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SEMILINEAR ELLIPTIC PROBLEMS INVOLVING HARDY-SOBOLEV-MAZ'YA POTENTIAL AND HARDY-SOBOLEV CRITICAL EXPONENTS

RUI-TING JIANG, CHUN-LEI TANG

Abstract. In this article, we study a class of semilinear elliptic equations involving Hardy-Sobolev critical exponents and Hardy-Sobolev-Maz'ya potential in a bounded domain. We obtain the existence of positive solutions using the Mountain Pass Lemma.

1. Introduction and statement of main results

The main purpose of this article is to investigate the existence of positive solution to the following semilinear elliptic problem with Dirichlet boundary value conditions

$$\begin{gathered} -\Delta u - \mu\frac{u}{|y|^2} = \frac{|u|^{2^*(s)-2}u}{|y|^s} + \lambda f(x,u), \quad \text{in } \Omega, \\ u > 0, \quad \text{in } \Omega, \\ u = 0, \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where $f \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$, Ω is a smooth bounded domain in $\mathbb{R}^N = \mathbb{R}^k \times \mathbb{R}^{N-k}$ with $N \geq 3$ and $2 \leq k < N$, a point $x \in \mathbb{R}^N$ is denoted as $x = (y,z) \in \mathbb{R}^k \times \mathbb{R}^{N-k}$ and $(0, z^0) \in \Omega$, $0 \leq \mu < \bar{\mu} = \frac{(k-2)^2}{4}$ for $k > 2$, $\mu = 0$ for $k = 2$. The so-called Hardy-Sobolev critical exponents are denoted by $2^*(s) = \frac{2(N-s)}{N-2}$ where $0 \leq s < 2$, $2^* = 2^*(0) = \frac{2N}{N-2}$ are the Sobolev critical exponents. $F(x,t)$ is a primitive function of $f(x,t)$ defined by $F(x,t) = \int_0^t f(x,s)ds$. $H_0^1(\Omega)$ is the Sobolev space with norm

$$\|u\| = \Big(\int_\Omega (|\nabla u|^2 - \mu\frac{u^2}{|y|^2})dx\Big)^{1/2},$$

which is equivalent to its general norm due to the Hardy inequality

$$C_k \int_{\mathbb{R}^N} \frac{u^2}{|y|^2}dx \leq \int_{\mathbb{R}^N} |\nabla u|^2 dx, \quad \forall u \in D^{1,2}(\mathbb{R}^N),$$

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where $C_k = \left(\frac{k-2}{2}\right)^2$ is the best constant and is not attained. Let S_μ be the best Sobolev constant, namely

$$S_\mu = \inf_{u\in D^{1,2}(\mathbb{R}^N\setminus(0,z^0)),u\neq 0} \frac{\int_{\mathbb{R}^N}\left(|\nabla u|^2 - \mu\frac{u^2}{|y|^2}\right)dx}{\left(\int_{\mathbb{R}^N}\frac{|u|^{2^*(s)}}{|y|^s}dx\right)^{\frac{2}{2^*(s)}}}. \tag{1.2}$$

When $k = N$, problem (1.1) becomes

$$\begin{gathered} -\Delta u - \mu\frac{u}{|x|^2} = \frac{|u|^{2^*(s)-2}u}{|x|^s} + \lambda f(x,u), \quad \text{in } \Omega, \\ u > 0, \quad \text{in } \Omega, \\ u = 0, \quad \text{on } \partial\Omega, \end{gathered} \tag{1.3}$$

where $0 \le \mu < \bar{\mu} = \frac{(N-2)^2}{4}$. There are many papers concerning the Dirichlet problem with critical exponents (see [1, 4, 6, 8, 11]) after the work of Breizis and Nirenberg [3]. When $\mu = 0$ and $s = 0$, problem (1.3) becomes the well-known Brezis-Nirenberg problem, and is studied extensively in [11]. When $\mu \neq 0$, the problem has its singularity at 0 and attracts much attention. Ding and Tang in [5] studied the existence of positive solutions with $N \ge 3$, $0 \le s < 2$ and $f(x,t)$ satisfying the (AR) condition in the case $\lambda = 1$. Kang in [7] showed the existence of positive solutions replacing $f(x,u)$ by $|u|^{q-2}u$ with $q > 2$ for $0 \le s < 2$.

When $2 \le k < N$, the problem has stronger singularity. Bhakta and Sandeep [2] studied the regularity, P.S. characterization and existence of solutions with $\lambda = 0$. Wang and Wang in [12] showed that the existence of infinitely many solutions replacing $f(x,u)$ by u for $N > 6 + s$. In [13], Yang studied (1.1) with Neumann boundary condition and obtained the existence of positive solution by the Mountain Pass Lemma. In order to estimate the mountain pass energy, the author use the extremal function of S_μ ([2]), which is achieved when

$$s = 2 - \frac{N-2}{N-k+\sqrt{(k-2)^2 - 4\mu}}.$$

In this article, we estimate the mountain pass energy for $0 \le s < 2$ through λ large enough instead of the extremal function, which makes the results more extensive and interested. Here is our main result.

Theorem 1.1. *Suppose $N \ge 3$, $2 \le k < N$ and $0 \le \mu < \bar{\mu}$. $f \in C(\overline{\Omega} \times \mathbb{R}^+, \mathbb{R})$ satisfies*

- (A1) *$f(x,t) = 0$ for $t \le 0$ uniformly for $x \in \overline{\Omega}$. There exists a nonempty open subset $\Omega_0 \subset \Omega$ with $(0,z^0) \in \Omega_0$, such that $f(x,t) \ge 0$ for almost everywhere $x \in \Omega$ and all $t > 0$, and $f(x,t) > 0$ for almost $x \in \Omega_0$ and all $t > 0$,*
- (A2) *$\lim_{t\to 0^+}\frac{f(x,t)}{t} = 0$ and $\lim_{t\to+\infty}\frac{f(x,t)}{t^{2^*(s)-1}} = 0$ uniformly for $x \in \overline{\Omega}$.*

Then there exists $\Lambda_* > 0$, such that problem (1.1) admits at least one positive solution for all $\lambda \ge \Lambda_*$.

Remark 1.2. First, there are many functions satisfying our assumptions of Theorem 1.1. For instance, $f(x,t) = t^q$ with $1 < q < 2^*(s) - 1$. Second, it is worth noting that, when $\mu = s = 0$, problem (1.1) reduces to the classical semilinear elliptic problem with critical exponents, [3] proved the existence of positive solution for $\lambda > 0$ large enough. In this paper, we obtain the similiar result as in [3] when

$2 \le k < N$ and $0 \le s < 2$. Our results complete the existence of positive solutions for elliptic problem with Hardy-Sobolev critical exponents.

2. Proof of Theorem 1.1

In this article, we use the following notation:
- The dual space of a Banach space E will be denoted by E'.
- $L^p(\Omega, |y|^{-s}dx)$ denotes the weighted Sobolev space.
- $\to$ denotes the strong convergence, while $\rightharpoonup$ denotes the weak convergence.
- C, C_i $(i = 0, 1, 2 \dots)$ will denote various positive constants and their values can vary from line to line.

To study the positive solutions of problem (1.1), we first consider the existence of nontrivial solutions to the problem

$$\begin{gathered} -\Delta u - \mu \frac{u}{|y|^2} = \frac{(u^+)^{2^*(s)-1}}{|y|^s} + \lambda f(x, u^+), \quad \text{in } \Omega, \\ u > 0, \quad \text{in } \Omega, \\ u = 0, \quad \text{on } \partial\Omega, \end{gathered} \tag{2.1}$$

where $u^+ = \max\{u, 0\}$. The energy functional corresponding to (2.1) is

$$I(u) = \frac{1}{2}\int_\Omega \Big(|\nabla u|^2 - \mu \frac{u^2}{|y|^2}\Big)dx - \frac{1}{2^*(s)}\int_\Omega \frac{(u^+)^{2^*(s)}}{|y|^s}dx - \lambda \int_\Omega F(x, u^+)dx, \tag{2.2}$$

for $u \in H_0^1(\Omega)$. Clearly, I is well defined and is C^1 smooth thanks to the Hardy-Sobolev-Maz'ya inequality [9]

$$\Big(\int_{\mathbb{R}^N} \frac{|u|^{2^*(s)}}{|y|^2}dx\Big)^{\frac{2}{2^*(s)}} \le S_\mu^{-1} \int_{\mathbb{R}^N} \Big(|\nabla u|^2 - \mu \frac{u^2}{|y|^2}\Big)dx, \tag{2.3}$$

where $S_\mu = S(\mu, N, k, s)$ is the best constant defined in (1.2). By the existence of the one to one correspondence between the critical points of I and the weak solutions of problem (2.1), we know that if u is a weak solution of problem (2.1), there holds

$$\langle I'(u), v\rangle = \int_\Omega \Big((\nabla u, \nabla v) - \mu \frac{uv}{|y|^2}\Big)dx - \int_\Omega \frac{(u^+)^{2^*(s)-1}v}{|y|^s}dx - \lambda \int_\Omega f(x, u^+)v dx = 0,$$

for any $v \in H_0^1(\Omega)$.

Before proving Theorem 1.1, it is necessary to prove the following lemmas.

Lemma 2.1. *Assume that conditions* (A1), (A2) *hold. Then for $0 \le \mu < \overline{\mu}$ and $\lambda > 0$, we can deduce that*
1. *there exist r, $\alpha > 0$ such that $I(u) \ge \alpha$ when $\|u\| = r$,*
2. *there exists $u_1 \in H_0^1(\Omega)$ such that $\|u_1\| > r$ and $I(u_1) < 0$.*

Proof. (1) From the continuity of embeddings

$$H_0^1(\Omega) \hookrightarrow L^q(\Omega)(1 \le q \le 2^*), \quad H_0^1(\Omega) \hookrightarrow L^{2^*(s)}(\Omega, |y|^{-s}dx),$$

there exist $C_1 > 0$ and $C_2 > 0$ such that

$$\int_\Omega |u|^q dx \le C_1\|u\|^q, \quad \int_\Omega \frac{|u|^{2^*(s)}}{|y|^s}dx \le C_2\|u\|^{2^*(s)}. \tag{2.4}$$

It follows from (A2) that for $\varepsilon > 0$, there exists $C_3 > 0$, such that

$$|F(x, t)| \le \varepsilon |t|^2 + C_3|t|^{2^*(s)}, \tag{2.5}$$

for all $t \in \mathbb{R}^+$ and $x \in \overline{\Omega}$. Combining (2.4) and (2.5), one has

$$
\begin{aligned}
I(u) &\geq \frac{1}{2}\|u\|^2 - \frac{1}{2^*(s)}\int_\Omega \frac{(u^+)^{2^*(s)}}{|y|^s}dx - \lambda\varepsilon\int_\Omega |u|^2dx - \lambda C_3\int_\Omega |u|^{2^*(s)}dx \\
&\geq \frac{1}{2}\|u\|^2 - \frac{C_4}{2^*(s)}\|u\|^{2^*(s)} - \lambda\varepsilon C_5\|u\|^2 - \lambda C_6\|u\|^{2^*(s)}.
\end{aligned}
$$

Therefore for a fixed $\lambda > 0$, there exists $\alpha > 0$ such that $I(u) \geq \alpha > 0$ for all $\|u\| = r$, where $r > 0$ small enough.

(2) Fix $v_0 \in C_0^\infty(\Omega) \setminus \{0\}$ with $v_0 \geq 0$ in Ω and $\|v_0\| = 1$. From $(A1)$, one has

$$
\begin{aligned}
I(tv_0) &= \frac{1}{2}t^2\|v_0\|^2 - \frac{1}{2^*(s)}t^{2^*(s)}\int_\Omega \frac{(v_0^+)^{2^*(s)}}{|y|^s}dx - \int_\Omega F(x, tv_0)dx \\
&\leq \frac{1}{2}t^2\|v_0\|^2 - \frac{1}{2^*(s)}t^{2^*(s)}\int_\Omega \frac{(v_0^+)^{2^*(s)}}{|y|^s}dx,
\end{aligned}
$$

then $\lim_{t\to+\infty} I(tv_0) \to -\infty$. Thus we can find $t' > 0$ such that $I(t'v_0) < 0$ when $\|t'v_0\| > r$. Set $u_1 = t'v_0$, the proof is complete. $\square$

According to the Mountain Pass Lemma without (PS) condition (see [10]), there exists a sequence $\{u_n\} \subset H_0^1(\Omega)$, such that

$$
I(u_n) \to c_\lambda > \alpha > 0, \quad I'(u_n) \to 0 \text{ in } \big(H_0^1(\Omega)\big)',
$$

as $n \to \infty$, where

$$
\begin{gathered}
c_\lambda = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I(\gamma(t)), \\
\Gamma = \{\gamma \in C([0,1], H_0^1(\Omega)) | \gamma(0) = 0, \gamma(1) = u_1\}.
\end{gathered}
$$

Lemma 2.2. *Suppose that* (A1) *holds, then* $\lim_{\lambda\to+\infty} c_\lambda = 0$.

Proof. If v_0 is the function given in the proof of Lemma 2.1, then one deduces that $I(tv_0) > 0$ for $t > 0$ small enough and $I(tv_0) \to -\infty$ as $t \to +\infty$. Thus there exists $t_\lambda > 0$ such that $I(t_\lambda v_0) = \max_{t\geq 0} I(tv_0)$. Hence

$$
t_\lambda^2\|v_0\|^2 = t_\lambda^{2^*(s)}\int_\Omega \frac{(v_0^+)^{2^*(s)}}{|y|^s}dx + \lambda\int_\Omega f(x, t_\lambda v_0^+)t_\lambda v_0^+dx.
$$

By (A1), one has

$$
t_\lambda^2\|v_0\|^2 \geq t_\lambda^{2^*(s)}\int_\Omega \frac{(v_0^+)^{2^*(s)}}{|y|^s}dx,
$$

which implies that $\{t_\lambda\}$ is bounded. Therefore there exist a sequence $\{\lambda_n\}$ and $t_0 \geq 0$, such that $\lambda_n \to +\infty$ and $t_{\lambda_n} \to t_0$ as $n \to \infty$. Consequently, there is $C_6 > 0$ such that $t_{\lambda_n}^2\|v_0\|^2 \leq C_6$ for all $n \in \mathbb{N}$, and thus

$$
\lambda_n\int_\Omega f(x, t_{\lambda_n}v_0^+)t_{\lambda_n}v_0^+dx + t_{\lambda_n}^{2^*(s)}\int_\Omega \frac{(v_0^+)^{2^*(s)}}{|y|^s}dx \leq C_6, \tag{2.6}
$$

for all $n \in \mathbb{N}$. If $t_0 > 0$, by $(A1)$, one obtains

$$
\lim_{n\to\infty} \lambda_n\int_{\Omega_0} f(x, t_{\lambda_n}v_0^+)t_{\lambda_n}v_0^+dx + t_{\lambda_n}^{2^*(s)}\int_{\Omega_0} \frac{(v_0^+)^{2^*(s)}}{|y|^s}dx = +\infty,
$$

then

$$
\lim_{n\to\infty} \lambda_n\int_\Omega f(x, t_{\lambda_n}v_0^+)t_{\lambda_n}v_0^+dx + t_{\lambda_n}^{2^*(s)}\int_\Omega \frac{(v_0^+)^{2^*(s)}}{|y|^s}dx = +\infty,
$$

which contradicts (2.6). Thus we conclude that $t_0 = 0$. Now, let us consider the path $\gamma_*(t) = tu_1$ for $t \in [0,1]$, which belongs to Γ, then we get the following estimate

$$0 < c_\lambda \le \max_{t\in[0,1]} I(\gamma_*(t)) \le I(t_\lambda v_0) \le \frac{1}{2} t_\lambda^2 \|v_0\|^2.$$

As $\lambda \to +\infty$, from the above inequality, we get $c_\lambda \to 0$. $\square$

Proof of Theorem 1.1. By Lemma 2.1, there exists a sequence $\{u_n\} \subset H_0^1(\Omega)$ such that $I(u_n) \to c_\lambda$ and $I'(u_n) \to 0$ as $n \to \infty$. According to Lemma 2.2, one deduces that there exists $\Lambda_* > 0$ such that

$$0 < c_\lambda < \frac{2-s}{2(N-s)} S_\mu^{\frac{N-s}{2-s}}$$

as $\lambda \ge \Lambda_*$. First, we prove that $\{u_n\}$ is bounded. Indeed, by $(A2)$ and the boundedness of Ω, for any $\varepsilon > 0$, there exists $M > 0$, such that

$$\begin{gathered}
|F(x,t)| \le \varepsilon |t|^{2^*(s)}, \quad x \in \Omega,\ t \ge M; \quad |F(x,t)| \le C_1(\varepsilon), \quad t \in (0,M];\\
|f(x,t)t| \le \varepsilon |t|^{2^*(s)}, \quad x \in \Omega,\ t \ge M; \quad |f(x,t)t| \le C_2(\varepsilon), \quad t \in (0,M].
\end{gathered}$$

Thus, we have

$$|F(x,t)| \le C_1(\varepsilon) + \varepsilon |t|^{2^*(s)}, \quad |f(x,t)t| \le C_2(\varepsilon) + \varepsilon |t|^{2^*(s)}, \tag{2.7}$$

for any $(x,t) \in \overline{\Omega} \times \mathbb{R}^+$. Then for $\xi \in (2, 2^*(s))$, one has

$$F(x,t) - \frac{1}{2} f(x,t)t \le F(x,t) - \frac{1}{\xi} f(x,t)t \le C_3(\varepsilon) + \varepsilon |t|^{2^*(s)}, \tag{2.8}$$

for any $(x,t) \in \overline{\Omega} \times \mathbb{R}^+$. Set $l(x,t) := |y|^{-s}|t|^{2^*(s)-1} + \lambda f(x,t)$, we claim that $l(x,t)$ satisfies the (AR) condition. By (2.8), one easily gets

$$\begin{aligned}
\xi L(x,t) - l(x,t)t &= \Big(\frac{\xi}{2^*(s)} - 1\Big)|y|^{-s}|t|^{2^*(s)} + \lambda\Big(\xi F(x,t) - f(x,t)t\Big)\\
&\le \Big(\frac{\xi}{2^*(s)} - 1\Big)|y|^{-s}|t|^{2^*(s)} + \xi\lambda C_4(\varepsilon) + \xi\lambda\varepsilon |t|^{2^*(s)}\\
&= \Big((\frac{\xi}{2^*(s)} - 1)|y|^{-s} + \lambda\xi\varepsilon\Big)|t|^{2^*(s)} + \xi\lambda C_4(\varepsilon).
\end{aligned}$$

Thus for a fixed $\lambda > 0$ and $\varepsilon > 0$ sufficiently small, there exists $M'_\lambda > 0$, such that

$$0 \le \xi L(x,t) \le l(x,t)t, \quad t \ge M'_\lambda,$$

where $L(x,t) = \int_0^t l(x,s)ds$. Moreover, by $(A2)$, we obtain

$$L(x,t) - \frac{1}{\xi} l(x,t)t \le \max_{x\in\overline{\Omega}, 0\le t\le M'_\lambda} \Big(F(x,t) - \frac{1}{\xi} f(x,t)t\Big) := M_\lambda.$$

It follows from the inequalities above that

$$L(x,t) - \frac{1}{\xi} l(x,t)t \le M_\lambda, \quad \text{for all } x \in \overline{\Omega} \setminus \{(0,z^0)\},\ t \ge 0. \tag{2.9}$$

Then, one has

$$\begin{aligned}
c + 1 + o(1)\|u_n\| &\ge I(u_n) - \frac{1}{\xi}\langle I'(u_n), u_n\rangle\\
&\ge \big(\frac{1}{2} - \frac{1}{\xi}\big)\|u_n\|^2 + \Big(\frac{1}{\xi} - \frac{1}{2^*(s)}\Big)\int_\Omega \frac{(u_n^+)^{2^*(s)}}{|y|^s} dx
\end{aligned}$$

$$
\begin{aligned}
&- \lambda \int_\Omega \Big(F(x,u_n^+) - \frac{1}{\xi} f(x,u_n^+)u_n^+ \Big) dx \\
&\geq \big(\frac{1}{2} - \frac{1}{\xi}\big)\|u_n\|^2 - \int_\Omega \Big(L(x,u_n^+) - \frac{1}{\xi} l(x,u_n^+)u_n^+ \Big) dx \\
&\geq \big(\frac{1}{2} - \frac{1}{\xi}\big)\|u_n\|^2 - M_\lambda |\Omega|.
\end{aligned}
$$

Thus, $\{u_n\}$ is bounded. Due to the continuity of embedding $H_0^1(\Omega) \hookrightarrow L^{2^*(s)}(\Omega)$, we have $\int_\Omega |u_n|^{2^*(s)} dx \leq C_7 < \infty$. Up to a subsequence, still denoted by $\{u_n\}$, there exists $u_0 \in H_0^1(\Omega)$ satisfying

$$
\begin{gathered}
u_n \rightharpoonup u_0, \quad \text{weakly in } H_0^1(\Omega), \\
u_n \to u_0, \quad \text{strongly in } L^p(\Omega),\ 1 \leq p < 2^*, \\
u_n(x) \to u_0(x), \quad \text{a.e. in } \Omega, \\
u_n^{2^*(s)-1} \rightharpoonup u_0^{2^*(s)-1}, \quad \text{weakly in } \big(L^{2^*(s)}(\Omega, |y|^{-s}dx)\big)',
\end{gathered} \tag{2.10}
$$

as $n \to \infty$. By (A2), for any $\varepsilon > 0$ there exists $a(\varepsilon) > 0$ such that

$$
|F(x,t)| \leq \frac{1}{2C_7} \varepsilon |t|^{2^*(s)} + a(\varepsilon) \quad \text{for } (x,t) \in \Omega \times \mathbb{R}^+.
$$

Set $\delta = \frac{\varepsilon}{2a(\varepsilon)} > 0$. When $E \subset \Omega$, $\operatorname{meas} E < \delta$, one gets

$$
\begin{aligned}
\Big| \int_E F(x,u_n^+) dx \Big| &\leq \int_E |F(x,u_n^+)| dx \\
&\leq \int_E a(\varepsilon) dx + \frac{1}{2C_7} \varepsilon \int_E |u_n|^{2^*(s)} dx \\
&\leq a(\varepsilon) \operatorname{meas} E + \frac{1}{2C_7} \varepsilon C_7 \leq \varepsilon.
\end{aligned}
$$

Hence $\big\{ \int_\Omega F(x,u_n^+) dx, n \in N \big\}$ is equi-absolutely-continuous. It follows from Vitali's Convergence Theorem that

$$
\int_\Omega F(x,u_n^+) dx \to \int_\Omega F(x,u_0^+) dx, \tag{2.11}
$$

as $n \to \infty$. Applying the same method, one has

$$
\int_\Omega f(x,u_n^+) u_n dx \to \int_\Omega f(x,u_0^+) u_0 dx, \tag{2.12}
$$

as $n \to \infty$. By (2.10) and (2.12) we have

$$
\begin{aligned}
\lim_{n \to +\infty} \langle I'(u_n), v \rangle = \int_\Omega \Big((\nabla u_0, \nabla v) - \mu \frac{u_0 v}{|y|^2} \Big) dx - \int_\Omega \frac{(u_0^+)^{2^*(s)-1} v}{|y|^s} dx \\
- \int_\Omega f(x,u_0^+) v dx = 0,
\end{aligned} \tag{2.13}
$$

for all $v \in H_0^1(\Omega)$. That is, $\langle I'(u_0), v \rangle = 0$ for any $v \in H_0^1(\Omega)$. Then u_0 is a critical point of I, thus u_0 is a solution of problem (2.1). Now we verify that $u_0 \not\equiv 0$. Let $v = u_0$ in (2.13), we get

$$
\|u_0\|^2 - \int_\Omega \frac{(u_0^+)^{2^*(s)}}{|y|^s} dx - \int_\Omega f(x,u_0^+) u_0 dx = 0. \tag{2.14}
$$

Set $w_n = u_n - u_0$, then we have

$$\int_\Omega |\nabla u_n|^2 dx = \int_\Omega |\nabla u_0|^2 dx + \int_\Omega |\nabla w_n|^2 dx + o(1). \tag{2.15}$$

And from Brézis-Lieb's lemma [3], it follows that

$$\int_\Omega \frac{u_n^2}{|y|^2} dx = \int_\Omega \frac{u_0^2}{|y|^2} dx + \int_\Omega \frac{w_n^2}{|y|^2} dx + o(1), \tag{2.16}$$

$$\int_\Omega \frac{(u_n^+)^{2^*(s)}}{|y|^s} dx = \int_\Omega \frac{(u_0^+)^{2^*(s)}}{|y|^s} dx + \int_\Omega \frac{(w_n^+)^{2^*(s)}}{|y|^s} dx + o(1). \tag{2.17}$$

By (2.11) and (2.15)-(2.17), one has

$$I(u_n) = I(u_0) + \frac{1}{2}\|w_n\|^2 - \frac{1}{2^*(s)} \int_\Omega \frac{(w_n^+)^{2^*(s)}}{|y|^s} dx = c_\lambda + o(1). \tag{2.18}$$

Since $\langle I'(u_n), u_n\rangle = o(1)$, combining with (2.12) and (2.14), one has

$$\|w_n\|^2 - \int_\Omega \frac{(w_n^+)^{2^*(s)}}{|y|^s} dx = o(1).$$

We may assume that $\|w_n\|^2 \to b$ and

$$\int_\Omega \frac{(w_n^+)^{2^*(s)}}{|y|^s} dx \to b$$

as $n \to \infty$. Clearly, $b \geq 0$. We now suppose that $u_0 \equiv 0$. On the one hand, if $b = 0$, together with (2.18), we get $c_\lambda = I(0) = 0$, which contradicts with $c_\lambda > 0$. On the other hand, if $b \neq 0$, we have from the definition of S_μ that

$$\|w_n\|^2 = \int_\Omega \Big(|\nabla w_n|^2 - \mu \frac{w_n^2}{|y|^2}\Big) dx \geq S_\mu \Big(\int_\Omega \frac{(w_n^+)^{2^*(s)}}{|y|^s} dx\Big)^{\frac{2}{2^*(s)}},$$

then $b \geq S_\mu^{\frac{N-s}{2-s}}$, together with (2.18) we deduce

$$\begin{aligned} c_\lambda + o(1) &= I(u_0) + \frac{1}{2}\|w_n\|^2 - \frac{1}{2^*(s)} \int_\Omega \frac{(w_n^+)^{2^*(s)}}{|y|^s} dx + o(1) \\ &\geq \frac{2-s}{2(N-s)} S_\mu^{\frac{N-s}{2-s}} + o(1), \end{aligned}$$

which contradicts $c_\lambda < \frac{2-s}{2(N-s)} S_\mu^{\frac{N-s}{2-s}}$. Therefore $u_0 \not\equiv 0$ and u_0 is a nontrivial solution of problem (2.1). Then by $\langle I'(u_0), u_0^-\rangle = 0$ where $u_0^- = \min\{u_0, 0\}$, one has $\|u_0^-\| = 0$, which implies that $u_0 \geq 0$. From (2.13), we get $\int_\Omega (\nabla u_0, \nabla v) dx \geq 0$ for any $v \in H_0^1(\Omega)$, which means $-\triangle u_0 \geq 0$ in Ω. By the strong maximum principle, we know u_0 is a positive solution of problem (1.1). Therefore Theorem 1.1 holds. □

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Rui-Ting Jiang
School of Mathematics and Statistics, Southwest University, Chongqing 400715, China
E-mail address: `1152373321@qq.com`

Chun-Lei Tang (corresponding author)
School of Mathematics and Statistics, Southwest University, Chongqing 400715, China
E-mail address: `tangcl@swu.edu.cn, phone +86 23 68253135, fax +86 23 68253135`