MULTIPlicity of CRITICAL POINTS FOR THE FRACTIONAL ALLEN-CAHN ENERGY

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Abstract. In this article we study the fractional analogue of the Allen-Cahn energy in bounded domains. We show that it admits a number of critical points which approaches infinity as the perturbation parameter tends to zero.

1. Introduction

The problems involving fractional operators attracted great attention during the previous years. Indeed these problems appear in areas such as optimization, finance, crystal dislocation, minimal surfaces, water waves, fractional diffusion; see for example [5, 6, 7, 8, 9, 10, 11, 12, 13]. In particular, from a probabilistic point of view, the fractional Laplacian is the infinitesimal generator of a Lévy process, see e.g. [1].

In this article we present some existence and multiplicity results for critical points of functionals of the form

\[ F_{\epsilon}(u) = \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} \, dx \, dy + \frac{1}{\epsilon^{2s}} \int_{\Omega} W(u) \, dx, \quad \text{if } s \in (0, 1/2), \]

\[ F_{\epsilon}(u) = \frac{1}{\log \epsilon} \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^2}{|x - y|^{n+1}} \, dx \, dy + \frac{1}{\epsilon \log \epsilon} \int_{\Omega} W(u) \, dx, \quad \text{if } s = 1/2, \]

\[ F_{\epsilon}(u) = \frac{\epsilon^{2s-1}}{2} \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} \, dx \, dy + \frac{1}{\epsilon} \int_{\Omega} W(u) \, dx, \quad \text{if } s \in (1/2, 1), \]

where \( \Omega \) is a smooth bounded domain of \( \mathbb{R}^n \), \( u \in H^s(\Omega; \mathbb{R}) \), \( W \in C^2(\mathbb{R}; \mathbb{R}^+) \) is the well known double well potential (see Section 2), and \( \epsilon \in \mathbb{R}^+ \).

\( F_{\epsilon} \) is the fractional energy of the Allen-Cahn equation. It is the fractional counterpart of the functionals studied by Modica-Mortola in [14, 15] where they proved the \( \Gamma \)-convergence of the energy to De Giorgi’s perimeter. In the same way, functionals (1.1), (1.2), (1.3) have been also considered by Valdinoci-Savin in [16], where it is discussed their \( \Gamma \)-convergence.

Moreover, as proved in [13] for the functional

\[ \int_{\Omega} [\epsilon |Du|^2 + \epsilon^{-1}(u^2 - 1)^2] \, dx, \]
we expect that the solutions have interesting geometric properties related to the interface minimality.

Some authors investigated multiplicity results of nontrivial solution for

$$
\epsilon^{2s}(-\Delta)^s u + u = h(u) \quad \text{in } \Omega
$$

$$
u > 0
$$

$$
u = 0 \quad \text{on } \partial \Omega
$$

where $\Omega$ is a bounded domain in $\mathbb{R}^n$, $n > 2s$, and $h(u)$ has a subcritical growth (see [12]), or for

$$
\epsilon^{2s}(-\Delta)^s u + V(z)u = f(u) \quad \text{in } \mathbb{R}^n, \; n > 2s
$$

$$
u \in H^s(\mathbb{R}^n)
$$

$$
u(z) > 0 \quad z \in \mathbb{R}^n
$$

where the potential $V : \mathbb{R}^n \rightarrow \mathbb{R}$ and the nonlinearity $f : \mathbb{R} \rightarrow \mathbb{R}$ satisfy suitable assumptions (see [11]).

Then Cabré and Sire in [5] studied the equation

$$
(-\Delta)^s u + G'(u) = 0 \quad \text{in } \mathbb{R}^n
$$

where $G$ denotes the potential associated to a nonlinearity $f$, and they proved existence, uniqueness and qualitative properties of solutions.

Indeed, Passaseo in [16] studied the analogue of our functional, with the classical Laplacian instead of the fractional one, i.e.,

$$
f_\epsilon(u) = \epsilon \int_{\Omega} |Du|^2 \, dx + \frac{1}{\epsilon} \int_{\Omega} G(u) \, dx
$$

where $\Omega$ is a bounded domain of $\mathbb{R}^n$, $u \in H^{1,2}(\Omega; \mathbb{R})$, $G \in C^2(\mathbb{R}; \mathbb{R}^+) \subset H^* \subset H^1 \subset L^1(\mathbb{R}^n)$ is a nonnegative function having exactly two zeros, $\alpha$ and $\beta$, and $\epsilon$ is a positive parameter: he proved that the number of critical points for $f_\epsilon$ goes to $\infty$ as $\epsilon \rightarrow 0$.

Passaseo was motivated by De Giorgi’s idea, contained in [9], i.e. if $u_\epsilon \rightarrow u_0$ in $L^1(\Omega)$ as $\epsilon \rightarrow 0$ and $\lim_{\epsilon \rightarrow 0} f_\epsilon(u_\epsilon) < \infty$, then the function $U_\epsilon(t)$, defined as steepest descent curves for $f_\epsilon$ starting from $u_\epsilon$, converge to a curve $U_0(t)$ in $L^1(\Omega)$ such that $U_0(t)$ is a function with values in $\{\alpha, \beta\}$ for every $t \geq 0$ and the interface between the sets $E_t = \{x \in \Omega : U_0(t)(x) = \alpha\}$ and $\Omega \setminus E_t$ moves by mean curvature. As a consequence the critical points $u_\epsilon$ of $f_\epsilon$ which satisfy

$$
\lim_{\epsilon \rightarrow \infty} f_\epsilon(u_\epsilon) < +\infty
$$

converge in $L^1(\Omega)$ to a function $u_0$ taking values in $\{\alpha, \beta\}$. De Giorgi considered also the problem of existence and multiplicity for nontrivial critical points of $f_\epsilon$ with the property (1.7), and Passaseo’s critical points verify this property and

$$
\lim_{\epsilon \rightarrow -\infty} f_\epsilon(u_\epsilon) > 0,
$$

so he can say that $u_0$ is nontrivial.

In this article we want to extend Passaseo’s results by replacing the function $G$ in (1.6) with the double well potential $W$, and Passaseo’s functional $f_\epsilon$ with its fractional counterpart.

The article is organized as follows: in Section 2 we give some preliminaries definitions and results. In Section 3, we define suitable functions and sets, then most of the work is dedicated to prove nonlocal estimates needful to obtain the bound from above of $F_\epsilon$, (see Lemma 3.5), and the (PS)-condition, Lemma 3.7. In
fact in particular for the first of these results, we had to split the domain in two
types of regions and estimate $F_\epsilon$ in the three possible interactions.

Finally, after recalling a technical result, Lemma 3.6, we can apply a classical
Krasnoselskii’s genus tool to show the existence and multiplicity results for solutions.

Hence, knowing that minimizers of $F_\epsilon$ $\Gamma$-converge to minimizers of the area
functional, we hope that also min-max solutions can pass to the limit as $\epsilon \to 0$
in a suitable sense, producing critical points of positive index for local, if $s \in [1/2, 1)$,
or nonlocal, if $s \in (0, 1/2)$, area functional.

2. Notation and preliminary results

In this section we introduce the framework that we will be used throughout this
article.

Let $\Omega$ be a bounded domain of $\mathbb{R}^n$, denote by $|\Omega|$ its Lebesgue measure and
consider $W$ the double well potential, that is an even function such that
\begin{align}
W: \mathbb{R} &\to [0, +\infty) \quad W \in C^2(\mathbb{R}; \mathbb{R}^+) \quad W(\pm 1) = 0 \\
W > 0 &\text{ in } (-1, 1) \quad W'(\pm 1) = 0 \quad W''(\pm 1) > 0.
\end{align}

Now we fix the fractional exponent $s \in (0, 1)$. For any $p \in [1, +\infty)$, we define
$$W^{s,p}(\Omega) := \left\{ u \in L^p(\Omega) : \frac{|u(x) - u(y)|}{|x - y|^{n+sp}} \in L^p(\Omega \times \Omega) \right\};$$
i.e. an intermediary Banach space between $L^p(\Omega)$ and $W^{1,p}(\Omega)$, endowed with the
natural norm
$$\|u\|_{W^{s,p}(\Omega)} := \left( \int_\Omega |u|^p \, dx + \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} \, dx \, dy \right)^{1/p}.$$

If $p = 2$ we define $W^{s,2}(\Omega) = H^s(\Omega)$ and it is a Hilbert space. Now let $\mathcal{F}(\mathbb{R}^n)$ be
the set of all tempered distributions, that is the topological dual of $\mathcal{S}(\mathbb{R}^n)$. As
usual, for any $\varphi \in \mathcal{S}(\mathbb{R}^n)$, we denote by
$$\mathcal{F} \varphi(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\xi \cdot x} \varphi(x) \, dx$$
the Fourier transform of $\varphi$ and we recall that one can extend $\mathcal{F}$ from $\mathcal{S}(\mathbb{R}^n)$ to
$\mathcal{F}'(\mathbb{R}^n)$. At this point we can define, for any $u \in \mathcal{S}(\mathbb{R}^n)$ and $s \in (0, 1)$, the
fractional Laplacian operator as
$$(-\Delta)^s u(x) = C(n, s) P.V. \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+2s}} \, dy.$$
Here P.V. stands for the Cauchy principal value and $C(n, s)$ is a normalizing constant (see [10] for more details). It is easy to prove that this definition is equivalent to the following two:
$$(-\Delta)^s u(x) = -\frac{1}{2} C(n, s) \int_{\mathbb{R}^n} \frac{u(x + y) + u(x - y) - 2u(x)}{|y|^{n+2s}} \, dy \quad \forall x \in \mathbb{R}^n,$$
and
$$(-\Delta)^s u(x) = \mathcal{F}^{-1}(|\xi|^{2s} (\mathcal{F} u)) \quad \forall \xi \in \mathbb{R}^n.$$
Proposition 2.1 ([II]). Let $p \in [1, +\infty)$ and $0 < s \leq s' \leq 1$. Let $\Omega$ be an open set of $\mathbb{R}^n$ and $u : \Omega \to \mathbb{R}$ be a measurable function. Then $W^{s',p}(\Omega)$ is continuously embedded in $W^{s,p}(\Omega)$, denoted by $W^{s',p}(\Omega) \hookrightarrow W^{s,p}(\Omega)$, and the following inequality holds
\[ \|u\|_{W^{s,p}(\Omega)} \leq C\|u\|_{W^{s',p}(\Omega)} \]
for some suitable positive constant $C = C(n, s, p) \geq 1$.

Moreover, if also $\Omega$ is an open set of $\mathbb{R}^n$ of class $C^{0,1}$ with bounded boundary, then $W^{1,p}(\Omega) \hookrightarrow W^{s,p}(\Omega)$ and we have
\[ \|u\|_{W^{s,p}(\Omega)} \leq C\|u\|_{W^{1,p}(\Omega)} \]
for some suitable positive constant $C = C(n, s, p) \geq 1$.

Definition 2.2 ([II]). For any $s \in (0, 1)$ and any $p \in [1, +\infty)$, we say that an open set $\Omega \subseteq \mathbb{R}^n$ is an extension domain for $W^{s,p}$ if there exists a positive constant $C = C(n, p, s, \Omega)$ such that: for every function $u \in W^{s,p}(\Omega)$ there exists $\tilde{u} \in W^{s,p}(\mathbb{R}^n)$ with $\tilde{u}(x) = u(x)$ for all $x \in \Omega$ and $\|\tilde{u}\|_{W^{s,p}(\mathbb{R}^n)} \leq C\|u\|_{W^{s,p}(\Omega)}$.

Theorem 2.3 ([II]). Let $s \in (0, 1)$ and $p \in [1, +\infty)$ be such that $sp < n$. Let $q \in [1, p^*)$, where $p^* = p^*(n, s) = np/(n - sp)$ is the so-called "fractional critical exponent". Let $\Omega \subseteq \mathbb{R}^n$ be a bounded extension domain for $W^{s,p}$ and $\mathcal{I}$ be a bounded subset of $L^p(\Omega)$. Suppose that
\[ \sup_{f \in \mathcal{I}} \int_{\Omega} \int_{\Omega} \frac{|f(x) - f(y)|^p}{|x - y|^{n+sp}} \, dx \, dy < \infty. \]
Then $\mathcal{I}$ is pre-compact in $L^p(\Omega)$.

We recall also the notion of Krasnoselskii’s genus, useful in the sequel.

Definition 2.4 ([II]). Let $H$ be a Hilbert space and $E$ be a closed subset of $H \setminus \{0\}$, symmetric with respect to 0 (i.e. $E = -E$).

We call genus of $E$ in $H$, indicated with $\text{gen}_H(E)$, the least integer $m$ such that there exists $\phi \in C(H; \mathbb{R})$ such that $\phi$ is odd and $\phi(x) \neq 0$ for all $x \in E$.

We set $\text{gen}_H(E) = +\infty$ if there are no integer with the above property and $\text{gen}_H(\emptyset) = 0$.

It is well known that $\text{gen}_H(S^k) = k + 1$ if $S^k$ is a $k$-dimensional sphere of $H$ with centre in zero.

Finally we recall a well known result:

Theorem 2.5 ([II]). Let $H$ be a Hilbert space and $f : H \to \mathbb{R}$ be an even $C^2$-functional satisfying the following Palais-Smale condition: given a sequence $(u_i)_i$ in $H$ such that the sequence $(f(u_i))_i$ is bounded and $f'(u_i) \to 0$, $(u_i)_i$ is relatively compact in $H$.

Set $f^c = \{u \in H : f(u) \leq c\}$ for all $c \in \mathbb{R}$. Then, for all $c_1, c_2 \in \mathbb{R}$, such that $c_1 \leq c_2 < f(0)$, we have
\[ \text{gen}_H(f^{c_2}) \leq \text{gen}_H(f^{c_1}) + \# \{(\pm u_i, u_i) : c_1 \leq f(u_i) \leq c_2, f'(u_i) = 0\}, \tag{2.2} \]
where, if $A$ is a set, we indicate with $\#A$ the cardinality of $A$.

For the rest of this article, we consider $H^*(\Omega)$ as Hilbert space and we shall write simply $\text{gen}(E)$ instead of $\text{gen}_{H^*(\Omega)}(E)$; then we refer to the Palais-Smale condition with the symbol ($PS$)-condition.
3. MULTIPlicity OF CRITICAL POINTS

Let us state the fundamental result of the paper.

**Theorem 3.1.** Let $\Omega$ be a smooth bounded domain of $\mathbb{R}^n$ and $W$ be a function satisfying (2.1). Then there exist two sequences of positive numbers $(c_k)_k$, $(\epsilon_k)_k$ such that for every $\epsilon \in (0, \epsilon_k)$, the functional $F_{\epsilon}$ has at least $k$ pairs $(-u_{1,\epsilon}, u_{1,\epsilon}), \ldots, (-u_{k,\epsilon}, u_{k,\epsilon})$ of critical points, all of them different from the constant pair $(-1, 1)$ satisfying

$$-1 \leq u_{i,\epsilon}(x) \leq 1 \quad \forall x \in \Omega, \forall \epsilon \in (0, \epsilon_k), \; i = 1, \ldots, k;$$

Moreover, for all $\epsilon \in (0, \epsilon_k)$ and all $i = 1, \ldots, k$ we have

$$F_{\epsilon}(u) \geq \min \{ F_{\epsilon}(u) : u \in H^s(\Omega), -1 \leq u(x) \leq 1 \text{ for } x \in \Omega, \int_{\Omega} u \, dx = 0 \}. \tag{3.1}$$

**Remark 3.2.** The constant function $u \equiv 0$ is obviously a critical point for the functional $F_{\epsilon}$ for every $\epsilon > 0$ but it is not included among the ones given by Theorem 3.1. Instead if $s \in (1/2, 1)$, but for the other cases it is similar,

$$F_{\epsilon}(0) = \frac{1}{\epsilon} \int_{\Omega} W(0) \, dx \rightarrow +\infty \quad \text{as } \epsilon \rightarrow 0.$$  

Moreover, since $\inf \{ W(t) : W'(t) = 0, -1 < t < 1 \} > 0$, one can say that the critical points given by Theorem 3.1 are not constant functions. In fact, if $u_{\epsilon} = c_{\epsilon}$ is a constant critical point for $F_{\epsilon}$ (distinct from $-1$ and $1$), it must be $W'(c_{\epsilon}) = 0$ and $-1 < c_{\epsilon} < 1$; therefore

$$W(c_{\epsilon}) \geq \inf \{ W(t) : W'(t) = 0, -1 < t < 1 \} > 0 \tag{3.2}$$

and so, for example by considering the functional related to $s \in (1/2, 1)$, but the other cases are similar,

$$F_{\epsilon}(c_{\epsilon}) = \frac{1}{\epsilon} \int_{\Omega} W(c_{\epsilon}) \, dx \rightarrow +\infty \quad \text{as } \epsilon \rightarrow 0, \tag{3.3}$$

in contradiction with $F_{\epsilon}(c_{\epsilon}) \leq c_k$ for all $\epsilon \in (0, \epsilon_k)$.

Notice that for all $\epsilon > 0$,

$$\min \{ F_{\epsilon}(u) : u \in H^s(\Omega), -1 \leq u(x) \leq 1 \forall x \in \Omega, \int_{\Omega} u \, dx = 0 \} > 0 \tag{3.4}$$

if we assume, without loss of generality, that $\Omega$ is a connected domain.

Let $\bar{u}$ be a minimizing function; if we assume $F_{\epsilon}(\bar{u}) = 0$, then

$$\int_{\Omega} \int_{\Omega} \frac{|\bar{u}(x) - \bar{u}(y)|^2}{|x - y|^{n+2s}} \, dx \, dy \equiv 0$$

and $W(\bar{u}) \equiv 0$. Therefore we must have $\bar{u} \equiv 0$ in contradiction with $W(0) > 0$.

**Definition 3.3.** Let $k$ be a fixed positive integer; for every $\lambda = (\lambda^{(0)}, \ldots, \lambda^{(k)}) \in \mathbb{R}^{k+1}$ define the function $\varphi_{\lambda} : \mathbb{R} \rightarrow \mathbb{R}$ by

$$\varphi_{\lambda}(t) = \sum_{m=0}^{k} \lambda^{(m)} \cos(mt).$$
For every $\lambda \in \mathbb{R}^{k+1}$ with $|\lambda|_{\mathbb{R}^{k+1}} = 1$ and $\epsilon > 0$, let $L_\epsilon(\varphi_\lambda) : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$L_\epsilon(\varphi_\lambda)(t) = \frac{1}{2\epsilon} \int_{t-\epsilon}^{t+\epsilon} \frac{\varphi_\lambda(\tau)}{|\varphi_\lambda(\tau)|} \, d\tau;$$

notice that $L_\epsilon(\varphi_\lambda)$ is well defined because $\varphi_\lambda$ has only isolated zeros $\forall \lambda \in \mathbb{R}^{k+1}$ with $|\lambda|_{\mathbb{R}^{k+1}} = 1$.

For $x = (x_1, \cdots, x_n) \in \Omega \subset \mathbb{R}^n$ we consider the projection onto the first component, $P_1(x) = x_1$, and the set

$$S^k_\epsilon = \{ L_\epsilon(\varphi_\lambda) \circ P_1 : \lambda \in \mathbb{R}^{k+1}, |\lambda|_{\mathbb{R}^{k+1}} = 1 \}.$$

**Lemma 3.4.** Let us fix $a, b \in \mathbb{R}$ with $a < b$ and set

$$\chi(\varphi_\lambda) = \{ t \in [a, b] : \varphi_\lambda(t) = 0 \}$$

for $\lambda \in \mathbb{R}^{k+1}$ with $|\lambda|_{\mathbb{R}^{k+1}} = 1$. Then for every $k \in \mathbb{N}$ we have

$$\sup \{ \chi(\varphi_\lambda) \subset \mathbb{R}^{k+1}, |\lambda|_{\mathbb{R}^{k+1}} = 1 \} < +\infty.$$
At this point it remains to analyze \( \int_{\Omega} \int_{\Omega} \frac{|u_{\lambda,\epsilon}(x) - u_{\lambda,\epsilon}(y)|^2}{|x-y|^{n+2s}} \, dx \, dy \). We split it in three cases:

**Case (a).** We have

\[
\int_{Y_{\lambda,\epsilon}} \int_{Y_{\lambda,\epsilon}} \frac{|u_{\lambda,\epsilon}(x) - u_{\lambda,\epsilon}(y)|^2}{|x-y|^{n+2s}} \, dx \, dy = \sum_{i,j=1}^{k} \int_{Y_{\lambda,\epsilon}} \int_{Y_{\lambda,\epsilon}} \frac{|u_{\lambda,\epsilon}(x) - u_{\lambda,\epsilon}(y)|^2}{|x-y|^{n+2s}} \, dx \, dy
\]

We denote \( Q_- = Q \cap P_1^{-1}\{x_1 < 0\} \), \( Q_+ = Q \cap P_1^{-1}\{y_1 > 2\} \) and we split \( Q_- \in N \) strips of width \( \epsilon \), with \( N \) of order \( 1/\epsilon \), so we obtain

\[
\sum_{i,j=1}^{k} \int_{Y_{\lambda,\epsilon}} \int_{Y_{\lambda,\epsilon}} \frac{|u_{\lambda,\epsilon}(x) - u_{\lambda,\epsilon}(y)|^2}{|x-y|^{n+2s}} \, dx \, dy
\]

\[
\leq k^2 \int_{Q_-} \int_{Q_+} \frac{4}{|x-y|^{n+2s}} \, dx \, dy
\]

\[
\leq 4Nk^2 \int_{-\epsilon}^{2\epsilon} \int_{-\epsilon}^{+\infty} y^{-2s-1} \, dr \, dx
\]

\[
= \frac{2}{s} Nk^2 \int_{-\epsilon}^{2\epsilon} (-2x_1)^{-2s} \, dx_1.
\]

Now we distinguish two cases:

(j) if \( s \neq 1/2 \), we have

\[
\frac{2}{s} Nk^2 \int_{-\epsilon}^{2\epsilon} (-2x_1)^{-2s} \, dx_1 = \frac{2^{1-2s}Nk^2}{s(1-2s)} \epsilon^{1-2s}(2^{1-2s} - 1);
\]

(jj) while, if \( s = 1/2 \),

\[
\frac{2}{s} Nk^2 \int_{-\epsilon}^{2\epsilon} (-2x_1)^{-2s} \, dx_1 = \frac{2^{1-2s}}{s} Nk^2 \log 2.
\]

**Case (b).** We note that \( Y_{\lambda,\epsilon} \subseteq Q \setminus \bar{Z}_{\lambda,\epsilon} \), so

\[
\int_{2\lambda,\epsilon} \int_{Y_{\lambda,\epsilon}} \frac{|u_{\lambda,\epsilon}(x) - u_{\lambda,\epsilon}(y)|^2}{|x-y|^{n+2s}} \, dx \, dy
\]

\[
\leq k \sum_{i=1}^{k} \int_{2\lambda,\epsilon} \int_{Q \setminus \bar{Z}_{\lambda,\epsilon}} \frac{|u_{\lambda,\epsilon}(x) - u_{\lambda,\epsilon}(y)|^2}{|x-y|^{n+2s}} \, dx \, dy
\]

\[
\leq 2\omega_{n-1} \rho^{-n-1} \epsilon \sum_{i=1}^{k} \sup_{x \in 2\lambda,\epsilon} \int_{Q \setminus \bar{Z}_{\lambda,\epsilon}} \min\{1/\epsilon^2|x-y|^2, 4\} \, dx \, dy
\]

\[
\leq 2k\epsilon\omega_{n-1} \rho^{-n-1}\left(\int_{0}^{2\epsilon} \frac{1}{\epsilon^2} r^{-1-2s} \, dr + \int_{2\epsilon}^{+\infty} 4r^{-1-2s} \, dr\right)
\]

\[
= k \left(\frac{2}{\epsilon} \frac{2^{-2s}}{2-2s} \epsilon^{2-2s} + 8 \epsilon^{-2s} \right) \omega_{n-1} \rho^{-n-1}
\]

\[
= k \epsilon^{2-2s} \left(\frac{2^{-2s}}{1-s} + \frac{2^{2-2s}}{s} \right) \omega_{n-1} \rho^{-n-1}.
\]
Case (c). It results
\[
\int_{\mathcal{Z}_{\lambda, \epsilon}} \int_{\mathcal{Z}_{\lambda, \epsilon}} \frac{|u_{\lambda, \epsilon}(x) - u_{\lambda, \epsilon}(y)|^2}{|x - y|^{n+2s}} \, dx \, dy \\
= \sum_{i=1}^{k} \int_{\mathcal{Z}_{\lambda, \epsilon}} \int_{\mathcal{Z}_{\lambda, \epsilon}} \frac{|u_{\lambda, \epsilon}(x) - u_{\lambda, \epsilon}(y)|^2}{|x - y|^{n+2s}} \, dx \, dy \\
+ \sum_{i,j=1 \atop i \neq j}^{k} \int_{\mathcal{Z}_{\lambda, \epsilon}} \int_{\mathcal{Z}_{\lambda, \epsilon}} \frac{|u_{\lambda, \epsilon}(x) - u_{\lambda, \epsilon}(y)|^2}{|x - y|^{n+2s}} \, dx \, dy. \tag{3.13}
\]

Concerning the first term of the right-hand side, we have
\[
\sum_{i=1}^{k} \int_{\mathcal{Z}_{\lambda, \epsilon}} \int_{\mathcal{Z}_{\lambda, \epsilon}} \frac{|u_{\lambda, \epsilon}(x) - u_{\lambda, \epsilon}(y)|^2}{|x - y|^{n+2s}} \, dx \, dy \\
\leq \frac{1}{\epsilon^2} \sum_{i=1}^{k} \left| \mathcal{Z}_{\lambda, \epsilon} \right| \int_{0}^{2s} r^{1-2s} \, dr \leq k \omega_{n-1} \rho^{n-1} \frac{2^2 - 2s}{1 - s} \epsilon^{1-2s}. \tag{3.14}
\]

The other term is estimated as in Case (b).

So we can obtain the estimates for the functionals \( F_{\epsilon} \). In fact, by (3.9), (3.10), (3.11), (3.12) and (3.14), if \( s \in (0, 1/2) \) we have
\[
F_{\epsilon}(u_{\lambda, \epsilon}) = \int_{\Omega} \int_{\Omega} \frac{|u_{\lambda, \epsilon}(x) - u_{\lambda, \epsilon}(y)|^2}{|x - y|^{n+2s}} \, dx \, dy + \frac{1}{\epsilon^2 s} \int_{\Omega} W(u_{\lambda, \epsilon}) \, dx \\
\leq 2k \omega_{n-1} \rho^{n-1} \left( \frac{2^2 - 2s}{1 - s} \epsilon^{1-2s} + \frac{2^2 - 2s}{s} \epsilon^{1-2s} \right) \\
+ \epsilon^{1-2s} \frac{2^2 - 2s}{1 - s} k \omega_{n-1} \rho^{n-1} + \frac{k M}{\epsilon^2 s} 2 \epsilon \omega_{n-1} \rho^{n-1} \\
+ \frac{2^{1-2s} N k^2}{s(1 - 2s)} \epsilon^{1-2s} (2^{1-2s} - 1) \\
\leq k \left( \frac{2^3 - 2s}{1 - s} + \frac{2^3 - 2s}{s} + \frac{2^2 - 2s}{1 - s} + 2M \right) \omega_{n-1} \rho^{n-1} \\
+ \frac{2^{1-2s} N k^2}{s(1 - 2s)} (2^{1-2s} - 1); \tag{3.15}
\]

if \( s = 1/2 \) we have
\[
F_{\epsilon}(u_{\lambda, \epsilon}) = \frac{1}{|\log \epsilon|} \int_{\Omega} \int_{\Omega} \frac{|u_{\lambda, \epsilon}(x) - u_{\lambda, \epsilon}(y)|^2}{|x - y|^{n+1}} \, dx \, dy + \frac{1}{|\log \epsilon|} \int_{\Omega} W(u_{\lambda, \epsilon}) \, dx \\
\leq \left( \frac{20k}{|\log \epsilon|} + \frac{2k M}{|\log \epsilon|} \right) \omega_{n-1} \rho^{n-1} + \frac{1}{s|\log \epsilon|} N k^2 \log 2 \\
\leq k(20 + 2M) \omega_{n-1} \rho^{n-1} + \frac{1}{s} N k^2 \log 2; \tag{3.16}
\]
Lemma 3.6. For every \( \epsilon > 0 \) and \( k \in \mathbb{N} \) the set \( S^k_\epsilon \) verifies the following properties:

(a) \( S^k_\epsilon \) is a compact subset of \( H^s(\Omega) \);
(b) \( S^k_\epsilon = -S^k_\epsilon \);
(c) for all \( k \in \mathbb{N} \) there exists \( \epsilon_k > 0 \) such that \( 0 \notin S^k_\epsilon \) \( \forall \epsilon \in (0, \epsilon_k) \);
(d) for all \( k \in \mathbb{N} \) and \( \forall \epsilon > 0 \) such that \( 0 \notin S^k_\epsilon \), it results \( \text{gen}(S^k_\epsilon) \geq k + 1 \).

Proof. The points (b), (c) and (d) are proved in [10]. For (a) we use [10, Lemma 2.8] and the continuous embedding of \( H^1(\Omega) \) in \( H^s(\Omega) \) for all \( s \in (0, 1) \), see Proposition 2.1.

Before proving the main theorem of this work, we point out a useful property of \( F_\epsilon \).

Lemma 3.7. The functionals \( W_1 \), \( W_2 \), \( W_3 \) satisfy the (PS)-condition.

Proof. We will prove the lemma for \( s \in (1/2, 1) \) being the other cases analogue. If \( W \) is quadratic, in particular there exist \( \alpha, \beta > 0 \) such that

\[
W(u) \geq \alpha u^2 + \beta \quad \forall u \in \mathbb{R}.
\]

(3.18)

Since \( (F_\epsilon(u_n)) \) is bounded, \( (3.18) \) implies that \( \|u_n\|_{H^s(\Omega)} \) is bounded, hence \( u_n \to u \) in \( H^s(\Omega) \), \( u_n \to u \) in \( L^q, \forall q \in [1, 2^* = \frac{2n}{n-2s}) \) from Theorem 2.3, therefore \( u_n \to u \) a.e. \( x \in \Omega \).

We claim that \( u \) is a critical point of \( F_\epsilon \). In fact for all \( v \in H^s(\Omega) \),

\[
F'_\epsilon(u)(v) = \epsilon^{2s-1} \int_\Omega \int_\Omega \frac{u(x) - u(y)}{|x-y|^{n+2s}} (v(x) - v(y)) \, dx \, dy + \frac{1}{\epsilon} \int_\Omega W'(u) v \, dx
\]

\[
= \epsilon^{2s-1} \lim_{n \to \infty} \int_\Omega \int_\Omega \frac{u_n(x) - u_n(y)}{|x-y|^{n+2s}} (v(x) - v(y)) \, dx \, dy + \frac{1}{\epsilon} \lim_{n \to \infty} \int_\Omega W'(u_n) v \, dx
\]

(3.19)

since \( u_n \to u \) in \( H^s(\Omega) \), \( u_n \to u \) in \( L^2(\Omega) \) and, by hypothesis, \( F'_\epsilon(u_n) \to 0 \).
This implies that \( F'(u_n)(u_n - u) + F'(u)(u_n - u) \to 0 \), but on the other hand
\[
F'(u_n)(u_n - u) + F'(u)(u_n - u) = e^{2s-1} \int_\Omega \int_\Omega \frac{u_n(x) - u_n(y)}{|x-y|^{n+2s}} (u_n(x) - u(x) - u_n(y) + u(y)) \, dx \, dy
\]
\[
- e^{2s-1} \int_\Omega \int_\Omega \frac{u(x) - u(y)}{|x-y|^{n+2s}} (u_n(x) - u(x) - u_n(y) + u(y)) \, dx \, dy
\]
\[
+ \frac{1}{\epsilon} \int_\Omega [W'(u_n) - W'(u)(u_n - u)] \, dx
\] (3.20)
and the second term on the right hand side approaches 0. In particular we obtain
\[
\int_\Omega \int_\Omega \frac{|u_n(x) - u_n(y)|^2}{|x-y|^{n+2s}} \, dx \, dy \to \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}} \, dx \, dy.
\]
Hence \( ||u_n||_{H^s(\Omega)} \to ||u||_{H^s(\Omega)} \) and since \( u_n \to u \) in \( H^s(\Omega) \), we have the result. \( \square \)

We are now able to prove our main result.

**Proof of Theorem 3.1.** As usual we prove the theorem only for \( s \in (1/2, 1) \). Consider \( W \in C^2(\mathbb{R}; \mathbb{R}^+) \) another even function, which satisfies the following properties:
\[
W = W \text{ for } t \in [-1, 1]; \quad tW'(t) > 0 \text{ for } |t| > 1,
\]
and the asymptotic behaviour guaranteeing that
\[
F_\epsilon(u) = \frac{e^{2s-1}}{2} \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}} \, dx \, dy + \frac{1}{\epsilon} \int_\Omega \mathcal{W}(u) \, dx
\]
is a \( C^2 \)-functional satisfying the (PS)-condition.

We prove now that for every critical point \( \bar{\eta} \in H^s(\Omega) \) which is a critical point for the functional \( F_\epsilon \), it results \( |\bar{\eta}(x)| \leq 1 \) for all \( x \in \Omega \), and so \( \bar{\eta} \) is a critical point for the functional \( F_\epsilon \) too: indeed we have that for all \( v \in H^s(\Omega) \),
\[
e^{2s-1} \int_\Omega \int_\Omega \frac{\bar{\eta}(x) - \bar{\eta}(y)}{|x-y|^{n+2s}} (v(x) - v(y)) \, dx \, dy + \frac{1}{\epsilon} \int_\Omega \mathcal{W}'(\bar{\eta}) v \, dx = 0.
\]
In particular, if we set \( \hat{\eta} = \max\{\min\{\bar{\eta}, 1\}, -1\} \), by choosing \( v = \bar{\eta} - \hat{\eta} \),
\[
e^{2s-1} \int_\Omega \int_\Omega \frac{\bar{\eta}(x) - \bar{\eta}(y)}{|x-y|^{n+2s}} (\bar{\eta}(x) - \hat{\eta}(x) - \bar{\eta}(y) + \hat{\eta}(y)) \, dx \, dy
\]
\[
+ \frac{1}{\epsilon} \int_\Omega \mathcal{W}'(\bar{\eta})(\bar{\eta} - \hat{\eta}) \, dx = 0
\]
with
\[
\int_\Omega \int_\Omega \frac{\bar{\eta}(x) - \bar{\eta}(y)}{|x-y|^{n+2s}} (\bar{\eta}(x) - \hat{\eta}(x) - \bar{\eta}(y) + \hat{\eta}(y)) \, dx \, dy
\]
\[
= \int_\Omega \int_\Omega \frac{|\bar{\eta}(x) - \bar{\eta}(y)|^2}{|x-y|^{n+2s}} \, dx \, dy \geq 0
\] (3.21)
and
\[
\int_\Omega \mathcal{W}'(\bar{\eta})(\bar{\eta} - \hat{\eta}) \, dx > 0 \quad \text{if } \bar{\eta} - \hat{\eta} \neq 0 \quad \text{in } \Omega
\]
since \( t\mathcal{W}'(t) > 0 \) for \( |t| > 1 \). It follows that \( \bar{\eta} = \hat{\eta} \), i.e., \( |\bar{\eta}(x)| \leq 1 \) for almost every \( x \in \Omega \).
Let \( \epsilon_k > 0 \) be such that \( \epsilon_k < \frac{1}{c_k} W(0) |\Omega| \), where \( c_k \) is the constant introduced in Lemma 3.5. Then, for every \( \epsilon \in (0, \epsilon_k) \) we can apply Theorem 2.5 to the functional \( \mathcal{F}_\epsilon \) with \( \tau_1 < 0 \) and \( c_2 = c_k \), because \( \mathcal{F}_\epsilon(0) = \frac{1}{\epsilon} W(0) |\Omega| > c_k \) for all \( \epsilon \in (0, \epsilon_k) \). In this way we can prove that for every \( \epsilon \in (0, \epsilon_k) \), \( \mathcal{F}_\epsilon \) has at least \((k+1)\) pairs \((-u_{0,\epsilon}, u_{0,\epsilon}), \ldots, (-u_{k,\epsilon}, u_{k,\epsilon})\) of critical points with \( \mathcal{F}_\epsilon(u_{i,\epsilon}) \leq c_k \) for all \( i = 0, 1, \ldots, k \). In fact \( \text{gen}(\mathcal{F}_\epsilon^{c_k}) = \text{gen}(\emptyset) = 0 \), while \( \text{gen}(\mathcal{F}_\epsilon^{c_k}) \geq k+1 \) because \( S_k^c \subseteq \mathcal{F}_\epsilon^{c_k} \subseteq H^s(\Omega) \setminus \{0\} \).

Note that these \((k+1)\) pairs of critical points include also the one implied by the minimizers \( \pm 1 \); so we can assume that \((-u_{0,\epsilon}, u_{0,\epsilon}) = (-1, +1)\).

On the contrary, the other solutions are not minimizers for the functional \( \mathcal{F}_\epsilon \) if \( \Omega \) is a connected domain. Indeed it results
\[
\mathcal{F}_\epsilon(u_{i,\epsilon}) > 0 \quad \forall \epsilon \in (0, \epsilon_k) \text{ and } i = 0, 1, \ldots, k
\]
because if \( F_\epsilon(u_{i,\epsilon}) = \mathcal{F}_\epsilon(u_{i,\epsilon}) = 0 \), then we should have
\[
\int_{\Omega} \int_{\Omega} \frac{|u_{i,\epsilon}(x) - u_{i,\epsilon}(y)|^2}{|x - y|^{n+2s}} \, dx \, dy = 0 \quad \text{and} \quad \mathcal{W}(u_{i,\epsilon}) \equiv 0 \quad \text{in } \Omega
\]
and so \( u_{i,\epsilon} \) should be a constant function with value \(+1\) or \(-1\).

Moreover let us remark that for all \( \epsilon \in (0, \epsilon_k) \) and \( i = 1, \ldots, k \) we have
\[
F_\epsilon(u_{i,\epsilon}) \geq \min \{ \mathcal{F}_\epsilon(u) : u \in H^s(\Omega), \int_{\Omega} u \, dx = 0 \}.
\]
(3.22)

In fact, we assume that
\[
\min \{ \mathcal{F}_\epsilon(u) : u \in H^s(\Omega), \int_{\Omega} u \, dx = 0 \} > 0,
\]
only otherwise (3.22) would be obvious. Then, for every \( \tau_1 > 0 \) such that
\[
\tau_1 < \min \{ \mathcal{F}_\epsilon(u) : u \in H^s(\Omega), \int_{\Omega} u \, dx = 0 \},
\]
we would have clearly \( \text{gen}(\mathcal{F}_\epsilon^{-\tau_1}) = 1 \) because below \( c_1 \) the mean is non zero and we can use it as odd function into \( \mathbb{R}^1 \) in the genus definition, see Definition 2.4 thus, if (3.22) were false, the solutions would belong to a set of genus one, in contradiction with their construction in Theorem 2.5.

Now, to prove (3.1), let us replace the function \( \mathcal{W} \) appearing in the definition of functional \( \mathcal{F}_\epsilon \) by a sequence of functions \( (\mathcal{W}_j)_j \) and denote by \( (\mathcal{F}_\epsilon^j)_j \) the corresponding sequence of new functionals. Assume moreover that the functions \( \mathcal{W}_j \) satisfy the same properties as \( \mathcal{W} \) for all \( j \in \mathbb{N} \) and that
\[
\lim_{j \to \infty} \mathcal{W}_j(t) = +\infty \quad \text{for } |t| > 1.
\]
(3.23)

Then property (3.22) holds for the higher critical values of the functional \( \mathcal{F}_\epsilon^j \) for all \( j \in \mathbb{N} \) and so (3.1) follows for \( j \) large enough, taking into account that
\[
\lim_{j \to \infty} \min \{ \mathcal{F}_\epsilon^j(u) : u \in H^s(\Omega), \int_{\Omega} u \, dx = 0 \}
\]
\[
= \min \{ F_\epsilon(u) : u \in H^s(\Omega), |u(x)| \leq 1 \quad \forall x \in \Omega, \int_{\Omega} u \, dx = 0 \}
\]
(3.23)
because of (3.23). \( \square \)
References


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