INFINITELY MANY SOLUTIONS VIA VARIATIONAL-HEMIVARIATIONAL INEQUALITIES UNDER NEUMANN BOUNDARY CONDITIONS

FARIBA FATTAHI, MOHSEN ALIMOHAMMADY

Abstract. In this article, we study the variational-hemivariational inequalities with Neumann boundary condition. Using a nonsmooth critical point theorem, we prove the existence of infinitely many solutions for boundary-value problems. Our technical approach is based on variational methods.

1. Introduction

In this article, we study following boundary-value problem, depending on the parameters $\lambda, \mu$ with nonsmooth Neumann boundary condition:

$$
\begin{align*}
-\Delta_{p(x)} u + a(x)|u|^{p(x)-2}u &= 0 \quad \text{in } \Omega \\
-|\nabla u|^{p(x)-2} \frac{\partial u}{\partial \nu} &\in -\lambda \theta(x) \partial F(u) - \mu \partial \vartheta(x) G(u) \quad \text{on } \partial \Omega,
\end{align*}
$$

where $\Omega \subset \mathbb{R}^N (N \geq 2)$ is a bounded smooth domain, $\frac{\partial u}{\partial \nu}$ is the outer unit normal derivative on $\partial \Omega$, $p : \bar{\Omega} \to \mathbb{R}$ is a continuous function satisfying $1 < p^- = \min_{x \in \Omega} p(x) \leq p(x) \leq p^+ = \max_{x \in \Omega} p(x) < +\infty$.

Here $\lambda, \mu$ are real parameters, $\lambda \in [0, \infty], \mu \in [0, \infty]$ and $\theta, \vartheta \in L^1(\partial \Omega)$, where $\theta(x), \vartheta(x) \geq 0$ for a.e. $x \in \partial \Omega$. $F, G : \mathbb{R} \to \mathbb{R}$ are locally Lipschitz functions given by $F(\omega) = \int_0^\omega f(t)dt$, $G(\omega) = \int_0^\omega g(t)dt$, $\omega \in \mathbb{R}$ such that $f, g : \mathbb{R} \to \mathbb{R}$ are locally essentially bounded functions. $\partial F(u), \partial G(u)$ denote the generalized Clarke gradient of $F(u), G(u)$.

Let $X$ be real Banach space. We assume that it is also given a functional $\chi : X \to \mathbb{R} \cup \{+\infty\}$ which is convex, lower semicontinuous, proper whose effective domain $\text{dom}(\chi) = \{x \in X : \chi(x) < +\infty\}$ is a (nonempty, closed, convex) cone in $X$. Our aim is to study the following variational-hemivariational inequalities problem: Find $u \in B$ which is called a weak solution of problem (1.1), i.e; if for all

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\( v \in \mathcal{B}, \)
\[
\int_{\Omega} |\nabla u|^{p(x)-2} \nabla u \nabla (v-u) dx + \int_{\Omega} a(x)|u|^{p(x)-2} u (v-u) dx \\
- \lambda \int_{\partial \Omega} \theta(x) F^0(u; u-v) d\sigma - \mu \int_{\partial \Omega} \vartheta(x) G^0(u; u-v) d\sigma \geq 0,
\]
where \( \mathcal{B} \) is a closed convex subset of \( W^{1,p(x)}_0(\Omega). \) For simplicity \( \mathcal{B} = W^{1,p(x)}_0(\Omega). \)

Recently, many researchers have paid attention to impulsive differential equations where \( p \) is a constant, the Luxemburg norm
\[
\|u\|_{p(\cdot)} = \inf \left\{ \lambda > 0 : \int_{\Omega} \frac{|u(x)|^{p(x)}}{\lambda} dx \leq 1 \right\}.
\]
Note that, when \( p \) is constant, the Luxemburg norm \( \| \cdot \|_{p(\cdot)} \) coincides with the standard norm \( \| \cdot \|_p \) of the Lebesgue space \( L^p(\Omega). \) \( (L^{p(\cdot)}(\Omega), \| \cdot \|_{p(\cdot)}) \) is a Banach space.

The generalized Lebesgue-Sobolev space \( W^{1,p(\cdot)}_{L,p(\cdot)}(\Omega) \) for \( L = 1, 2, \ldots \) is defined by
\[
W^{1,p(\cdot)}_{L,p(\cdot)}(\Omega) = \{ u \in L^{p(\cdot)}(\Omega) : D^\alpha u \in L^{p(\cdot)}(\Omega), |\alpha| \leq L \},
\]
where \( D^\alpha u = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} u \) with \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N) \) is a multi-index and \( |\alpha| = \sum_{i=1}^N \alpha_i. \) The space \( W^{1,p(\cdot)}_{L,p(\cdot)}(\Omega) \) with the norm
\[
\|u\|_{W^{1,p(\cdot)}_{L,p(\cdot)}(\Omega)} = \sum_{|\alpha| \leq L} \|D^\alpha u\|_{p(\cdot)},
\]
is a separable reflexive Banach space \([9]\).

\(W_{0}^{L,p(x)}(\Omega)\) denotes the closure in \(W^{L,p(x)}(\Omega)\) of the set of functions in \(W^{L,p(x)}(\Omega)\) with compact support.

For every \(u \in W_{0}^{L,p(x)}(\Omega)\) the Poincaré inequality holds, where \(C_{p} > 0\) is a constant

\[\|u\|_{L^{p(x)}(\Omega)} \leq C_{p}\|\nabla u\|_{L^{p(x)}(\Omega)}\],

(see \([12]\)). Hence, an equivalent norm for the space \(W_{0}^{L,p(x)}(\Omega)\) is given by

\[\|u\|_{W_{0}^{L,p(x)}(\Omega)} = \sum_{|\alpha|=L} \|D^{\alpha}u\|_{p(x)}\].

Given \(p(x)\), let \(p_{L}^{*}(x)\) denote the critical variable exponent related to \(p\), defined for all \(x \in \Omega\) by the pointwise relation

\[p_{L}^{*}(x) = \begin{cases} \frac{Np(x)}{N-Lp(x)} & Lp(x) < N, \\ +\infty & Lp(x) \geq N, \end{cases} \quad (2.1)\]

is the critical exponent related to \(p\). Let

\[\mathcal{K} = \sup_{u \in X \setminus \{0\}} \frac{\max\{u(x)\}^{p}}{\|u\|^{p}}, \quad \mathcal{M} = \inf_{u \in X \setminus \{0\}} \frac{\min\{u(x)\}^{p}}{\|u\|^{p}}. \quad (2.2)\]

Since \(p > N\), \(X\) are compactly embedded in \(C^{0}(\Omega)\), it follows that \(\mathcal{K}, \mathcal{M} < \infty\).

**Proposition 2.1.** For \(\Phi(u) = \int_{\Omega}|\nabla u|^{p(x)} + a(x)|u(x)|^{p(x)}\,dx\), and \(u, u_{n} \in X\), we have

(i) \(\|u\| < (=, >)1 \iff \Phi(u) < (=, >)1\),

(ii) \(\|u\| \leq 1 \Rightarrow \|u\|^{p_{L}^{*}} \leq \Phi(u) \leq \|u\|^{p}\),

(iii) \(\|u\| \geq 1 \Rightarrow \|u\|^{p_{L}^{*}} \leq \Phi(u) \leq \|u\|^{p}\),

(iv) \(\|u_{n}\| \to 0 \iff \Phi(u_{n}) \to 0\),

(v) \(\|u_{n}\| \to \infty \iff \Phi(u_{n}) \to \infty\).

The proof of the above proposition is similar to that in \([11]\).

**Proposition 2.2** \((11) [14]\). For \(p, q \in C_{+}(\Omega)\) in which \(q(x) \leq p_{L}^{*}(x)\) for all \(x \in \Omega\), there is a continuous embedding

\[W^{L,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega).\]

If we replace \(\leq w ith <\), the embedding is compact.

**Remark 2.3.** (i) By the proposition \([22]\) there is a continuous and compact embedding of \(W_{0}^{1,p(x)}(\Omega)\) into \(L^{q(x)}\) where \(q(x) < p^{*}(x)\) for all \(x \in \Omega\). \(W_{0}^{1,p(x)}(\Omega)\) is continuously embedded in \(W^{1,p^{-}}(\Omega)\) and since \(p^{-} > N\), we deduce that \(W_{0}^{1,p^{-}}(\Omega)\) is compactly embedded in \(C^{0}(\Omega)\). So, there exists a constant \(c > 0\) such that

\[\|u\|_{\infty} \leq c\|u\|, \quad \forall u \in X, \quad (2.3)\]

where \(\|u\|_{\infty} := \sup_{x \in \Omega} |u(x)|\).

(ii) Denote

\[\|u\| = \inf \{\lambda > 0 : \int_{\Omega} \frac{\nabla u}{\lambda} \frac{|p(x) + a(x)u(x)|^{p}}{\lambda^{p(x)}} \,dx \leq 1\},\]

which is a norm on \(W_{0}^{1,p(x)}(\Omega)\).
Let $\eta : \partial \Omega \to \mathbb{R}$ be a measurable. Define the weighted variable exponent Lebesgue space by

$$L^{p(x)}_{\eta(x)}(\partial \Omega) = \{ u : \partial \Omega \to \mathbb{R} \text{ is measurable and } \int_{\partial \Omega} |\eta(x)||u|^{p(x)}d\sigma < \infty \},$$

with the norm

$$|u|_{(p(x), \eta(x))} = \inf\{ \tau > 0 : \int_{\partial \Omega} |\eta(x)| \frac{|u|^{p(x)}}{\tau} d\sigma \leq 1 \},$$

where $d\sigma$ is the measure on the boundary.

**Lemma 2.4** ([8]). Let $\rho(x) = \int_{\partial \Omega} |\eta(x)||u|^{p(x)}d\sigma$ for $u \in L^{p(x)}_{\eta(x)}(\partial \Omega)$ we have

$$|u|_{(p(x), \eta(x))} \geq 1 \Rightarrow |u|_{(p(x), \eta(x))} \leq \rho(u) \leq |u|_{(p(x), \eta(x))}^+,$$

$$|u|_{(p(x), \eta(x))} \leq 1 \Rightarrow |u|_{(p(x), \eta(x))} \leq \rho(u) \leq |u|_{(p(x), \eta(x))}^-.$$

For $A \subseteq \Omega$ denote by $\inf_{x \in A} p(x) = p^-$, $\sup_{x \in A} p(x) = p^+$. Define

$$p_0^0(x) = (p(x))^0 = \begin{cases} \frac{(N-1)p(x)}{N-p(x)} & p(x) < N, \\
\infty & p(x) \geq N, \end{cases}$$

$$p_0^0(x)_{r(x)} := \frac{r(x)-1}{r(x)} p_0^0(x),$$

where $x \in \partial \Omega, r \in C(\partial \Omega, \mathbb{R})$ and $r(x) > 1$.

**Proposition 2.5** ([10] [14]). If $q \in C_+(\overline{\Omega})$ and $q(x) < p^0(x)$ for any $x \in \overline{\Omega}$, then the embedding $W^{1,p^0} : (\Omega) \to L^{q}^{0} (\partial \Omega)$ is compact and continuous.

In this part we introduce the definitions and basic properties from the theory of generalized differentiation for locally Lipschitz functions. Let $X$ be a Banach space and $X^*$ its topological dual. By $\| \cdot \|$ we will denote the norm in $X$ and by $\langle \cdot, \cdot \rangle_X$ the duality brackets for the pair $(X, X^*)$. A function $h : X \to \mathbb{R}$ is said to be locally Lipschitz, if for every $x \in X$ there exists a neighbourhood $U$ of $x$ and a constant $K > 0$ depending on $U$ such that $|h(y) - h(z)| \leq K\|y - z\|$ for all $y, z \in U$. For a locally Lipschitz function $h : X \to \mathbb{R}$ is defined by the generalized directional derivative of $h$ at $u \in X$ in the direction $\gamma \in X$ by

$$h^0(u; \gamma) = \limsup_{w \to u, t \to 0^+} \frac{h(w + t\gamma) - h(w)}{t}.$$

The generalized gradient of $h$ at $u \in X$ is defined by

$$\partial h(u) = \{ x^* \in X^* : \langle x^*, \gamma \rangle_X \leq h^0(u; \gamma), \forall \gamma \in X \},$$

which is non-empty, convex and $w^*$-compact subset of $X^*$, where $\langle \cdot, \cdot \rangle_X$ is the duality pairing between $X^*$ and $X$.

**Proposition 2.6** ([11]). Let $h, g : X \to \mathbb{R}$ be locally Lipschitz functions. Then:

(i) $h^0(u; \cdot)$ is subadditive, positively homogeneous.

(ii) $(-h)^0(u; v) = h^0(u; -v)$ for all $u, v \in X$.

(iii) $h^0(u; v) = \max \{ \langle \xi, v \rangle_X : \xi \in \partial h(u) \}$ for all $u, v \in X$.

(iv) $(h + g)^0(u; v) \leq h^0(u; v) + g^0(u; v)$ for all $u, v \in X$. 

Theorem 2.9. We recall the following nonsmooth version of a critical point result.

Definition 2.7. Let $X$ be a Banach space, $\mathcal{I} : X \to (-\infty, +\infty]$ is called a Motreanu-Panagiotopoulos-type functional, if $\mathcal{I} = h + \chi$, where $h : X \to \mathbb{R}$ is locally Lipschitz and $\chi : X \to (-\infty, +\infty]$ is convex, proper and lower semicontinuous.

Definition 2.8. An element $u \in X$ is said to be a critical point of $\mathcal{I} = h + \chi$ if

$$h^0(u; v - u) + \chi(v) - \chi(u) \geq 0, \quad \forall v \in X.$$ 

Let $X$ is a reflexive real Banach space, $\phi : X \to \mathbb{R}$ is a sequentially weakly lower semicontinuous and coercive, $\Upsilon : X \to \mathbb{R}$ is a sequentially weakly upper semicontinuous, $\lambda$ is a positive real parameter, $\chi : X \to (-\infty, +\infty]$ is a convex, proper, lower semicontinuous functional and $D(\chi)$ is the effective domain of $\chi$. Assuming also that $\phi$ and $\Upsilon$ are locally Lipschitz continuous functionals. Set

$$\mathcal{E} := \Upsilon - \chi, \quad \mathcal{L}_\lambda := \phi - \lambda \mathcal{E} = (\phi - \lambda \Upsilon) + \lambda \chi.$$ 

We assume that

$$\phi^{-1}(-\infty, r] \cap D(\chi) \neq \emptyset, \quad \forall r > \inf_\chi \phi,$$

and define for every $r > \inf_\chi \phi$,

$$\varphi(r) = \inf_{u \in \phi^{-1}(-\infty, r]} \frac{\left(\sup_{v \in \phi^{-1}(-\infty, r]} \mathcal{E}(v)\right) - \mathcal{E}(u)}{r - \phi(u)}$$

and

$$\gamma := \liminf_{r \to +\infty} \varphi(r), \quad \delta := \liminf_{r \to -\inf_\chi \phi^+} \varphi(r).$$

We recall the following nonsmooth version of a critical point result.

Theorem 2.9 (I). Under the above assumptions on $X$, $\phi$ and $\mathcal{E}$, we have

(a) For every $r > \inf_\chi \phi$, and every $\lambda \in (0, \frac{1}{\varphi(r)})$, the restriction of the functional

$$\mathcal{L}_\lambda = \phi - \lambda \mathcal{E}$$

to $\phi^{-1}(-\infty, r)$ admits a global minimum, which is a critical point (local minimum) of $\mathcal{L}_\lambda$ in $X$.

(b) If $\gamma < +\infty$, then for each $\lambda \in (0, 1/\gamma)$, the following alternative holds: either (b1) $\mathcal{L}_\lambda$ possesses a global minimum, or (b2) there is a sequence $\{u_n\}$ of critical points (local minima) of $\mathcal{L}_\lambda$ such that

$$\lim_{n \to +\infty} \phi(u_n) = +\infty.$$

(c) If $\delta < +\infty$, then for each $\lambda \in (0, \frac{1}{\delta})$, the following alternative holds: either (c1) there is a global minimum of $\phi$ which is a local minimum of $\mathcal{L}_\lambda$, or (c2) there is a sequence $\{u_n\}$ of pairwise distinct critical points (local minima) of $\mathcal{L}_\lambda$ that converges weakly to a global minimum of $\phi$.

Consider $\phi, \mathcal{F}, \mathcal{G} : X \to \mathbb{R}$, as follows

$$\phi(u) = \int_{\Omega} \frac{1}{p(x)} ||\nabla u||^p + a(x)|u|^p dx, \quad u \in W^{1,p}(\Omega),$$

$$\mathcal{F}(u) = \int_{\partial \Omega} F(u(x)) d\sigma, \quad u \in W^{1,p}(\Omega),$$

$$\mathcal{G}(u) = \int_{\partial \Omega} G(u(x)) d\sigma, \quad u \in W^{1,p}(\Omega).$$
The next lemma characterizes some properties of $\phi$ [2].

**Lemma 2.10.** Let
\[
\phi(u) = \int_{\Omega} \frac{1}{p(x)} [|\nabla u|^{p(x)} + a(x)|u|^{p(x)}] dx.
\]
Then
(i) $\phi : X \to \mathbb{R}$ is sequentially weakly lower semicontinuous;
(ii) $\phi'$ is of $(S_+)$ type;
(iii) $\phi'$ is a homeomorphism.

**Proposition 2.11 ([15]).** Let $F, G : \mathbb{R} \to \mathbb{R}$ be locally Lipschitz functions. Then $F$ and $G$ are well-defined and
\[
F^0(u; v) \leq \int_{\partial \Omega} F^0(u(x); v(x)) d\sigma, \quad \forall u, v \in W^{1,p}_0(\Omega),
\]
\[
G^0(u; v) \leq \int_{\partial \Omega} G^0(u(x); v(x)) d\sigma, \quad \forall u, v \in W^{1,p}_0(\Omega).
\]

3. **Main results**

Let $f : \mathbb{R} \to \mathbb{R}$ be a locally essentially bounded function whose potential $F(t) = \int_0^t f(\omega)d\omega$ for all $t \in \mathbb{R}$. Set
\[
\alpha := \lim \inf_{\omega \to +\infty} \frac{\max_{|t| \leq \omega} F(t)}{|\omega|^{p^*}}; \quad \beta := \lim \sup_{\omega \to +\infty} \frac{F(\omega)}{|\omega|^{p^*}}.
\]

**Theorem 3.1.** Let $\theta, \vartheta \in L^1(\partial \Omega)$ be non-negative and non-zero identically zero functions. Assume that
\[
\alpha < \frac{p^- M^{*\theta} \beta}{p^+ K}, \quad (3.1)
\]
for each $\lambda \in (\lambda_1, \lambda_2)$, where
\[
\lambda_1 = \frac{1}{p^+ M^{*\theta} \beta}, \quad \lambda_2 = \frac{1}{p^+ K \alpha},
\]
and $\theta^* = \int_{\partial \Omega} \theta(x)d\sigma$. Also assume that for each locally essentially bounded function $g : \mathbb{R} \to \mathbb{R}$ with potential $G(t) = \int_0^t g(\omega)d\omega$, for all $t \in \mathbb{R}$, satisfies
\[
G_\infty = \lim \sup_{\omega \to +\infty} \frac{\max_{|t| \leq \omega} G(t)}{|\omega|^{p^-}} < +\infty, \quad (3.2)
\]
for every $\mu \in [0, \mu_{G, \lambda})$, where
\[
\mu_{G, \lambda} = \frac{1}{p^+ K G_\infty}(1 - p^+ K \lambda \alpha).
\]
Then [11] has a sequence of weak solutions for every $\mu \in [0, \mu_{G, \lambda})$ in $X$ such that
\[
\int_{\Omega} \frac{1}{p(x)} [|\nabla u_n|^{p^*(x)} + a(x)|u_n|^{p(x)}] dx \to +\infty.
\]

**Proof.** Our strategy is to apply Theorem 2.9 (b).

**Case 1.** Assume that $\|u\| \geq 1$. Let $\lambda \in (\lambda_1, \lambda_2)$ and $G$ satisfy our assumptions. Since $\lambda < \lambda_2$, it follows that
\[
\mu_{G, \lambda} = \frac{1}{p^+ K G_\infty}(1 - p^+ K \lambda \alpha).
\]
Fix $\bar{\mu} \in (0, \mu_{c, \lambda})$ and define the functionals $\phi, \mathcal{E} : X \to \mathbb{R}$ for each $u \in X$ as follows:

$$
\phi(u) = \int_\Omega \frac{1}{p(x)} [||\nabla u||^{p(x)} + a(x)|u|^{p(x)}] \, dx,
$$

$$
\Upsilon(u) = \int_{\partial \Omega} \theta(x)[F(u(x))] \, d\sigma + \frac{\bar{\mu}}{\lambda} \int_{\partial \Omega} \vartheta(x)[G(u(x))] \, d\sigma,
$$

$$
\chi(u) = \begin{cases}
0 & u \in \mathcal{B}, \\
+\infty & u \notin \mathcal{B},
\end{cases}
$$

$$
\mathcal{E}(u) = \Upsilon(u) - \chi(u).
$$

Then define the functional

$$
\mathcal{L}_\lambda(u) := \phi(u) - \bar{\lambda} \mathcal{E}(u)
$$

whose critical points are the weak solutions of (1.1).

To apply Lemma 2.10, we assume that $\phi$ satisfies the regularity assumptions of Theorem 2.9. By standard argument, $\Upsilon$ is sequentially weakly continuous. First, we claim that $\bar{\lambda} < 1/\gamma$. Note that $\phi(0) = \mathcal{E}(0) = 0$, then for every $n$ large enough, one has

$$
\varphi(r) = \inf_{u \in \phi^{-1}([-\infty, r])} \left( \frac{\sup_{v \in \phi^{-1}([-\infty, r])} \mathcal{E}(v)}{r - \phi(u)} \right)
\leq \sup_{v \in \phi^{-1}([-\infty, r])} \mathcal{E}(v).
$$

Coercivity of $\phi$ implies that $\inf_\chi \varphi = \phi(0) = 0$. Since $\mathcal{B}$ contains constant functions, $0 \in \mathcal{B} = D(\chi)$, thus

$$
0 \in \phi^{-1}([-\infty, r]) \cap D(\chi), \quad \forall r > \inf_\chi \varphi.
$$

For $v \in X$ with $\phi(v) < r$ and in view of (2.2),

$$\phi^{-1}([-\infty, r]) := \{v \in X : \phi(v) < r\} = \{v \in X : \frac{1}{p} ||v||^{p^-} < r\}
\subseteq \{v \in X : |v(x)| < (p^+ \mathcal{K} r)^{-\frac{1}{p^+}}\}.
$$

Then

$$
\varphi(r) \leq \frac{\left( \sup_{\{v \in X : |v(x)| < (p^+ \mathcal{K} r)^{-\frac{1}{p^+}}\}} \mathcal{E}(v) - \chi(v) \right)}{r}.
$$

Let $\{\omega_n\}$ be a sequence of positive numbers in $X$ such that $\lim_{n \to +\infty} \omega_n = +\infty$ and

$$
\alpha = \lim_{n \to +\infty} \frac{\max_{|t| \leq \omega_n} F(t)}{|\omega_n|^{p^-}}.
$$

Set

$$
r_n = \frac{|\omega_n|^{p^-}}{\mathcal{K} p^+}, \quad n \in \mathbb{N}.
$$

Take $v \in \phi^{-1}([-\infty, r_n])$, from (3.4), we have $|v(x)| < (p^+ \mathcal{K} r_n)^{-\frac{1}{p^+}}$. Hence,

$$
\varphi(r_n) \leq \sup_{\{v \in X : |v(x)| < \omega_n, \forall x \in \partial \Omega\}} \int_{\partial \Omega} \left[ \theta(x) F(v) + \frac{\bar{\mu}}{\lambda} \vartheta(x) G(v) \right] \, d\sigma
\leq \frac{\omega_n^{p^-}}{\mathcal{K} p^+} \omega_n^{p^-}.
$$
for every large enough \( n \). Let \( \bar{\lambda} \) be a sequence in \( \mathbb{N} \) such that
\[
\xi_n(x) = \tau_n \quad \text{for all} \quad x \in \Omega.
\]
Fix \( n \in \mathbb{N} \), by proposition \( 2.1 \)
\[
\phi(\xi_n) = \int_{\partial \Omega} \frac{1}{p(x)} \left[ |\nabla \xi_n|^p + a(x)|\xi_n|^p \right] dx \leq \frac{1}{p} \|\tau_n\|^p \leq \frac{1}{M_0} |\tau_n|^p. \tag{3.6}
\]
Since \( G \) is non-negative and from the definition of \( E \)
\[
E(\xi_n) = \int_{\partial \Omega} \left[ \theta(x)F(\xi_n) + \frac{\bar{\mu}}{\lambda} \vartheta(x)G(\xi_n) \right] d\sigma - \chi(\xi_n)
\geq \int_{\partial \Omega} \theta(x)F(\xi_n) d\sigma = \theta^* F(\tau_n). \tag{3.7}
\]
According to \( \xi_n(x) = \tau_n \) in \( \Omega \) and \( \lambda \)
\[
L_\lambda(\xi_n) \leq \frac{1}{p} |\tau_n|^p - \bar{\lambda} \int_{\partial \Omega} \theta(x)F(\tau_n) d\sigma < \frac{1}{M_p} |\tau_n|^p - \frac{1}{M_p} \lambda |\tau_n|^p \eta,
\]
for every large enough \( n \in \mathbb{N} \). Since \( \bar{\lambda} \eta > 1 \) and \( \lim_{n \to +\infty} \tau_n = +\infty \), it results that
\[
\lim_{n \to +\infty} L_\lambda(\xi_n) = -\infty.
\]
Hence, the functional \( L_\lambda \) is unbounded from below, and it follows that \( L_\lambda \) has no global minimum. Therefore, applying \( 3.5 \) we deduce that there is a sequence \( u_n \in X \) of critical points of \( L_\lambda \) such that
\[
\int_{\Omega} \frac{1}{p(x)} \left[ |\Delta u_n|^p(x) + a(x)|u_n|^p(x) \right] dx \to +\infty.
\]

**Case 2.** If \( \|u\| \leq 1 \) the proof is similar to the first case and the proof of theorem is complete.

**Lemma 3.2.** Every critical point of the functional \( L_\lambda \) is a solution of \( (1.1) \).
Proof. By definition $2.7 \quad \mathcal{L}_\lambda = (\phi - \lambda \mathcal{Y}) + \lambda \chi$ is a Motreanu-Panagiotopoulos type functional. Let $\{u_n\} \subset X$ be a critical sequence of $\mathcal{L}_\lambda = \phi - \lambda \mathcal{F} - \mu \mathcal{G} + \lambda \chi$ then $u_n \in \mathcal{B}$, definition $2.8$ and proposition $2.6$ imply that

$$
(\phi - \lambda \mathcal{Y})^0(u_n; v - u_n) \geq 0, \quad \forall v \in \mathcal{B}.
$$

Using proposition $2.11$

$$
\int |\nabla u_n|^p(x) - 2 \nabla u_n \nabla (v - u_n) dx + \int a(x) |u_n|^p(x) - 2 u_n(v - u_n) dx
$$

$$
- \lambda \int \partial(x) F^0(u_n; v - u_n) d\mu - \mu \int \partial(x) G^0(u_n; v - u_n) d\sigma \geq 0.
$$

for every $v \in \mathcal{B}$. This completes the proof. \qed

Now, we give a concrete application of Theorem $3.1$

**Theorem 3.3.** Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a non-negative, continuous function and set $F(\omega) = \int_0^\omega f(t) dt$ for $\omega \in \mathbb{R}$. Assume that

$$
\liminf_{\omega \rightarrow +\infty} \frac{F(\omega)}{\omega} < \frac{M(\theta(1) + \theta(0))}{2K} \limsup_{\omega \rightarrow +\infty} \frac{F(\omega)}{\omega^2}.
$$

Then, for each

$$
\lambda \in ] \frac{1}{\mu(\theta(1) + \theta(0))} \limsup_{\omega \rightarrow +\infty} \frac{F(\omega)}{\omega}, \frac{1}{2K \liminf_{\omega \rightarrow +\infty} \frac{F(\omega)}{\omega} [},
$$

for each non-negative, continuous function $g : \mathbb{R} \rightarrow \mathbb{R}$, whose potential $G(\omega) = \int_0^\omega g(t) dt$ satisfies

$$
\limsup_{\omega \rightarrow +\infty} \frac{G(\omega)}{\omega} < +\infty
$$

and for every $\mu \in [0, \mu_{G, \lambda}]$, where

$$
\mu_{G, \lambda} := \frac{1}{2K G_\infty} (1 - 2K \lambda \liminf_{\xi \rightarrow +\infty} \frac{F(\xi)}{\omega}),
$$

there is a sequence of pairwise distinct functions $\{u_n\} \subset W^{1,2}_0[0,1]$ such that for all $n \in \mathbb{N}$ one has

$$
-|u'(x)|^{-\alpha} u'(x) + |u(x)|^{-\alpha} u(x) = 0 \quad x \in ]0,1[, \\
|u_n'(1)|^{-1} u_n'(1) = \lambda \theta(1) f(u_n(1)) + \tilde{\mu} \theta(1) g(u_n(1)), \\
|u_n'(0)|^{-1} u_n'(0) = \tilde{\lambda} \theta(0) f(u_n(0)) + \tilde{\mu} \theta(0) g(u_n(0)).
$$

**Proof.** The first step is the inequality

$$
\int_0^1 \theta(x) [F(u(x))] + \vartheta(x) [G(u(x))] d\sigma
$$

$$
\leq (\theta(1) + \theta(0)) \max_{|\omega| \leq \omega_n} F(\omega) + (\vartheta(1) + \vartheta(0)) \max_{|\omega| \leq \omega_n} G(\omega).
$$

It results that

$$
\gamma \leq \liminf_{n \rightarrow +\infty} \varphi(\tau_n) \leq K p^+ \alpha(\theta(1) + \theta(0)) + K p^+ (\vartheta(1) + \vartheta(0)) \frac{\tilde{\mu}}{\lambda} G_\infty < +\infty.
$$

The second step is the inequality

$$
\int_0^1 \vartheta(x) [G(\xi_n(x))] d\sigma = (\vartheta(1) + \vartheta(0)) G(\tau_n) \geq (\vartheta(1) + \vartheta(0)) \liminf_{\omega \rightarrow +\infty} G(\omega) \geq 0,
$$
which implies that $\lim_{n \to +\infty} L_{\lambda}(\xi_n) = -\infty$. The last one is
\[
\left[ \int_{\partial \Omega} \theta(x) F(u_n(x); v(x) - u_n(x))d\sigma + \int_{\partial \Omega} \vartheta(x) G(u_n(x); v(x) - u_n(x))d\sigma \right]^{\circ}
\leq \left[ \int_{\partial \Omega} \theta(x) F(u_n(x); v(x) - u_n(x))d\sigma \right]^{\circ} + \left[ \int_{\partial \Omega} \vartheta(x) G(u_n(x); v(x) - u_n(x))d\sigma \right]^{\circ}
\leq \left[ \theta(1) F(u_n(1); v(1) - u_n(1)) + \theta(0) F(u_n(0); v(0) - u_n(0)) \right]^{\circ}
+ \left[ \vartheta(1) G(u_n(1); v(1) - u_n(1)) + \vartheta(0) G(u_n(0); v(0) - u_n(0)) \right]^{\circ}
\leq \left[ \theta(1) F^{\circ}(u_n(1); v(1) - u_n(1)) + \theta(0) F^{\circ}(u_n(0); v(0) - u_n(0)) \right]
+ \left[ \vartheta(1) G^{\circ}(u_n(1); v(1) - u_n(1)) + \vartheta(0) G^{\circ}(u_n(0); v(0) - u_n(0)) \right].
\]

Choosing $X = W^{1,2-x}[0,1]$, $\Omega = [0,1]$, $p(x) = 2 - x$ and $a(x) = 1$, then the conditions of Theorem 3.1 hold. Hence,
\[
\int_{0}^{1} \left[ |u''_n(x)|^{-x} u'_n(x)(v' - u'_n) + |u_n(x)|^{-x} u_n(x)(v - u_n) \right]dx
- \lambda \left[ \theta(1)f(u_n(1))v(1) + \theta(0)f(u_n(0))v(0) \right]
- \mu \left[ \vartheta(1)g(u_n(1))v(1) + \vartheta(0)g(u_n(0))v(0) \right] \geq 0.
\]

There exists an unbounded sequence $\{u_n\} \subset W^{1,2-x}[0,1]$ such that
\[
\int_{0}^{1} \left[ |u''_n(x)|^{-x} u'_n(x)v'(x) + |u_n(x)|^{-x} u_n(x)v(x) \right]dx
- \left( \lambda \theta(1)f(u_n(1)) + \mu \vartheta(1)g(u_n(1)) \right)
+ \lambda \theta(0)f(u_n(0)) + \mu \vartheta(0)g(u_n(0)) \right] \geq 0.
\]

Therefore $\{u_n\}$ is the unique solution of the problem (3.10). \hfill \square

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**References**

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Fariba Fattahi
Department of Mathematics, University of Mazandaran, Babolsar, Iran
E-mail address: F.Fattahi@stu.umz.ac.ir

Mohsen Alimohammady
Department of Mathematics, University of Mazandaran, Babolsar, Iran
E-mail address: Amohsen@umz.ac.ir