\geq e^{-b_1 t} \bar{\varphi}_1 > 0, \\
    v(t, x) &\geq e^{-b_2 t} \bar{\varphi}_2 > 0, \\
    w(t, x) &\geq e^{-b_3 t} \bar{\varphi}_3 > 0.
\end{align*}
\]

(2) \[
\begin{align*}
    u(t, x) &\geq \min \left( \sigma_1 / b_1, \bar{\varphi}_1 \right) = m_1, \\
    v(t, x) &\geq \min \left( \sigma_2 / b_2, \bar{\varphi}_2 \right) = m_2, \\
    w(t, x) &\geq \min \left( \sigma_3 / b_3, \bar{\varphi}_3 \right) = m_3.
\end{align*}
\]

The proof of the above lemma follows immediate from the maximum principle.

3. Boundedness of solutions

For proving the existence of global solutions for (1.1)–(1.3), it suffices to prove that the solutions remains bounded in $(0, T) \times \Omega$. One of the main results of this paper reads as follows.

**Theorem 3.1.** Assume that (1.4) holds. Let $(u, v, w)$ be a solution to (1.1)–(1.3), and let

$$
L(t) = \int_{\Omega} \frac{u^\alpha(t, x) v^\beta(t, x) w^\gamma(t, x)}{v^\gamma(t, x) w^\gamma(t, x)} dx, \quad \text{for all } t \in (0, T),
$$

where $\alpha, \beta$ and $\gamma$ are positive constants satisfying the following conditions:

$$
\alpha > 2 \max \left( 1, \frac{3b_2 + b_3}{b_1} \right), \quad \frac{1}{\beta} > \frac{(a_1 + a_2)^2}{2a_1 a_2}, \quad (3.2)
$$

and

$$
\left( \frac{1}{2\beta} \left( \frac{(a_1 + a_2)^2}{4a_1 a_2} \right) \right)^2 > \left( \frac{(\alpha - 1)(a_2 + a_3)}{2\alpha \sqrt{a_2 a_3}} - \frac{(a_1 + a_2)(a_1 + a_3)}{4\sqrt{a_1^2 a_2 a_3}} \right)^2. \quad (3.3)
$$
Then there exists a positive constant $C$ such that for all $t \in (0, T)$,
\[
\frac{d}{dt} L(t) \leq -(ab_1 - 3b_2\beta - \gamma b_3)L(t) + C. 
\tag{3.4}
\]

**Corollary 3.2.** Under the assumptions of Theorem 3.1, all solutions of (1.1)–(1.3) with positive initial data in $C(\Omega)$ are global and uniformly bounded on $(0, +\infty) \times \Omega$.

Before proving the above theorem we first need the following technical lemma.

**Lemma 3.3.** Suppose that $x > 0$, $y > 0$ and $z > 0$, then for each group of indices $\alpha, \beta, \gamma, \delta, \theta, \lambda$ and $\xi$ satisfies $\lambda < p < \delta$ (not necessarily positive), and any constant $\Lambda > 0$, we have
\[
\frac{x^\alpha}{y^\beta z^\gamma} \leq \Lambda \frac{x^\delta}{y^\varepsilon z^\xi} + \Lambda \frac{x^\lambda}{y^{\eta_1} z^{\eta_2}},
\]
where
\[
\eta_1 = [q(\delta - \lambda) - \theta(p - \lambda)](\delta - p)^{-1},
\]
\[
\eta_2 = [r(\delta - \lambda) - \xi(p - \lambda)](\delta - p)^{-1}.
\]

**Proof.** We can write
\[
\frac{x^p}{y^q z^r} = \left( \frac{\alpha(\delta - \lambda)}{\alpha} \frac{\beta(p - \lambda)}{\beta} \frac{\gamma(\delta - \lambda)}{\gamma} \right) \left( \frac{\delta - p}{\delta - p} \frac{\xi(\delta - \lambda)}{\xi} \frac{\eta_1(\delta - \lambda)}{\eta_1} \right).
\]
By using Young’s inequality we obtain
\[
\frac{x^p}{y^q z^r} \leq \varepsilon \frac{x^\delta}{y^\varepsilon z^\xi} + \varepsilon \frac{x^\lambda}{y^{\eta_1} z^{\eta_2}},
\]
where
\[
\eta_1 = [q(\delta - \lambda) - \theta(p - \lambda)](\delta - p)^{-1},
\]
\[
\eta_2 = [r(\delta - \lambda) - \xi(p - \lambda)](\delta - p)^{-1}.
\]
Then Lemma 3.3 is proved. \qed

**Proof of Theorem 3.1.** Let $(u, v, w)$ be the solution of system (1.1)–(1.3) in $(0, T)$. Differentiating $L(t)$ respect to $t$, we obtain $L'(t) = I + J$, where
\[
I = a_1 \alpha \int_{\Omega} \frac{u^{\alpha - 1}}{v^\beta w^\gamma} \Delta u \, dx - a_2 \beta \int_{\Omega} \frac{u^\alpha}{v^\beta w^\gamma + 1} \Delta v \, dx - a_3 \gamma \int_{\Omega} \frac{u^\alpha}{v^\beta w^\gamma + 1} \Delta w \, dx,
\]
\[
J = (-ab_1 + \beta b_2 + \gamma b_3) L(t) + \alpha \int_{\Omega} \rho_1(x, u, v, w) \frac{u^{\alpha - 1 + p_1}}{v^\beta + \eta_1 w^{\gamma + r_1}} \, dx
\]
\[
- \beta \int_{\Omega} \rho_2(x, u, v, w) \frac{u^{\alpha + p_2}}{v^\beta + q_2 w^{\gamma + r_2}} \, dx - \gamma \int_{\Omega} \rho_3(x, u, v, w) \frac{u^{\alpha + p_3}}{v^\beta + q_3 w^{\gamma + r_3}} \, dx
\]
\[
+ \alpha \int_{\Omega} \sigma_1(x) \frac{u^{\alpha - 1}}{v^\beta w^\gamma} \, dx - \beta \int_{\Omega} \sigma_2(x) \frac{u^\alpha}{v^\beta w^\gamma + 1} \, dx - \gamma \int_{\Omega} \sigma_3(x) \frac{u^\alpha}{v^\beta w^{\gamma + 1}} \, dx.
\]
Using Green’s formula, for all $t \in (0, T)$, we obtain (see [1])
\[
I \leq 0. 
\tag{3.5}
\]
Now let us estimate the term \( J \). For all \( t \in (0, T) \) we have

\[
J \leq \left( - \alpha b_1 + \beta b_2 + \gamma b_3 \right) L(t) + \alpha \rho_1 \int_{\Omega} \frac{u^{\alpha-1+p_1}}{v^{\beta+q_1}w^{r_1}} dx - \beta \rho_2 \int_{\Omega} \frac{u^{\alpha+p_2}}{v^{\beta+q_2}w^{r_2}} dx
- \rho_3 \gamma \int_{\Omega} \frac{u^{\alpha+p_3}}{v^{\beta+q_3}w^{r_3}} dx + \alpha \sigma_1 \int_{\Omega} u^{-1} \gamma \omega dx.
\]  

Applying Lemma 3.3 with \( p = \alpha - 1, q = \theta = \beta, r = \gamma, \delta = \alpha, \xi = \gamma \) and \( \lambda = 0 \), one gets

\[
\alpha \sigma_1 \int_{\Omega} u^{\alpha-1} \omega dx \leq \beta \rho_2 \int_{\Omega} \frac{u^{\alpha}}{v^{\beta}w^{\gamma}} dx + C_1 \int_{\Omega} \frac{1}{v^{\gamma}} dx,
\]

where \( C_1 = \alpha \sigma_1 \left( \frac{\beta \rho_2}{\alpha \sigma_1} \right)^{1-\alpha} \).

Now, we choose \( \epsilon_1 \in (0, \alpha) \) such that

\[
\beta + \alpha \left( \frac{q_1 p_2 - (p_1 - 1)(1 + q_2)}{\epsilon_1 (p_2 + 1 - p_1)} \right) + \alpha \left( \frac{q_1 - 1 - q_2}{p_2 + 1 - p_1} \right) \geq 0,
\]

\[
\gamma + \alpha \left( \frac{r_1 p_2 - r_2 (p_1 - 1)}{\epsilon_1 (p_2 - p_1 + 1)} \right) + \alpha \left( \frac{r_1 - r_2}{p_2 - p_1 + 1} \right) \geq 0.
\]

Again, applying Lemma 3.3 for \( p = \alpha - 1 + p_1, q = \beta + q_1, r = \gamma + r_1, \delta = \alpha + p_2, \theta = \beta + 1 + q_2, \xi = \gamma + r_2 \) and \( \lambda = \alpha - \epsilon_1 \), we obtain

\[
\alpha \rho_1 \int_{\Omega} \frac{u^{\alpha-1+p_1}}{v^{\beta+q_1}w^{r_1}} dx \leq \beta \rho_2 \int_{\Omega} \frac{u^{\alpha+p_2}}{v^{\beta+q_2}w^{r_2}} dx + C_2 \int_{\Omega} \frac{1}{v^{\gamma}} dx,
\]

where

\[
\eta_1 = \beta + \left( q_1 p_2 - (q_2 + 1)(p_1 - 1) + \epsilon_1 (q_1 - q_2 - 1) \right) (p_2 - p_1 + 1)^{-1},
\]

\[
\eta_2 = \gamma + \left( \left[ r_1 p_2 - r_2 (p_1 - 1) + \epsilon_1 (r_1 - r_2) \right] (p_2 - p_1 + 1)^{-1} \right),
\]

and \( C_2 = \alpha \rho_1 \left( \frac{\beta \rho_2}{\alpha \sigma_1} \right)^{\frac{p_1 - 1 + q_1}{\epsilon_1}} \).

In an analogous way, we have

\[
C_3 \int_{\Omega} \frac{1}{v^{\eta_3}} dx \leq \beta \int_{\Omega} \frac{u^{\alpha}}{v^{\beta}w^{\gamma}} dx + C_3 \int_{\Omega} \frac{1}{v^{\gamma}} dx,
\]

where

\[
\eta_3 = \beta + \alpha \left[ q_1 p_3 - q_3 (p_1 - 1) + q_1 - q_2 - 1 \right] (p_2 - p_1 + 1)^{-1} \geq 0,
\]

\[
\eta_4 = \gamma + \alpha \left[ r_1 p_3 - r_3 (p_1 - 1) + r_1 - r_2 - 1 \right] (p_2 - p_1 + 1)^{-1} \geq 0,
\]

and \( C_3 = C_2 \left( \frac{\beta \rho_2}{\alpha \sigma_1} \right)^{\frac{p_1 - 1 + q_1}{\epsilon_1}} \).

Or, we choose \( \epsilon_2 \in (0, \alpha) \) such that

\[
\beta + \alpha \left( \frac{q_1 p_3 - q_3 (p_1 - 1)}{\epsilon_2 (p_3 - p_1 + 1)} \right) + \alpha \left( \frac{q_1 - q_3}{p_3 - p_1 + 1} \right) \geq 0,
\]

\[
\gamma + \alpha \left( \frac{r_1 p_3 - r_3 (p_1 - 1)}{\epsilon_2 (p_3 - p_1 + 1)} \right) + \alpha \left( \frac{r_1 - r_2 - 1}{p_3 - p_1 + 1} \right) \geq 0.
\]

Now, applying Lemma 3.3 with \( p = p_1 + \alpha - 1, q = q_1 + \beta, r = r_1 + \gamma, \delta = p_3 + \alpha, \theta = q_3 + \beta, \xi = r_3 + \gamma + 1 \) and \( \lambda = \alpha - \epsilon_2 \), we find that

\[
\alpha \rho_1 \int_{\Omega} \frac{u^{\alpha-1+p_1}}{v^{\beta+q_1}w^{r_1}} dx \leq \gamma \rho_3 \int_{\Omega} \frac{u^{\alpha+p_3}}{v^{\beta+q_3}w^{r_3}} dx + C_4 \int_{\Omega} \frac{1}{v^{\gamma}} dx,
\]
where
\[ \eta_5 = \beta + [q_1p_3 - q_3(p_1 - 1) + \epsilon_2(q_1 - q_3)](p_3 - p_1 + 1)^{-1}, \]
\[ \eta_6 = \gamma + [r_1p_3 - (r_3 + 1)(p_1 - 1) + \epsilon_2(r_1 - r_3 - 1)](p_3 - p_1 + 1)^{-1}, \]
and \( C_4 = \alpha \tilde{\rho}_1\left(\frac{\gamma_0}{\alpha_0^1}\right)^{-\frac{p_3 - 1 + \epsilon_2}{r_3 - r_2 + 1}}. \) In the same way, we obtain
\[ C_4 \int_{\Omega} \frac{u^{\alpha - \epsilon_2}}{\rho^{\eta_5} w^\eta_6} \, dx \leq b_2 \beta \int_{\Omega} \frac{u^{\gamma}}{\rho^{\eta_2} w^\eta_7} \, dx + C_5 \int_{\Omega} \frac{1}{\rho^{\eta_4} w^\eta_8} \, dx, \] (3.11)
where
\[ \eta_7 = \beta + \alpha (q_1 - q_3)(p_3 - p_1 - 1) + q_1 - q_3(p_3 - p_1 + 1)^{-1} \geq 0, \]
\[ \eta_8 = \gamma + \alpha (r_1 - r_3 - 1)(p_3 - p_1 - 1) + r_1 - r_3 - 1(p_3 - p_1 + 1)^{-1} \geq 0, \]
and \( C_5 = C_4 \left(\frac{b_2 \beta}{\alpha_0^1}\right)^{-\frac{\alpha - \epsilon_2}{r_2 - r_1}}. \)

From (3.6)–(3.11) there exists a positive constant \( C \) such that
\[ L'(t) \leq -(b_1 \alpha - 3b_2 \gamma - \gamma b_3)L(t) + C, \quad \forall t \in (0, T). \]

Then the proof is complete. \( \square \)

**Proof of Corollary 3.2** Since
\[ L(t) \leq L(0) + \frac{C}{\alpha b_1 - 3b_2 \beta - \gamma b_3} \quad \text{for all } t \in (0, T), \]
then there exist non-negative constants \( C_0, C_7 \) and \( C_8 \) independent of \( t \) such that
\[ \| f(u, v, w) - b_1 u \|_N \leq C_0, \]
\[ \| g(u, v, w) - b_2 v \|_N \leq C_7, \]
\[ \| h(u, v, w) - b_3 w \|_N \leq C_8. \]

Since \((\varphi_1, \varphi_2, \varphi_3) \in (C(\bar{\Omega}))^3\), we conclude from the \( L^p\)-\( L^q \)-estimate (see Henry 3, Haraux and Kirane 4) that
\[ u \in L^\infty((0, T), L^\infty(\Omega)), \quad v \in L^\infty((0, T), L^\infty(\Omega)), \quad w \in L^\infty((0, T), L^\infty(\Omega)). \]
Finally, we deduce that the solutions of the system (1.1)–(1.3) are global and uniformly bounded on \((0, +\infty) \times \bar{\Omega}\). \( \square \)

**Remark 3.4.** It is clear that the results of this section are valid when \( \sigma_1 \equiv \sigma_2 \equiv \sigma_3 \equiv 0. \)

4. **Asymptotic behavior of the solutions**

In this section, we study the asymptotic behavior of the solutions for the system
\[ \frac{\partial u}{\partial t} - a_1 \Delta u = -b_1 u + f(u, v, w), \quad \text{in } \mathbb{R}^+ \times \Omega, \]
\[ \frac{\partial v}{\partial t} - a_2 \Delta v = -b_2 v + g(u, v, w), \quad \text{in } \mathbb{R}^+ \times \Omega, \]
\[ \frac{\partial w}{\partial t} - a_3 \Delta w = -b_3 w + h(u, v, w), \quad \text{in } \mathbb{R}^+ \times \Omega, \]
where
\[ f(u, v, w) = \rho_1(x, u, v, w)\frac{u^{p_1}}{v^{q_2} + w^{p_3} + \sigma_1}, \]
\[ g(u, v, w) = \rho_2(x, u, v, w)\frac{v^{q_2}}{u^{p_1} + w^{p_3} + \sigma_2}, \]
\[ h(u, v, w) = \rho_3(x, u, v, w)\frac{w^{p_3}}{u^{p_1} + v^{q_2} + \sigma_3}. \]
where and initial data
\[ u(0, x) = \varphi_1(x), \quad v(0, x) = \varphi_2(x), \quad w(0, x) = \varphi_3(x) \quad \text{in } \Omega. \] (4.3)

Here \( \sigma_1, \sigma_2 \) and \( \sigma_3 \) are non negative constants.

Before stating the results, let us expose some simple facts concluded from the result of the previous section. From Theorem 3.1 and by using classical method of a semi group and a power fractional (see [5]) we can find the positive constants \( M_1, M_2 \) and \( M_3 \) explicitly (see [9]) such that
\[ \|u(t, \cdot)\|_\infty M_1, \quad \|v(t, \cdot)\|_\infty \leq M_2, \quad \|w(t, \cdot)\|_\infty \leq M_3. \]

Let us consider the same function as in Theorem 3.1,
\[ L(t) = \int_\Omega \frac{u^\alpha(t, x)}{v^\beta(t, x)w^\gamma(t, x)} \, dx, \quad \forall t \in (0, +\infty), \]
where \( \alpha, \beta \) and \( \gamma \) are positive constants satisfying the following conditions
\[ \alpha > 2 \max(1, \frac{3b_2 + b_3}{b_1}), \quad \frac{1}{\beta} > \frac{(a_1 + a_2)^2}{2a_1a_2}, \]
and
\[ \left( \frac{1}{2\beta} \frac{(a_1 + a_2)^2}{4a_1a_2} \right) \left( \frac{1}{2\gamma} \frac{(a_1 + a_3)^2}{4a_1a_3} \right) > \left( \frac{(\alpha - 1)(a_2 + a_3)}{2\alpha a_2 a_3} - \frac{(a_1 + a_2)(a_1 + a_3)}{4\sqrt{a_1^2a_2a_3}} \right)^2. \]

The main result in this section reads as follows.

**Theorem 4.1.** Assume (1.4) holds. Let \((u, v, w)\) be the solution of (4.1)-(4.3) in \((0, +\infty)\). Suppose that \( \sigma_1 = 0 \), and
\[ b_1 > \frac{\beta b_2 + \gamma b_3 + K}{2}, \] (4.4)
where
\[ K = \frac{\alpha \rho_1(\frac{\beta p_2}{\alpha b_1}) - \rho_1^{p_1 - 1}}{m_2 (q_1 p_2 - (q_2 + 1)(p_1 - 1)) (p_2 - p_1 + 1)^{p_1 - r_2(p_1 - 1)} (p_2 - p_1 + 1)^{r_2}}, \]
or
\[ K = \frac{\alpha \rho_1(\frac{\gamma p_3}{\alpha b_1}) - \rho_1^{p_1 - 1}}{m_2 (q_1 p_3 - q_2(p_1 - 1)) (p_2 - p_1 + 1)^{p_1 - r_2(p_1 - 1)} (p_2 - p_1 + 1)^{r_2}}. \]

Then for all \( t \in (0, +\infty) \) we have
\[ L(t) \leq \int_\Omega \frac{\varphi_1^\alpha(x)}{\varphi_2^\beta(x)\varphi_3^\gamma(x)} \, dx. \]
Corollary 4.2. Under the assumptions of Theorem 4.1, for all positive initial data in \( C(\Omega) \) we have

\[
\|u(t,.)\|_\infty \to 0 \quad \text{as} \quad t \to +\infty, \\
\|v(t,.) - \frac{\sigma_2}{b_2}\|_\infty \to 0 \quad \text{as} \quad t \to +\infty, \\
\|w(t,.) - \frac{\sigma_3}{b_3}\|_\infty \to 0 \quad \text{as} \quad t \to +\infty.
\]

Proof of Theorem 4.1. From (3.5) and (3.6), we obtain for all \( t \in (0, +\infty) \)

\[
L'(t) \leq - (ab_1 - \beta b_2 - \gamma b_2)L(t) + \alpha \tilde{\rho}_1 \int_{\Omega} \frac{u^{\alpha-1+p_1}}{v^{\beta+q_1}w^{\gamma+r_1}} dx \tag{4.5}
\]

Now, we apply Lemma 3.3 for \( p = \alpha - 1 + p_1, \quad q = \beta + q_1, \quad r = \gamma + r_1, \quad \delta = \alpha + p_2, \quad \theta = \beta + q_2, \quad \xi = \gamma + \rho_2 \) and \( \lambda = \alpha \) we obtain

\[
\alpha \tilde{\rho}_1 \int_{\Omega} \frac{u^{\alpha-1+p_1}}{v^{\beta+q_1}w^{\gamma+r_1}} dx \leq \beta \tilde{\rho}_2 \int_{\Omega} \frac{u^{\alpha+p_2}}{v^{\beta+q_1}w^{\gamma+r_2}} dx + A_1 \int_{\Omega} \frac{u^{\alpha}}{v^{\beta+q_2}w^{\gamma+r_2}} dx, \tag{4.6}
\]

where

\[
\eta_9 = \beta + [q_1 p_2 - (q_2 + 1)(p_1 - 1)](p_2 - p_1 + 1)^{-1} > 0,
\]

\[
\eta_{10} = \gamma + [\gamma + q_2 - r_2(p_1 - 1)](p_2 - p_1 + 1)^{-1} > 0,
\]

and \( A_1 = \alpha \tilde{\rho}_1 \left( \frac{\beta p_2}{\alpha p_1} \right)^{- \frac{p_1 - 1}{p_2 - p_1 + 1}} \).

Or, applying Lemma 3.3 for \( p = \alpha - 1 + p_1, \quad q = \beta + q_1, \quad r = \gamma + r_1, \quad \delta = \alpha + p_3, \quad \theta = \beta + q_3, \quad \xi = \gamma + 1 + r_3 \) and \( \lambda = \alpha \), we obtain

\[
\alpha \tilde{\rho}_1 \int_{\Omega} \frac{u^{\alpha-1+p_1}}{v^{\beta+q_1}w^{\gamma+r_1}} dx \leq \gamma \tilde{\rho}_3 \int_{\Omega} \frac{u^{\alpha+p_3}}{v^{\beta+q_3}w^{\gamma+1+r_3}} dx + A_2 \int_{\Omega} \frac{u^{\alpha}}{v^{\beta+q_3}w^{\gamma+1+r_3}} dx, \tag{4.7}
\]

where

\[
\eta_{11} = \beta + [q_1 p_3 - q_3(p_1 - 1)](p_3 - p_1 + 1)^{-1} > 0,
\]

\[
\eta_{12} = \gamma + [\gamma + q_3 - (r_3 + 1)(p_1 - 1)](p_3 - p_1 + 1)^{-1} > 0,
\]

and \( A_2 = \alpha \tilde{\rho}_1 \left( \frac{\gamma p_3}{\alpha p_1} \right)^{- \frac{p_3 - 1}{p_3 - p_1 + 1}} \).

By combining (4.5) with (4.6) and (4.7) we obtain

\[
L'(t) \leq - (ab_1 - \beta b_2 - \gamma b_3 - K)L(t), \quad \forall t \in (0, +\infty), \tag{4.8}
\]

where

\[
K = \frac{\alpha \tilde{\rho}_1 \left( \frac{\beta p_2}{\alpha p_1} \right)^{- \frac{p_1 - 1}{p_2 - p_1 + 1}}}{m_2} \frac{[q_1 p_2 - (q_2 + 1)(p_1 - 1)](p_2 - p_1 + 1)^{-1}}{m_3} \frac{[r_3 p_2 - r_2(p_1 - 1)](p_2 - p_1 + 1)^{-1}},
\]

or

\[
K = \frac{\alpha \tilde{\rho}_1 \left( \frac{\gamma p_3}{\alpha p_1} \right)^{- \frac{p_3 - 1}{p_3 - p_1 + 1}}}{m_2} \frac{[q_1 p_3 - q_3(p_1 - 1)](p_3 - p_1 + 1)^{-1}}{m_3} \frac{[r_3 p_3 - (r_3 + 1)(p_1 - 1)](p_3 - p_1 + 1)^{-1}}{m_3}.
\]

Using (4.4) we deduce that the function \( t \mapsto L(t) \) is a non-increasing function. This completes the proof of Theorem 4.1. \( \square \)
Proof of Corollary 4.2. Setting for all \((t, x) \in (0, +\infty) \times \Omega:\)

\[
\begin{align*}
    h_1(t, x) &= u(t, x), \\
    h_2(t, x) &= v(t, x) - \frac{\sigma_2}{b_2}, \\
    h_3(t, x) &= w(t, x) - \frac{\sigma_3}{b_3}.
\end{align*}
\]

For \(i = 1, 2, 3\) we have

\[
\frac{dh_i}{dt} - a_i \Delta h_i = -b_i h_i + \rho_i(x, u, v, w) \frac{u^{p_i}}{v^{q_i} w^{r_i}}.
\]

(4.9)

Multiplying (4.9) by \(h_i(t, x), i = 1, 2, 3\) and integrating over \([0, t] \times \Omega\) we obtain

\[
\begin{align*}
\frac{1}{2} \int_\Omega h_i^2(x) dx + a_i \int_0^t \int_\Omega |\nabla h_i|^2 dx ds + b_i \int_0^t \int_\Omega h_i^2 dx ds \\
= \frac{1}{2} \int_\Omega h_i^2(0) dx + \int_0^t \int_\Omega h_i \rho_i(x, u, v) \frac{u^{p_i}}{v^{q_i} w^{r_i}} dx ds.
\end{align*}
\]

From (4.8), for all \(t \in (0, +\infty)\), and for \(i = 1, 2, 3\) we obtain

\[
\int_0^t \int_\Omega h_i \rho_i(x, u, v) \frac{u^{p_i}}{v^{q_i} w^{r_i}} dx ds \leq \bar{\rho} M_i M_1^{p_i} M_2^2 M_3^3 \int_0^t \int_\Omega u^{\alpha} v^{\beta} w^{\gamma} dx ds < +\infty.
\]

One obviously deduces that for \(i = 1, 2, 3,\)

\[
\begin{align*}
    h_i(t, .) \in L^2(\Omega), & \quad \int_0^{+\infty} \int_\Omega |\nabla h_i|^2 dx ds < +\infty, \\
    \int_0^{+\infty} \int_\Omega h_i^2 dx ds < +\infty,
\end{align*}
\]

so that Barbalate’s lemma [3, Lemma 1.2.2] permits to conclude that

\[
\lim_{t \to +\infty} \|h_i(t, .)\|_2 = 0, \quad i = 1, 2, 3.
\]

On the other hand, since the orbits \(\{h_i(t, .)/t \geq 0, i = 1, 2, 3\}\) are relatively compact in \(C(\Omega)\) (see [4]), it follows readily that

\[
\lim_{t \to +\infty} \|h_i(t, .)\|_\infty = 0, \quad i = 1, 2, 3.
\]

Then proof of Corollary 4.2 is complete. \(\square\)

Acknowledgments. The authors want to thank Prof. M. Kirane and the anonymous referee for their suggestions that improved the quality of this article.

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