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MULTIPLICITY OF CRITICAL POINTS FOR THE FRACTIONAL ALLEN-CAHN ENERGY

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Abstract. In this article we study the fractional analogue of the Allen-Cahn energy in bounded domains. We show that it admits a number of critical points which approaches infinity as the perturbation parameter tends to zero.

1. Introduction

The problems involving fractional operators attracted great attention during the previous years. Indeed these problems appear in areas such as optimization, finance, crystal dislocation, minimal surfaces, water waves, fractional diffusion; see for example [8, 6, 3, 4, 7, 19, 18]). In particular, from a probabilistic point of view, the fractional Laplacian is the infinitesimal generator of a Lévy process, see e.g. [2].

In this article we present some existence and multiplicity results for critical points of functionals of the form

$$F_\epsilon(u) = \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy + \frac{1}{\epsilon^{2s}}\int_\Omega W(u)\,dx, \quad \text{if } s \in (0,1/2), \tag{1.1}$$

$$F_\epsilon(u) = \frac{1}{|\log\epsilon|}\int_\Omega \int_\Omega \frac{|u(x)-u(y)|^2}{|x-y|^{n+1}}\,dx\,dy + \frac{1}{|\epsilon\log\epsilon|}\int_\Omega W(u)\,dx, \quad \text{if } s = 1/2, \tag{1.2}$$

$$F_\epsilon(u) = \frac{\epsilon^{2s-1}}{2}\int_\Omega \int_\Omega \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy + \frac{1}{\epsilon}\int_\Omega W(u)\,dx, \quad \text{if } s \in (1/2,1), \tag{1.3}$$

where Ω is a smooth bounded domain of $\mathbb{R}^n$, $u \in H^s(\Omega;\mathbb{R})$, $W \in C^2(\mathbb{R};\mathbb{R}^+)$ is the well known double well potential (see Section 2), and $\epsilon \in \mathbb{R}^+$.

F_ϵ is the fractional energy of the Allen-Cahn equation. It is the fractional counterpart of the functionals studied by Modica-Mortola in [14, 15] where they proved the Γ-convergence of the energy to De Giorgi's perimeter. In the same way, functionals (1.1), (1.2), (1.3) have been also considered by Valdinoci-Savin in [17], where it is discussed their Γ-convergence.

Moreover, as proved in [13] for the functional

$$\int_\Omega [\epsilon|Du|^2 + \epsilon^{-1}(u^2-1)^2]\,dx,$$

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we expect that the solutions have interesting geometric properties related to the interface minimality.

Some authors investigated multiplicity results of nontrivial solution for

$$\begin{gathered} \epsilon^{2s}(-\Delta)^s u + u = h(u) \quad \text{in } \Omega \\ u > 0 \\ u = 0 \quad \text{on } \partial\Omega \end{gathered} \tag{1.4}$$

where Ω is a bounded domain in $\mathbb{R}^n$, $n > 2s$, and $h(u)$ has a subcritical growth (see [12]), or for

$$\begin{gathered} \epsilon^{2s}(-\Delta)^s u + V(z)u = f(u) \quad \text{in } \mathbb{R}^n,\ n > 2s \\ u \in H^s(\mathbb{R}^n) \\ u(z) > 0 \quad z \in \mathbb{R}^n \end{gathered} \tag{1.5}$$

where the potential $V : \mathbb{R}^n \to \mathbb{R}$ and the nonlinearity $f : \mathbb{R} \to \mathbb{R}$ satisfy suitable assumptions (see [11]).

Then Cabré and Sire in [5] studied the equation

$$(-\Delta)^s u + G'(u) = 0 \quad \text{in } \mathbb{R}^n$$

where G denotes the potential associated to a nonlinearity f, and they proved existence, uniqueness and qualitative properties of solutions.

Indeed, Passaseo in [16] studied the analogue of our functional, with the classical Laplacian instead of the fractional one, i.e.,

$$f_\epsilon(u) = \epsilon \int_\Omega |Du|^2\,dx + \frac{1}{\epsilon}\int_\Omega G(u)\,dx \tag{1.6}$$

where Ω is a bounded domain of $\mathbb{R}^n$, $u \in H^{1,2}(\Omega;\mathbb{R})$, $G \in C^2(\mathbb{R};\mathbb{R}^+)$ is a nonnegative function having exactly two zeros, α and β, and ϵ is a positive parameter: he proved that the number of critical points for f_ϵ goes to ∞ as $\epsilon \to 0$.

Passaseo was motivated by De Giorgi's idea, contained in [9], i.e. if $u_\epsilon \to u_0$ in $L^1(\Omega)$ as $\epsilon \to 0$ and $\lim_{\epsilon\to 0} f_\epsilon(u_\epsilon) < \infty$, then the function $U_\epsilon(t)$, defined as steepest descent curves for f_ϵ starting from u_ϵ, converge to a curve $U_0(t)$ in $L^1(\Omega)$ such that $U_0(t)$ is a function with values in $\{\alpha,\beta\}$ for every $t \ge 0$ and the interface between the sets $E_t = \{x \in \Omega : U_0(t)(x) = \alpha\}$ and $\Omega \setminus E_t$ moves by mean curvature. As a consequence the critical points u_ϵ of f_ϵ which satisfy

$$\liminf_{\epsilon\to\infty} f_\epsilon(u_\epsilon) < +\infty \tag{1.7}$$

converge in $L^1(\Omega)$ to a function u_0 taking values in $\{\alpha,\beta\}$. De Giorgi considered also the problem of existence and multiplicity for nontrivial critical points of f_ϵ with the property (1.7), and Passaseo's critical points verify this property and

$$\liminf_{\epsilon\to\infty} f_\epsilon(u_\epsilon) > 0,$$

so he can say that u_0 is nontrivial.

In this article we want to extend Passaseo's results by replacing the function G in (1.6) with the double well potential W, and Passaseo's functional f_ϵ with its fractional counterpart.

The article is organized as follows: in Section 2 we give some preliminaries definitions and results. In Section 3, we define suitable functions and sets, then most of the work is dedicated to prove nonlocal estimates needful to obtain the bound from above of F_ϵ, (see Lemma 3.5), and the (PS)-condition, Lemma 3.7. In fact in particular for the first of these results, we had to split the domain in two types of regions and estimate F_ϵ in the three possible interactions.

Finally, after recalling a technical result, Lemma 3.6, we can apply a classical Krasnoselsii's genus tool to show the existence and multiplicity results for solutions.

Hence, knowing that minimizers of F_ϵ Γ-converge to minimizers of the area functional, we hope that also min-max solutions can pass to the limit as $\epsilon \to 0$ in a suitable sense, producing critical points of positive index for local, if $s \in [1/2, 1)$, or nonlocal, if $s \in (0, 1/2)$, area functional.

2. Notation and preliminary results

In this section we introduce the framework that we will be used throughout this article.

Let Ω be a bounded domain of $\mathbb{R}^n$, denote by $|\Omega|$ its Lebesgue measure and consider W the double well potential, that is an even function such that

$$\begin{gathered} W : \mathbb{R} \to [0, +\infty) \quad W \in C^2(\mathbb{R}; \mathbb{R}^+) \quad W(\pm 1) = 0 \\ W > 0 \text{ in } (-1, 1) \quad W'(\pm 1) = 0 \quad W''(\pm 1) > 0. \end{gathered} \tag{2.1}$$

Now we fix the fractional exponent $s \in (0, 1)$. For any $p \in [1, +\infty)$, we define

$$W^{s,p}(\Omega) := \Big\{ u \in L^p(\Omega) : \frac{|u(x) - u(y)|}{|x - y|^{s+n/p}} \in L^p(\Omega \times \Omega) \Big\};$$

i.e. an intermediary Banach space between $L^p(\Omega)$ and $W^{1,p}(\Omega)$, endowed with the natural norm

$$\|u\|_{W^{s,p}(\Omega)} := \Big(\int_\Omega |u|^p \, dx + \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} \, dx \, dy \Big)^{1/p}.$$

If $p = 2$ we define $W^{s,2}(\Omega) = H^s(\Omega)$ and it is a Hilbert space. Now let $\mathscr{S}'(\mathbb{R}^n)$ be the set of all tempered distributions, that is the topological dual of $\mathscr{S}(\mathbb{R}^n)$. As usual, for any $\varphi \in \mathscr{S}(\mathbb{R}^n)$, we denote by

$$\mathscr{F}\varphi(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\xi \cdot x} \varphi(x) \, dx$$

the Fourier transform of φ and we recall that one can extend $\mathscr{F}$ from $\mathscr{S}(\mathbb{R}^n)$ to $\mathscr{S}'(\mathbb{R}^n)$. At this point we can define, for any $u \in \mathscr{S}(\mathbb{R}^n)$ and $s \in (0, 1)$, the fractional Laplacian operator as

$$(-\Delta)^s u(x) = C(n, s) P.V. \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+2s}} \, dy.$$

Here P.V. stands for the Cauchy principal value and $C(n, s)$ is a normalizing constant (see [10] for more details). It is easy to prove that this definition is equivalent to the following two:

$$(-\Delta)^s u(x) = -\frac{1}{2} C(n, s) \int_{\mathbb{R}^n} \frac{u(x + y) + u(x - y) - 2u(x)}{|y|^{n+2s}} \, dy \quad \forall x \in \mathbb{R}^n,$$

and

$$(-\Delta)^s u(x) = \mathscr{F}^{-1}(|\xi|^{2s} (\mathscr{F} u)) \quad \forall \xi \in \mathbb{R}^n.$$

Now we recall some embedding's results for the fractional spaces:

Proposition 2.1 ([10]). *Let $p \in [1,+\infty)$ and $0 < s \le s' \le 1$. Let Ω be an open set of $\mathbb{R}^n$ and $u : \Omega \to \mathbb{R}$ be a measurable function. Then $W^{s',p}(\Omega)$ is continuously embedded in $W^{s,p}(\Omega)$, denoted by $W^{s',p}(\Omega) \hookrightarrow W^{s,p}(\Omega)$, and the following inequality holds*

$$\|u\|_{W^{s,p}(\Omega)} \le C\|u\|_{W^{s',p}(\Omega)}$$

for some suitable positive constant $C = C(n,s,p) \ge 1$.

Moreover, if also Ω is an open set of $\mathbb{R}^n$ of class $C^{0,1}$ with bounded boundary, then $W^{1,p}(\Omega) \hookrightarrow W^{s,p}(\Omega)$ and we have

$$\|u\|_{W^{s,p}(\Omega)} \le C\|u\|_{W^{1,p}(\Omega)}$$

for some suitable positive constant $C = C(n,s,p) \ge 1$.

Definition 2.2 ([10]). For any $s \in (0,1)$ and any $p \in [1,+\infty)$, we say that an open set $\Omega \subseteq \mathbb{R}^n$ is an extension domain for $W^{s,p}$ if there exists a positive constant $C = C(n,p,s,\Omega)$ such that: for every function $u \in W^{s,p}(\Omega)$ there exists $\tilde{u} \in W^{s,p}(\mathbb{R}^n)$ with $\tilde{u}(x) = u(x)$ for all $x \in \Omega$ and $\|\tilde{u}\|_{W^{s,p}(\mathbb{R}^n)} \le C\|u\|_{W^{s,p}(\Omega)}$.

Theorem 2.3 ([10]). *Let $s \in (0,1)$ and $p \in [1,+\infty)$ be such that $sp < n$. Let $q \in [1,p^*)$, where $p^* = p^*(n,s) = np/(n-sp)$ is the so-called "fractional critical exponent". Let $\Omega \subseteq \mathbb{R}^n$ be a bounded extension domain for $W^{s,p}$ and $\mathscr{I}$ be a bounded subset of $L^p(\Omega)$. Suppose that*

$$\sup_{f\in\mathscr{I}} \int_\Omega \int_\Omega \frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}}\,dx\,dy < \infty.$$

Then $\mathscr{I}$ is pre-compact in $L^q(\Omega)$.

We recall also the notion of Krasnoselskii's genus, useful in the sequel.

Definition 2.4 ([1]). Let H be a Hilbert space and E be a closed subset of $H \setminus \{0\}$, symmetric with respect to 0 (i.e. $E = -E$).

We call genus of E in H, indicated with $\mathrm{gen}_H(E)$, the least integer m such that there exists $\phi \in C(H;\mathbb{R})$ such that ϕ is odd and $\phi(x) \ne 0$ for all $x \in E$.

We set $\mathrm{gen}_H(E) = +\infty$ if there are no integer with the above property and $\mathrm{gen}_H(\emptyset) = 0$.

It is well known that $\mathrm{gen}_H(S^k) = k+1$ if S^k is a k-dimensional sphere of H with centre in zero.

Finally we recall a well known result:

Theorem 2.5 ([1]). *Let H be a Hilbert space and $f : H \to \mathbb{R}$ be an even C^2-functional satisfying the following Palais-Smale condition: given a sequence $(u_i)_i$ in H such that the sequence $(f(u_i))_i$ is bounded and $f'(u_i) \to 0$, $(u_i)_i$ is relatively compact in H.*

Set $f^c = \{u \in H : f(u) \le c\}$ for all $c \in \mathbb{R}$. Then, for all c_1, $c_2 \in \mathbb{R}$, such that $c_1 \le c_2 < f(0)$, we have

$$\mathrm{gen}_H(f^{c_2}) \le \mathrm{gen}_H(f^{c_1}) + \#\{(-u_i,u_i) : c_1 \le f(u_i) \le c_2,\ f'(u_i) = 0\}, \tag{2.2}$$

where, if A is a set, we indicate with $\#A$ the cardinality of A.

For the rest of this article, we consider $H^s(\Omega)$ as Hilbert space and we shall write simply $\mathrm{gen}(E)$ instead of $\mathrm{gen}_{H^s(\Omega)}(E)$; then we refer to the Palais-Smale condition with the symbol (PS)-condition.

3. Multiplicity of critical points

Let us state the fundamental result of the paper.

Theorem 3.1. *Let Ω be a smooth bounded domain of $\mathbb{R}^n$ and W be a function satisfying* (2.1). *Then there exist two sequences of positive numbers $(\epsilon_k)_k$, $(c_k)_k$ such that for every $\epsilon \in (0, \epsilon_k)$, the functional F_ϵ has at least k pairs*

$$(-u_{1,\epsilon}, u_{1,\epsilon}), \dots, (-u_{k,\epsilon}, u_{k,\epsilon})$$

of critical points, all of them different from the constant pair $(-1, 1)$ satisfying

$$\begin{gathered} -1 \le u_{i,\epsilon}(x) \le 1 \quad \forall x \in \Omega,\ \forall \epsilon \in (0, \epsilon_k),\ i = 1, \dots k; \\ F_\epsilon(u_{i,\epsilon}) \le c_k \quad \forall \epsilon \in (0, \epsilon_k),\ i = 1, \dots, k. \end{gathered}$$

Moreover, for all $\epsilon \in (0, \epsilon_k)$ and all $i = 1, \dots, k$ we have

$$F_\epsilon(u) \ge \min \big\{ F_\epsilon(u) : u \in H^s(\Omega), -1 \le u(x) \le 1 \text{ for } x \in \Omega, \int_\Omega u\, dx = 0 \big\}. \tag{3.1}$$

Remark 3.2. The constant function $u \equiv 0$ is obviously a critical point for the functional F_ϵ for every $\epsilon > 0$ but it is not included among the ones given by Theorem 3.1. Instead if $s \in (1/2, 1)$, but for the other cases it is similar,

$$F_\epsilon(0) = \frac{1}{\epsilon} W(0)|\Omega| \to +\infty \quad \text{as } \epsilon \to 0.$$

Moreover, since $\inf\{W(t) : W'(t) = 0, -1 < t < 1\} > 0$, one can say that the critical points given by Theorem 3.1 are not constant functions. In fact, if $u_\epsilon = c_\epsilon$ is a constant critical point for F_ϵ (distinct from -1 and 1), it must be $W'(c_\epsilon) = 0$ and $-1 < c_\epsilon < 1$; therefore

$$W(c_\epsilon) \ge \inf\{W(t) : W'(t) = 0, -1 < t < 1\} > 0 \tag{3.2}$$

and so, for example by considering the functional related to $s \in (1/2, 1)$, but the other cases are similar,

$$F_\epsilon(c_\epsilon) = \frac{1}{\epsilon} W(c_\epsilon)|\Omega| \to +\infty \quad \text{as } \epsilon \to 0, \tag{3.3}$$

in contradiction with $F_\epsilon(c_\epsilon) \le c_k$ for all $\epsilon \in (0, \epsilon_k)$.

Notice that for all $\epsilon > 0$,

$$\min \big\{ F_\epsilon(u) : u \in H^s(\Omega), -1 \le u(x) \le 1 \quad \forall x \in \Omega, \int_\Omega u\, dx = 0 \big\} > 0 \tag{3.4}$$

if we assume, without loss of generality, that Ω is a connected domain.

Let $\bar{u}$ be a minimizing function; if we assume $F_\epsilon(\bar{u}) = 0$, then

$$\int_\Omega \int_\Omega \frac{|\bar{u}(x) - \bar{u}(y)|^2}{|x - y|^{n+2s}}\, dx\, dy \equiv 0$$

and $W(\bar{u}) \equiv 0$. Therefore we must have $\bar{u} \equiv 0$ in contradiction with $W(0) > 0$.

Definition 3.3. Let k be a fixed positive integer; for every $\lambda = (\lambda^{(0)}, \dots, \lambda^{(k)}) \in \mathbb{R}^{k+1}$ define the function $\varphi_\lambda : \mathbb{R} \to \mathbb{R}$ by

$$\varphi_\lambda(t) = \sum_{m=0}^{k} \lambda^{(m)} \cos(mt).$$

For every $\lambda \in \mathbb{R}^{k+1}$ with $|\lambda|_{\mathbb{R}^{k+1}} = 1$ and $\epsilon > 0$, let $L_\epsilon(\varphi_\lambda) : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$L_\epsilon(\varphi_\lambda)(t) = \frac{1}{2\epsilon} \int_{t-\epsilon}^{t+\epsilon} \frac{\varphi_\lambda(\tau)}{|\varphi_\lambda(\tau)|}\, d\tau;$$

notice that $L_\epsilon(\varphi_\lambda)$ is well defined because φ_λ has only isolated zeros $\forall \lambda \in \mathbb{R}^{k+1}$ with $|\lambda|_{\mathbb{R}^{k+1}} = 1$.

For $x = (x_1, \cdots, x_n) \in \Omega \subset \mathbb{R}^n$ we consider the projection onto the first component, $P_1(x) = x_1$, and the set

$$S^k_\epsilon = \{L_\epsilon(\varphi_\lambda) \circ P_1 : \lambda \in \mathbb{R}^{k+1}, |\lambda|_{\mathbb{R}^{k+1}} = 1\}.$$

Lemma 3.4. *Let us fix $a, b \in \mathbb{R}$ with $a < b$ and set*

$$\chi(\varphi_\lambda) = \#\{t \in [a, b] : \varphi_\lambda(t) = 0\}$$

for $\lambda \in \mathbb{R}^{k+1}$ with $|\lambda|_{\mathbb{R}^{k+1}} = 1$. Then for every $k \in \mathbb{N}$ we have

$$\sup\{\chi(\varphi_\lambda) : \lambda \in \mathbb{R}^{k+1}, |\lambda|_{\mathbb{R}^{k+1}} = 1\} < +\infty.$$

For a proof of the above lemma, see [16].

Lemma 3.5. *Let Ω be a bounded domain of $\mathbb{R}^n$ and W be a function satisfying (2.1). Then, for every $k \in \mathbb{N}$ there exists a positive constant c_k such that*

$$\max F_\epsilon(f) \le c_k \quad \forall f \in S^k_\epsilon. \tag{3.5}$$

Proof. Let $u_{\lambda,\epsilon} = L_\epsilon(\varphi_\lambda) \circ P_1 \in S^k_\epsilon$ and set

$$a = \inf P_1(\Omega), \quad b = \sup P_1(\Omega),$$
$$Z_\lambda = \{t \in [a, b] : \varphi_\lambda(t) = 0\},$$
$$Z_{\lambda,\epsilon} = \{t \in \mathbb{R} : \mathrm{dist}(t, Z_\lambda) < \epsilon\}.$$

Note that

(i) If $P_1(x) \notin Z_{\lambda,\epsilon}$, then $|u_{\lambda,\epsilon}(x)| = 1$ and $Du_{\lambda,\epsilon}(x) = 0$, while

(ii) if $P_1(x) \in Z_{\lambda,\epsilon}$, then $|u_{\lambda,\epsilon}(x)| \le 1$ and $|Du_{\lambda,\epsilon}(x)| \le \frac{1}{\epsilon}$.

Since Ω is bounded, we can suppose it is included in a cube Q of side large enough. We will denote with $Y_{\lambda,\epsilon} = Z^C_{\lambda,\epsilon}$ the complement to $Z_{\lambda,\epsilon}$, then we have to distinguish three cases:

(a) if $x \in Y_{\lambda,\epsilon}$ and $y \in Y_{\lambda,\epsilon}$;

(b) if $x \in Z_{\lambda,\epsilon}$ and $y \in Y_{\lambda,\epsilon}$;

(c) if $x \in Z_{\lambda,\epsilon}$ and $y \in Z_{\lambda,\epsilon}$.

We set $k = \max\{\chi(\varphi_\lambda) : \lambda \in \mathbb{R}^{k+1}, |\lambda|_{\mathbb{R}^{k+1}} = 1\}$, then

$$Z_{\lambda,\epsilon} = \sum_{i=1}^k Z^i_{\lambda,\epsilon} \quad \text{and} \quad Y_{\lambda,\epsilon} = \sum_{i=1}^k Y^i_{\lambda,\epsilon}.$$

Now we call $\check{Z}_{\lambda,\epsilon} = P_1^{-1}(Z_{\lambda,\epsilon}) \cap \Omega$, $\check{Y}_{\lambda,\epsilon} = P_1^{-1}(Y_{\lambda,\epsilon}) \cap \Omega$ and we observe that

$$\int_{\check{Y}_{\lambda,\epsilon}} W(u_{\lambda,\epsilon})\, dx = 0, \tag{3.6}$$

while if we set $\rho = \sup\{|x| : x \in \Omega\}$, $M = \max\{W(t) : |t| \le 1\}$ and denote by ω_{n-1} the $(n-1)$-dimensional measure of the unit sphere of $\mathbb{R}^{n-1}$, it results

$$\int_{\check{Z}_{\lambda,\epsilon}} W(u_{\lambda,\epsilon})\, dx \le M|\check{Z}_{\lambda,\epsilon}| \le 2\epsilon M \omega_{n-1} \rho^{n-1}. \tag{3.7}$$

At this point it remains to analyze $\int_\Omega\int_\Omega \frac{|u_{\lambda,\epsilon}(x)-u_{\lambda,\epsilon}(y)|^2}{|x-y|^{n+2s}}\,dxdy$. We split it in three cases:

Case (a). We have

$$\int_{\check{Y}_{\lambda,\epsilon}}\int_{\check{Y}_{\lambda,\epsilon}} \frac{|u_{\lambda,\epsilon}(x)-u_{\lambda,\epsilon}(y)|^2}{|x-y|^{n+2s}}\,dxdy = \sum_{\substack{i,j=1\\ i\neq j}}^{k}\int_{\check{Y}^i_{\lambda,\epsilon}}\int_{\check{Y}^j_{\lambda,\epsilon}} \frac{|u_{\lambda,\epsilon}(x)-u_{\lambda,\epsilon}(y)|^2}{|x-y|^{n+2s}}\,dx\,dy \tag{3.8}$$

We denote $Q_-=Q\cap P_1^{-1}(\{x_1<0\})$, $Q_+=Q\cap P_1^{-1}(\{y_1>2\epsilon\})$ and we split Q_- in N strips of width ϵ, with N of order $1/\epsilon$, so we obtain

$$\begin{aligned}
&\sum_{\substack{i,j=1\\ i\neq j}}^{k}\int_{\check{Y}^i_{\lambda,\epsilon}}\int_{\check{Y}^j_{\lambda,\epsilon}} \frac{|u_{\lambda,\epsilon}(x)-u_{\lambda,\epsilon}(y)|^2}{|x-y|^{n+2s}}\,dx\,dy\\
&\leq k^2\int_{Q_-}\int_{Q_+}\frac{4}{|x-y|^{n+2s}}\,dx\,dy\\
&\leq 4Nk^2\int_{-\epsilon}^{-2\epsilon}\int_{-2x_1}^{+\infty} r^{-2s-1}\,dr\,dx_1\\
&= \frac{2}{s}Nk^2\int_{-\epsilon}^{-2\epsilon}(-2x_1)^{-2s}\,dx_1.
\end{aligned}\tag{3.9}$$

Now we distinguish two cases:

(j) if $s\neq 1/2$, we have

$$\frac{2}{s}Nk^2\int_{-\epsilon}^{-2\epsilon}(-2x_1)^{-2s}\,dx_1 = \frac{2^{1-2s}Nk^2}{s(1-2s)}\epsilon^{1-2s}(2^{1-2s}-1); \tag{3.10}$$

(jj) while, if $s=1/2$,

$$\frac{2}{s}Nk^2\int_{-\epsilon}^{-2\epsilon}(-2x_1)^{-2s}\,dx_1 = \frac{2^{1-2s}}{s}Nk^2\log 2. \tag{3.11}$$

Case (b). We note that $\check{Y}^i_{\lambda,\epsilon}\subseteq Q\setminus\check{Z}^i_{\lambda,\epsilon}$, so

$$\begin{aligned}
&\int_{\check{Z}_{\lambda,\epsilon}}\int_{\check{Y}_{\lambda,\epsilon}} \frac{|u_{\lambda,\epsilon}(x)-u_{\lambda,\epsilon}(y)|^2}{|x-y|^{n+2s}}\,dxdy\\
&\leq \sum_{i=1}^{k}\int_{\check{Z}^i_{\lambda,\epsilon}}\int_{Q\setminus\check{Z}^i_{\lambda,\epsilon}} \frac{|u_{\lambda,\epsilon}(x)-u_{\lambda,\epsilon}(y)|^2}{|x-y|^{n+2s}}\,dxdy\\
&\leq 2\omega_{n-1}\rho^{n-1}\epsilon\sum_{i=1}^{k}\sup_{x\in\check{Z}^i_{\lambda,\epsilon}}\int_{Q\setminus\check{Z}^i_{\lambda,\epsilon}}\frac{\min\{1/\epsilon^2|x-y|^2,4\}}{|x-y|^{n+2s}}\,dy\\
&\leq 2k\epsilon\omega_{n-1}\rho^{n-1}\Big(\int_0^{2\epsilon}\frac{1}{\epsilon^2}r^{1-2s}\,dr+\int_{2\epsilon}^{+\infty}4r^{-1-2s}\,dr\Big)\\
&= k\Big(\frac{2}{\epsilon}\frac{r^{2-2s}}{2-2s}\Big|_0^{2\epsilon}+8\epsilon\frac{r^{-2s}}{-2s}\Big|_{2\epsilon}^{+\infty}\Big)\omega_{n-1}\rho^{n-1}\\
&= k\epsilon^{1-2s}\Big(\frac{2^{2-2s}}{1-s}+\frac{2^{2-2s}}{s}\Big)\omega_{n-1}\rho^{n-1}.
\end{aligned}\tag{3.12}$$

Case (c). It results

$$
\begin{aligned}
&\int_{\check{Z}_{\lambda,\epsilon}}\int_{\check{Z}_{\lambda,\epsilon}} \frac{|u_{\lambda,\epsilon}(x)-u_{\lambda,\epsilon}(y)|^2}{|x-y|^{n+2s}}\,dxdy\\
&=\sum_{i=1}^{k}\int_{\check{Z}^i_{\lambda,\epsilon}}\int_{\check{Z}^i_{\lambda,\epsilon}} \frac{|u_{\lambda,\epsilon}(x)-u_{\lambda,\epsilon}(y)|^2}{|x-y|^{n+2s}}\,dxdy\\
&\quad+\sum_{\substack{i,j=1\\ i\neq j}}^{k}\int_{\check{Z}^j_{\lambda,\epsilon}}\int_{\check{Z}^i_{\lambda,\epsilon}} \frac{|u_{\lambda,\epsilon}(x)-u_{\lambda,\epsilon}(y)|^2}{|x-y|^{n+2s}}\,dxdy.
\end{aligned}
\tag{3.13}
$$

Concerning the first term of the right-hand side, we have

$$
\begin{aligned}
&\sum_{i=1}^{k}\int_{\check{Z}^i_{\lambda,\epsilon}}\int_{\check{Z}^i_{\lambda,\epsilon}} \frac{|u_{\lambda,\epsilon}(x)-u_{\lambda,\epsilon}(y)|^2}{|x-y|^{n+2s}}\,dxdy\\
&\le \frac{1}{\epsilon^2}\sum_{i=1}^{k}|\check{Z}^i_{\lambda,\epsilon}|\int_0^{2\epsilon} r^{1-2s}\,dr \le k\omega_{n-1}\rho^{n-1}\frac{2^{2-2s}}{1-s}\epsilon^{1-2s}.
\end{aligned}
\tag{3.14}
$$

The other term is estimated as in Case (b).

So we can obtain the estimates for the functionals F_ϵ. In fact, by (3.9), (3.10), (3.11), (3.12) and (3.14), if $s\in(0,1/2)$ we have

$$
\begin{aligned}
F_\epsilon(u_{\lambda,\epsilon}) &= \int_\Omega\int_\Omega \frac{|u_{\lambda,\epsilon}(x)-u_{\lambda,\epsilon}(y)|^2}{|x-y|^{n+2s}}\,dx\,dy + \frac{1}{\epsilon^{2s}}\int_\Omega W(u_{\lambda,\epsilon})\,dx\\
&\le 2k\omega_{n-1}\rho^{n-1}\Big(\frac{2^{2-2s}}{1-s}\epsilon^{1-2s}+\frac{2^{2-2s}}{s}\epsilon^{1-2s}\Big)\\
&\quad+\epsilon^{1-2s}\frac{2^{2-2s}}{1-s}k\omega_{n-1}\rho^{n-1}+\frac{kM}{\epsilon^{2s}}2\epsilon\omega_{n-1}\rho^{n-1}\\
&\quad+\frac{2^{1-2s}Nk^2}{s(1-2s)}\epsilon^{1-2s}(2^{1-2s}-1)\\
&\le k\Big(\frac{2^{3-2s}}{1-s}+\frac{2^{3-2s}}{s}+\frac{2^{2-2s}}{1-s}+2M\Big)\omega_{n-1}\rho^{n-1}\\
&\quad+\frac{2^{1-2s}Nk^2}{s(1-2s)}(2^{1-2s}-1);
\end{aligned}
\tag{3.15}
$$

if $s=1/2$ we have

$$
\begin{aligned}
F_\epsilon(u_\lambda,\epsilon) &= \frac{1}{|\log\epsilon|}\int_\Omega\int_\Omega \frac{|u_{\lambda,\epsilon}(x)-u_{\lambda,\epsilon}(y)|^2}{|x-y|^{n+1}}\,dx\,dy + \frac{1}{|\epsilon\log\epsilon|}\int_\Omega W(u_{\lambda,\epsilon})\,dx\\
&\le \Big(\frac{20k}{|\log\epsilon|}+\frac{2kM}{|\log\epsilon|}\Big)\omega_{n-1}\rho^{n-1}+\frac{1}{s|\log\epsilon|}Nk^2\log 2\\
&\le k(20+2M)\omega_{n-1}\rho^{n-1}+\frac{1}{s}Nk^2\log 2;
\end{aligned}
\tag{3.16}
$$

and, if $s \in (1/2, 1)$ we get

$$\begin{aligned} F_\epsilon(u_\lambda, \epsilon) &= \frac{\epsilon^{2s-1}}{2} \int_\Omega \int_\Omega \frac{|u_{\lambda,\epsilon}(x) - u_{\lambda,\epsilon}(y)|^2}{|x-y|^{n+2s}}\, dx\, dy + \frac{1}{\epsilon} \int_\Omega W(u_{\lambda,\epsilon})\, dx \\ &\leq k\Big(\frac{2^{2-2s}}{1-s} + \frac{2^{2-2s}}{s} + \frac{2^{1-2s}}{1-s} + 2M\Big)\omega_{n-1}\rho^{n-1} \\ &\quad + \frac{2^{-2s}Nk^2}{s(1-2s)}(2^{1-2s} - 1). \end{aligned} \tag{3.17}$$

□

Now we show a technical lemma, that we will used for proving our main result.

Lemma 3.6. *For every $\epsilon > 0$ and $k \in \mathbb{N}$ the set S_ϵ^k verifies the following properties:*

(a) *S_ϵ^k is a compact subset of $H^s(\Omega)$;*
(b) *$S_\epsilon^k = -S_\epsilon^k$;*
(c) *for all $k \in \mathbb{N}$ there exists $\bar\epsilon_k > 0$ such that $0 \notin S_\epsilon^k \quad \forall \epsilon \in (0, \bar\epsilon_k)$;*
(d) *for all $k \in \mathbb{N}$ and $\forall\, \epsilon > 0$ such that $0 \notin S_\epsilon^k$, it results $gen(S_\epsilon^k) \geq k+1$.*

Proof. The points (b), (c) and (d) are proved in [16]. For (a) we use [16, Lemma 2.8] and the continuous embedding of $H^1(\Omega)$ in $H^s(\Omega)$ for all $s \in (0,1)$, see Proposition 2.1. □

Before proving the main theorem of this work, we point out a useful property of F_ϵ.

Lemma 3.7. *The functionals* (1.1), (1.2), (1.3) *satisfy the (PS)-condition.*

Proof. We will prove the lemma for $s \in (1/2, 1)$ being the other cases analogue. If W is quadratic, in particular there exist α, $\beta > 0$ such that

$$W(u) \geq \alpha u + \beta \quad \forall u \in \mathbb{R}. \tag{3.18}$$

Since $(F_\epsilon(u_n))_n$ is bounded, (3.18) implies that $\|u_n\|_{H^s(\Omega)}$ is bounded, hence $u_n \rightharpoonup u$ in $H^s(\Omega)$, $u_n \to u$ in L^q, $\forall q \in [1, 2^* = \frac{2n}{n-2s})$ from Theorem 2.3, therefore $u_n \to u$ a.e. $x \in \Omega$.

We claim that u is a critical point of F_ϵ. In fact for all $v \in H^s(\Omega)$,

$$\begin{aligned} F'_\epsilon(u)(v) &= \epsilon^{2s-1} \int_\Omega \int_\Omega \frac{u(x)-u(y)}{|x-y|^{n+2s}}(v(x)-v(y))\, dx\, dy \\ &\quad + \frac{1}{\epsilon}\int_\Omega W'(u)v\, dx \\ &= \epsilon^{2s-1} \lim_{n\to\infty} \int_\Omega \int_\Omega \frac{u_n(x)-u_n(y)}{|x-y|^{n+2s}}(v(x)-v(y))\, dx\, dy \\ &\quad + \frac{1}{\epsilon}\lim_{n\to\infty}\int_\Omega W'(u_n)v\, dx \end{aligned} \tag{3.19}$$

since $u_n \rightharpoonup u$ in $H^s(\Omega)$, $u_n \to u$ in $L^2(\Omega)$ and, by hypothesis, $F'_\epsilon(u_n) \to 0$.

This implies that $F'_\epsilon(u_n)(u_n-u)+F'_\epsilon(u)(u_n-u)\to 0$, but on the other hand

$$
\begin{aligned}
&F'_\epsilon(u_n)(u_n-u)+F'_\epsilon(u)(u_n-u)\\
&=\epsilon^{2s-1}\int_\Omega\int_\Omega \frac{u_n(x)-u_n(y)}{|x-y|^{n+2s}}(u_n(x)-u(x)-u_n(y)+u(y))\,dx\,dy\\
&\quad-\epsilon^{2s-1}\int_\Omega\int_\Omega \frac{u(x)-u(y)}{|x-y|^{n+2s}}(u_n(x)-u(x)-u_n(y)+u(y))\,dx\,dy\\
&\quad+\frac{1}{\epsilon}\int_\Omega [W'(u_n)-W'(u)(u_n-u)]\,dx,
\end{aligned}
\tag{3.20}
$$

and the second term on the right hand side appraoches 0. In particular we obtain

$$
\int_\Omega\int_\Omega \frac{|u_n(x)-u_n(y)|^2}{|x-y|^{n+2s}}\,dx\,dy\to\int_\Omega\int_\Omega \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy.
$$

Hence $\|u_n\|_{H^s(\Omega)}\to\|u\|_{H^s(\Omega)}$ and since $u_n\rightharpoonup u$ in $H^s(\Omega)$, we have the result. $\square$

We are now able to prove our main result.

Proof of Theorem 3.1. As usual we prove the theorem only for $s\in(1/2,1)$. Consider $\overline{W}\in C^2(\mathbb{R};\mathbb{R}^+)$ another even function, which satisfies the following properties:

$$
\overline{W}=W\ \forall t\in[-1,1];\quad t\overline{W}'(t)>0 \text{ for } |t|>1,
$$

and the asymptotic behaviour guaranteeing that

$$
\overline{F}_\epsilon(u)=\frac{\epsilon^{2s-1}}{2}\int_\Omega\int_\Omega \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy+\frac{1}{\epsilon}\int_\Omega \overline{W}(u)\,dx
$$

is a C^2-functional satisfying the (PS)-condition.

We prove now that for every critical point $\overline{u}\in H^s(\Omega)$ which is a critical point for the functional $\overline{F}_\epsilon$, it results $|\overline{u}(x)|\le 1$ for all $x\in\Omega$, and so $\overline{u}$ is a critical point for the functional F_ϵ too: indeed we have that for all $v\in H^s(\Omega)$,

$$
\epsilon^{2s-1}\int_\Omega\int_\Omega \frac{\overline{u}(x)-\overline{u}(y)}{|x-y|^{n+2s}}(v(x)-v(y))dxdy+\frac{1}{\epsilon}\int_\Omega \overline{W}'(\overline{u})v\,dx=0.
$$

In particular, if we set $\hat{u}=\max\{\min\{\overline{u},1\},-1\}$, by choosing $v=\overline{u}-\hat{u}$,

$$
\begin{aligned}
&\epsilon^{2s-1}\int_\Omega\int_\Omega \frac{\overline{u}(x)-\overline{u}(y)}{|x-y|^{n+2s}}(\overline{u}(x)-\hat{u}(x)-\overline{u}(y)+\hat{u}(y))\,dx\,dy\\
&+\frac{1}{\epsilon}\int_\Omega \overline{W}'(\overline{u})(\overline{u}-\hat{u})\,dx=0
\end{aligned}
$$

with

$$
\begin{aligned}
&\int_\Omega\int_\Omega \frac{\overline{u}(x)-\overline{u}(y)}{|x-y|^{n+2s}}(\overline{u}(x)-\hat{u}(x)-\overline{u}(y)+\hat{u}(y))\,dx\,dy\\
&=\int_\Omega\int_\Omega \frac{|\overline{u}(x)-\overline{u}(y)|^2}{|x-y|^{n+2s}}\,dx\,dy\ge 0
\end{aligned}
\tag{3.21}
$$

and

$$
\int_\Omega \overline{W}'(\overline{u})(\overline{u}-\hat{u})\,dx>0\quad\text{if } \overline{u}-\hat{u}\not\equiv 0\quad\text{in } \Omega
$$

since $t\overline{W}'(t)>0$ for $|t|>1$. It follows that $\overline{u}=\hat{u}$, i.e., $|\overline{u}(x)|\le 1$ for almost every $x\in\Omega$.

Let $\epsilon_k > 0$ be such that $\epsilon_k < \frac{1}{c_k} W(0)|\Omega|$, where c_k is the constant introduced in Lemma 3.5. Then, for every $\epsilon \in (0, \epsilon_k)$ we can apply Theorem 2.5 to the functional $\overline{F}_\epsilon$ with $\bar{c}_1 < 0$ and $c_2 = c_k$, because $\overline{F}_\epsilon(0) = \frac{1}{\epsilon} W(0)|\Omega| > c_k$ for all $\epsilon \in (0, \epsilon_k)$. In this way we can prove that for every $\epsilon \in (0, \epsilon_k)$, $\overline{F}_\epsilon$ has at least $(k+1)$ pairs $(-u_{0,\epsilon}, u_{0,\epsilon}), \dots, (-u_{k,\epsilon}, u_{k,\epsilon})$ of critical points with $\overline{F}_\epsilon(u_{i,\epsilon}) \le c_k$ for all $i = 0, 1, \dots, k$. In fact $\operatorname{gen}(\overline{F}_\epsilon^{\bar{c}_1}) = \operatorname{gen}(\emptyset) = 0$, while $\operatorname{gen}(\overline{F}_\epsilon^{\bar{c}_k}) \ge \operatorname{gen}(S_\epsilon^k) \ge k+1$ because $S_\epsilon^k \subseteq \overline{F}_\epsilon^{\bar{c}_k} \subseteq H^s(\Omega) \setminus \{0\}$.

Note that these $(k+1)$ pairs of critical points include also the one implied by the minimizers ± 1; so we can assume that $(-u_{0,\epsilon}, u_{0,\epsilon}) = (-1, +1)$.

On the contrary, the other solutions are not minimizers for the functional $\overline{F}_\epsilon$ if Ω is a connected domain. Indeed it results

$$\overline{F}_\epsilon(u_{i,\epsilon}) > 0 \quad \forall \epsilon \in (0, \epsilon_k) \text{ and } i = 0, 1, \dots, k$$

because if $F_\epsilon(u_{i,\epsilon}) = \overline{F}_\epsilon(u_{i,\epsilon}) = 0$, then we should have

$$\int_\Omega \int_\Omega \frac{|u_{i,\epsilon}(x) - u_{i,\epsilon}(y)|^2}{|x-y|^{n+2s}} \, dx \, dy = 0 \quad \text{and} \quad \overline{W}(u_{i,\epsilon}) \equiv 0 \quad \text{in } \Omega$$

and so $u_{i,\epsilon}$ should be a constant function with value $+1$ or -1.

Moreover let us remark that for all $\epsilon \in (0, \epsilon_k)$ and $i = 1, \dots k$ we have

$$F_\epsilon(u_{i,\epsilon}) \ge \min \big\{ \overline{F}_\epsilon(u) : u \in H^s(\Omega), \int_\Omega u \, dx = 0 \big\}. \tag{3.22}$$

In fact, we assume that

$$\min \big\{ \overline{F}_\epsilon(u) : u \in H^s(\Omega), \int_\Omega u \, dx = 0 \big\} > 0,$$

otherwise (3.22) would be obvious. Then, for every $\bar{c}_1 > 0$ such that

$$\bar{c}_1 < \min \big\{ \overline{F}_\epsilon(u) : u \in H^s(\Omega), \int_\Omega u \, dx = 0 \big\},$$

we would have clearly $\operatorname{gen}(\overline{F}_\epsilon^{\bar{c}_1}) = 1$ because below c_1 the mean is non zero and we can use it as odd function into $\mathbb{R}^1$ in the genus definition, see Definition 2.4; thus, if (3.22) were false, the solutions would belong to a set of genus one, in contradiction with their construction in Theorem 2.5.

Now, to prove (3.1), let us replace the function $\overline{W}$ appearing in the definition of functional $\overline{F}_\epsilon$ by a sequence of functions $(\overline{W}_j)_j$ and denote by $(\overline{F}_\epsilon^j)_j$ the corresponding sequence of new functionals. Assume moreover that the functions $\overline{W}_j$ satisfy the same properties as $\overline{W}$ for all $j \in \mathbb{N}$ and that

$$\lim_{j \to \infty} \overline{W}_j(t) = +\infty \quad \text{for } |t| > 1. \tag{3.23}$$

Then property (3.22) holds for the higher critical values of the functional $\overline{F}_\epsilon^j$ for all $j \in \mathbb{N}$ and so (3.1) follows for j large enough, taking into account that

$$\begin{aligned} &\lim_{j \to \infty} \min \big\{ \overline{F}_\epsilon^j(u) : u \in H^s(\Omega), \int_\Omega u \, dx = 0 \big\} \\ &= \min \big\{ F_\epsilon(u) : u \in H^s(\Omega), |u(x)| \le 1 \ \forall x \in \Omega, \int_\Omega u \, dx = 0 \big\} \end{aligned}$$

because of (3.23). $\square$

References

[1] A. Ambrosetti, A. Malchiodi; *Nonlinear analysis and semilinear elliptic problems*, Cambridge Studies in Advanced Mathematics, 104. Cambridge University Press, Cambridge, (2007).

[2] J. Bertoin; *Lévy processes*, Cambridge Tracts in Mathematics, 121, Cambridge University Press, Cambridge, (1996).

[3] P. Biler, G. Karch, R.Monneau; *Nonlinear diffusion of dislocation density and self-similar solutions*, Comm. Math. Phys. 294, (2010), no. 1, 145–168.

[4] L. Caffarelli, J.-M. Roquejoffre, O. Savin; *Nonlocal minimal surfaces*, Comm. Pure Appl. Math. 63, (2010), 1111–1144.

[5] X. Cabré, Y. Sire; *Nonlinear equations for fractional Laplacians II: Existence, uniqueness, and qualitative properties of solutions*, Trans. Amer. Math. Soc. 367 (2015), no. 2, 911–941.

[6] R. Cont, P. Tankov; *Financial Modelling with Jump processes*, Chapman & Hall/CRC Financial Mathematics Series Chapman & Hall/CRC, Boca Raton, FL, (2004).

[7] L. Caffarelli, E. Valdinoci; *Uniform estimates and limiting arguments for nonlocal minimal surfaces*, Calc. Var. Partial Differential Equations 41, (2011), no. 1-2, 203–240.

[8] G. Duvaut, J.-L. Lions; *Inequalities in Mechanics and Physics*, Grundlehren der Mathematischen Wissenschaften, 219, Springer-Verlag, Berlin-New York, (1976).

[9] E. De Giorgi; *Conjectures on limits of some semilinear parabolic equations and flow by mean curvature*, Partial differential equations and related subjects, (Trento, 1990), Pitman Res. Notes Math. Ser., 269, 85–95.

[10] E. Di Nezza, G. Palatucci, E. Valdinoci; *Hitchhiker's guide to the fractional Sobolev spaces*, Bull. Sci. Math. 136 (2012), no. 5, 521–573.

[11] G. M. Figueiredo, G. Siciliano; *A multiplicity result via Ljusternick-Schnirelmann category and Morse theory for a fractional Schrödinger equation in* $\mathbb{R}^n$, preprint.

[12] G. M. Figueiredo, M. T. O. Pimenta, G. Siciliano; *Multiplicity results for the fractional Laplacian in expanded domains*, preprint.

[13] R. Kohn, P. Sternberg; *Local minimisers and singular perturbations*, Proc. Roy. Soc. Edinburgh Sect. A 111 (1989), no. 1-2, 69–84.

[14] L. Modica, S. Mortola; *Un esempio di* Γ*-convergenza*, Boll. Un. Mat. Ital B (5) 14, (1977), no. 1 285–299.

[15] L. Modica, S. Mortola; *The* Γ*-convergence of some functionals*, "Istituto Matematico L. Tonelli" Univ. Pisa, preprint n. 77-7, (1977).

[16] D. Passaseo; *Multiplicity of critical points for some functionals related to the minimal surfaces problem*, Calc. Var. Partial Differential Equations 6 (1998), no. 2, 105–121.

[17] O. Savin, E. Valdinoci; Γ*-convergence for nonlocal phase transitions*, Ann. Inst. H. Poincare Anal. Non Lináire 29 (2012), no. 4, 479–500.

[18] J. L. Vázquez; *Nonlinear diffusion with fractional Laplacian operators*, Nonlinear partial differential equations, Abel symp. 7, Springer, Heidelberg, (2012), 271–298.

[19] V. E. Zakharov; *Stability of periodic waves of finite amplitude on the surface of a deep fluid*, Zh. Prikl. Mekh. Tekh. Fiz. 9, (1968), 86–94.

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