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EXISTENCE OF PERIODIC SOLUTIONS FOR HIGHER-ORDER NONLINEAR DIFFERENCE EQUATIONS

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Abstract. In this article, we study a higher-order nonlinear difference equation. By using critical point theory, we establish sufficient conditions for the existence of periodic solutions.

1. Introduction

Difference equations, the discrete analogs of differential equations, have attracted the interest of many researchers in the past twenty years since they provided a natural description of several discrete models. Such discrete models occur in numerous settings and forms, both in mathematics and in its applications to computer science, economics, neural networks, ecology, cybernetics, biological systems, optimal control, and population dynamics. These studies cover many of the branches of difference equations, such as stability, attractivity, periodicity, oscillation, homoclinic orbits, and boundary value problems [1, 2, 3, 4, 6, 11, 12, 13, 14, 16, 17, 18, 19, 20, 22, 24, 25, 26, 27, 28, 30]. Only a few papers discuss the periodic solutions of higher-order difference equations. Therefore, it is worthwhile to explore this topic.

Let $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{R}$ denote the sets of all natural numbers, integers and real numbers respectively. For any a, b in $\mathbb{Z}$, define $\mathbb{Z}(a) = \{a, a+1, \dots\}$, $\mathbb{Z}(a, b) = \{a, a+1, \dots, b\}$ when $a < b$. Let the symbol $*$ denote the transpose of a vector. Moreover, for all $n \in \mathbb{N}$, $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^n$ defined by

$$|X| = \Big(\sum_{i=1}^{n} X_i^2\Big)^{1/2}, \quad \forall X = (X_1, X_2, \dots, X_n) \in \mathbb{R}^n.$$

This article considers the higher order nonlinear difference equation

$$\sum_{i=0}^{n} r_i(X_{k-i} + X_{k+i}) + f(k, X_{k+\Gamma}, \dots, X_k, \dots, X_{k-\Gamma}) = 0, \quad n \in \mathbb{N},\ k \in \mathbb{Z}, \tag{1.1}$$

where r_i is real valued for $i \in \mathbb{Z}$, Γ is a nonnegative integer, m is a positive integer, $f = (f_1, f_2, \dots, f_m)^* \in C(\mathbb{R}^{2\Gamma+2} \times \mathbb{R}^m, \mathbb{R})$, $f(k, Y_\Gamma, \dots, Y_0, \dots, Y_{-\Gamma})$ is T-periodic in k for a given positive integer T.

As usual, a solution X_k of (1.1) is said to be periodic of period T if

$$X_{k+T} = X_k, \quad \forall k \in \mathbb{Z}.$$

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If $m = 1$, $n = 1$, $\Gamma = 1$, $r_0 = -1$, $r_1 = 1$, then (1.1) can be reduced to the second-order difference equation

$$\Delta^2 u_{k-1} = f(k, u_{k+1}, u_k, u_{k-1}), \quad k \in \mathbb{Z}. \tag{1.2}$$

This equation can be seen as an analogue discrete form of the second-order functional differential equation

$$\frac{d^2 u(t)}{dt^2} = f(t, u(t+1), u(t), u(t-1)), \quad t \in \mathbb{R}. \tag{1.3}$$

Equations similar in structure to (1.3) arise in the study of the existence of solitary waves of lattice differential equations, periodic solutions and homoclinic orbits of functional differential equations, see [8, 9, 29].

Migda [22] in 2004 studied the existence of nonoscillatory solutions of a higher order linear difference equation of the form,

$$\Delta^m u_k + \delta a_k u_{k+1} = 0, \quad k \in \mathbb{Z}. \tag{1.4}$$

In 2007, Cai and Yu [2] obtained some criteria for the existence of periodic solutions of a $2n$th-order difference equation

$$\Delta^n (r_{k-n} \Delta^n u_{k-n}) + f(k, u_k) = 0, \quad n \in \mathbb{Z}(3),\ k \in \mathbb{Z}, \tag{1.5}$$

by using the critical point theory.

Shi and Zhang [27] considered the existence of periodic solutions for the $2n$th-order nonlinear difference equation

$$\Delta^n (r_{k-n} \Delta^n u_{k-n}) = (-1)^n f(k, u_{k+1}, u_k, u_{k-1}), \quad n \in \mathbb{Z}(3),\ k \in \mathbb{Z}, \tag{1.6}$$

by using the Saddle Point Theorem in combination with variational technique. (1.6) can be seen a special form of system (1.1) with $m = 1$ and $\Gamma = 1$.

When the nonlinear term of (1.6) is neither superlinear nor sublinear, Xia, Zhang and Shi [18] obtained some criteria for the existence and multiplicity of periodic and subharmonic solutions of (1.6).

If $\Gamma = 0$, Hu [13] in 2014 and Hu, Huang [14] in 2008 applied the critical point theorem and Lyapunov-Schmidt reduction respectively to prove the existence of periodic solution of a higher order difference equation as the type

$$\sum_{i=0}^{n} r_i (X_{k-i} + X_{k+i}) + f(k, X_k) = 0, \quad n \in \mathbb{N},\ k \in \mathbb{Z}. \tag{1.7}$$

Fixed point theorems in cones have been used widely for the existence of periodic solutions of difference equations, see [1]. Also critical point theory which is a powerful tool have been used for differential equations, see [8, 9, 10, 21, 23]. Only since 2003, critical point theory has been employed to establish sufficient conditions on the existence of periodic solutions of difference equations. Compared to first-order or second-order difference equations, the study of higher-order equations has received considerably less attention; see [1, 2, 3, 4, 5, 6, 7, 11, 12, 13, 14, 16, 17, 18, 19, 20, 22, 24, 25, 26, 27, 28, 30]. However, to the best of our knowledge, results obtained in the literature on the periodic solutions of (1.1) are very scarce. Since f in (1.1) depends on $X_{k+\Gamma}, \dots, X_k, \dots, X_{k-\Gamma}$, the traditional ways of establishing the functional in [5, 11, 12, 13, 14, 30] are not applicable to our case. The main purpose of this article is to establish sufficient conditions for the existence of periodic solutions to (1.1). Also some nonexistence conditions of nontrivial periodic solutions to (1.1) are also presented. We remark that such results are scarce in the literature.

On the one hand, we demonstrate the usefulness of critical point theory in the study of the existence of periodic solutions of difference equations. On the other hand, we extend existing results, as stated in Remarks 1.2 and 1.3. The motivation for the present work stems from the recent papers [6, 18, 27]. For basic knowledge of variational methods, the reader is referred to [21, 23].

In this article we use the following hypotheses:

(H1) $r_0 + \sum_{s=1}^{n} |r_s| \leq 0$, and there exists $i \in \{1, 2, \dots, T\}$ such that
$$\sum_{s=0}^{n} r_s \cos \frac{2is\pi}{T} = 0;$$

(H2) there exists a function $F(t, Y_\Gamma, \dots, Y_0) \in C^1(\mathbb{R}^{\Gamma+2} \times \mathbb{R}^m, \mathbb{R})$ such that
$$F(t + T, Y_\Gamma, \dots, Y_0) = F(t, Y_\Gamma, \dots, Y_0),$$
$$\sum_{i=-\Gamma}^{0} F'_{2+\Gamma+i}(t + i, Y_{\Gamma+i}, \dots, Y_i) = f(t, Y_\Gamma, \dots, Y_0, \dots, Y_{-\Gamma});$$

(H3) there exists a constant $K_0 > 0$ for all $(t, Y_\Gamma, \dots, Y_0) \in \mathbb{R}^{\Gamma+2}$ such that
$$\Big|\frac{\partial F(t, Y_\Gamma, \dots, Y_0)}{\partial Y_j}\Big| \leq K_0, \ j = 1, 2, \dots, \Gamma;$$

(H4) $F(t, Y_\Gamma, \dots, Y_0) \to +\infty$ uniformly for $t \in \mathbb{R}$ as $\sqrt{|Y_\Gamma|^2 + \dots + |Y_0|^2} \to +\infty$.

Theorem 1.1. *Assume* (H1)–(H4) *and that* $T \geq 2n + 1$. *Then* (1.1) *has at least one* T*-periodic solution.*

Remark 1.2. Assumption (H3) implies that there exists a constant $K_1 > 0$ such that

(H3') $|F(t, Y_\Gamma, \dots, Y_0)| \leq K_1 + K_0(|Y_\Gamma| + \dots + |Y_0|)$ for all $(t, Y_\Gamma, \dots, Y_0) \in \mathbb{R}^{\Gamma+2}$.

Remark 1.3. Theorem 1.1 extends [12, Theorem 1.1] which is the special the case when $m = 1$, $n = 1$, $\Gamma = 0$, $r_0 = -1$ and $r_1 = 1$.

Theorem 1.4. *Suppose that* (H2) *and the following assumptions are satisfied:*

(H1') $-r_0 + \sum_{s=1}^{n} |r_s| > 0$;

(H5) $Y_0 f(t, Y_\Gamma, \dots, Y_0, \dots, Y_{-\Gamma}) > 0$, *for* $Y_0 \neq 0$ *and all* $t \in \mathbb{R}$.

Then (1.1) *has no nontrivial* T*-periodic solution.*

The rest of this article organized as follows. In Section 2, we shall establish the variational framework associated with (1.1) and transfer the problem of the existence of periodic solutions of (1.1) into that of the existence of critical points of the corresponding functional. In Section 3, we shall present some lemmas which will play important roles in the proofs of our main results. In Section 4, we shall complete the proof of the results by using the critical point method.

2. Variational structure

To apply the critical point theory to study the existence of periodic solutions of equation (1.1), we shall construct suitable variational structure. At first, we shall state some basic notation and lemmas which will be used in the proofs of our main results.

Let S be the set of sequences $X = (\dots, X_{-k}, \dots, X_{-1}, X_0, X_1, \dots, X_k, \dots) = \{X_k\}_{k=-\infty}^{+\infty}$, where $X_k = (X_{k,1}, X_{k,2}, \dots, X_{k,m}) \in \mathbb{R}^m$.

For any $X, Y \in S$, $a, b \in \mathbb{R}$, $aX + bY$ is defined by

$$aX + bY := \{aX_k + bY_k\}_{k=-\infty}^{+\infty}.$$

Then S is a vector space. For any positive integer T, we define a subspace of S by

$$E_T = \{X \in S : X_{k+T} = X_k,\ \forall k \in \mathbb{Z}\}.$$

This subspace is equipped with the inner product

$$\langle X, Y\rangle := \sum_{j=1}^{T} X_j \cdot Y_j,\ \forall X, Y \in E_T, \tag{2.1}$$

and the norm

$$\|X\| := \Big(\sum_{j=1}^{T} |X_j|^2\Big)^{1/2}. \tag{2.2}$$

where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^m$, and $X_j \cdot Y_j$ denotes the usual scalar product in $\mathbb{R}^m$.

We define the linear map $M : E_T \to \mathbb{R}^{mT}$ by

$$MX := \big(X_{1,1}, \dots, X_{T,1}, X_{1,2}, \dots, X_{T,2}, \dots, X_{1,m}, \dots, X_{T,m}\big)^*, \tag{2.3}$$

where $X = \{X_k\}$, $X_k = (X_{k,1}, X_{k,2}, \dots, X_{k,m})^*$, $k \in \mathbb{Z}(1, T)$. It is easy to see that the map M defined in (2.3) is a linear homeomorphism with $\|X\| = |MX|$, and $(E_T, \langle\cdot,\cdot\rangle)$ is a Hilbert space, which can be identified with $\mathbb{R}^{mT}$.

For $X \in E_T$, define the functional J on E_T as follows

$$J(X) := \frac{1}{2}\sum_{k=1}^{T}\sum_{i=0}^{n} r_i(X_{k-i} + X_{k+i})X_k + \sum_{k=1}^{T} F(k, X_{k+\Gamma}, \dots, X_k).$$

Since E_T is linearly homeomorphic to $\mathbb{R}^{mT}$, J can be viewed as a continuously differentiable functional defined on a finite dimensional Hilbert space. That is, $J \in C^1(E_T, \mathbb{R})$. Furthermore, $J'(X) = 0$ if and only if

$$\frac{\partial J(X)}{\partial X_{k,l}} = 0, \quad l \in \mathbb{Z}(1, m),\ k \in \mathbb{Z}(1, T).$$

If we define $X_0 := X_T$, then

$$\frac{\partial J(X)}{\partial X_{k,l}} = \sum_{i=0}^{n} r_i(X_{k-i,l} + X_{k+i,l}) + f_l(k, X_{k+\Gamma}, \dots, X_k, \dots, X_{k-\Gamma}),$$

for all $l \in \mathbb{Z}(1, m)$ and $k \in \mathbb{Z}(1, T)$. Therefore, $X \in E_T$ is a critical point of J, i.e., $J'(X) = 0$ if and only if

$$\sum_{i=0}^{n} r_i(X_{k-i,l} + X_{k+i,l}) + f_l(k, X_{k+\Gamma}, \dots, X_k, \dots, X_{k-\Gamma}) = 0,$$

for all $l \in \mathbb{Z}(1, m)$ and $k \in \mathbb{Z}(1, T)$. That is,

$$\sum_{i=0}^{n} r_i(X_{k-i} + X_{k+i}) + f(k, X_{k+\Gamma}, \dots, X_k, \dots, X_{k-\Gamma}) = 0, \quad k \in \mathbb{Z}(1, T).$$

On the other hand, $\{X_k\}_{k\in\mathbb{Z}} \in E_T$ is T-periodic in k and $f(k, Y_\Gamma, \dots, Y_0, \dots, Y_{-\Gamma})$ is T-periodic in k. So $X \in E_T$ is a critical point of J if and only if

$$\sum_{i=0}^{n} r_i(X_{k-i} + X_{k+i}) + f(k, X_{k+\Gamma}, \dots, X_k, \dots, X_{k-\Gamma}) = 0, \quad \forall k \in \mathbb{Z}.$$

Thus, we reduce the problem of finding T-periodic solutions of (1.1) to that of seeking critical points of the functional J in E_T.

For all $X \in E_T$ and $T \geq 2n+1$, J can be rewritten as

$$J(X) = -\frac{1}{2}\langle DMX, MX\rangle + \sum_{k=1}^{T} F(k, X_{k+\Gamma}, \dots, X_k),$$

where $X = \{X_k\} \in E_T$, $X_k = (X_{k,1}, X_{k,2}, \dots, X_{k,m})^*$, $k \in \mathbb{Z}(1,T)$, and

$$D = \begin{pmatrix} P & & & 0 \\ & P & & \\ & & \ddots & \\ 0 & & & P \end{pmatrix}_{mT\times mT},$$

$$-P = \begin{pmatrix}
2r_0 & r_1 & \dots & r_n & 0 & \dots & 0 & r_n & \dots & r_1 \\
r_1 & 2r_0 & r_1 & & & & & & & \vdots \\
\vdots & \ddots & \ddots & \ddots & & & & & & \\
r_n & & & & & & & & & r_{n-1} \\
0 & & & & & & & & & r_n \\
\vdots & & & & & & & & & \\
\vdots & & & & & & & & & \\
0 & & & & & & & & & r_n \\
r_n & & & & & & & & & r_{n-1} \\
\vdots & & & & & \ddots & \ddots & \ddots & \ddots & r_1 \\
r_1 & \dots & r_n & 0 & \dots & 0 & r_n & \dots & r_1 & 2r_0
\end{pmatrix}$$

is a $T \times T$ matrix. Assume that the eigenvalues of P are $\lambda_1, \lambda_2, \dots, \lambda_T$ respectively, and P is a circulant matrix [15] denoted by

$$P := \operatorname{Circ}\big\{-2r_0, -r_1, -r_2, \dots, -r_n, 0, \dots, 0, -r_n, -r_{n-1}, \dots, -r_2, -r_1\big\}.$$

By [15], the eigenvalues of P are

$$\begin{aligned}
\lambda_j &= -2r_0 - \sum_{s=1}^{n} r_s\{\exp i\frac{2j\pi}{T}\}^s - \sum_{s=1}^{n} r_s\{\exp i\frac{2j\pi}{T}\}^{T-s} \\
&= -2\sum_{s=0}^{n} r_s \cos\big(\frac{2js\pi}{T}\big),
\end{aligned} \tag{2.4}$$

where $j = 1, 2, \dots, T$. By (2.4), we know that

$$-2r_0 - 2\sum_{s=1}^{n}|r_s| \leq \lambda_j \leq -2r_0 + 2\sum_{s=1}^{n}|r_s|, \quad j = 1, 2, \dots, T. \tag{2.5}$$

It follows from (H1) that the matrix P is semi-positive and $\lambda_j \geq 0$ for all $j \in \mathbb{Z}(1, T)$. Denote

$$\lambda_{\max} = \max\{\lambda_j : \lambda_j \neq 0, j = 1, 2, \dots, T\},$$
$$\lambda_{\min} = \min\{\lambda_j : \lambda_j \neq 0, j = 1, 2, \dots, T\}.$$

Let

$$H = \ker DM = \{X \in E_T | DMX = 0 \in \mathbb{R}^{mT}\}.$$

Then

$$H = \{X \in E_T : X = \{B\},\ B \in \mathbb{R}^m\}.$$

Let G be the direct orthogonal complement of E_T to W, i.e., $E_T = G \oplus H$. For convenience, we identify $X \in E_T$ with $X = (X_1, X_2, \dots, X_T)^*$.

3. Lemmas

In this section, we give two lemmas which will play important roles in the proofs of our main results.

Let E be a real Banach space, $J \in C^1(E, \mathbb{R})$, i.e., J is a continuously Fréchet-differentiable functional defined on E. J is said to satisfy the Palais-Smale condition (PS condition for short) if any sequence $\{X^{(n)}\}_{n\in\mathbb{N}} \subset E$ for which $\{J(X^{(n)})\}_{n\in\mathbb{N}}$ is bounded and $J'(X^{(n)}) \to 0$ $(n \to \infty)$ possesses a convergent subsequence in E.

Let B_ρ denote the open ball in E about 0 of radius ρ and let ∂B_ρ denote its boundary.

Lemma 3.1 (Saddle Point Theorem [21, 23])**.** *Let E be a real Banach space, $E = E_1 \oplus E_2$, where $E_1 \neq \{0\}$ and is finite dimensional. Suppose that $J \in C^1(E, \mathbb{R})$ satisfies the PS condition and*

- (H6) *there exist constants $\sigma,\ \rho > 0$ such that $J|_{\partial B_\rho \cap E_1} \leq \sigma$;*
- (H7) *there exists $e \in B_\rho \cap E_1$ and a constant $\omega \geq \sigma$ such that $J_{e+E_2} \geq \omega$.*

Then J possesses a critical value $c \geq \omega$, where

$$c = \inf_{h\in\Gamma} \max_{u\in B_\rho\cap E_1} J(h(u)),\ \Gamma = \{h \in C(\bar{B}_\rho \cap E_1, E) | h|_{\partial B_\rho \cap E_1} = \mathrm{id}\}$$

and id *denotes the identity operator.*

Lemma 3.2. *Assume that* (H1)–(H4) *are satisfied. Then J satisfies the PS condition.*

Proof. Let $\{X^{(n)}\}_{n\in\mathbb{N}} \subset E_T$ be such that $\{J(X^{(n)})\}_{n\in\mathbb{N}}$ is bounded and $J'(X^{(n)}) \to 0$ as $n \to \infty$. Then there exists a positive constant K_2 such that $|J(X^{(n)})| \leq K_2$.

Let $X^{(n)} = V^{(n)} + W^{(n)} \in G + H$. For n large enough, since

$$\begin{aligned} -\|X\| &\leq \langle J'(X^{(n)}), MX\rangle \\ &= -\langle DM(X^{(n)}), MX\rangle + \sum_{k=1}^{T} f(k, X_{k+\Gamma}^{(n)}, \dots, X_k^{(n)}, \dots, X_{k-\Gamma}^{(n)}) X_k, \end{aligned}$$

combining (H3) with (H4), we have

$$\langle DM(X^{(n)}), MV^{(n)}\rangle \leq \sum_{k=1}^{T} f(k, X_{k+\Gamma}^{(n)}, \dots, X_k^{(n)}, \dots, X_{k-\Gamma}^{(n)}) V_k^{(n)} + \|V^{(n)}\|$$

$$
\begin{aligned}
&\le (\Gamma+1)K_0 \sum_{k=1}^{T} |V_k^{(n)}| + \|V^{(n)}\| \\
&\le \big[(\Gamma+1)K_0\sqrt{T}+1\big]\|V^{(n)}\|.
\end{aligned}
$$

On the other hand, we know that

$$
\langle DM(X^{(n)}), MV^{(n)}\rangle = \langle DM(V^{(n)}), MV^{(n)}\rangle \ge \lambda_{\min}\|V^{(n)}\|^2.
$$

Thus, we have

$$
\lambda_{\min}\|V^{(n)}\|^2 \le [(\Gamma+1)K_0\sqrt{T}+1]\|V^{(n)}\|.
$$

The above inequality implies that $\{V^{(n)}\}$ is bounded.

Next, we shall prove that $\{W^{(n)}\}$ is bounded. Since

$$
\begin{aligned}
K_2 \ge J(X^{(n)}) &= -\frac{1}{2}\langle DMX^{(n)}, MX^{(n)}\rangle + \sum_{k=1}^{T} F\big(k, X_{k+\Gamma}^{(n)}, \dots, X_k^{(n)}\big) \\
&= -\frac{1}{2}\langle DMV^{(n)}, MV^{(n)}\rangle + \sum_{k=1}^{T}\Big[F(k, X_{k+\Gamma}^{(n)}, \dots, X_k^{(n)}) \\
&\quad - F\Big(k, W_{k+\Gamma}^{(n)}, \dots, W_k^{(n)}\Big)\Big] + \sum_{k=1}^{T} F(k, W_{k+\Gamma}^{(n)}, \dots, W_k^{(n)}),
\end{aligned}
$$

we obtain

$$
\begin{aligned}
&\sum_{k=1}^{T} F\Big(k, W_{k+\Gamma}^{(n)}, \dots, W_k^{(n)}\Big) \\
&\le K_2 + \frac{1}{2}\langle DMV^{(n)}, MV^{(n)}\rangle + \sum_{k=1}^{T}\big|F(k, X_{k+\Gamma}^{(n)}, \dots, X_k^{(n)}) \\
&\quad - F(k, W_{k+\Gamma}^{(n)}, \dots, W_k^{(n)})\big| \\
&\le K_2 + \frac{1}{2}\lambda_{\max}\|V^{(n)}\|^2 + \sum_{k=1}^{T}\Big|\frac{\partial F(k, W_{k+\Gamma}^{(n)} + \theta V_{k+\Gamma}^{(n)}, \dots, W_k^{(n)} + \theta V_k^{(n)})}{\partial Y_\Gamma} V_{k+\Gamma}^{(n)} \\
&\quad + \dots + \frac{\partial F(k, W_{k+\Gamma}^{(n)} + \theta V_{k+\Gamma}^{(n)}, \dots, W_k^{(n)} + \theta V_k^{(n)})}{\partial Y_0} V_k^{(n)}\Big| \\
&\le K_2 + \frac{1}{2}\lambda_{\max}\|V^{(n)}\|^2 + (\Gamma+1)K_0\sqrt{T}\|V^{(n)}\|,
\end{aligned}
$$

where $\theta \in (0,1)$. It is not difficult to see that $\{\sum_{k=1}^{T} F(k, W_{k+\Gamma}^{(n)}, \dots, W_k^{(n)})\}$ is bounded.

By (H4), $\{W^{(n)}\}$ is bounded. Otherwise, assume that $\|W^{(n)}\| \to +\infty$ as $i \to \infty$. Since there exist $B^{(n)} \in \mathbb{R}^m$, $n \in \mathbb{N}$, such that $W^{(n)} = (B^{(n)}, B^{(n)}, \dots, B^{(n)})^* \in E_T$, then

$$
\|W^{(n)}\| = (\sum_{k=1}^{T} |W_k^{(n)}|^2)^{1/2} = (\sum_{k=1}^{T} |B^{(n)}|^2)^{1/2} = \sqrt{T}|B^{(n)}| \to +\infty
$$

as $n \to \infty$. Since $F(k, W_{k+\Gamma}^{(n)}, \dots, W_k^{(n)}) = F(k, B_{k+\Gamma}^{(n)}, \dots, B_k^{(n)})$, it follows that $F(k, W_{k+\Gamma}^{(n)}, \dots, W_k^{(n)}) \to +\infty$. This contradicts that $\{\sum_{k=1}^{T} F(k, W_{k+\Gamma}^{(n)}, \dots, W_k^{(n)})\}$ is bounded. Thus the PS condition is satisfied. $\square$

4. Proof of main results

In this Section, we prove Theorems 1.1 and 1.4, by using the critical point method.

Proof of Theorem 1.1. By Lemma 3.2, we know that J satisfies the PS condition. To prove Theorem 1.1 by using the Saddle Theorem, we shall prove the conditions (H6) and (H7).

From (2.5) and (H3'), for any $V \in G$,

$$\begin{aligned} J(V) &= -\frac{1}{2}\langle DMV, MV\rangle + \sum_{k=1}^{T} F(k, V_{k+\Gamma}, \dots, V_k) \\ &\leq -\frac{1}{2}\lambda_{\min}\|V\|^2 + TK_1 + K_0\sum_{k=1}^{T}(|V_{k+\Gamma}| + \dots + |V_k|) \\ &\leq -\frac{1}{2}\lambda_{\min}\|V\|^2 + TK_1 + (\Gamma+1)K_0\sqrt{T}\|V\| \to -\infty \end{aligned}$$

as $\|V\| \to +\infty$. Therefore, it is easy to see that (H6) is satisfied.

The rest of the proof is similar to that of [27, Theorem 1.1], but for the sake of completeness, we give the details.

In the following, we shall verify the condition (H7). For any $W \in H$, $W = (W_1, W_2, \dots, W_T)^*$, there exists $B \in \mathbb{R}^m$ such that $W_k = B$, for all $k \in \mathbb{Z}(1, T)$. By (H4), we know that there exists a constant $C_0 > 0$ such that $F(k, B, \dots, B) > 0$ for $k \in \mathbb{Z}$ and $|B| > \frac{C_0}{\sqrt{\Gamma+1}}$. Let $K_3 = \min\big\{F(k, B, \dots, B) : k \in \mathbb{Z},\ |B| \leq C_0/\sqrt{\Gamma+1}\big\}$, $K_4 = \min\{0, K_3\}$. Then

$$F(k, B, \dots, B) \geq K_4, \quad \forall (k, B, \dots, B) \in \mathbb{Z} \times \mathbb{R}^{\Gamma+1}.$$

So we have

$$J(W) = \sum_{k=1}^{T} F(k, W_{k+\Gamma}, \dots, W_k) = \sum_{k=1}^{T} F(k, B, \dots, B) \geq TK_4, \quad \forall W \in H.$$

Conditions of (H6) and (H7) are satisfied. □

Proof of Theorem 1.4. It follows from (H1') that the matrix P is negative semi-positive and $\lambda_j \leq 0$ for all $j \in \mathbb{Z}(1, T)$. For the sake of contradiction, assume that (1.1) has a nontrivial T-periodic solution. Then J has a nonzero critical point $X^\star$. Since

$$\frac{\partial J}{\partial X_k^\star} = \sum_{i=0}^{n} r_i(X_{k-i}^\star + X_{k+i}^\star) + f(k, X_{k+\Gamma}^\star, \dots, X_k^\star, \dots, X_{k-\Gamma}^\star),$$

we obtain

$$\begin{aligned} &\sum_{k=1}^{T} f(k, X_{k+\Gamma}^\star, \dots, X_k^\star, \dots, X_{k-\Gamma}^\star)X_k^\star \\ &= -\sum_{k=1}^{T}\sum_{i=0}^{n} r_i(X_{k-i}^\star + X_{k+i}^\star)X_k^\star \\ &= \langle DMX^\star, MX^\star\rangle \leq 0. \end{aligned} \tag{4.1}$$

On the other hand, from (H5) it follows that

$$\sum_{i=1}^{T} f(k, X^{\star}_{k+\Gamma}, \dots, X^{\star}_{k}, \dots, X^{\star}_{k-\Gamma}) X^{\star}_{k} > 0. \tag{4.2}$$

This contradicts (4.1) and hence the proof is complete. □

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