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EXISTENCE OF SOLUTIONS FOR KIRCHHOFF EQUATIONS INVOLVING p-LINEAR AND p-SUPERLINEAR THERMS AND WITH CRITICAL GROWTH

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Abstract. In this article we establish the existence of a nontrivial weak solution to a class of nonlinear boundary-value problems of Kirchhoff type involving p-linear and p-superlinear terms and with critical Caffaearelli-Kohn-Nirenberg exponent.

1. Introduction

In this article we study the existence of nontrivial solutions for the nonlocal boundary-value problem of Kirchhoff type

$$\begin{gathered} L(u) = \lambda |x|^{-\delta} f(x,u) + |x|^{-bp^*}|u|^{p^*-2}u \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where

$$L(u) := -\Big[M\Big(\int_\Omega |x|^{-ap}|\nabla u|^p\,dx\Big)\Big]\operatorname{div}\big(|x|^{-ap}|\nabla u|^{p-2}\nabla u\big),$$

and $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain with $N \geq 3$, $1 < p < N$, $a \leq \frac{N-p}{p}$, $p^* = \frac{Np}{N-dp}$ is the critical Caffarelli-Kohn-Nirenberg exponent, where $d = 1 + a - b$ with $a \leq b \leq a+1$, $M : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^+$ is a continuous function, and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Caratheodory function.

Because of the integral over Ω in $L(u)$, (1.1) is no longer a pointwise equation, so it is called nonlocal problem. The mathematical difficulties that comes with this phenomenon is what makes the study of such problems particularly interesting. Also the physical motivation makes this problem interesting. Indeed, (1.1) is related to the stationary version of the Kirchhoff equation

$$\begin{gathered} u_{tt} - M\Big(\int_\Omega |\nabla u|^2 dx\Big)\Delta u = g(x,u) \quad \text{in } \Omega \times (0,T) \\ u = 0 \quad \text{on } \partial\Omega \times (0,T) \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \end{gathered} \tag{1.2}$$

where $M(s) = a + bs$, $a, b > 0$. It was proposed by Kirchhoff [18] as an extension of the classical D'Alembert's wave equation for free vibrations of elastic strings to

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describe the transversal oscillations of a stretched string, particularly, taking into account the subsequent change in string length caused by oscillations.

Some early classical studies of Kirchhoff equations were done by Bernstein [5] and Pohozaev [27]. However, (1.2) received great attention only after Lions [19] proposed an abstract framework for the problem. After that, the study on nonlocal problems of the type (1.2) grew exponentially. Some interesting results can be found, for example, in [1, 4, 8, 9, 10, 15, 16, 20, 22, 23, 24, 25, 26], and the references therein.

Problems involving a Kirchhoff equation with critical growth can be seen, for example, in [2, 12, 13]. In [7], the authors studied a problem involving the p-Laplacian operator with weights, but with subcritical growth. A version of a Kirchhoff type problem involving the p-Laplacian operator with weights and critical growth was studied in [14].

In our work we intent to complement the results obtained in [14]. There the authors studied problem (1.1) involving p-sublinear and p-superlinear therms. We treat the case in which (1.1) has a p-linear therm. Also, we extend the results for the p-superlinar case by finding a weak solution for each $\lambda > 0$. We use the mountain pass theorem to find weak solutions for the problem. Different from the techniques in [14] and the other articles listed above, we work with extremal functions to control the level of the Palais-Smale sequence obtained with the mountain pass theorem. The lack of compactness due to the critical therm in the first equation of (1.1) was bypassed using a technique in common with some of the above papers: a version of the concentration-compactness principle due to Lions [21].

Because of the nonlocal terms in the equation (1.1), it was necessary to make a truncation on the Kirchhoff type function that appear on the operator, creating an auxiliary problem. By finding solutions of the auxiliary problem we can find solutions for (1.1). This truncation argument is similar to the one used in [12].

For enunciating the main result, we need to give some hypotheses on the continuous function $M : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^+$, and on the Caratheodory function $f : \Omega \times \mathbb{R} \to \mathbb{R}$:

- (H1) There exists $m_0 > 0$ such that $M(t) \geq m_0$ for all $t \geq 0$.
- (H2) The function M is increasing.
- (H3) $f(x,-t) = -f(x,t)$ for all $(x,t) \in \Omega \times \mathbb{R}$.
- (H4) There exist $r \in [p, p^*)$ and C_1, C_2 positive constants with $C_1 < C_2$, such that
$$C_1|t|^{r-1} \leq f(x,t) \leq C_2|t|^{r-1}, \quad \forall (x,t) \in \Omega \times (\mathbb{R}^+ \cup \{0\}).$$
Moreover, $\delta \leq (a+1)r + N(1 - \frac{r}{p})$.
- (H5) The well known Ambrosetti-Rabinowitz superlinear condition holds,
$$0 < \xi \int_0^t f(x,s)ds \leq tf(x,t), \quad \forall (x,t) \in \Omega \times \mathbb{R}^+, \text{ and some } \xi \in (p, p^*).$$

We denote by λ_1 the first eigenvalue of the problem

$$\begin{gathered} -\operatorname{div}(|x|^{-ap}|\nabla u|^{p-2}\nabla u) = \lambda \int_\Omega |x|^{-\delta}|u|^{p-2}u dx \qquad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.3}$$

Note that the first eigenvalue of (1.3) is given by

$$\lambda_1 = \inf\Big\{ \int_\Omega |x|^{-ap}|\nabla u|^p dx; u \in \mathcal{D}_a^{1,p}, \int_\Omega |x|^{-\delta}|u|^p dx = 1 \Big\}, \tag{1.4}$$

and it is positive (see for instance [29]). The main results of our paper are read as follows.

Theorem 1.1. *Assume* (H1)–(H5) *hold, and* $r = p$. *Then* (1.1) *has a nontrivial solution for each* $\lambda \in (0, \frac{m_0}{C_2}\lambda_1)$.

Theorem 1.2. *Assume* (H1)–(H5) *hold, and and* $p < r < p^*$. *Then* (1.1) *has a nontrivial solution for each* $\lambda > 0$.

This article is organized as follows. In section 2 we provide some preliminary results and the variational framework. In section 3 we constructed the auxiliary problem. Section 4 is devoted to the Palais-Smale condition for the Euler-Lagrange functional associated to problem (1.1). In sections 5 and 6 we prove Theorems 1.1 and 1.2, respectively.

2. Preliminary results and variational framework

Consider $\Omega \subset \mathbb{R}^N$ a smooth domain with $0 \in \Omega$, $N \geq 3$, $1 < p < N$, $a < (N-p)/p$, $a \leq b < a+1$, and $p^* = Np/(N-dp)$, where $d = 1+a-b$. From [6, 30] we have

$$\Big(\int_\Omega |x|^{-\alpha}|u|^r dx\Big)^{p/r} \leq C\int_\Omega |x|^{-ap}|\nabla u|^p dx, \quad \forall u \in \mathcal{D}_a^{1,p}, \tag{2.1}$$

where $1 \leq r \leq Np/(N-p)$, $\alpha \leq (a+1)r + N(1-\frac{r}{p})$, $\mathcal{D}_a^{1,p}$ is the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\| = \Big(\int_\Omega |x|^{-ap}|\nabla u|^p dx\Big)^{1/p};$$

thus we have the continuous embedding of $\mathcal{D}_a^{1,p}$ in the weighted space $L^r(\Omega, |x|^{-\alpha})$. This space is $L^r(\Omega)$ with the norm

$$\|u\|_{r,\alpha} = \Big(\int_\Omega |x|^{-\alpha}|u|^r dx\Big)^{1/r}.$$

Moreover, this embedding is compact if $1 \leq r < Np/(N-p)$ and $\alpha < (a+1)r + N(1-\frac{r}{p})$. The best constant of the weighted Caffarelli-Kohn-Nirenberg type (see [6]) inequality will be denoted by $C^*_{a,p}$, which is characterized by

$$C^*_{a,p} = \inf_{u \in \mathcal{D}_a^{1,p}\setminus\{0\}} \frac{\int_\Omega |x|^{-ap}|\nabla u|^p dx}{\big(\int_\Omega |x|^{-bp^*}|u|^{p^*} dx\big)^{p/p^*}}.$$

We will look for solutions of (1.1) by finding critical points of the Euler-Lagrange functional $I : \mathcal{D}_a^{1,p} \to \mathbb{R}$ given by

$$I(u) = \frac{1}{p}\widehat{M}(\|u\|^p) - \lambda\int_\Omega |x|^{-\delta}F(x,u)\,dx - \frac{1}{p^*}\int_\Omega |x|^{-bp^*}|u|^{p^*}\,dx,$$

where $\widehat{M}(t) := \int_0^t M(s)ds$ and $F(x,t) = \int_0^t f(x,s)ds$. Note that $I \in C^1$ and

$$\begin{aligned} I'(u)(\phi) &= M(\|u\|^p)\int_\Omega |x|^{-ap}|\nabla u|^{p-2}\nabla u\nabla\phi\,dx \\ &\quad - \lambda\int_\Omega |x|^{-\delta}f(x,u)\phi\,dx - \int_\Omega |x|^{-bp^*}|u|^{p^*-2}u\phi\,dx, \end{aligned}$$

for all $\phi \in \mathcal{D}_a^{1,p}$.

The next Lemma will be useful, and can be easily proved by using [11, Lemma 4.1].

Lemma 2.1 (S_+ condition). *Suppose that $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $0 \in \Omega$, $1 < p < N$, $-\infty < a < \frac{N-p}{p}$, and $(u_n) \subset \mathcal{D}_a^{1,p}$ such that*

$$u_n \rightharpoonup u, \quad \text{as } n \to \infty,$$

$$\limsup_{n\to\infty} \int_\Omega |x|^{-ap} |\nabla u_n|^{p-2} \nabla u_n \nabla(u_n - u) dx \le 0,$$

then there exists a subsequence strongly convergent in $\mathcal{D}_a^{1,p}$.

3. Auxiliary problem

To proof Theorems 1.1 and 1.2, we will use a version of the mountain pass theorem due to Ambrosetti and Rabinowitz [3], but since we are working with critical growth and a nonlocal operator without more information about the behavior of the function M at infinity, we need to make a truncation on function M. So we will prove that the Euler-Lagrange functional associated to (1.1) has the Mountain Pass Geometry.

From (H2), there exists $t_0 > 0$ such that $m_0 = M(0) < M(t_0) < \frac{\xi}{p} m_0$, where ξ is given by (H5). We set

$$M_0(t) := \begin{cases} M(t), & \text{if } 0 \le t \le t_0, \\ M(t_0), & \text{if } t \ge t_0. \end{cases}$$

From (H2) we obtain

$$m_0 \le M_0(t) \le \frac{\xi}{p} m_0, \quad \forall t \ge 0. \tag{3.1}$$

The proofs of the Theorems 1.1 and 1.2 are based on a careful study of solutions of the auxiliary problem

$$\begin{gathered} L_0(u) = \lambda |x|^{-\delta} f(x,u) + |x|^{-bp^*} |u|^{p^*-2} u \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{3.2}$$

where

$$L_0(u) := -\Big[M_0\Big(\int_\Omega |x|^{-ap} |\nabla u|^p \, dx \Big)\Big] \operatorname{div}\big(|x|^{-ap} |\nabla u|^{p-2} \nabla u \big).$$

We will look for solutions of (3.2) by finding critical points of the Euler-Lagrange functional $J : \mathcal{D}_a^{1,p} \to \mathbb{R}$ given by

$$J(u) = \frac{1}{p} \widehat{M_0}(\|u\|^p) - \lambda \int_\Omega |x|^{-\delta} F(x,u) \, dx - \frac{1}{p^*} \int_\Omega |x|^{-bp^*} |u|^{p^*} dx,$$

where $\widehat{M_0}(t) := \int_0^t M_0(s) ds$. Note that J is C^1 and

$$\begin{aligned} J'(u)(\phi) &= M_0(\|u\|^p) \int_\Omega |x|^{-ap} |\nabla u|^{p-2} \nabla u \nabla \phi \, dx \\ &\quad - \lambda \int_\Omega |x|^{-\delta} f(x,u) \phi \, dx - \int_\Omega |x|^{-bp^*} |u|^{p^*-2} u \phi \, dx, \end{aligned}$$

for all $\phi \in \mathcal{D}_a^{1,p}$.

4. Palais-Smale Condition

In this section we verify that, under the hypotheses (H1)–(H4), the functional J satisfies the Palais-Smale condition below a given level.

Lemma 4.1. *Let* (u_n) *be a bounded sequence in* $\mathcal{D}_a^{1,p}$ *such that*

$$J(u_n) \to c \quad \text{and} \quad J'(u_n) \to 0 \text{ in } (\mathcal{D}_a^{1,p})^{-1}, \quad \text{as } n \to \infty.$$

Suppose (H1)–(H5) *hold, and*

$$c < \Big(\frac{1}{\xi} - \frac{1}{p^*}\Big)\big(m_0 C_{a,p}^*\big)^{\frac{p^*}{p^*-p}}.$$

Then there exists a subsequence strongly convergent in $\mathcal{D}_a^{1,p}$.

Proof. Since (u_n) is bounded in $\mathcal{D}_a^{1,p}$, passing to a subsequence, if necessary, we have

$$\begin{gathered} u_n \rightharpoonup u \quad \text{in } \mathcal{D}_a^{1,p}, \\ u_n \to u \quad \text{in} \quad L^s(\Omega, |x|^{-\sigma}), \\ u_n(x) \to u(x) \quad \text{a.e. in } \Omega, \\ \|u_n\| \to t_0 \geq 0, \end{gathered}$$

as $n \to +\infty$, where $1 \leq s < p^*$ and $\sigma < (a+1)s + N(1 - s/p)$. Moreover, using the concentration-compactness principle due to Lions (cf. [21, 30]), we obtain at most countable index set Λ, sequences $(x_i) \subset \mathbb{R}^N$, $(\mu_i), (\nu_i) \subset (0, \infty)$, such that

$$|x|^{-ap}|\nabla u_n|^p \rightharpoonup |x|^{-ap}|\nabla u|^p + \mu \quad \text{and} \quad |x|^{-bp^*}|u_n|^{p^*} \rightharpoonup |x|^{-bp^*}|u|^{p^*} + \nu, \tag{4.1}$$

as $n \to +\infty$, in weak*-sense of measures where

$$\nu = \sum_{i\in\Lambda} \nu_i \delta_{x_i}, \quad \mu \geq \sum_{i\in\Lambda} \mu_i \delta_{x_i}, \quad C_{a,p}^* \nu_i^{p/p^*} \leq \mu_i, \tag{4.2}$$

for all $i \in \Lambda$, where δ_{x_i} is the Dirac mass at $x_i \in \Omega$.

Now let $k \in \mathbb{N}$. Without loss of generality we can suppose $B_2(0) \subset \Omega$, then for every $\varrho > 0$, we set $\psi_\varrho(x) := \psi((x - x_k)/\varrho)$ where $\psi \in C_0^\infty(\Omega, [0, 1])$ is such that $\psi \equiv 1$ on $B_1(0)$, $\psi \equiv 0$ on $\Omega \setminus B_2(0)$, and $|\nabla\psi| \leq 1$. Observe that $(\psi_\varrho u_n)$ is bounded in $\mathcal{D}_a^{1,p}$. So we have $J'(u_n)(\psi_\varrho u_n) \to 0$; that is,

$$\begin{aligned} &M_0(\|u_n\|^p)\int_\Omega \frac{u_n|\nabla u_n|^{p-2}\nabla u_n \nabla\psi_\varrho}{|x|^{ap}}dx + o_n(1) \\ &= -M_0(\|u_n\|^p)\int_\Omega \frac{|\nabla u_n|^p\psi_\varrho}{|x|^{ap}}dx + \lambda\int_\Omega \frac{f(x,u_n)\psi_\varrho u_n}{|x|^\delta}dx + \int_\Omega \frac{\psi_\varrho|u_n|^{p^*}}{|x|^{bp^*}}dx. \end{aligned}$$

Since $u_n \to u$ in $L^r\left(\Omega, |x|^{-\delta}\right)$, it follows from (4.1), (H1), (H4) and the Dominated Convergence Theorem, that

$$\begin{aligned} &\limsup_{n\to\infty}\Big[M_0(\|u_n\|^p)\int_\Omega \frac{u_n|\nabla u_n|^{p-2}\nabla u_n\nabla\psi_\varrho}{|x|^{ap}}dx\Big] \\ &\leq -m_0\int_\Omega \frac{|\nabla u|^p\psi_\varrho}{|x|^{ap}}dx - m_0\int_\Omega \psi_\varrho d\mu + \lambda\int_\Omega \frac{f(x,u)\psi_\varrho u}{|x|^\delta}dx + \int_\Omega \frac{\psi_\varrho|u|^{p^*}}{|x|^{bp^*}}dx + \int_\Omega \psi_\varrho d\nu. \end{aligned}$$

Using the Dominated Convergence Theorem again, we obtain

$$\int_\Omega \frac{|\nabla u|^p\psi_\varrho}{|x|^{ap}}dx = o_\varrho(1), \quad \int_\Omega \frac{f(x,u)\psi_\varrho u}{|x|^\delta}dx = o_\varrho(1), \quad \int_\Omega \frac{\psi_\varrho |u|^{p^*}}{|x|^{bp^*}}dx = o_\varrho(1),$$

where $\lim_{\varrho\to 0^+} o_\varrho(1) = 0$. So, we obtain

$$\begin{aligned} &\lim_{\varrho\to 0^+}\Big\{\limsup_{n\to\infty}\Big[M_0(\|u_n\|^p)\int_\Omega \frac{u_n|\nabla u_n|^{p-2}\nabla u_n\nabla\psi_\varrho}{|x|^{ap}}dx\Big]\Big\} \\ &\le \lim_{\varrho\to 0^+}\Big[\int_\Omega \psi_\varrho d\nu - m_0\int_\Omega \psi_\varrho d\mu\Big]. \end{aligned} \tag{4.3}$$

Now, we show that

$$\lim_{\varrho\to 0^+}\Big[\limsup_{n\to\infty} M_0(\|u_n\|^p)\int_\Omega |x|^{-ap}u_n|\nabla u_n|^{p-2}\nabla u_n\nabla\psi_\varrho dx\Big] = 0. \tag{4.4}$$

Indeed, by Hölder's Inequality,

$$\Big|\int_\Omega \frac{u_n|\nabla u_n|^{p-2}\nabla u_n\nabla\psi_\varrho}{|x|^{ap}}dx\Big| \le \|u_n\|^{p-1}\Big(\int_\Omega \frac{|u_n\nabla\psi_\varrho|^p}{|x|^{ap}}dx\Big)^{1/p}.$$

Since u_n is bounded in $\mathcal{D}_a^{1,p}$, M_0 is continuous, and $\operatorname{supp}(\psi_\varrho) \subset B(x_k;2\varrho)$, there exists $L > 0$ such that

$$M_0(\|u_n\|^p)\Big|\int_\Omega \frac{u_n|\nabla u_n|^{p-2}\nabla u_n\nabla\psi_\varrho}{|x|^{ap}}dx\Big| \le L\Big(\int_{B(x_k;2\varrho)} \frac{|u_n\nabla\psi_\varrho|^p}{|x|^{ap}}dx\Big)^{1/p}.$$

Using the dominated convergence theorem and Hölder's inequality, we obtain

$$\begin{aligned} &\limsup_{n\to\infty}\Big[M_0(\|u_n\|^p)\Big|\int_\Omega \frac{u_n|\nabla u_n|^{p-2}\nabla u_n\nabla\psi_\varrho}{|x|^{ap}}dx\Big|\Big] \\ &\le L\Big(\int_{B(x_k;2\varrho)} \frac{|u|^p|\nabla\psi_\varrho|^p}{|x|^{ap}}dx\Big)^{1/p} \\ &\le L\Big(\int_{B(x_k;2\varrho)} |\nabla\psi_\varrho|^N dx\Big)^{1/N}\Big(\int_{B(x_k;2\varrho)} \big(|x|^{-ap}|u|^p\big)^{\frac{N}{N-p}} dx\Big)^{\frac{N-p}{Np}} \\ &\le L|B(x_k;2\varrho)|^{1/N}\Big(\int_\Omega \chi_{B(x_k;2\varrho)}\big(|x|^{-ap}|u|^p\big)^{\frac{N}{N-p}} dx\Big)^{\frac{N-p}{Np}}. \end{aligned}$$

Letting $\varrho \to 0^+$ on the above expression, we obtain (4.4). Thus, we conclude from (4.3) that

$$0 \le \lim_{\rho\to 0^+}\Big[\int_\Omega \psi_\varrho d\nu - m_0\int_\Omega \psi_\varrho d\mu\Big].$$

That is,

$$\begin{aligned} 0 &\le \lim_{\rho\to 0^+}\Big[\int_{B(x_k;2\varrho)} \psi_\varrho d\nu - m_0\int_{B(x_k;2\varrho)} \psi_\varrho d\mu\Big] \\ &= \nu(\{x_k\}) - m_0\mu(\{x_k\}) \\ &\le \sum_{i\in\Lambda}\nu_i\delta_{x_i}(\{x_k\}) - m_0\sum_{i\in\Lambda}\mu_i\delta_{x_i}(\{x_k\}) \\ &= \nu_k - m_0\mu_k. \end{aligned}$$

So, we have $m_0\mu_k \le \nu_k$. It follows from (4.2) that

$$\nu_k \ge (m_0 C^*_{a,p})^{\frac{p^*}{p^*-p}} \ge \big(\frac{1}{\theta} - \frac{1}{p^*}\big)(m_0 C^*_{a,p})^{p^*/(p^*-p)}. \tag{4.5}$$

Now we shall prove that the above expression can not occur, and therefore the set Λ is empty. Indeed, arguing by contradiction, let us suppose that (4.5) hold for some $k \in \Lambda$. Thus, once that $m_0 \le M_0(t) \le \frac{\xi}{p} m_0$, for all $t \in \mathbb{R}$, and by using (f_3) we have

$$\begin{aligned} c &= J(u_n) - \frac{1}{\xi} J'(u_n)(u_n) + o_n(1) \\ &\ge \big(\frac{m_0}{p} - \frac{\xi m_0}{\xi p}\big)\|u_n\|^p - \lambda \int_\Omega \frac{F(x,u_n) - \frac{1}{\xi} f(x,u_n)u_n}{|x|^\delta} dx \\ &\quad + \big(\frac{1}{\xi} - \frac{1}{p^*}\big)\int_\Omega \frac{|u_n|^{p^*}}{|x|^{bp^*}} dx + o_n(1) \\ &\ge \big(\frac{1}{\xi} - \frac{1}{p^*}\big)\int_\Omega \frac{|u_n|^{p^*}\psi_\varrho}{|x|^{bp^*}} dx + o_n(1). \end{aligned}$$

Letting $n \to +\infty$, we obtain

$$c \ge \big(\frac{1}{\xi} - \frac{1}{p^*}\big)(m_0 C^*_{a,p})^{\frac{p^*}{p^*-p}}.$$

But this is a contradiction. Thus Λ is empty and it follows that $u_n \to u$ in $L^{p^*}\big(\Omega, |x|^{-bp^*}\big)$.

Now we will prove that $u_n \to u$ in $\mathcal{D}^{1,p}_a$. Indeed, since $u_n \to u$ in $L^r(\Omega, |x|^{-\delta})$ and in $L^{p^*}(\Omega, |x|^{-bp^*})$, it follows from the dominated convergence theorem that

$$\lim_{n\to+\infty} \int_\Omega \frac{f(x,u_n)(u_n-u)}{|x|^\delta} dx = \lim_{n\to+\infty} \int_\Omega \frac{|u_n|^{p^*-2}u_n(u_n-u)}{|x|^{bp^*}} dx = 0.$$

Therefore, as (u_n) is bounded in $\mathcal{D}^{1,p}_a$, $J'(u_n)(u_n-u) \to 0$ in $(\mathcal{D}^{1,p}_a)^{-1}$, $\|u_n\| \to t_0$, as $n \to \infty$, and as M is continuous and positive, we conclude that

$$\lim_{n\to\infty} \int_\Omega |x|^{-ap}|\nabla u_n|^{p-2}\nabla u_n \nabla(u_n - u) dx = 0.$$

It follows from Lemma 2.1 that $u_n \to u$ in $\mathcal{D}^{1,p}_a$. □

5. Proof of Theorem 1.1

In this section we prove Theorem 1.1, which concerns to problem (1.1) when $r = p$. The next two lemmas show that the functional J has the Mountain Pass geometry.

Lemma 5.1. *Suppose that $r = p$ and let λ_1 be as in* (1.4)*. Assume that the conditions* (H1)–(H4) *hold. Then, there exist positive numbers ρ and α such that*

$$J(u) \ge \alpha > 0, \forall u \in \mathcal{D}^{1,p}_a, \quad \text{with } \|u\| = \rho,$$

for all $\lambda \in (0, \frac{m_0}{C_2}\lambda_1)$.

Proof. Let $\lambda \in (0, \frac{m_0}{C_2}\lambda_1)$. From (H1), (H3), (H4), (1.4), and Caffarelli-Khon-Nirenberg inequality, we obtain

$$J(u) \ge \big(m_0 - \frac{\lambda C_2}{\lambda_1}\big)\frac{1}{p}\|u\|^p - \frac{1}{p^*}\tilde{C}\|u\|^{p^*}.$$

Since $p < p^*$ and $\lambda < \frac{m_0}{C_2}\lambda_1$. The result follows by choosing $\rho > 0$ small enough. □

Lemma 5.2. *Suppose that $r = p$. Assume that the conditions* (H1), (H3), (H4) *hold. For each $\lambda > 0$, there exists $e \in \mathcal{D}^{1,p}_a$ with $J(e) < 0$ and $\|e\| > \rho$.*

Proof. Fix $v_0 \in \mathcal{D}_a^{1,p}\backslash\{0\}$, with $v_0 > 0$ in Ω. Using (3.1) and (H4) we obtain

$$J(tv_0) \leq \frac{\xi}{p^2} m_0 t^p \|v_0\|^p - \frac{\lambda C_1}{p} t^p \int_\Omega \frac{|v_0|^p}{|x|^\delta} dx - \frac{t^{p^*}}{p^*} \int_\Omega \frac{|v_0|^{p^*}}{|x|^{bp^*}} dx.$$

Since $p < p^*$, we have $\lim_{t\to+\infty} J(tv_0) = -\infty$. Thus, there exists $\bar{t} > 0$ large enough, such that $\bar{t}\|v_0\| > \rho$ and $J(\bar{t}v_0) < 0$. The result follows by considering $e = \bar{t}v_0$. □

Using a version of the mountain pass theorem without the (PS) condition (see [28]), for each $\lambda \in (0, \frac{m_0}{C_2}\lambda_1)$, there exists a sequence $(u_n) \in \mathcal{D}_a^{1,p}$, satisfying

$$J(u_n) \to c_\lambda \text{ and } J'(u_n) \to 0 \text{ in } (\mathcal{D}_a^{1,p})^{-1},$$

where

$$0 < c_\lambda = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J(\gamma(t)),$$

$$\Gamma := \big\{\gamma \in C([0,1], \mathcal{D}_a^{1,p}) : \gamma(0) = 0, \gamma(1) = \bar{t}v_0\big\},$$

and $v_0 \in \mathcal{D}_a^{1,p}$ is such that $v_0 > 0$.

To obtain the level c_λ below the level given by Lemma 4.1, we will give some estimates. We define the Sobolev space

$$W_{a,b}^{1,p}(\Omega) = \big\{u \in L^{p^*}(\Omega, |x|^{-bp^*}) : |\nabla u| \in L^p(\Omega, |x|^{-ap})\big\},$$

with respect to the norm

$$\|u\|_{W_{a,b}^{1,p}(\Omega)} = \|u\|_{p^*,bp^*} + \|\nabla u\|_{p,ap}.$$

We consider the best constant of the weighted Caffarelli-Kohn-Nirenberg type given by

$$\tilde{S}_{a,p} = \inf_{u\in W_{a,b}^{1,p}(\mathbb{R}^N)\backslash\{0\}} \left\{ \frac{\int_{\mathbb{R}^N} |x|^{-ap}|\nabla u|^p dx}{\big(\int_{\mathbb{R}^N} |x|^{-bp^*}|u|^{p^*} dx\big)^{p/p^*}} \right\}.$$

We also set $R_{a,b}^{1,p}(\Omega)$ as the subspace of $W_{a,b}^{1,p}(\Omega)$ of the radial functions, more precisely

$$R_{a,b}^{1,p}(\Omega) = \Big\{u \in W_{a,b}^{1,p}(\Omega) : u(x) = u(|x|)\Big\},$$

with respect to the induced norm

$$\|u\|_{R_{a,b}^{1,p}(\Omega)} = \|u\|_{W_{a,b}^{1,p}(\Omega)}.$$

Horiuchi [17] proved that

$$\tilde{S}_{a,p,R} = \inf_{u\in R_{a,b}^{1,p}(\mathbb{R}^N)\backslash\{0\}} \left\{ \frac{\int_{\mathbb{R}^N} |x|^{-ap}|\nabla u|^p dx}{\big(\int_{\mathbb{R}^N} |x|^{-bp^*}|u|^{p^*} dx\big)^{p/p^*}} \right\}$$

is achieved by functions of the form

$$u_\varepsilon(x) = k_{a,p}(\varepsilon)v_\varepsilon(x), \quad \forall\varepsilon > 0,$$

where

$$k_{a,p}(\varepsilon) = c\varepsilon^{(N-dp)/dp^2} \quad v_\varepsilon(x) = \Big(\varepsilon + |x|^{\frac{dp(N-p-ap)}{(p-1)(N-dp)}}\Big)^{-(\frac{N-dp}{dp})}.$$

Moreover, u_ε satisfies

$$\int_{\mathbb{R}^N} |x|^{-ap}|\nabla u_\varepsilon|^p dx = \int_{\mathbb{R}^N} |x|^{-bp^*}|u_\varepsilon|^{p^*} dx = (\tilde{S}_{a,p,R})^{\frac{p^*}{p^*-p}}. \tag{5.1}$$

From (5.1) we obtain

$$\int_{\mathbb{R}^N} |x|^{-ap}|\nabla v_\varepsilon|^p dx = [k_{a,p}(\varepsilon)]^{-p}(\tilde{S}_{a,p,R})^{\frac{p^*}{p^*-p}}, \tag{5.2}$$

$$\int_{\mathbb{R}^N} |x|^{-bp^*}|v_\varepsilon|^{p^*} dx = [k_{a,p}(\varepsilon)]^{-p^*}(\tilde{S}_{a,p,R})^{\frac{p^*}{p^*-p}}. \tag{5.3}$$

Let R_0 be a positive constant and set $\Psi(x) \in C_0^\infty(\mathbb{R}^N)$ such that $0 \le \Psi(x) \le 1$, $\Psi(x) = 1$, for all $|x| \le R_0$, and $\Psi(x) = 0$, for all $|x| \ge 2R_0$. Set

$$\tilde{v}_\varepsilon(x) = \Psi(x)v_\varepsilon(x), \tag{5.4}$$

for all $x \in \mathbb{R}^N$ and for all $\varepsilon > 0$. Without loss of generality we can consider $B(0; 2R_0) \subset \Omega$.

Lemma 5.3. *With the above notation we have*

$$\lim_{\varepsilon\to 0^+} \frac{\|\tilde{v}_\varepsilon\|^p}{\big(\int_\Omega |x|^{-bp^*}|\tilde{v}_\varepsilon|^{p^*} dx\big)^{p/p^*}} = 0.$$

Proof. By a straightforward computation we obtain

$$\|\tilde{v}_\varepsilon\|^p \le [k_{a,p}(\varepsilon)]^{-p}(\tilde{S}_{a,p,R})^{\frac{p^*}{p^*-p}} + C, \tag{5.5}$$

$$\int_\Omega |x|^{-bp^*}|\tilde{v}_\varepsilon|^{p^*} dx = \varepsilon^{-\frac{N-dp}{dp}p^*} \cdot C, \quad \forall \varepsilon \in (0,1), \tag{5.6}$$

where C denotes a positive constant. Therefore, for all $\varepsilon \in (0,1)$, from (5.5) and (5.6) we obtain

$$\begin{aligned}
\frac{\|\tilde{v}_\varepsilon\|^p}{\big(\int_\Omega |x|^{-bp^*}|\tilde{v}_\varepsilon|^{p^*} dx\big)^{p/p^*}} &\le \frac{[k_{a,p}(\varepsilon)]^{-p}(\tilde{S}_{a,p,R})^{\frac{p^*}{p^*-p}} + C}{\Big(\varepsilon^{-\frac{N-dp}{dp}p^*} \cdot C\Big)^{p/p^*}} \\
&= \frac{c^{-p}(\tilde{S}_{a,p,R})^{\frac{p^*}{p^*-p}}\varepsilon^{\frac{N-dp}{dp}(p-1)} + C\varepsilon^{\frac{N-dp}{dp}p}}{C}.
\end{aligned}$$

Since $p > 1$, we have

$$\lim_{\varepsilon\to 0^+} \frac{\|\tilde{v}_\varepsilon\|^p}{\big(\int_\Omega |x|^{-bp^*}|\tilde{v}_\varepsilon|^{p^*} dx\big)^{p/p^*}} = 0.$$

□

Lemma 5.4. *Let $\lambda \in (0, \frac{m_0}{C_2}\lambda_1)$. Assume that* (H1)–(H5) *hold. Set*

$$l^* = \min\Big\{\Big(\frac{1}{p}m_0 - \frac{1}{\xi}M_0(t_0)\Big)t_0, \big(\frac{1}{\xi} - \frac{1}{p^*}\big)(m_0C^*_{a,p})^{\frac{p^*}{p^*-p}}\Big\}.$$

Then, there exists $\varepsilon_1 \in (0,1)$ such that

$$\sup_{t\ge 0} J(t\tilde{v}_\varepsilon) < l^*,$$

for all $\varepsilon \le \varepsilon_1$.

Proof. Let $0 < \varepsilon < 1$ and $\tilde{v}_\varepsilon$ be as in (5.4). Since from Lemmas 5.1 and 5.2 the functional J satisfies the Mountain Pass geometry, for each $\lambda \in (0, \frac{m_0}{C_2}\lambda_1)$, there exists t_ε such that

$$\sup_{t\ge 0} J(t\tilde{v}_\varepsilon) = J(t_\varepsilon\tilde{v}_\varepsilon),$$

for each $\lambda \in (0, \frac{m_0}{C_2}\lambda_1)$. So, we have

$$\begin{aligned}\sup_{t\geq 0} J(t\tilde{v}_\varepsilon) &= \frac{1}{p}\widehat{M}(\|t_\varepsilon \tilde{v}_\varepsilon\|^p) - \lambda \int_\Omega |x|^{-\delta} F(x, t_\varepsilon \tilde{v}_\varepsilon)dx - \frac{1}{p^*}\int_\Omega |x|^{-bp^*} t_\varepsilon^{p^*} |\tilde{v}_\varepsilon|^{p^*} dx \\ &\leq \frac{\xi}{p^2} m_0 t_\varepsilon^p \|\tilde{v}_\varepsilon\|^p - \frac{1}{p^*} t_\varepsilon^{p^*} \int_\Omega |x|^{-bp^*} |\tilde{v}_\varepsilon|^{p^*} dx,\end{aligned}$$

for each $\lambda \in (0, \frac{m_0}{C_2}\lambda_1)$. Now we consider the function $g : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^+ \cup \{0\}$, given by

$$g(s) = \Big(\frac{\xi}{p^2} m_0 \|\tilde{v}_\varepsilon\|^p\Big) s^p - \Big(\frac{1}{p^*}\int_\Omega |x|^{-bp^*} |\tilde{v}_\varepsilon|^{p^*} dx\Big) s^{p^*}.$$

It is easy to see that

$$\bar{s} = \Big(\frac{\frac{\xi}{p} m_0 \|\tilde{v}_\varepsilon\|^p}{\int_\Omega |x|^{-bp^*} |\tilde{v}_\varepsilon|^{p^*} dx}\Big)^{\frac{1}{p^*-p}}$$

is a maximum of g and we have

$$g(\bar{s}) = (\frac{1}{p} - \frac{1}{p^*})\Big(\frac{\xi}{p} m_0\Big)^{\frac{p^*}{p^*-p}} \Big(\frac{\|\tilde{v}_\varepsilon\|^p}{\big(\int_\Omega |x|^{-bp^*} |\tilde{v}_\varepsilon|^{p^*} dx\big)^{p/p^*}}\Big)^{\frac{p^*}{p^*-p}}.$$

So, we have

$$\sup_{t\geq 0} J(t\tilde{v}_\varepsilon) \leq (\frac{1}{p} - \frac{1}{p^*})\Big(\frac{\xi}{p} m_0\Big)^{\frac{p^*}{p^*-p}} \Big(\frac{\|\tilde{v}_\varepsilon\|^p}{\big(\int_\Omega |x|^{-bp^*} |\tilde{v}_\varepsilon|^{p^*} dx\big)^{p/p^*}}\Big)^{\frac{p^*}{p^*-p}},$$

for each $\lambda \in (0, \frac{m_0}{C_2}\lambda_1)$.

It follows from Lemma 5.3 that there exists $0 < \varepsilon_1 < 1$ such that

$$\sup_{t\geq 0} J(t\tilde{v}_\varepsilon) < l^*,$$

for all $\varepsilon \leq \varepsilon_1$ and for each $\lambda \in (0, \frac{m_0}{C_2}\lambda_1)$. □

Remark 5.5. Let $\lambda \in (0, \frac{m_0}{C_2}\lambda_1)$ and let us consider the path $\gamma_*(t) = t(\bar{t}v_{\varepsilon_1})$, for $t \in [0, 1]$, which belongs to Γ. It follows from Lemma 5.4 that we obtain the following estimate

$$0 < c_\lambda = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J(\gamma(t)) \leq \sup_{s\geq 0} J(s\tilde{v}_{\varepsilon_1}) < l^*,$$

for all $\lambda \in (0, \frac{m_0}{C_2}\lambda_1)$.

Lemma 5.6. *Suppose that $r = p$, $\lambda \in (0, \frac{m_0}{C_2}\lambda_1)$, and* (H1), (H2), (H4), (H5) *hold. Let $(u_n) \in \mathcal{D}_a^{1,p}$ be a sequence such that*

$$J(u_n) \to c_\lambda \quad \textit{and} \quad J'(u_n) \to 0 \quad \textit{in } (\mathcal{D}_a^{1,p})^{-1}, \quad \textit{as } n \to +\infty.$$

Then, for all $n \in \mathbb{N}$, we have $\|u_n\|^p \leq t_0$.

Proof. Suppose by contradiction that for some $n \in \mathbb{N}$ we have $\|u_n\|^p > t_0$. From the definition of $M_0(t)$, (H5), and (3.1) we have that (u_n) bounded. Thus, we obtain

$$|J'(u_n) \cdot (u_n)| \leq |J'(u_n)| \, \|(u_n)\| \to 0,$$

as $n \to +\infty$. Which implies

$$\begin{aligned} c_\lambda &= J(u_n) - \frac{1}{\xi} J'(u_n)(u_n) + o_n(1) \\ &\geq \frac{1}{p} \widehat{M_0}(\|u_n\|^p) - \frac{1}{\xi} M_0(t_0)\|u_n\|^p + o_n(1) \\ &\geq \Big(\frac{1}{p} m_0 - \frac{1}{\xi} M_0(t_0)\Big)\|u_n\|^p + o_n(1). \end{aligned} \tag{5.7}$$

Since $m_0 < M(t_0) < \frac{\xi}{p} m_0$ we have $\frac{1}{p} m_0 - \frac{1}{\xi} M_0(t_0) > 0$. So we obtain

$$c_\lambda \geq \Big(\frac{1}{p} m_0 - \frac{1}{\xi} M_0(t_0)\Big) t_0 > 0.$$

Since $\lambda \in (0, \frac{m_0}{C_2}\lambda_1)$, this contradicts the Remark 5.5. Hence we conclude that $\|u_n\|^p \leq t_0$. □

Proof of Theorem 1.1. Let $\lambda \in (0, \frac{m_0}{C_2}\lambda_1)$. It follows from Remark 5.5 that

$$c_\lambda < \big(\frac{1}{\xi} - \frac{1}{p^*}\big)(m_0 C^*_{a,p})^{\frac{p^*}{p^*-p}}. \tag{5.8}$$

From Lemmas 5.1 and 5.2, there exists a bounded sequence $(u_n) \subset \mathcal{D}^{1,p}_a$ such that $J(u_n) \to c_\lambda$ and $J'(u_n) \to 0$, $(\mathcal{D}^{1,p}_a)^{-1}$, as $n \to \infty$. Since (5.8) holds, it follows from Lemma 4.1 that, up to a subsequence, $u_n \to u_\lambda$ strongly in $\mathcal{D}^{1,p}_a$. Thus u_λ is a weak solution of problem (3.2). By Lemma 5.6, we conclude that u_λ is a weak solution of problem (1.1). □

6. Proof of Theorem 1.2

Here we consider the case $p < r < p^*$. The main idea of the proof is essentially the same as in Theorem 1.2, we apply the mountain pass theorem and use Lemma 4.1. The next two lemmas show that the functional J has the Mountain Pass geometry.

Lemma 6.1. *Suppose that $p < r < p^*$. Assume that the conditions* (H1)–(H4) *hold. There exist positive numbers ρ and α such that*

$$J(u) \geq \alpha > 0, \forall u \in \mathcal{D}^{1,p}_a, \quad \textit{with } \|u\| = \rho.$$

Proof. From (H1), (H3), (H4), and Caffarelli-Kohn-Nirenberg inequality, we obtain

$$J(u) \geq \frac{m_0}{p}\|u\|^p - \lambda \tilde{C}_2 \|u\|^r - \frac{1}{p^*}\tilde{C}\|u\|^{p^*}.$$

Since $p < r < p^*$, the result follows by choosing $\rho > 0$ small enough. □

Lemma 6.2. *Suppose that $p < r < p^*$. For all $\lambda > 0$, there exists $e \in \mathcal{D}^{1,p}_a$ with $J(e) < 0$ and $\|e\| > \rho$.*

Proof. Fix $v_0 \in \mathcal{D}^{1,p}_a \backslash \{0\}$, with $v_0 > 0$ in Ω. Using (3.1) and (H4) we obtain

$$J(tv_0) \leq \frac{\xi}{p^2} m_0 t^p \|v_0\|^p - \frac{\lambda C_1}{r} t^r \int_\Omega \frac{|v_0|^r}{|x|^\delta} dx - \frac{t^{p^*}}{p^*}\int_\Omega \frac{|v_0|^{p^*}}{|x|^{bp^*}} dx.$$

Since $p < r < p^*$, we have $\lim_{t \to +\infty} J(tv_0) = -\infty$. Thus, there exists $\bar{t} > 0$ large enough, such that $\bar{t}\|v_0\| > \rho$ and $J(\bar{t}v_0) < 0$. The result follows by considering $e = \bar{t}v_0$. □

Using a version of the mountain pass theorem without the (PS) condition (see [28]), there exists a sequence $(u_n) \in \mathcal{D}_a^{1,p}$, satisfying

$$J(u_n) \to c_\lambda \quad \text{and} \quad J'(u_n) \to 0 \quad \text{in } (\mathcal{D}_a^{1,p})^{-1},$$

where

$$0 < c_\lambda = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J(\gamma(t)),$$
$$\Gamma := \big\{\gamma \in C([0,1], \mathcal{D}_a^{1,p}) : \gamma(0) = 0, \gamma(1) = \bar{t}v_0\big\},$$

and $v_0 \in \mathcal{D}_a^{1,p}$ is such that $v_0 > 0$.

Remark 6.3. From Lemmas 6.1 and 6.2, Lemma 5.4 holds for all $\lambda > 0$, when $p < r < p^*$. So, if we consider the path $\gamma_*(t) = t(\bar{t}v_{\varepsilon_1})$, for $t \in [0,1]$, which belongs to Γ, we obtain the estimate

$$0 < c_\lambda = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J(\gamma(t)) \le \sup_{s\ge 0} J(s\tilde{v}_{\varepsilon_1}) < l^*$$

for all $\lambda > 0$.

The next Lemma is a version of the Lemma 5.6 when $p < r < p^*$. By hypothesis (H5) and Remark 6.3, its proof is similar to the proof of Lemma 5.6.

Lemma 6.4. *Suppose that $p < r < p^*$, and* (H1), (H2), (H4), (H5) *hold. Let $(u_n) \in \mathcal{D}_a^{1,p}$ be a sequence such that*

$$J(u_n) \to c_\lambda \quad \textit{and} \quad J'(u_n) \to 0 \quad \textit{in } (\mathcal{D}_a^{1,p})^{-1}, \quad \textit{as } n \to +\infty.$$

Then, for all $n \in \mathbb{N}$, we have $\|u_n\|^p \le t_0$.

Proof of Theorem 1.2. It follows from Remark 6.3 that

$$c_\lambda < \big(\frac{1}{\xi} - \frac{1}{p^*}\big)(m_0 C_{a,p}^*)^{\frac{p^*}{p^*-p}}. \tag{6.1}$$

From Lemmas 6.1 and 6.2, there exists a bounded sequence $(u_n) \subset \mathcal{D}_a^{1,p}$ such that $J(u_n) \to c_\lambda$ and $J'(u_n) \to 0$, $(\mathcal{D}_a^{1,p})^{-1}$, as $n \to \infty$. Since (6.1) holds, it follows from Lemma 4.1 that, up to a subsequence, $u_n \to u_\lambda$ strongly in $\mathcal{D}_a^{1,p}$. Thus u_λ is a weak solution of problem (3.2). Moreover, by Lemma 5.6 we conclude that u_λ is a weak solution of problem (1.1). $\square$

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References

[1] C. Alves, F. J. S. A. Corrêa, T. F. Ma; *Positive solutions for a quasilinear elliptic equation of Kirchhoff type*, Comput. Math. Appl. 49 (2005), 85-93.

[2] C. Alves, F. J. S. A. Corrêa, G. M. Figueiredo; *On a class of nolocal elliptic problems with critical growth*, Differential Equations Applications 3 (2010), 409-417.

[3] A. Ambrosetti, P. H. Rabinowitz; *Dual variational methods in critical point theory and apllications*, J. Functional Analysis 14 (1973), 349-381.

[4] G. Anelo; *A uniqueness result for a nonlocal equation of Kirchhoff type and some related open problem*, J. Math. Anal. Appl. 373 (2011), 248-251.

[5] S. Bernstein; *Sur une classe d'équations fonctionnelles aux dérivés partielles*, Bull. Acad. Sci. URSS. Sér. Math. (Izvestia Akad. Nauk SSSR) 4 (1940), 17-26.

[6] L. Caffarelli, R. Kohn, L. Nirenberg; *First order interpolation inequalities with weights*, Compos. Math. 53 (1984), 259-275.

[7] N. T. Chung, T. H. Quoc; *Existence and multiplicity of solutions for a degenerate nonlocal elliptic differential equation,* EJDE 148 (2013), 13 pp.
[8] F. J. S. A. Corrêa, G. M. Figueiredo; *On an elliptic equation of p-Kirchhoff-type via variational methods*, Bull. Austral. Math. Soc. 77 (2006), 263-277.
[9] F. J. S. A. Corrêa, G. M. Figueiredo; *On a p-Kirchhoff equation via Krasnoselskii's genus*, Applied Mathematics Letters 22 (2009), 819-822.
[10] G. Dai, D. Liu; *Infinitely many positive solutions for a p(x)-Kirchhoff-type equation*, J. Math. Anal. Apll. 359 (2009), 764-710.
[11] N. Ghoussoub, C. Yuan; *Multiple solutions for quasi-linear PDEs involving the critical Sobolev and Hardy exponents* Trans. Amer. Math. Soc. 352 (1998), 5703-5743.
[12] G. M. Figueiredo; *Existence of positive solution for a Kirchhoff problem type with critical growth via truncation argument*, J. Math, Anal. Appl. 401 (2013), 706-713.
[13] G. M. Figueiredo, J. R. dos Santos Junior; *Multiplicity of solutions for a Kirchhoff equation with subcritical or critical growth*, DIE-Diff. Int. Equations 25 (2012), 853-868.
[14] G. M. Figueiredo, M. B. Guimarães, R. S. Rodrigues; *Solutions for a Kirchhoff equation with weight and nonlinearity with subcritical and critical Caffarelli-Kohn-Nirenberg growth*, Proceedings of Edinburgh Mathematical Society (2015), DOI:10.1017/S0013091515000395, to appear.
[15] G. M. Figueiredo, G. Molica Bisci, R. Servadei; *On a fractional Kirchhoff-type equation via Krasnoselskii's genus,* Asymptot. Anal. 94 (2015), 347-361.
[16] X. He, W. Zou; *Infinitely many positive solutions for Kirchhoff-type problems*, Nonlinear Anal. 70 (2009), 1407-1414.
[17] T. Horiuchi; *Best constant in weighted Sobolev inequality with weights being powers of distance from origin*, J. Inequal. Appl. **1** (1997), 275-292.
[18] G. Kirchhoff; *Mechanik, Teubner*, Leipzig, 1883.
[19] J.-L. Lions; *On some questions in boundary value problems of mathematical physics*, in: Contemporary Developments in Continuum Mechanics and Partial Differential Equations (Proc. Internat. Sympos., Inst. Mat., Univ. Fed. Rio de Janeiro, Rio de Janeiro, 1977), in: North-Holland Math. Stud., vol. 30, North-Holland, Amsterdam, New York, (1978), 284-346.
[20] D. C. Liu; *On a p-Kirhhoff equation via fountain theorem and dual fountain theorem*, Nonlinear Anal. 72 (2010), 302-208.
[21] P.L. Lions; *The concentration-compactness principle in the calculus of variations. The limit case*, Rev. Mat. Iberoamericana **1** (1985), 145-201.
[22] T. F. Ma, J. E. Muñoz Rivera; *Positive solutions for a nonlinear nonlocal elliptic transmission problem*, Applied Mathematics Letters 16 (2) (2003), 243-248.
[23] Z. Binlin, G. Molica Bisci, R. Servadei; *Superlinear nonlocal fractional problems with infinitely many solutions,* Nonlinearity 28 (2015), 2247-2264.
[24] G. Molica Bisci, V. Radulescu; *Bifurcation analysis of a singular elliptic problem modelling the equilibrium of anisotropic continuous media,* Topol. Methods Nonlinear Anal. 45 (2015), 493-508.
[25] G. Molica Bisci, P. Pizzimenti; *Sequences of weak solutions for non-local elliptic problems with Dirichlet boundary condition,* Proc. Edinb. Math. Soc. 57 (2014), 779-809.
[26] K. Perera, Z. Zhang; *Nontrivial solutions of Kirchhoff-type problems via the Yang index*, J. Differential Equations 221 (2006), 246-255
[27] S. I. Pohozaev; *A certain class of quasilinear hyperbolic equations*, Mat. Sb. (N.S.) 96 (138) (1975), 152-166, 168.
[28] M. Willem; *Minimax theorems*, Birkhäuser, 1996.
[29] B. Xuan; *The eigenvalue problem of a singular quasilinear elliptic equation*, Electron. J. Diff. Equ. 2004, no. 16 (2004), 1-11.
[30] B. Xuan; *The solvability of quasilinear Brezis-Nirenberg-type problems with singular weights*, Nonlinear Anal. 62 (2005), 703-725.

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