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MINIMAL WAVE SPEEDS OF DELAYED DISPERSAL PREDATOR-PREY SYSTEMS WITH STAGE STRUCTURE

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Abstract. This article concerns the minimal wave speed of delayed predator-prey systems with nonlocal dispersal and stage structure. By the method of upper and lower solutions, we prove the existence of positive traveling wave solutions. With the help of a contracting rectangle, we establish the limit behavior of traveling wave solutions. The nonexistence of traveling wave solutions is obtained using the theory of asymptotic spreading, and therefore, the minimal wave speed is obtained.

1. Introduction

In this article, we sutdy the delayed predator-prey systems with nonlocal dispersal and stage structure,

$$\begin{aligned}\frac{\partial u(x,t)}{\partial t} &= (D_1u)(x,t) + \alpha e^{-\gamma\tau_1}u(x,t-\tau_1) - mu^2(x,t) - a_1u(x,t)v(x,t),\\ \frac{\partial v(x,t)}{\partial t} &= (D_2v)(x,t) + r_1v(x,t) + a_2u(x,t-\tau_2)v(x,t-\tau_2) - bv^2(x,t),\end{aligned} \tag{1.1}$$

in which all the parameters are positive and

$$\begin{aligned}(D_1u)(x,t) &= \int_{\mathbb{R}} J_1(x-y)[u(y,t)-u(x,t)]dy,\\ (D_2v)(x,t) &= \int_{\mathbb{R}} J_2(x-y)[v(y,t)-v(x,t)]dy,\end{aligned}$$

herein $J_1, J_2 : \mathbb{R} \to \mathbb{R}^+$ are integrable functions satisfying some conditions specified later.

Zhang et al [29] gave this model with state structure and nonlocal dispersal. Moreover, they also established the existence of traveling wave solutions connecting the trivial steady state with the positive equilibrium if the wave speed is larger than a threshold. Such a traveling wave solution could formulate the existence of a transition zone moving from the steady state with no species to the steady state with the coexistence of both species in mathematical biology [29].

Although the existence of traveling wave solutions could reflect some phenomena of population dynamics, the minimal wave speed depending on the existence and nonexistence of traveling wave solutions is one of the most important thresholds

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in mathematical biology. However, the estimation of minimal wave speed is not an easy job. Before presenting our methods and results of minimal wave speeds, we first recall some important results on the topic. After the pioneer works of Fisher [8] and Kolmogorov et al [9] on traveling wave solutions of reaction-diffusion equations, Aronson and Weinberger [1] studied the asymptotic spreading of some population models with reaction and diffusion, which describes some dynamical results different from those in [8, 9]. Besides some results for reaction-diffusion systems, integral equations and integrodifference equations, there are some results appealing to abstract monotone semiflows, see some results by Chen [3], Fang and Zhao [7], Liang and Zhao [18], Weinberger [25], Weinberger et al [26], Yi et al [28] and a survey paper by Zhao [30].

However, for non-cooperation systems, it is difficult to obtain the minimal wave speed due to the deficiency of comparison principle appealing to cooperative systems. On the traveling wave solutions of predator-prey systems, some classical conclusions were established about three decades ago by Dunbar [4, 5, 6], Gardner and Smoller [10], Gardner and Jones [11]. After 2000, several investigators further studied the problem by phase analysis, perturbation theory and fixed point theory, we refer to some results by Huang et al [12], Huang [13], Hsu et al [14], Liang et al [17], Lin [19], Lin et al [21] and Wang et al [24]. In particular, Zhang et al [29] proved the existence of traveling wave solutions by constructing upper and lower solutions if the wave speed is larger than the threshold, and we shall investigate the existence or nonexistence of traveling wave solutions when the wave speed is the threshold and smaller than the threshold.

To further study the existence of traveling wave solutions, we shall first present a result via generalized upper and lower solutions motivated by Lin and Ruan [20]. Then the asymptotic behavior will be established by the idea of contracting rectangles [20] (see the definition of contracting rectangle for functional differential equations by Smith [23]) as well as the theory of asymptotic spreading given by Fang and Zhao [7], Jin and Zhao [15]. Finally, the nonexistence of traveling wave solutions is confirmed by combining the asymptotic behavior of traveling wave solutions with the theory of asymptotic spreading.

2. Main Results

In this section, we shall present our main results. We first give some notation and definitions. In what follows, we use the standard partial ordering and order intervals in $\mathbb{R}$ or $\mathbb{R}^2$, and apply $\|\cdot\|$ to denote the norm in $\mathbb{R}^2$. That is, for $u=(u_1,u_2)$ and $v=(v_1,v_2)$, we denote $u\le v$ if $u_i\le v_i$ for $i=1,2$, and $u<v$ if $u\le v$ but $u\ne v$. In particular, we denote $u\ll v$ if $u\le v$ but $u_i\ne v_i$ for $i=1,2$. If $u\le v$, we denote $(u,v]=\{w\in\mathbb{R}^2, u<w\le v\}$, $[u,v)=\{w\in\mathbb{R}^2, u\le w<v\}$, and $[u,v]=\{w\in\mathbb{R}^2, u\le w\le v\}$.

Define

$$X=\{U: U \text{ is a bounded and uniformly continuous function from } \mathbb{R} \text{ to } \mathbb{R}^2\},$$

then X is a Banach space equipped with the standard supremum norm. If $\mathbf{a},\mathbf{b}\in\mathbb{R}^2$ with $\mathbf{a}\le\mathbf{b}$, then

$$X_{[\mathbf{a},\mathbf{b}]}=\{U\in X:\mathbf{a}\le U(\xi)\le\mathbf{b},\xi\in\mathbb{R}\}.$$

$C^1(\mathbb{R}, \mathbb{R}^2)$ is defined by

$$C^1(\mathbb{R}, \mathbb{R}^2) = \{(u, v) : (u, v), (u', v') \in X\}.$$

By scaling, it suffices to investigate

$$\begin{aligned} \frac{\partial u(x,t)}{\partial t} &= (D_1 u)(x,t) + \alpha e^{-\gamma\tau_1}[u(x, t-\tau_1) - u^2(x,t) - au(x,t)v(x,t)], \\ \frac{\partial v(x,t)}{\partial t} &= (D_2 v)(x,t) + r_1[v(x,t) + bu(x, t-\tau_2)v(x, t-\tau_2) - v^2(x,t)], \end{aligned} \tag{2.1}$$

A *traveling wave solution* of (2.1) is a special translation invariant solution of the form

$$(u(x,t), v(x,t)) = (\phi(\xi), \psi(\xi)), \quad \xi = x + ct,$$

in which $(\phi, \psi) \in C^1$ is the profiles of the wave that propagate through the one-dimensional spatial domain at a constant velocity $c > 0$. If we substitute (ϕ, ψ) into (2.1), then

$$\begin{aligned} c\phi'(\xi) &= \int_{\mathbb{R}} J_1(\xi - y)[\phi(y) - \phi(\xi)]dy \\ &\quad + \alpha e^{-\gamma\tau_1}[\phi(\xi - c\tau_1) - \phi^2(\xi) - a\phi(\xi)\psi(\xi)], \\ c\psi'(\xi) &= \int_{\mathbb{R}} J_2(\xi - y)[\psi(y) - \psi(\xi)]dy \\ &\quad + r_1[\psi(\xi) + b\phi(\xi - c\tau_2)\psi(\xi - c\tau_2) - \psi^2(\xi)], \end{aligned} \tag{2.2}$$

where $\xi \in \mathbb{R}$. Same as that in [29], we also require that (ϕ, ψ) satisfy the asymptotic boundary conditions

$$\lim_{\xi \to -\infty} (\phi(\xi), \psi(\xi)) = (0, 0) \quad \text{and} \quad \lim_{\xi \to \infty} (\phi(\xi), \psi(\xi)) = (k_1, k_2), \tag{2.3}$$

where

$$k_1 = \frac{1-a}{1+ab}, \quad k_2 = \frac{1+b}{1+ab}$$

provided that $a < 1$ which will be imposed throughout this paper.

For J_1, J_2, we assume that

(J1) $J_i : \mathbb{R} \to \mathbb{R}^+$ is symmetric and Lebesgue measurable for each $i = 1, 2$;
(J2) for any $\lambda \in \mathbb{R}$, $0 < \int_{\mathbb{R}} J_i(y)e^{\lambda y}dy < \infty$, $i = 1, 2$.

Define

$$\Delta_1(\lambda, c) = \int_{-\infty}^{+\infty} J_1(y)(e^{\lambda y} - 1)dy - c\lambda + \alpha e^{-\gamma\tau_1}e^{-\lambda c\tau_1},$$

$$\Delta_2(\lambda, c) = \int_{-\infty}^{+\infty} J_2(y)(e^{\lambda y} - 1)dy - c\lambda + r_1.$$

Using (J1) and (J2), we have the following results.

Lemma 2.1. *There exists $c^* > 0$ such that the following four items hold.*

(i) *For any given $c > c^*$, $\Delta_1(\lambda, c)$ has two distinct positive roots $\lambda_1(c)$ and $\lambda_3(c)$. Moreover, assume that $0 < \lambda_1(c) < \lambda_3(c)$ holds. Then*

$$\Delta_1(\lambda, c) \begin{cases} > 0 & \text{for } 0 < \lambda < \lambda_1(c) \text{ or } \lambda > \lambda_3(c), \\ < 0 & \text{for } \lambda_1(c) < \lambda < \lambda_3(c). \end{cases}$$

(ii) *For any given $c > c^*$, $\Delta_2(\lambda, c)$ has two distinct positive roots $\lambda_2(c)$ and $\lambda_4(c)$. Moreover, assume that $0 < \lambda_2(c) < \lambda_4(c)$ holds. Then*
$$\Delta_2(\lambda, c) \begin{cases} > 0 & \text{for } 0 < \lambda < \lambda_2(c) \text{ or } \lambda > \lambda_4(c) \\ < 0 & \text{for } \lambda_2(c) < \lambda < \lambda_4(c). \end{cases}$$
(iii) *If $c = c^*$, then at least one of $\Delta_1(\lambda, c) = 0, \Delta_2(\lambda, c) = 0$ has a double root.*
(iv) *If $c < c^*$, then at least one of $\Delta_1(\lambda, c)$ and $\Delta_2(\lambda, c)$ has no real root.*

Remark 2.2. By Fang and Zhao [7], Liang and Zhao [18], Jin and Zhao [15], c^* can also be defined as
$$\begin{aligned} c^* = \max\Big\{ &\inf_{\lambda>0} \Big[\frac{\int_{-\infty}^{+\infty} J_1(y)(e^{\lambda y} - 1)dy + \alpha e^{-\gamma\tau_1} e^{-\lambda c \tau_1}}{\lambda}\Big], \\ &\inf_{\lambda>0} \Big[\frac{\int_{-\infty}^{+\infty} J_2(y)(e^{\lambda y} - 1)dy + r_1}{\lambda}\Big]\Big\}. \end{aligned}$$

Our main results reads as follows.

Theorem 2.3. *Assume that* (J1)–(J2) *and*
$$a(1 + b) < 1. \tag{2.4}$$
(1) *If $c > c^*$, then* (2.2) *has a positive solution satisfying* (2.3).
(2) *If*
$$\inf_{\lambda>0} \Big[\frac{\int_{-\infty}^{+\infty} J_1(y)(e^{\lambda y} - 1)dy + \alpha e^{-\gamma\tau_1} e^{-\lambda c \tau_1}}{\lambda}\Big] < c^*$$
and $\overline{\lambda}_2 \le \lambda_1(c^*)$, where $\overline{\lambda}_2$ is the positive root of $\Delta_2(\lambda, c^*) = 0$, then (2.2) *has a positive solution satisfying* (2.3).
(3) *If $c < c^*$, then* (2.2) *does not have a positive solution satisfying* (2.3).

Remark 2.4. Zhang et al [29, Condition (3.2)] proved the existence of traveling wave solutions when
$$1 - a > a(1 + b). \tag{2.5}$$
Clearly, (2.4) is weaker than (2.5).

Remark 2.5. Theorem 2.3 implies that c^* is the minimal wave speed. However, when $c = c^*$, the result needs further investigation.

3. Existence of traveling wave solutions: $c \ge c^*$.

In this section, we shall prove the existence of positive solutions of (2.2) by several lemmas throughout which (J1)-(J2) hold without further illustration.

Lemma 3.1. *Assume that there exist $\overline{\Phi} = (\overline{\phi}, \overline{\psi}) \in C_{[0,M]}$ and $\underline{\Phi} = (\underline{\phi}, \underline{\psi}) \in C_{[0,M]}$ with $M = (1, 1 + b)$ satisfy*

(1) *for $\mathbb{T} = \{T_i \in \mathbb{R}, i = 1, \dots, m\}$, $\overline{\Phi}'$ and $\underline{\Phi}'$ exist and are bounded for $t \in \mathbb{R}\backslash\mathbb{T}$;*

(2) *for* $\xi \in \mathbb{R}\backslash\mathbb{T}$, $\overline{\Phi}'$ *and* $\underline{\Phi}'$ *satisfy*

$$\begin{aligned} c\overline{\phi}'(\xi) \geq & \int_{\mathbb{R}} J_1(\xi-y)[\overline{\phi}(y)-\overline{\phi}(\xi)]dy \\ & + \alpha e^{-\gamma\tau_1}[\overline{\phi}(\xi-c\tau_1)-\overline{\phi}^2(\xi)-a\overline{\phi}(\xi)\underline{\psi}(\xi)], \\ c\overline{\psi}'(\xi) \geq & \int_{\mathbb{R}} J_2(\xi-y)[\overline{\psi}(y)-\overline{\psi}(\xi)]dy \\ & + r_1[\overline{\psi}(\xi)+b\overline{\phi}(\xi-c\tau_2)\overline{\psi}(\xi-c\tau_2)-\overline{\psi}^2(\xi)] \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} c\underline{\phi}'(\xi) \leq & \int_{\mathbb{R}} J_1(\xi-y)[\underline{\phi}(y)-\underline{\phi}(\xi)]dy \\ & + \alpha e^{-\gamma\tau_1}[\underline{\phi}(\xi-c\tau_1)-\underline{\phi}^2(\xi)-a\underline{\phi}(\xi)\overline{\psi}(\xi)], \\ c\underline{\psi}'(\xi) \leq & \int_{\mathbb{R}} J_2(\xi-y)[\underline{\psi}(y)-\underline{\psi}(\xi)]dy \\ & + r[\underline{\psi}(\xi)+b\underline{\phi}(\xi-c\tau_2)\underline{\psi}(\xi-c\tau_2)-\underline{\psi}^2(\xi)]. \end{aligned} \tag{3.2}$$

Then (2.2) *has a positive solution* $(\phi(\xi),\psi(\xi))$ *satisfying*

$$(\underline{\phi}(\xi),\underline{\psi}(\xi)) \leq (\phi(\xi),\psi(\xi)) \leq (\overline{\phi}(\xi),\overline{\psi}(\xi)), \xi \in \mathbb{R}.$$

The proof of the above lemma is similar to that in Pan [22, Theorem 3.2], so we omit it here. Different from that in Zhang et al [29], we do not require the asymptotic behavior when $\xi \to \infty$. Of course, this leads to a weaker result than that in [29].

Lemma 3.2. *If* $c > c^*$, *then* (2.2) *has a positive solution* $(\phi(\xi),\psi(\xi)) \in C_{[0,M]}$.

Proof. Define continuous functions as follows

$$\overline{\phi}(\xi) = \min\{e^{\lambda_1(c)\xi}, 1\}, \overline{\psi}(\xi) = \min\{e^{\lambda_2(c)\xi} + p_1 e^{\eta\lambda_2(c)\xi}, 1+b\},$$
$$\underline{\phi}(\xi) = \max\{e^{\lambda_1(c)\xi} - p_2 e^{\eta\lambda_1(c)\xi}, 0\}, \underline{\psi}(\xi) = \max\{e^{\lambda_2(c)\xi} - p_3 e^{\eta\lambda_3(c)\xi}, 0\},$$

where p_1, p_2, p_3 are constants which will be defined later, and η is a constant satisfying

$$1 < \eta < \min\big\{\frac{\lambda_3(c)}{\lambda_1(c)}, \frac{\lambda_4(c)}{\lambda_2(c)}, 2\big\}.$$

We shall prove that these functions satisfy (3.1) and (3.2) by eight steps.

Step 1. If $\overline{\phi}(\xi) = e^{\lambda_1(c)\xi} < 1$, then

$$\begin{aligned} & \int_{\mathbb{R}} J_1(\xi-y)[\overline{\phi}(y)-\overline{\phi}(\xi)]dy + \alpha e^{-\gamma\tau_1}\left[\overline{\phi}(\xi-c\tau_1)-\overline{\phi}^2(\xi)-a\overline{\phi}(\xi)\underline{\psi}(\xi)\right] \\ & \leq \int_{\mathbb{R}} J_1(\xi-y)[\overline{\phi}(y)-\overline{\phi}(\xi)]dy + \alpha e^{-\gamma\tau_1}\overline{\phi}(\xi-c\tau_1) \\ & \leq \int_{\mathbb{R}} J_1(\xi-y)[e^{\lambda_1(c)y}-e^{\lambda_1(c)\xi}]dy + \alpha e^{-\gamma\tau_1}e^{\lambda_1(c)(\xi-c\tau_1)} \\ & = e^{\lambda_1(c)\xi}\big[\int_{\mathbb{R}} J_1(y)[e^{\lambda_1(c)y}-1]dy + \alpha e^{-\gamma\tau_1}e^{-\lambda_1(c)c\tau_1}\big] \\ & = c\lambda_1(c)e^{\lambda_1(c)\xi} \\ & = c\overline{\phi}'(\xi). \end{aligned}$$

Step 2. If $\overline{\phi}(\xi) = 1 < e^{\lambda_1(c)\xi}$, then

$$
\begin{aligned}
&\int_{\mathbb{R}} J_1(\xi - y)[\overline{\phi}(y) - \overline{\phi}(\xi)]dy + \alpha e^{-\gamma\tau_1}\big[\overline{\phi}(\xi - c\tau_1) - \overline{\phi}^2(\xi) - a\overline{\phi}(\xi)\underline{\psi}(\xi)\big] \\
&\le \int_{\mathbb{R}} J_1(\xi - y)[\overline{\phi}(y) - \overline{\phi}(\xi)]dy + \alpha e^{-\gamma\tau_1}\big[\overline{\phi}(\xi - c\tau_1) - \overline{\phi}^2(\xi)\big] \\
&\le \alpha e^{-\gamma\tau_1}\big[\overline{\phi}(\xi - c\tau_1) - \overline{\phi}^2(\xi)\big] \\
&\le 0 = c\overline{\phi}'(\xi).
\end{aligned}
$$

Step 3. If $\overline{\psi}(\xi) = 1 + b < e^{\lambda_2(c)\xi} + p_1 e^{\eta\lambda_2(c)\xi}$, then

$$
\begin{aligned}
&\int_{\mathbb{R}} J_2(\xi - y)[\overline{\psi}(y) - \overline{\psi}(\xi)]dy + r_1\big[\overline{\psi}(\xi) + b\overline{\phi}(\xi - c\tau_2)\overline{\psi}(\xi - c\tau_2) - \overline{\psi}^2(\xi)\big] \\
&\le r_1\big[\overline{\psi}(\xi) + b\overline{\phi}(\xi - c\tau_2)\overline{\psi}(\xi - c\tau_2) - \overline{\psi}^2(\xi)\big] \\
&\le r_1\big[\overline{\psi}(\xi) + b\overline{\psi}(\xi - c\tau_2) - \overline{\psi}^2(\xi)\big] \\
&\le 0 = c\overline{\psi}'(\xi).
\end{aligned}
$$

Step 4. If $\overline{\psi}(\xi) = e^{\lambda_2(c)\xi} + p_1 e^{\eta\lambda_2(c)\xi} < 1 + b$, then

$$
\xi < \frac{\ln\frac{1+b}{p_1}}{\eta\lambda_2(c)}
$$

and

$$
\begin{aligned}
&\int_{\mathbb{R}} J_2(\xi - y)[\overline{\psi}(y) - \overline{\psi}(\xi)]dy + r_1\big[\overline{\psi}(\xi) + b\overline{\phi}(\xi - c\tau_2)\overline{\psi}(\xi - c\tau_2) - \overline{\psi}^2(\xi)\big] \\
&\le \int_{\mathbb{R}} J_2(\xi - y)[e^{\lambda_2(c)y} + p_1 e^{\eta\lambda_2(c)y} - e^{\lambda_2(c)\xi} - p_1 e^{\eta\lambda_2(c)\xi}]dy \\
&\quad + r_1\Big[\big[e^{\lambda_2(c)\xi} + p_1 e^{\eta\lambda_2(c)\xi}\big] + b e^{\lambda_1(c)\xi}\big[e^{\lambda_2(c)\xi} + p_1 e^{\eta\lambda_2(c)\xi}\big] \\
&\quad - \Big(e^{\lambda_2(c)\xi} + p_1 e^{\eta\lambda_2(c)\xi}\Big)^2\Big] \\
&\le \int_{\mathbb{R}} J_2(\xi - y)[e^{\lambda_2(c)y} + p_1 e^{\eta\lambda_2(c)y} - e^{\lambda_2(c)\xi} - p_1 e^{\eta\lambda_2(c)\xi}]dy \\
&\quad + r_1\big[e^{\lambda_2(c)\xi} + p_1 e^{\eta\lambda_2(c)\xi}\big] + b e^{\lambda_1(c)\xi}\big[e^{\lambda_2(c)\xi} + p_1 e^{\eta\lambda_2(c)\xi}\big] \\
&= \big[\Delta_2(\lambda_2(c), c) + c\lambda_2(c)\big]e^{\lambda_2(c)\xi} + p_1\big[\Delta_2(\eta\lambda_2(c), c) + c\eta\lambda_2(c)\big]e^{\eta\lambda_2(c)\xi} \\
&\quad + b e^{(\lambda_1(c)+\lambda_2(c))\xi} + b p_1 e^{(\eta+1)\lambda_2(c)\xi} \\
&= c\lambda_2(c)e^{\lambda_2(c)\xi} + p_1 c\eta\lambda_2(c)e^{\eta\lambda_2(c)\xi} \\
&\quad + p_1\Delta_2(\eta\lambda_2(c), c)e^{\eta\lambda_2(c)\xi} + b e^{(\lambda_1(c)+\lambda_2(c))\xi} + b p_1 e^{(\eta+1)\lambda_2(c)\xi} \\
&\le c\lambda_2(c)e^{\lambda_2(c)\xi} + p_1 c\eta\lambda_2(c)e^{\eta\lambda_2(c)\xi} \\
&= c\overline{\psi}'(\xi)
\end{aligned}
$$

if

$$
p_1\Delta_2(\eta\lambda_2(c), c)e^{\eta\lambda_2(c)\xi} + b e^{(\lambda_1(c)+\lambda_2(c))\xi} + b p_1 e^{(\eta+1)\lambda_2(c)\xi} \le 0. \tag{3.3}
$$

Clearly, (3.3) holds provided that

$$p_1\Delta_2(\eta\lambda_2(c),c)e^{\eta\lambda_2(c)\xi}+2be^{(\lambda_1(c)+\lambda_2(c))\xi}\le 0, \tag{3.4}$$

$$p_1\Delta_2(\eta\lambda_2(c),c)e^{\eta\lambda_2(c)\xi}+2bp_1e^{(\eta+1)\lambda_2(c)\xi}\le 0. \tag{3.5}$$

Note that $\eta\lambda_2(c)<\lambda_1(c)+\lambda_2(c)$, then (3.4) is true if

$$p_1>1-\frac{2b}{\Delta_2(\eta\lambda_2(c),c)}>1.$$

At the same time, (3.5) is true if $\xi<0$ and

$$\lambda_2(c)\xi\le\ln\frac{2b}{-\Delta_2(\eta\lambda_2(c),c)},$$

which holds provided that

$$\ln\frac{1+b}{p_1}\le 0\le\eta\ln\frac{2b}{-\Delta_2(\eta\lambda_2(c),c)};$$

that is,

$$p_1\ge(1+b)\Big[\Big(\frac{2b}{-\Delta_2(\eta\lambda_2(c),c)}\Big)^{\eta}+1\Big]+1-\frac{2b}{\Delta_2(\eta\lambda_2(c),c)}:=\overline{p}_1.$$

What we have done implies that if $p_1=\overline{p}_1$, then (3.1) is true.

Step 5. If $\underline{\phi}(\xi)=e^{\lambda_1(c)\xi}-p_2e^{\eta\lambda_1(c)\xi}>0$, then

$$\begin{aligned}
&\int_{\mathbb{R}}J_1(\xi-y)[\underline{\phi}(y)-\underline{\phi}(\xi)]dy+\alpha e^{-\gamma\tau_1}\left[\underline{\phi}(\xi-c\tau_1)-\underline{\phi}^2(\xi)-a\underline{\phi}(\xi)\overline{\psi}(\xi)\right]\\
&\ge\int_{\mathbb{R}}J_1(\xi-y)[e^{\lambda_1(c)y}-p_2e^{\eta\lambda_1(c)y}-e^{\lambda_1(c)\xi}+p_2e^{\eta\lambda_1(c)\xi}]dy\\
&\quad+\alpha e^{-\gamma\tau_1}\left[e^{\lambda_1(c)(\xi-c\tau_1)}-(e^{\lambda_1(c)\xi}-p_2e^{\eta\lambda_1(c)\xi})^2\right]\\
&\quad-a\alpha e^{-\gamma\tau_1}(e^{\lambda_1(c)\xi}-p_2e^{\eta\lambda_1(c)\xi})(e^{\lambda_2(c)\xi}+\overline{p}_1e^{\eta\lambda_2(c)\xi})\\
&\ge c\lambda_1(c)e^{\lambda_1(c)\xi}-c\eta p_2\lambda_1(c)e^{\eta\lambda_1(c)\xi}-p_2\Delta_1(\eta\lambda_1(c),c)e^{\eta\lambda_1(c)\xi}\\
&\quad-\alpha e^{-\gamma\tau_1}e^{2\lambda_1(c)\xi}-a\alpha e^{-\gamma\tau_1}e^{(\lambda_1(c)+\lambda_2(c))\xi}-a\alpha e^{-\gamma\tau_1}\overline{p}_1e^{(\lambda_1(c)+\eta\lambda_2(c))\xi}\\
&\ge c\lambda_1(c)e^{\lambda_1(c)\xi}-c\eta p_2\lambda_1(c)e^{\eta\lambda_1(c)\xi}=c\underline{\phi}'(\xi)
\end{aligned}$$

provided that

$$\begin{aligned}
&-p_2\Delta_1(\eta\lambda_1(c),c)e^{\eta\lambda_1(c)\xi}\\
&\ge e^{2\lambda_1(c)\xi}+a\alpha e^{-\gamma\tau_1}e^{(\lambda_1(c)+\lambda_2(c))\xi}+a\alpha e^{-\gamma\tau_1}\overline{p}_1e^{(\lambda_1(c)+\eta\lambda_2(c))\xi},
\end{aligned}$$

which holds when

$$p_2=\frac{1+a\alpha e^{-\gamma\tau_1}+a\alpha e^{-\gamma\tau_1}\overline{p}_1}{-\Delta_1(\eta\lambda_1(c),c)}+1>1.$$

Step 6. If $\underline{\phi}(\xi)=0>e^{\lambda_1(c)\xi}-p_2e^{\eta\lambda_1(c)\xi}$, then the result is clear.

Step 7. If $\underline{\psi}(\xi)=e^{\lambda_2(c)\xi}-p_3e^{\eta\lambda_3(c)\xi}>0$, then

$$\begin{aligned}
&\int_{\mathbb{R}}J_2(\xi-y)[\underline{\psi}(y)-\underline{\psi}(\xi)]dy+r_1[\underline{\psi}(\xi)+b\underline{\phi}(\xi-c\tau_2)\underline{\psi}(\xi-c\tau_2)-\underline{\psi}^2(\xi)]\\
&\ge\int_{\mathbb{R}}J_2(\xi-y)[\underline{\psi}(y)-\underline{\psi}(\xi)]dy+r_1[\underline{\psi}(\xi)-\underline{\psi}^2(\xi)]
\end{aligned}$$

$$\begin{aligned}
&\geq \int_{\mathbb{R}} J_2(\xi-y)[e^{\lambda_2(c)y} - p_3 e^{\eta\lambda_3(c)y} - e^{\lambda_2(c)\xi} + p_3 e^{\eta\lambda_3(c)\xi}]dy \\
&\quad + r_1[e^{\lambda_2(c)\xi} - p_3 e^{\eta\lambda_3(c)\xi}] - r_1(e^{\lambda_2(c)\xi} - p_3 e^{\eta\lambda_3(c)\xi})^2 \\
&\geq c\lambda_2(c)e^{\lambda_2(c)\xi} - p_3 c\eta\lambda_2(c)e^{\eta\lambda_2(c)\xi} - p_3\Delta_2(\eta\lambda_2(c),c)e^{\eta\lambda_2(c)\xi} - r_1 e^{2\lambda_2(c)\xi} \\
&\geq c\lambda_2(c)e^{\lambda_2(c)\xi} - p_3 c\eta\lambda_2(c)e^{\eta\lambda_2(c)\xi} \\
&= c\underline{\psi}'(\xi)
\end{aligned}$$

provided that $p_3 = \frac{r_1}{-\Delta_2(\eta\lambda_2(c),c)} + 1$, and so (3.2) holds.

Step 8. If $\underline{\psi}(\xi) = 0 > e^{\lambda_2(c)\xi} - p_3 e^{\eta\lambda_3(c)\xi}$, then the result is clear.

By Lemma 3.1, the proof is complete. □

By Carr and Chmaj [2], Li et al. [16] and Wu and Ruan [27], we have the following result of scalar equations.
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Lemma 3.3. *Assume that*

$$\inf_{\lambda>0}\Big[\frac{\int_{-\infty}^{+\infty} J_1(y)(e^{\lambda y}-1)dy + \alpha e^{-\gamma\tau_1}e^{-\lambda c\tau_1}}{\lambda}\Big] < \inf_{\lambda>0}\Big[\frac{\int_{-\infty}^{+\infty} J_2(y)(e^{\lambda y}-1)dy + r_1}{\lambda}\Big].$$

Then when $c = c^*$, the scalar equation

$$c\psi'(\xi) = \int_{\mathbb{R}} J_2(\xi-y)[\psi(y)-\psi(\xi)]dy + r_1\psi(\xi) - \psi^2(\xi), \xi \in \mathbb{R} \tag{3.6}$$

has a strictly positive solution satisfying

$$\lim_{\xi\to-\infty}\psi(\xi) = 0, \quad \lim_{\xi\to\infty}\psi(\xi) = r_1, \quad \lim_{\xi\to-\infty}\frac{\psi(\xi)}{\xi e^{-\overline{\lambda}_2\xi}} = -1.$$

Lemma 3.4. *Assume that*

$$\begin{aligned}
&\inf_{\lambda>0}\Big[\frac{\int_{-\infty}^{+\infty} J_1(y)(e^{\lambda y}-1)dy + \alpha e^{-\gamma\tau_1}e^{-\lambda c\tau_1}}{\lambda}\Big] \\
&< \inf_{\lambda>0}\Big[\frac{\int_{-\infty}^{+\infty} J_2(y)(e^{\lambda y}-1)dy + r_1}{\lambda}\Big] = c^*.
\end{aligned}$$

If $\overline{\lambda}_2 \leq \lambda_1(c^*)$, then (2.2) with $c = c^*$ has a positive solution $(\phi(\xi),\psi(\xi)) \in C_{[0,M]}$.

Proof. We shall construct continuous functions satisfying (3.1) and (3.2). Let

$$\overline{\phi}(\xi) = \min\{e^{\lambda_1(c)\xi}, 1\}, \quad \underline{\phi}(\xi) = \max\{e^{\lambda_1(c)\xi} - p_2 e^{\eta\lambda_1(c)\xi}, 0\},$$

where $p_2 > 1$ is a positive constants specified later and η satisfies

$$1 < \eta\lambda_1(c) < \min\{\lambda_3(c), \lambda_1(c) + \overline{\lambda}_2/4, 3\lambda_1(c)/2\}.$$

Define

$$\underline{\psi}(\xi) = \widetilde{\psi}(\xi),$$

where $\widetilde{\psi}(\xi)$ is the positive solution of (3.6) and satisfies Lemma 3.3. Further define

$$\overline{\psi}(\xi) = \begin{cases} 1+b, & \xi \geq \xi_1, \\ (M-2\xi)e^{\overline{\lambda}_2\xi}, & \xi < \xi_1, \end{cases}$$

where $M > 0$ is a constant clarified later. Clearly, if $M > 1 + 1/b$ is large, then $(M-2\xi)e^{\overline{\lambda}_2\xi} = 1 + b$ has two real roots, and here ξ_1 is the smaller root.

We now verify that these functions satisfy (3.1) and (3.2). In particular, the inequalities about $\overline{\phi}(\xi), \underline{\psi}(\xi)$ are clear. Moreover, if $\xi < \xi_1$, then

$$b\phi(\xi - c\tau_2)\psi(\xi - c\tau_2) - \psi^2(\xi) < 0$$

and the inequalities on $\underline{\psi}(\xi)$ is true.

Moreover $p_2 > 1$ such that

$$e^{\lambda_1(c)\xi} - p_2 e^{\eta\lambda_1(c)\xi} > 0$$

implies that $\xi < \xi_1$ and

$$0 < (M - 2\xi)e^{\overline{\lambda}_2\xi} < e^{\overline{\lambda}_2\xi/2}.$$

Similar to that in Lemma 3.2, we see that

$$\int_{\mathbb{R}} J_1(\xi - y)[\underline{\phi}(y) - \underline{\phi}(\xi)]dy + \alpha e^{-\gamma\tau_1}\left[\underline{\phi}(\xi - c\tau_1) - \underline{\phi}^2(\xi) - a\underline{\phi}(\xi)\overline{\psi}(\xi)\right] \geq c^*\underline{\phi}'(\xi)$$

if $p_2 > 1$ is large enough. By Lemma 3.1, the result follows. □

4. Nonexistence of traveling wave solutions: $c < c^*$

In this section, we shall prove that if $c < c^*$, then (2.2) does not have a positive solution satisfying (2.3). We first consider the following initial value problem by Fang and Zhao [7], Jin and Zhao [15]

$$\begin{aligned} \frac{\partial w(x,t)}{\partial t} &= \int_{\mathbb{R}} J(x-y)[w(y,t) - w(x,t)] \\ &\quad + dw(x,t-\tau) + fw(x,t) - gw^2(x,t), \quad t > 0, \\ w(x,s) &= \varphi(x,s), \quad s \in [-\tau, 0], \end{aligned} \tag{4.1}$$

where $x \in \mathbb{R}, \tau \geq 0, d \geq 0, d + f > 0, g > 0$ and $\varphi(x,s)$ is bounded and uniformly continuous in $(x,s) \in \mathbb{R} \times [-\tau, 0]$.

Lemma 4.1. *Assume that J satisfies (J1) and (J2). Define*

$$c_0 = \inf_{\lambda > 0}\Big[\frac{\int_{-\infty}^{+\infty} J(y)(e^{\lambda y} - 1)dy + de^{-\lambda c\tau} + f}{\lambda}\Big].$$

If $\varphi(x,s)$ has nonempty support for each $s \in [-\tau, 0]$, then

$$\lim_{t\to\infty} \sup_{|x|\leq ct} w(x,t) = \lim_{t\to\infty} \inf_{|x|\leq ct} w(x,t) = \frac{g}{d+f}$$

for each $c < c_0$.

Lemma 4.2. *Assume that J satisfies* (J1) *and* (J2). *If $\overline{w}(x,t)$ satisfies*

$$\begin{aligned} \frac{\partial \overline{w}(x,t)}{\partial t} &\geq \int_{\mathbb{R}} J(x-y)[\overline{w}(y,t) - \overline{w}(x,t)] \\ &\quad + d\overline{w}(x,t-\tau) + f\overline{w}(x,t) - g\overline{w}^2(x,t), \quad t > 0, \\ \overline{w}(x,s) &\geq \varphi(x,s), \quad s \in [-\tau, 0] \end{aligned} \tag{4.2}$$

for $x \in \mathbb{R}$, then

$$\overline{w}(x,t) \geq w(x,t), \quad x \in \mathbb{R},\ t > 0.$$

By analysis, we have the following result.

Lemma 4.3. *Assume that $(\phi(\xi),\psi(\xi))$ is a bounded positive solution of (2.2). If $\phi(\xi_1)>0$ for some $\xi_1\in\mathbb{R}$, then $\phi(\xi)>0$ for all $\xi\in\mathbb{R}$, if $\psi(\xi_2)>0$ for some $\xi_2\in\mathbb{R}$, then $\psi(\xi)>0$ for all $\xi\in\mathbb{R}$. Moreover, $\phi(\xi),\psi(\xi)$ satisfy*

$$0\le\phi(\xi)\le 1, 0\le\psi(\xi)\le 1+b, \xi\in\mathbb{R}.$$

Theorem 4.4. *If $c<c^*$, then (2.2) does not have a positive solution satisfying (2.3).*

Proof. Were the statement false, then for some $c_1<c^*$, (2.2) has a positive solution satisfying (2.3). That is, there exist $(\phi(\xi),\psi(\xi))$ satisfying

$$\begin{aligned} c_1\phi'(\xi)&=\int_{\mathbb{R}}J_1(\xi-y)[\phi(y)-\phi(\xi)]dy\\ &\quad+\alpha e^{-\gamma\tau_1}[\phi(\xi-c\tau_1)-\phi^2(\xi)-a\phi(\xi)\psi(\xi)],\\ c_1\psi'(\xi)&=\int_{\mathbb{R}}J_2(\xi-y)[\psi(y)-\psi(\xi)]dy\\ &\quad+r_1[\psi(\xi)+b\phi(\xi-c\tau_2)\psi(\xi-c\tau_2)-\psi^2(\xi)], \end{aligned}\tag{4.3}$$

and

$$\lim_{\xi\to-\infty}(\phi(\xi),\psi(\xi))=(0,0),\quad \lim_{\xi\to\infty}(\phi(\xi),\psi(\xi))=(k_1,k_2).\tag{4.4}$$

If

$$c^*=\inf_{\lambda>0}\Big[\frac{\int_{-\infty}^{+\infty}J_1(y)(e^{\lambda y}-1)dy+\alpha e^{-\gamma\tau_1}e^{-\lambda c\tau_1}}{\lambda}\Big],$$

then there exists $\epsilon\in(0,\alpha e^{-\gamma\tau_1})$ such that

$$c_1<\inf_{\lambda>0}\Big[\frac{\int_{-\infty}^{+\infty}J_1(y)(e^{\lambda y}-1)dy+\alpha e^{-\gamma\tau_1}e^{-\lambda c\tau_1}-2\epsilon}{\lambda}\Big]=:c_2.$$

By (4.4), there exists $T\in\mathbb{R}$ such that

$$a\alpha e^{-\gamma\tau_1}\psi(\xi)<\epsilon,\quad \xi\le T,$$

and so

$$\begin{aligned}&\alpha e^{-\gamma\tau_1}[\phi(\xi-c\tau_1)-\phi^2(\xi)-a\phi(\xi)\psi(\xi)]\\ &\ge\alpha e^{-\gamma\tau_1}\phi(\xi-c\tau_1)-\epsilon\phi(\xi)-\alpha e^{-\gamma\tau_1}\phi^2(\xi),\quad \xi\le T.\end{aligned}$$

If $\xi>T$, then (4.4) and Lemma 4.3 imply that there exists $M>0$ such that

$$a\alpha e^{-\gamma\tau_1}\phi(\xi)\psi(\xi)<M\phi^2(\xi).$$

Therefore, $\psi(\xi)$ satisfies

$$c_1\phi'(\xi)\ge\int_{\mathbb{R}}J_1(\xi-y)[\phi(y)-\phi(\xi)]dy+\alpha e^{-\gamma\tau_1}\phi(\xi-c\tau_1)-\epsilon\phi(\xi)-(M+\alpha e^{-\gamma\tau_1})\phi^2(\xi)$$

for all $\xi\in\mathbb{R}$. Since $\xi=x+c_1t$, we have

$$\begin{gathered}\frac{\partial u(x,t)}{\partial t}\ge(D_1u)(x,t)+\alpha e^{-\gamma\tau_1}u(x,t-\tau_1)-\epsilon u(x,t)\\ -(M+\alpha e^{-\gamma\tau_1})u^2(x,t)],\quad t>0,\\ u(x,-s)=\phi(x+c_1s),\quad s\in[-\tau_1,0].\end{gathered}\tag{4.5}$$

By Lemmas 4.1 and 4.2, we have

$$\lim_{t\to\infty}\inf_{|x|\le c_2t}u(x,t)\ge\frac{e^{-\gamma\tau_1}-\epsilon}{M+\alpha e^{-\gamma\tau_1}}>0.$$

On the other hand, letting $-x = c_2 t$, we have

$$x + c_1 t = (c_1 - c_2)t \to -\infty, \quad t \to \infty$$

and so $u(-c_2 t, t) = \phi(-c_2 t + c_1 t) \to 0$, as $t \to \infty$, which is a contradiction.

If

$$c^* = \inf_{\lambda > 0} \Big[\frac{\int_{-\infty}^{+\infty} J_2(y)(e^{\lambda y} - 1)dy + r_1}{\lambda} \Big],$$

then there exists $\iota \in (0,1)$ such that

$$c_1 < \inf_{\lambda > 0} \Big[\frac{\int_{-\infty}^{+\infty} J_2(y)(e^{\lambda y} - 1)dy + r_1(1-\iota)}{\lambda} \Big] := c_3.$$

At the same time, $\psi(x + c_1 t) = v(x,t)$ satisfies

$$\begin{gathered} \frac{\partial v(x,t)}{\partial t} \geq (D_2 v)(x,t) + r_1[v(x,t) - v^2(x,t)], \quad t > 0, \\ v(x,0) = \psi(x), \end{gathered} \tag{4.6}$$

where $x \in \mathbb{R}$. By Lemmas 4.1 and 4.2, we see that

$$\lim_{t \to \infty} \inf_{|x| \leq c_3 t} v(x,t) \geq 1 > 0.$$

On the other hand, letting $-x = c_3 t$, we have

$$x + c_1 t = (c_1 - c_3)t \to -\infty, t \to \infty$$

and so $v(-c_3 t, t) = \phi(-c_3 t + c_1 t) \to 0$, as $t \to \infty$, which is also a contradiction. The proof is complete. $\square$

By the same process as above, we can obtain the following result.

Corollary 4.5. *If $c < c^*$, then* (2.2) *does not have a positive solution satisfying*

$$\lim_{\xi \to -\infty} (\phi(\xi), \psi(\xi)) = (0,0), \quad \liminf_{\xi \to \infty} (\phi(\xi), \psi(\xi)) \gg (0,0).$$

5. Asymptotic behavior of traveling wave solutions

In this section, we study the asymptotic behavior of the traveling wave solutions obtained in Section 3. The method is based on the idea of contracting rectangles, which was earlier used by Lin and Ruan [20] in studying the asymptotic behavior of traveling wave solutions of delayed reaction-diffusion systems. For $s \in [0,1]$, define

$$\begin{gathered} \underline{a}(s) = sk_1 + (1-s)(1-ab)(1-a)(1-\varepsilon_1), \\ \overline{a}(s) = sk_1 + (1-s)(1-a)(1+\varepsilon_2), \\ \underline{b}(s) = sk_2 + (1-s)(1-\varepsilon_3), \\ \overline{b}(s) = sk_2 + (1-s)(1+b(1-a))(1+\varepsilon_4), \end{gathered}$$

where $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \in (0,1)$ with

$$(1-ab)(1-a)\varepsilon_1 = 2a(1+b(1-a))\varepsilon_4, \tag{5.1}$$

$$(1+b(1-a))\varepsilon_4 = 2b(1-a)\varepsilon_2, \tag{5.2}$$

$$(1-a)\varepsilon_2 = 2a\varepsilon_3. \tag{5.3}$$

We now illustrate that $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \in (0,1)$ are admissible. Let $\overline{\varepsilon}_1 = 1$ and

$$\overline{\varepsilon}_4 = \frac{(1-ab)(1-a)}{2a(1+b(1-a))}, \quad \overline{\varepsilon}_2 = \frac{(1+b(1-a))}{2b(1-a)} \overline{\varepsilon}_4, \quad \overline{\varepsilon}_3 = \frac{1-a}{2a} \overline{\varepsilon}_2.$$

For any $c > 0$, then
$$(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) = (c, c\bar{\varepsilon}_2, c\bar{\varepsilon}_3, c\bar{\varepsilon}_4)$$
satisfy (5.1)-(5.3). Clearly, $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \in (0, 1)$ if $c > 0$ is small enough.

Lemma 5.1. *For each $s \in (0, 1)$, we have*
$$1 - \underline{a}(s) - a\bar{b}(s) > 0, \tag{5.4}$$
$$1 - \bar{a}(s) - a\underline{b}(s) < 0, \tag{5.5}$$
$$1 + b\underline{a}(s) - \underline{b}(s) > 0, \tag{5.6}$$
$$1 + b\bar{a}(s) - \bar{b}(s) < 0. \tag{5.7}$$

Proof. If $s \in (0, 1)$, then
$$\begin{aligned}
1 - \underline{a}(s) - a\bar{b}(s) &= 1 - sk_1 - (1 - s)(1 - ab)(1 - a)(1 - \varepsilon_1) \\
&\quad - ask_2 - a(1 - s)(1 + b(1 - a))(1 + \varepsilon_4) \\
&= (1 - s)[1 - (1 - ab)(1 - a)(1 - \varepsilon_1) - a(1 + b(1 - a))(1 + \varepsilon_4)] \\
&> (1 - s)[(1 - ab)(1 - a)\varepsilon_1 - a(1 + b(1 - a))\varepsilon_4] \\
&= (1 - s)a(1 + b(1 - a))\varepsilon_4 > 0,
\end{aligned}$$
by $(1 - ab)(1 - a)\varepsilon_1 = 2a(1 + b(1 - a))\varepsilon_4$. The above inequality implies (5.4).

Since $2a\varepsilon_3 = (1 - a)\varepsilon_2$, we have
$$\begin{aligned}
1 - \bar{a}(s) - a\underline{b}(s) &= 1 - sk_1 - (1 - s)(1 - a)(1 + \varepsilon_2) - ask_2 - a(1 - s)(1 - \varepsilon_3) \\
&= (1 - s)[1 - (1 - a)(1 + \varepsilon_2) - a(1 - \varepsilon_3)] \\
&= (1 - s)\left[a\varepsilon_3 - (1 - a)\varepsilon_2\right] \\
&= -(1 - s)a\varepsilon_3 < 0,
\end{aligned}$$
which implies (5.5).

Moreover, (5.6) holds since
$$\begin{aligned}
1 + b\underline{a}(s) - \underline{b}(s) &= 1 - sk_2 - (1 - s)(1 - \varepsilon_3) + bsk_1 \\
&\quad + b(1 - s)(1 - ab)(1 - a)(1 - \varepsilon_1) \\
&= (1 - s)(\varepsilon_3 + b(1 - ab)(1 - a)(1 - \varepsilon_1)) > 0.
\end{aligned}$$

Note that $2b(1 - a)\varepsilon_2 = (1 + b(1 - a))\varepsilon_4$. Then
$$\begin{aligned}
1 + b\bar{a}(s) - \bar{b}(s) &= 1 - sk_2 - (1 - s)(1 + b(1 - a))(1 + \varepsilon_4) + sbk_1 \\
&\quad + b(1 - s)(1 - a)(1 + \varepsilon_2) \\
&= (1 - s)(1 - (1 + b(1 - a))(1 + \varepsilon_4) + b(1 - a)(1 + \varepsilon_2)) \\
&= (1 - s)(b(1 - a)\varepsilon_2 - (1 + b(1 - a))\varepsilon_4) \\
&= -(1 - s)b(1 - a)\varepsilon_2 < 0,
\end{aligned}$$
which implies (5.7). The proof is complete. □

Lemma 5.2. *If $(\phi(\xi), \psi(\xi))$ is a positive solution of (2.2), then*
$$(1 - ab)(1 - a) \le \liminf_{\xi\to\infty} \phi(\xi) \le \limsup_{\xi\to\infty} \phi(\xi) \le 1 - a,$$
$$1 \le \liminf_{\xi\to\infty} \psi(\xi) \le \limsup_{\xi\to\infty} \psi(\xi) \le 1 + b(1 - a).$$

Proof. By the definition, $\psi(x+ct)=v(x,t)$ satisfies

$$\begin{gathered}\frac{\partial v(x,t)}{\partial t}\geq (D_2v)(x,t)+r_1[v(x,t)-v^2(x,t)],\quad t>0,\\ v(x,0)=\psi(x),\end{gathered}\tag{5.8}$$

where $x\in\mathbb{R}$. By Lemmas 4.1 and 4.2, we have

$$\liminf_{t\to\infty} v(0,t)\geq 1>0.$$

which implies

$$\liminf_{\xi\to\infty}\psi(\xi)\geq 1.$$

Let $\beta>0$. Note that $\phi(\xi)$ and $\psi(\xi)$ are bounded and positive, then there exists $\beta>0$ such that

$$\beta\phi(s)-\phi(s)\int_{\mathbb{R}}J_1(y)dy+\alpha e^{-\gamma\tau_1}[\phi(s-c\tau_1)-\phi^2(s)-a\phi(s)\psi(s)]$$

is monotone increasing in $\phi(s)$ and

$$\beta\psi(s)-\psi(s)\int_{\mathbb{R}}J_2(y)dy+r_1[\psi(s)-\psi^2(s)+b\phi(s-c\tau_2)\psi(s-c\tau_2)]$$

is monotone increasing in $\psi(s)$. Moreover, $\phi(\xi)$ and $\psi(\xi)$ also satisfy

$$\begin{aligned}\phi(\xi)=\frac{1}{c}\int_{-\infty}^{\xi}&e^{-\frac{\beta(\xi-s)}{c}}\int_{\mathbb{R}}J_1(s-y)[\phi(y)-\phi(s)]dyds\\ &+\int_{-\infty}^{\xi}\big\{\beta\phi(s)+\alpha e^{-\gamma\tau_1}[\phi(s-c\tau_1)-\phi^2(s)-a\phi(s)\psi(s)]\big\}ds,\\ \psi(\xi)=\frac{1}{c}\int_{-\infty}^{\xi}&e^{-\frac{\beta(\xi-s)}{c}}\int_{\mathbb{R}}J_2(s-y)[\psi(y)-\psi(s)]dyds\\ &+\int_{-\infty}^{\xi}\big\{\beta\psi(s)+r_1[\psi(s)-\psi^2(s)+b\phi(s-c\tau_2)\psi(s-c\tau_2)]\big\}ds.\end{aligned}$$

Since $\liminf_{\xi\to\infty}\psi(\xi)\geq 1$. Applying Fatou's lemma in the integral equation of $\phi(\xi)$, we see that

$$\alpha e^{-\gamma\tau_1}\Big[\limsup_{\xi\to\infty}\phi(\xi)-(\limsup_{\xi\to\infty}\phi(\xi))^2-a\limsup_{\xi\to\infty}\phi(\xi)\Big]\geq 0;$$

then the boundedness of $\limsup_{\xi\to\infty}\phi(\xi)$ indicates that

$$\limsup_{\xi\to\infty}\phi(\xi)\leq 1-a.$$

Further applying Fatou's lemma in the integral equation of $\psi(\xi)$, we see that $\limsup_{\xi\to\infty}\psi(\xi)\geq 1$, and

$$\limsup_{\xi\to\infty}\psi(\xi)-\Big(\limsup_{\xi\to\infty}\psi(\xi)\Big)^2+b(1-a)\limsup_{\xi\to\infty}\psi(\xi)\geq 0,$$

which leads to

$$\limsup_{\xi\to\infty}\psi(\xi)\leq 1+b(1-a).$$

Returning to the integral equation of $\phi(\xi)$, we see that

$$\liminf_{\xi\to\infty}\phi(\xi)\geq(1-ab)(1-a)$$

if $\liminf_{\xi\to\infty}\phi(\xi)>0$. In fact, by Lemma 4.3, we see that $\phi(\xi)$ satisfies

$$c\phi'(\xi)\ge\int_{\mathbb{R}}J_1(\xi-y)[\phi(y)-\phi(\xi)]dy+\alpha e^{-\gamma\tau_1}[\phi(\xi-c\tau_1)-a(1+b)\phi(\xi)-\phi^2(\xi)].$$

That is, $u(x,t)=\phi(x+ct)$ satisfies

$$\begin{gathered}\frac{\partial u(x,t)}{\partial t}=(D_1u)(x,t)+\alpha e^{-\gamma\tau_1}[u(x,t-\tau_1)-a(1+b)u(x,t)-u^2(x,t)],\quad t>0,\\ u(x,s)=\phi(x+cs),\quad s\in[-\tau_1,0],\end{gathered}$$

where $x\in\mathbb{R}$. By Lemmas 4.1 and 4.2, we see that

$$\liminf_{t\to\infty}u(0,t)\ge 1-a(1+b)>0,$$

which implies that

$$\liminf_{\xi\to\infty}\phi(\xi)>1-a(1+b)>0$$

by the invariant form of traveling wave solutions. The proof is complete. □

Lemma 5.3. *If $(\phi(\xi),\psi(\xi))$ is a positive solution of (2.2), then*

$$\lim_{\xi\to\infty}(\phi(\xi),\psi(\xi))=(k_1,k_2).$$

Proof. By Lemma 5.2, we see that there exists $s_1\in(0,1)$ such that

$$\begin{gathered}\underline{a}(s)\le\liminf_{\xi\to\infty}\phi(\xi)\le\limsup_{\xi\to\infty}\phi(\xi)\le\overline{a}(s),\\ \underline{b}(s)\le\liminf_{\xi\to\infty}\psi(\xi)\le\limsup_{\xi\to\infty}\psi(\xi)\le\overline{b}(s)\end{gathered}\tag{5.9}$$

for all $s\le s_1$ since $\underline{a}(s),\overline{a}(s),\underline{b}(s),\overline{b}(s)$ are continuous and monotone, and

$$\begin{gathered}\underline{a}(0)<(1-ab)(1-a)\le 1-a<\overline{a}(0),\\ \underline{b}(0)<1<1+b(1-a)<\overline{b}(0).\end{gathered}$$

Define

$$s_0=\sup_{s\in(0,1]}\{(5.9)\text{ hold}\}.$$

Then s_0 is well defined.

If $s_0=1$, then the result is true. We now assume that $s_0<1$. Without loss of generality, we suppose that

$$\begin{gathered}\underline{a}(s_0)=\liminf_{\xi\to\infty}\phi(\xi),\\ \underline{a}(s_0)=\liminf_{\xi\to\infty}\phi(\xi)\le\limsup_{\xi\to\infty}\phi(\xi)\le\overline{a}(s_0),\\ \underline{b}(s_0)\le\liminf_{\xi\to\infty}\psi(\xi)\le\limsup_{\xi\to\infty}\psi(\xi)\le\overline{b}(s_0).\end{gathered}\tag{5.10}$$

By the definition of lim inf, there exist a sequence $\{\xi_m\}$ such that

$$\begin{gathered}\lim_{m\to\infty}\xi_m=\infty,\quad\lim_{m\to\infty}\phi(\xi_m)=\underline{a}(s_0),\quad\lim_{m\to\infty}\phi'(\xi_m)=0,\\ \liminf_{m\to\infty}\Big[\int_{\mathbb{R}}J_1(\xi_m-y)[\phi(y)-\phi(\xi_m)]dy\Big]\ge 0.\end{gathered}$$

At the same time, (5.4) implies that

$$\liminf_{m\to\infty}\alpha e^{-\gamma\tau_1}[\phi(\xi_m-c\tau_1)-\phi^2(\xi_m)-a\phi(\xi_m)\psi(\xi_m)]$$

$$\begin{aligned}&\geq \alpha e^{-\gamma\tau_1}[\underline{a}(s_0) - \underline{a}^2(s_0) - a\underline{a}(s_0)\bar{b}(s_0)]\\ &= \alpha e^{-\gamma\tau_1}\underline{a}(s_0)[1 - \underline{a}(s_0) - a\bar{b}(s_0)] > 0\,.\end{aligned}$$

This is a contradiction, so $s_0 = 1$. The proof is complete. $\square$

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