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QUENCHING PHENOMENON OF SINGULAR PARABOLIC PROBLEMS WITH L^1 INITIAL DATA

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Abstract. We extend some previous existence results for quenching type parabolic problems involving a negative power of the unknown in the equation to the case of merely integrable initial data. We show that $L^1(\Omega)$ is the suitable framework to obtain the continuous dependence with respect to some norm of the initial datum. This way we answer to the question raised by several authors in the previous literature. We also show the complete quenching phenomena for such a L^1-initial datum.

1. Introduction

The main purpose of this paper is to study the existence of nonnegative mild solution and the "quenching phenomenon" of the singular parabolic equation

$$
\begin{gathered}
\partial_t u - \Delta u + \chi_{\{u>0\}} u^{-\beta} = 0 \quad \text{in } \Omega \times (0,T), \\
u = 0 \quad \text{on } \partial\Omega \times (0,T), \\
u(\cdot,0) = u_0(\cdot) \quad \text{on } \Omega,
\end{gathered}
\tag{1.1}
$$

where $\beta \in (0,1)$, Ω is a smooth bounded domain in $\mathbb{R}^N$, $0 \le u_0$ and $\chi_{\{u>0\}}$ denotes the characteristic function of the set of points (x,t) where $u(x,t) > 0$. Parabolic equations involving as zero order term a negative exponent of the unknown are quite common in the literature since 1960. The pioneering paper by Fulks and Maybee [18] was motivated by the study of the heat conduction in an electric medium but in the modelling the singular term was of a sourcing nature and so in the right hand side of the equation: the differences between the behavior of solutions of such model with respect to our problem (1.1) are today well-known. Perhaps, one of the first papers dealing with the equation of (1.1) was [24] in the study of Electric Current Transient in Polarized Ionic Conductors (in fact for $\beta = 1$). The literature on this type of problems increased then very quickly and models arising in other contexts were mentioned by different authors, specially when regarding the equation (1.1) as the limit case of models in chemical catalyst kinetics (Langmuir-Hinshelwood model) or of models in enzyme kinetics (see [13, 15] for the elliptic case and [2, 30] for the parabolic equation). See also many references in the survey [22] and the monograph [20]). Obviously, what makes specially interesting equations like (1.1) is the fact that the solutions may raise to a free boundary defined as $\partial\{(x,t)\colon\ u(x,t) > 0\}$ (see e.g. [14] and its references). In many contexts the

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boundary conditions are not zero but, for instance $u = 1$ and thus the terminology of quenching problem was used in the literature to denote the appearance of blow-up result on $\partial_t u$ for the first time in which $u = 0$ (see, e.g., [24, 28, 30]).

In spite of such a long list of references, most of the theory in the literature deals with bounded (quite often even assumed continuous) initial data. We must add that even so, it is today well-known that the uniqueness of solution fails (see [33]), except for the case in which there is not a free boundary, see ([11]). The main purpose of this work is to deal with initial data satisfying merely

$$0 \leq u_0 \in L^1(\Omega).$$

Let us introduce the notion of solution we shall use in this paper:

Definition 1.1. A function $u \in \mathcal{C}([0,T); L^1(\Omega))$ is called a mild solution of (1.1) if $\chi_{\{u>0\}} u^{-\beta} \in L^1(\Omega \times (0,T))$ and u fulfills

$$u(\cdot,t) = S(t)u_0(\cdot) - \int_0^t S(t-s)\chi_{\{u>0\}} u^{-\beta}(\cdot,s)ds, \quad \text{in } L^1(\Omega), \tag{1.2}$$

where $S(t)$ is the semigroup corresponding to the Laplace operator with homogeneous Dirichlet boundary conditions.

We recall that the $L^1(\Omega)$-semigroup $S(t)$ corresponding to the Laplace operator with homogeneous Dirichlet boundary conditions was considered by many authors since the seventies (or even earlier of the past century and that the associated weak solutions $S(t)u_0$ can be characterized by multiplying by suitable test functions (see, e.g., [3, 5, 8] and the exposition made in Chapter 4 of [13]). In particular, we know that any mild solution u belongs to the space $L^s(0,T; W_0^{1,s}(\Omega))$, for any $s \in (1, \frac{N+2}{N+1})$, and satisfies

$$\begin{aligned}
&\int_\Omega u(x,t)\psi(x,t)dx + \int_0^t \int_\Omega \nabla u(x,s) \cdot \nabla\psi(x,s)\,dx\,ds \\
&+ \int_0^t \int_\Omega \chi_{\{u>0\}} u^{-\beta}(x,s)\psi(x,s)\,dx\,ds \\
&= \int_0^t \int_\Omega u(x,s)\partial_t\psi(x,s)\,dx\,ds + \int_\Omega u_0(x)\psi(x,0)dx,
\end{aligned}$$

for any test function $\psi \in W^{1,\infty}(0,T; L^1(\Omega)) \cap L^\infty(0,T; W_0^{1,\infty}(\Omega))$, and for every $t \in (0,T)$.

The main results of this article are the following:

Theorem 1.2. *Let $0 \leq u_0 \in L^1(\Omega)$. Then, there exists the a maximal nonnegative mild solution u of (1.1) in $\Omega \times (0,\infty)$, i.e., for any other mild solution v of (1.1) we have $0 \leq v \leq u$ in $\Omega \times [0,\infty)$.*

Concerning the quenching phenomenon, we recall that since there is lack of uniqueness of solutions, it seems to be difficult to apply, directly, super and subsolutions methods to study it. Our approach is to use the energy methods, but with the new fact that our initial datum does not need to be in the natural energy space defined over $L^2(\Omega)$.

Theorem 1.3. *Let $0 \leq u_0 \in L^1(\Omega)$. Then, if v is any nonnegative mild solution of (1.1), there exists a finite time $T^* > 0$ such that*

$$v(x,t) = 0, \quad \textit{for a.e. } (x,t) \in \Omega \times (T^*, \infty).$$

Moreover, T^* only depends on $\|u_0\|_{L^1(\Omega)}$, N and $|\Omega|$.

This article is organized as follows: Section 2 is devoted to the proof of Theorem 1.2. In Section 3, we will consider the quenching phenomenon. We also prove the uniqueness result under additional assumption.

2. Proof of Theorem 1.2

We shall follow a scheme of approximation similar to the one used in [33]. We start by considering the problem

$$\begin{gathered} \partial_t u_\varepsilon - \Delta u_\varepsilon + g_\varepsilon(u_\varepsilon) = 0 \quad \text{in } \Omega \times (0,\infty), \\ u_\varepsilon = 0 \quad \text{on } \partial\Omega \times (0,\infty), \\ u_\varepsilon(\cdot,0) = u_0(\cdot) \quad \text{on } \Omega \end{gathered} \tag{2.1}$$

with

$$g_\varepsilon(s) = \begin{cases} 0 & \text{if} s \le 0, \\ \psi_\varepsilon(s)s^{-\beta} & \text{if } s > 0. \end{cases}$$

where $\psi_\varepsilon(s) = \psi(\frac{s}{\varepsilon})$ and $\psi \in \mathcal{C}^\infty(\mathbb{R})$ is a non-decreasing function on $\mathbb{R}$ such that $\psi(s) = 0$ for $s \le 1$, $\psi(s) = 1$ for $s \ge 2$. The main idea of the proof is to pass to the limit in equation (2.1) as $\varepsilon \to 0$ to obtain a solution of (1.1), which is the maximal solution.

First of all, we observe that for any fixed $\varepsilon > 0$, g_ε is a global Lipschitz function. Then, we have the following result.

Theorem 2.1. *There exists a unique nonnegative mild solution to problem* (2.1), *$u_\varepsilon \in \mathcal{C}([0,+\infty); L^1(\Omega))$; i.e. satisfying that for any $t > 0$,*

$$u_\varepsilon(t) = S(t)u_0 - \int_0^t S(t-s)g_\varepsilon(u_\varepsilon(s))ds. \tag{2.2}$$

Moreover, for any $0 < \tau < T < \infty$, and for some $\alpha \in (0,1)$, we have $u_\varepsilon \in \mathcal{C}^{2+\alpha,1+\frac{\alpha}{2}}_{x,t}(\overline{\Omega} \times (\tau,T))$.

Proof. The existence of solutions is a classical result, and we put its proof in the Appendix. Now, we focus on the proof of uniqueness of solution. The proof is an immediate consequence from the lemma below.

Lemma 2.2. *For any $0 < \tau < T$, let $v_1 \in L^\infty(\Omega \times (\tau,T)) \cap L^2(\tau,T; W^{1,2}_0(\Omega))$ (resp. v_2) be a mild sub-solution (resp super-solution) of* (2.1). *Then, we have $v_1 \le v_2$, in $\Omega \times (0,T)$.*

We introduce the truncation function

$$T_k(s) := \begin{cases} s & \text{if } |s| \le k, \\ \operatorname{sign}(s)k & \text{if } |s| > k, \end{cases}$$

and its primitive integral

$$S_k(u) := \int_0^u T_k(s)ds = \frac{1}{2}|u|^2\chi_{\{|u|<k\}} + k\Big(|u| - \frac{1}{2}k\Big)\chi_{\{|u|\ge k\}}.$$

Let us consider the equation satisfied by the difference between v_1 and v_2,

$$\partial_t(v_1 - v_2) - \Delta(v_1 - v_2) + g_\varepsilon(v_1) - g_\varepsilon(v_2) \le 0.$$

Then, using the test function $T_1(v_+)$, with $v = v_1 - v_2$, we obtain that for any $0 < \tau < t$,

$$\int_\Omega S_1(v_+(t))dx + \int_\tau^t \int_\Omega |\nabla v_+|^2 \,dx\,ds + \int_\tau^t \int_\Omega \big(g_\varepsilon(v_1) - g_\varepsilon(v_2)\big)T_1(v_+)\,dx\,ds \\ \leq \int_\Omega S_1(v_+(\tau))dx.$$

Since g_ε is a global Lipschitz function, it follows from the last inequality that

$$\int_\Omega S_1(v_+(t))dx \leq C(\varepsilon) \int_\tau^t \int_\Omega |v| T_1(v_+)\,dx\,ds + \int_\Omega S_1(v_+(\tau))dx. \tag{2.3}$$

Passing $\tau \to 0$ in (2.3), and noting that $\int_\Omega S_1(v_+(\tau))dx \to 0$, as $\tau \to 0$, we obtain

$$\int_\Omega S_1(v_+(t))dx \leq C(\varepsilon) \int_0^t \int_\Omega |v| T_1(v_+)\,dx\,ds. \tag{2.4}$$

On the other hand, we observe that

$$|v| T_1(v_+) \leq 2S_1(v_+). \tag{2.5}$$

Combining (2.4) and (2.5) we deduce

$$\int_\Omega S_1(v_+(t))dx \leq 2C(\varepsilon) \int_0^t \int_\Omega S_1(v_+)\,dx\,ds.$$

Let

$$y(t) = \int_\Omega S_1(v_+(t))dx.$$

We have the ordinary differential inequality:

$$y'(t) \leq 2C(\varepsilon)y(t), \quad t > 0, \\ y(0) = 0.$$

Gronwall's inequality implies $y(t) = 0$, and so $v_+(t) = 0$. This completes the proof. ☐

Next, we shall show the existence of solution of (1.1) by passing to the limit as $\varepsilon \to 0$.

Theorem 2.3. *The sequence $\{u_\varepsilon\}_{\varepsilon>0}$ is nondecreasing, so u_ε converges to a function u in $L^r(0,T;W_0^{1,r}(\Omega))$, which is a solution of (1.1), for $r \in (1, \frac{N+2}{N+1})$. Furthermore, u is a mild solution of (1.1).*

Proof. It follows from (2.2) that for any $t > 0$,

$$0 \leq u_\varepsilon(x,t) \leq S(t)u_0(x) \leq Ct^{-\frac{N}{2}} \|u_0\|_{L^1(\Omega)}. \tag{2.6}$$

The constant C in (2.6) merely depends on $N, |\Omega|$, see [4, 9]. Then u_ε is bounded locally in time.

For any $0 < \tau < T$, integrating equation (2.1) on $\Omega \times (\tau, T)$ yields

$$\int_\Omega u_\varepsilon(x,T)dx - \int_\tau^T \int_{\partial\Omega} \nabla u_\varepsilon.\mathbf{n}\,d\sigma\,ds + \int_\tau^T \int_\Omega g_\varepsilon(u_\varepsilon)\,dx\,ds = \int_\Omega u(x,\tau)dx,$$

where $\mathbf{n}$ is the unit outward normal vector of $\partial\Omega$. Since $\nabla u_\varepsilon.\mathbf{n} \leq 0$, we obtain

$$\int_\Omega u_\varepsilon(x,T)dx + \int_\tau^T \int_\Omega g_\varepsilon(u_\varepsilon)\,dx\,ds \leq \int_\Omega u(x,\tau)dx,$$

Passing to the limit as $\tau \to 0$ in the above inequality asserts that

$$\int_\Omega u_\varepsilon(x,T)dx + \int_0^T \int_\Omega g_\varepsilon(u_\varepsilon)\,dx\,ds \le \|u_0\|_{L^1(\Omega)}. \tag{2.7}$$

Using [3, Lemma 3.3], we obtain

$$\|u_\varepsilon\|_{L^s(0,T;W_0^{1,r}(\Omega))} \le C(s,r,T,\Omega)\left(\|g_\varepsilon(u_\varepsilon)\|_{L^1(\Omega\times(0,T))} + \|u_0\|_{L^1(\Omega)}\right), \tag{2.8}$$

with $s, r \ge 1$ such that $\frac{2}{s} + \frac{N}{r} > N+1$. Combining (2.7) and (2.8) we obtain

$$\|u_\varepsilon\|_{L^r(0,T;W_0^{1,r}(\Omega))} \le C(r,T,\Omega)\|u_0\|_{L^1(\Omega)}, \tag{2.9}$$

with $r = s \in [1, \frac{N+2}{N+1})$. Thus, for any $r \in (1, \frac{N+2}{N+1})$, $\{\partial_t u_\varepsilon\}_{\varepsilon>0}$ is bounded in $L^1(0,T;W^{-1,r'}(\Omega)) + L^1(\Omega\times(0,T))$ by a constant independent of ε. Then, the sequence $\{u_\varepsilon\}_\varepsilon$ is relatively compact in $L^1(\Omega\times(0,T))$ (see [31]) and there is a subsequence of $\{u_\varepsilon\}_\varepsilon$ (still denoted as $\{u_\varepsilon\}_\varepsilon$) such that

$$u_\varepsilon \to u, \quad \text{in} \quad L^1(\Omega\times(0,T)). \tag{2.10}$$

Next, we claim that

$$u_\varepsilon(x,t) \downarrow u(x,t), \quad \text{for a.e. } (x,t) \in \Omega\times(0,T). \tag{2.11}$$

It is sufficient to show that $\{u_\varepsilon\}_\varepsilon$ is a non-decreasing sequence. Indeed, for any $\varepsilon > \varepsilon' > 0$, we have

$$g_\varepsilon(s) \le g_{\varepsilon'}(s), \quad \forall s \in \mathbb{R}.$$

Then

$$\partial_t u_\varepsilon - \Delta u_\varepsilon + g_{\varepsilon'}(u_\varepsilon) \ge \partial_t u_\varepsilon - \Delta u_\varepsilon + g_\varepsilon(u_\varepsilon) = 0.$$

This implies that u_ε is a super-solution of the equation satisfied by $u_{\varepsilon'}$. Thanks to Lemma 2.2, we obtain $u_\varepsilon(x,t) \ge u_{\varepsilon'}(x,t)$, for a.e. $(x,t) \in \Omega\times(0,T)$, thereby we obtain the claim (2.11).

Next, we shall show the convergence of the gradients. Let us first demonstrate that

$$\nabla u_\varepsilon \xrightarrow[\varepsilon\to 0]{} \nabla u \quad \text{in } L^1(\Omega\times(0,T)), \tag{2.12}$$

For any $\varepsilon, \varepsilon' > 0$, we consider function $v_{\varepsilon,\varepsilon'} = u_\varepsilon - u_{\varepsilon'}$, and the difference between the equations satisfied by u_ε and $u_{\varepsilon'}$

$$\partial_t v_{\varepsilon,\varepsilon'} - \Delta v_{\varepsilon,\varepsilon'} + g_\varepsilon(u_\varepsilon) - g_{\varepsilon'}(u_{\varepsilon'}) = 0. \tag{2.13}$$

For any $\delta > 0$, and any $0 < T_0 < \infty$, we take $T_\delta(v_{\varepsilon,\varepsilon'})$ as a test function for (2.13). Then, we obtain

$$\begin{aligned} &\int_\Omega S_\delta(v_{\varepsilon,\varepsilon'}(T_0))dx + \int_0^{T_0}\int_\Omega |\nabla T_\delta(v_{\varepsilon,\varepsilon'})|^2\,dx\,ds \\ &+ \int_0^{T_0}\int_\Omega (g_\varepsilon(u_\varepsilon) - g_\varepsilon(u_\varepsilon))\,T_\delta(v_{\varepsilon,\varepsilon'})\,dx\,ds \\ &= \int_\Omega S_\delta(v_{\varepsilon,\varepsilon'}(0))dx. \end{aligned} \tag{2.14}$$

It follows from (2.14) that

$$\int_0^{T_0}\int_\Omega |\nabla T_\delta(v_{\varepsilon,\varepsilon'})|^2\,dx\,ds \le \delta\int_0^{T_0}\int_\Omega g_\varepsilon(u_\varepsilon) + g_{\varepsilon'}(u_\varepsilon)\,dx\,ds. \tag{2.15}$$

Combining (2.7) and (2.15) yields

$$\int_{\{|v_{\varepsilon,\varepsilon'}(x,t)|<\delta\}\cap\Omega\times(0,T_0)} |\nabla v_{\varepsilon,\varepsilon'}|^2\,dx\,ds \le 2\delta\|u_0\|_{L^1(\Omega)}. \tag{2.16}$$

On the one hand, Holder's inequality yields

$$\begin{aligned} &\int_{\{|v_{\varepsilon,\varepsilon'}(x,t)|<\delta\}\cap\Omega\times(0,T_0)} |\nabla v_{\varepsilon,\varepsilon'}|\,dx\,ds \\ &\le \operatorname{meas}\big(\{\Omega\times(0,T_0)\}\big)^{1/2}\Big(\int_{\{|v_{\varepsilon,\varepsilon'}(x,t)|<\delta\}\cap\Omega\times(0,T_0)} |\nabla v_{\varepsilon,\varepsilon'}|^2\,dx\,ds\Big)^{1/2}. \end{aligned} \tag{2.17}$$

From (2.16)and(2.17), we obtain

$$\int_{\{|v_{\varepsilon,\varepsilon'}(x,t)|<\delta\}\cap\Omega\times(0,T_0)} |\nabla v_{\varepsilon,\varepsilon'}|\,dx\,ds \le C\sqrt{\delta}, \tag{2.18}$$

where $C = C(|\Omega|, T_0, \|u_0\|_{L^1(\Omega)})$.

on the other hand, Holder's inequality again yields

$$\begin{aligned} &\int_{\{|v_{\varepsilon,\varepsilon'}(x,t)|\ge\delta\}\cap\Omega\times(0,T_0)} |\nabla v_{\varepsilon,\varepsilon'}|\,dx\,ds \\ &\le \Big(\int_{\{|v_{\varepsilon,\varepsilon'}(x,t)|\ge\delta\}\cap\Omega\times(0,T_0)} |\nabla v_{\varepsilon,\varepsilon'}|^r\,dx\,ds\Big)^{1/r} \\ &\quad\times \operatorname{meas}\big(\{|v_{\varepsilon,\varepsilon'}(x,t)|\ge\delta\}\cap\Omega\times(0,T_0)\big)^{1-\frac{1}{r}}, \end{aligned}$$

with some value $r \in (1, \frac{N+2}{N+1})$.

Inserting (2.9) into the above inequality, we obtain

$$\begin{aligned} &\int_{\{|v_{\varepsilon,\varepsilon'}(x,t)|\ge\delta\}\cap\Omega\times(0,T_0)} |\nabla v_{\varepsilon,\varepsilon'}|\,dx\,ds \\ &\le C(r,|\Omega|,T_0,\|u_0\|_{L^1(\Omega)})\operatorname{meas}\big(\{|v_{\varepsilon,\varepsilon'}(x,t)|\ge\delta\}\cap\Omega\times(0,T_0)\big)^{1-\frac{1}{r}}. \end{aligned} \tag{2.19}$$

Combining (2.16) and (2.19) induces

$$\int_0^{T_0}\int_\Omega |\nabla v_{\varepsilon,\varepsilon'}|\,dx\,ds \le C\Big(\sqrt{\delta}+\operatorname{meas}\big(\{|v_{\varepsilon,\varepsilon'}(x,t)|\ge\delta\}\cap\Omega\times(0,T_0)\big)^{1-\frac{1}{r}}\Big). \tag{2.20}$$

Clearly, $v_{\varepsilon,\varepsilon'}$ converges to 0 in measure by (2.11). Then, letting $\varepsilon,\varepsilon' \to 0$ in (2.20) leads to

$$\limsup_{\varepsilon,\varepsilon'\to 0}\int_0^{T_0}\int_\Omega |\nabla v_{\varepsilon,\varepsilon'}|dx d\tau \le C\sqrt{\delta}.$$

The above inequality holds for any $\delta > 0$, so we obtain (2.12).

As a consequence of (2.12), there is a sub-sequence of $\{u_\varepsilon\}_{\varepsilon>0}$ such that

$$\nabla u_\varepsilon \to \nabla u, \quad \text{for a.e } (x,t) \in \Omega\times(0,\infty). \tag{2.21}$$

Let us show now a sharper convergence: for any $r \in (1, \frac{N+2}{N+1})$,

$$u_\varepsilon \to u, \quad \text{in } L^r(0,T_0; W_0^{1,r}(\Omega)). \tag{2.22}$$

Indeed, conclusion (2.22) just follows from (2.9), (2.10), (2.12) and Vitali's theorem.

Next, we show that there is a subsequence of $\{g_\varepsilon(u_\varepsilon)\}_{\varepsilon>0}$ such that

$$g_\varepsilon(u_\varepsilon) \to u^{-\beta}\chi_{\{u>0\}}, \quad \text{in } L^1(\Omega\times(0,T_0)). \tag{2.23}$$

More precisely, we claim that the above subsequence satisfies, from Fatou's lemma, that

$$\liminf_{\varepsilon\to 0} g_\varepsilon(u_\varepsilon) = u^{-\beta}\chi_{\{u>0\}}, \quad \text{in } L^1(\Omega\times(0,T_0)), \tag{2.24}$$

this conclusion allows us to obtain

$$u \in \mathcal{C}([0,T_0];L^1(\Omega)). \tag{2.25}$$

Let us skip the proof of (2.23) (or (2.24)) at the moment and we will show (2.25) if (2.23) holds. For any $0<t<T_0$, we use the argument of (2.14) with $\delta=1$ to obtain

$$\begin{aligned}
&\int_\Omega S_1(v_{\varepsilon,\varepsilon'})(t)dx + \int_0^t\int_\Omega |\nabla T_1(v_{\varepsilon,\varepsilon'})|^2\,dx\,ds\\
&+\int_0^t\int_\Omega (g_\varepsilon(u_\varepsilon)-g_{\varepsilon'}(u_{\varepsilon'}))\,T_1(v_{\varepsilon,\varepsilon'})\,dx\,ds = 0;
\end{aligned}$$

and so

$$\int_\Omega S_1(v_{\varepsilon,\varepsilon'})(t)dx \le \int_0^T\int_\Omega |g_\varepsilon(u_\varepsilon)-g_{\varepsilon'}(u_{\varepsilon'})|\,dx\,ds. \tag{2.26}$$

On the other hand, we observe from the expression of S_1 that

$$\int_\Omega |v_{\varepsilon,\varepsilon'}(t)|\chi_{\{|v_{\varepsilon,\varepsilon'}(t)|\ge 1\}}dx \le 2\int_\Omega S_1(v_{\varepsilon,\varepsilon'})(t)dx;$$

using Holder's inequality yields

$$\begin{aligned}
\int_\Omega |v_{\varepsilon,\varepsilon'}(t)|\chi_{\{|v_{\varepsilon,\varepsilon'}(t)|<1\}}dx &\le |\Omega|^{1/2}\Big(\int_\Omega |v_{\varepsilon,\varepsilon'}(t)|^2\chi_{\{|v_{\varepsilon,\varepsilon'}(t)|<1\}}dx\Big)^{1/2}\\
&\le \Big(2|\Omega|\int_\Omega S_1(v_{\varepsilon,\varepsilon'})(t)dx\Big)^{1/2}.
\end{aligned}$$

Therefore,

$$\int_\Omega |v_{\varepsilon,\varepsilon'}(t)|dx \le 2\int_\Omega S_1(v_{\varepsilon,\varepsilon'}(t)dx + \Big(2|\Omega|\int_\Omega S_1(v_{\varepsilon,\varepsilon'})(t)dx\Big)^{1/2}. \tag{2.27}$$

It follows from (2.23), (2.26) and (2.27) that

$$\lim_{\varepsilon,\varepsilon'\to 0}\|v_{\varepsilon,\varepsilon'}(t)\|_{L^1(\Omega)} = 0, \quad \text{uniformly on} \quad [0,T_0].$$

Or

$$\lim_{\varepsilon\to 0}\|u_\varepsilon(t)-u(t)\|_{L^1(\Omega)} = 0, \quad \text{uniformly on} \quad [0,T_0]. \tag{2.28}$$

This implies the conclusion of (2.25).

To prove (2.24), we shall use a gradient estimate, obtained by Winkler [33, Lemma 3.1] (see also Davila and Montenegro [11, Lemma 2.4]).

Lemma 2.4. *There is a positive constant $C>0$ such that for any $\tau>0$ fixed, we have*

$$|\nabla u(x,t)| \le Cu^{\frac{1-\beta}{2}}\Big(1+\big(\tau^{-\frac{N}{2}}\|u_0\|_{L^1(\Omega)}\big)^{\frac{\beta+1}{2}}\Big)\Big(1+(t-\tau)^{-\frac12}+d(x)^{-1}\Big), \tag{2.29}$$

with $C=C(N,|\Omega|,\beta)>0$, and $d(x)=\inf_{y\in\partial\Omega}\{\|x-y\|\}$, the distance from x to the boundary of the domain Ω.

In fact, we observe that, for any $\tau > 0$, $u_\varepsilon(\tau) \in \mathcal{C}_0(\Omega)$. Then, we apply [33, Lemma 3.3] to u_ε, by considering $u_\varepsilon(\tau)$ as the initial condition instead of $u_\varepsilon(0)$ to obtain

$$|\nabla u_\varepsilon(x,t)| \le C(\beta) u_\varepsilon^{\frac{1-\beta}{2}} \Big(1 + \|u_\varepsilon(\tau)^{\frac{\beta+1}{2}}\|_{L^\infty(\Omega)}\Big)\Big((t-\tau)^{-\frac{1}{2}} + d(x)^{-1}\Big). \tag{2.30}$$

Combining (2.6) and (2.30) deduce that there is a positive constant $C = C(N, \beta, |\Omega|)$ such that

$$|\nabla u_\varepsilon(x,t)| \le C u_\varepsilon^{\frac{1-\beta}{2}} \Big(1 + \big(\tau^{-\frac{N}{2}}\|u_0\|_{L^1(\Omega)}\big)^{\frac{\beta+1}{2}}\Big)\Big(1 + (t-\tau)^{-\frac{1}{2}} + d(x)^{-1}\Big). \tag{2.31}$$

Now let $\varepsilon \to 0$ in (2.31). Then, inequality (2.29) follows from (2.21), and the monotonicity of u_ε. This implies

$$\nabla u_\varepsilon \to \nabla u, \quad \text{in } L^q_{loc}(\Omega \times (0,+\infty)),\ \forall q \in (1,\infty).$$

Now, it remains to show claim (2.24). Indeed, using (2.7) and Fatou's lemma asserts that for any $T_0 \in (0,\infty)$, there is a non-negative function $\Phi \in L^1(\Omega \times (0,T_0))$ such that

$$\liminf_{\varepsilon \to 0} g_\varepsilon(u_\varepsilon) = \Phi, \quad \text{in } L^1(\Omega \times (0,T_0)). \tag{2.32}$$

Furthermore, we observe that

$$g_\varepsilon(u_\varepsilon)(x,t) \ge g_\varepsilon(u_\varepsilon)\chi_{\{u>0\}}(x,t), \quad \text{for a.e. } (x,t) \in \Omega \times (0,T_0),$$

which implies

$$\liminf_{\varepsilon \to 0} g_\varepsilon(u_\varepsilon)(x,t) \ge u^{-\beta}\chi_{\{u>0\}}(x,t), \quad \text{for a.e. } (x,t) \in \Omega \times (0,T_0). \tag{2.33}$$

It follows from the Lebesgue dominated convergence theorem that

$$u^{-\beta}\chi_{\{u>0\}} \le \Phi \quad \text{and} \quad u^{-\beta}\chi_{\{u>0\}} \in L^1(\Omega \times (0,T_0)). \tag{2.34}$$

For any $\eta > 0$ fixed, we use the test function $\psi_\eta(u_\varepsilon)\phi$, $\phi \in \mathcal{C}_c^\infty(\Omega \times (0,T_0))$ to the equation satisfied by u_ε. Then, integration by parts gives us

$$\begin{aligned}\int_{\operatorname{supp}\phi)} \Big(&- \Psi_\eta(u_\varepsilon)\partial_t\phi + \frac{1}{\eta}|\nabla u_\varepsilon|^2\psi'\big(\frac{u_\varepsilon}{\eta}\big)\phi \\ &+ \nabla u_\varepsilon \cdot \nabla\phi\psi_\eta(u_\varepsilon) + g_\varepsilon(u_\varepsilon)\psi_\eta(u_\varepsilon)\phi \Big)\,dx\,ds = 0,\end{aligned}$$

where

$$\Psi_\eta(u) = \int_0^u \psi_\eta(s)ds.$$

By (2.11) and (2.29), we can pass to the limit as $\varepsilon \to 0$ in the above inequality to obtain

$$\begin{aligned}\int_{\operatorname{supp}\phi)} \Big(&- \Psi_\eta(u)\partial_t\phi + \frac{1}{\eta}|\nabla u|^2\psi'\big(\frac{u}{\eta}\big)\phi \\ &+ \nabla u \cdot \nabla\phi\psi_\eta(u) + u^{-\beta}\psi_\eta(u)\phi\Big)\,dx\,ds = 0,\end{aligned} \tag{2.35}$$

From (2.29), (2.34), and the dominated convergence theorem, it is not difficult to verify that

$$\begin{aligned}&\lim_{\eta \to 0}\int_{\operatorname{supp}\phi)} \Big(- \Psi_\eta(u)\partial_t\phi + \nabla u \cdot \nabla\phi\psi_\eta(u) + u^{-\beta}\psi_\eta(u)\phi\Big)\,dx\,ds \\ &= \int_{\operatorname{supp}\phi)} \Big(- u\partial_t\phi + \nabla u \cdot \nabla\phi + u^{-\beta}\chi_{\{u>0\}}\phi\Big)\,dx\,ds,\end{aligned} \tag{2.36}$$

with any term of the left-hand side converges to any term of the right-hand side in order.

On the other hand, it follows from (2.29) that

$$\begin{aligned}\frac{1}{\eta}\int_{\operatorname{supp}\phi)}|\nabla u|^2\Big|\psi'(\frac{u}{\eta})\phi\Big|\,dx\,ds &\le C(\phi)\frac{1}{\eta}\int_{\operatorname{supp}\phi)\cap\{\eta<u<2\eta\}}u^{1-\beta}\,dx\,ds\\ &\le 2C(\phi)\int_{\operatorname{supp}\phi)\cap\{\eta<u<2\eta\}}u^{-\beta}\,dx\,ds.\end{aligned}$$

By (2.34), we obtain

$$\lim_{\eta\to 0}\int_{\operatorname{supp}\phi)\cap\{\eta<u<2\eta\}}u^{-\beta}\,dx\,ds=0,$$

thereby it proves

$$\frac{1}{\eta}\int_{\operatorname{supp}\phi)}|\nabla u|^2\big|\psi'(\frac{u}{\eta})\phi\big|\,dx\,ds=0. \tag{2.37}$$

Combining (2.35), (2.36) and (2.37) yields

$$\int_{\operatorname{supp}\phi)}\big(-u\partial_t\phi+\nabla u\cdot\nabla\phi+u^{-\beta}\chi_{\{u>0\}}\phi\big)\,dx\,ds=0. \tag{2.38}$$

Note that (2.38) says that u is a weak solution of (1.1) in $\Omega\times(0,T_0)$. However, this is not sufficient to conclude that u is a mild solution of (1.1).

Next, since u_ε is a weak solution of (2.1), we have

$$\int_{\operatorname{supp}\phi)}\Big(-u_\varepsilon\partial_t\phi+\nabla u_\varepsilon\cdot\nabla\phi+g_\varepsilon(u_\varepsilon)\phi\Big)\,dx\,ds=0.$$

The passage to the limit as $\varepsilon\to 0$ provides us

$$\int_{\operatorname{supp}\phi)}\big(-u\partial_t\phi+\nabla u\cdot\nabla\phi\big)\,dx\,ds+\lim_{\varepsilon\to 0}\int_{\operatorname{supp}\phi)}g_\varepsilon(u_\varepsilon)\phi\;dx\,ds=0. \tag{2.39}$$

By (2.38) and (2.39), we obtain

$$\lim_{\varepsilon\to 0}\int_0^\infty\int_\Omega g_\varepsilon(u_\varepsilon)\phi\,dx\,ds=\int_0^\infty\int_\Omega u^{-\beta}\chi_{\{u>0\}}\phi\;dx\,ds. \tag{2.40}$$

Thanks to Fatou's lemma, (2.32) and (2.40), we obtain for any non-negative $\phi\in\mathcal{C}_c^\infty(\Omega\times(0,\infty))$,

$$\int_0^\infty\int_\Omega u^{-\beta}\chi_{\{u>0\}}\phi\,dx\,ds\ge\int_0^\infty\int_\Omega\Phi\phi\,dx\,ds.$$

We deduce from the last inequality and (2.34) that

$$u^{-\beta}\chi_{\{u>0\}}=\Phi,\quad\text{a.e. in }\Omega\times(0,\infty).$$

In other words, we obtain the claim (2.24).

Since u_ε is a mild solution of equation (2.1), we have

$$u_\varepsilon(t)=S(t)u_0-\int_0^t S(t-s)g_\varepsilon\big(u_\varepsilon(s)\big)ds. \tag{2.41}$$

By (2.24), we can pass to the limit as $\varepsilon\to 0$ in (2.41) to obtain

$$u(t)=S(t)u_0-\int_0^t S(t-s)u^{-\beta}\chi_{\{u>0\}}(s)ds,$$

or u is a mild solution of equation (1.1).

Finally, we prove that the solution u constructed above is the maximal solution of (1.1).

Proposition 2.5. *Let v be any mild solution of equation* (1.1)*. Then, we have*

$$v \le u, \quad \text{in } \Omega \times (0, \infty).$$

First of all, we observe that any mild solution v of (1.1) satisfies

$$v \in L^2(\tau, T; W_0^{1,2}(\Omega)) \cap L^\infty(\Omega \times (\tau, \infty)), \quad \text{for } 0 < \tau < T < \infty. \tag{2.42}$$

This result is classical, so we give its proof in the Appendix. Then, we have that for any $\varepsilon > 0$,

$$0 = \partial_t v - \Delta v + v^{-\beta} \chi_{\{v>0\}} \ge \partial_t v - \Delta v + +g_\varepsilon(v).$$

This implies that v is a sub-solution of (2.1). Applying Lemma 2.2 to v and u_ε we obtain

$$v \le u_\varepsilon, \quad \text{in } \Omega \times (0, \infty).$$

Letting $\varepsilon \to 0$, we obtain the desired conclusion. □

3. Quenching phenomenon in a finite time

Since u above is the maximal solution, then it is sufficient to show the quenching property for u.

Theorem 3.1. *Let u be the maximal solution of equation* (1.1)*, see Theorem 1.2. Then, there exists a finite time $T^* > 0$ such that*

$$u(x, t) = 0, \quad \forall (x, t) \in \Omega \times (T^*, \infty).$$

Moreover, T^* only depends on $\|u_0\|_{L^1(\Omega)}, N, \beta$, and $|\Omega|$.

Proof. First of all, we establish the energy equation for u (local in time). By multiplying (2.1) by u_ε, and integrating by parts, we obtain that for any $0 < \tau < t < +\infty$,

$$\frac{1}{2} \int_\Omega (|u_\varepsilon(t)|^2 - |u_\varepsilon(\tau)|^2) dx + \int_\tau^t \int_\Omega |\nabla u_\varepsilon|^2 \, dx \, ds + \int_\tau^t \int_\Omega g_\varepsilon(u_\varepsilon) u_\varepsilon \, dx \, ds = 0.$$

By passing to the limit in the above equation as $\varepsilon \to 0$, we deduce

$$\frac{1}{2} \int_\Omega (|u(t)|^2 - |u(\tau)|^2) dx + \int_\tau^t \int_\Omega |\nabla u|^2 \, dx \, ds + \int_\tau^t \int_\Omega u^{1-\beta} \, dx \, ds = 0. \tag{3.1}$$

Then, the variational arguments lead to the fact that

$$\frac{d}{dt}\Big(\frac{1}{2} \int_\Omega |u(t)|^2 dx\Big) + \int_\Omega |\nabla u(t)|^2 dx + \int_\Omega u^{1-\beta}(t) dx = 0, \quad \text{for } t \in (0, \infty). \tag{3.2}$$

On the other hand, from the Gagliardo-Nirenberg inequality, we have

$$\|u(t)\|_{L^2(\Omega)} \le C(N, \theta) \|\nabla u(t)\|^\theta_{L^2(\Omega)} \|u(t)\|^{1-\theta}_{L^1(\Omega)}, \tag{3.3}$$

with $\theta = \frac{N}{N+2}$, and $C(N, \theta) = C(N)$. Moreover, for any $\tau > 0$ fixed, (2.6) yields

$$\sup_{t \ge \tau} \|u(t)\|_{L^\infty(\Omega)} \le C(N, |\Omega|) \tau^{-\frac{N}{2}} \|u_0\|_{L^1(\Omega)} := M_\tau.$$

Thus, we have that for any $t \ge \tau$,

$$\int_\Omega u^{1-\beta}(t) dx \ge M_\tau^{-\beta} \int_\Omega u(t) dx. \tag{3.4}$$

Combining (3.3) and (3.4), we deduce

$$\begin{aligned} M_\tau^{-\beta(1-\theta)}\|u(t)\|_{L^2(\Omega)} &\le C(N)\Big(\int_\Omega |\nabla u(t)|^2dx\Big)^{\theta/2}\Big(M_\tau^{-\beta}\int_\Omega u(t)dx\Big)^{1-\theta}\\ &\le C(N)\Big(\int_\Omega |\nabla u(t)|^2dx\Big)^{\theta/2}\Big(\int_\Omega u^{1-\beta}(t)dx\Big)^{1-\theta}\\ &\le C(N)\Big(\int_\Omega |\nabla u(t)|^2dx+\int_\Omega u^{1-\beta}(t)dx\Big)^{\frac{\theta}{2}+1-\theta}. \end{aligned}$$

Then,

$$M_\tau^{-\frac{2\beta(1-\theta)}{2-\theta}}\Big(\int_\Omega |u(t)|^2dx\Big)^{\gamma} \le C(N)\Big(\int_\Omega |\nabla u(t)|^2dx+\int_\Omega u^{1-\beta}(t)dx\Big), \tag{3.5}$$

with $\gamma=\frac{1}{2-\theta}=\frac{N+2}{N+4}$. Hence, from (3.2) and (3.5), we obtain

$$\frac{d}{dt}w(t)+K(\tau)w^{\gamma}(t)\le 0,\quad \text{for any } t\ge\tau. \tag{3.6}$$

where

$$w(t)=\int_\Omega |u(t)|^2dx,\quad \text{and}\quad K(\tau)=2C(N)^{-1}M_\tau^{-\frac{2\beta(1-\theta)}{2-\theta}}.$$

Clearly, if there is a finite time $\tau_0>\tau$, such that $w(\tau_0)=0$, it follows from (3.6) that

$$w(t)=0,\quad \forall t>\tau_0.$$

If not, $w(t)>0$, for $t>\tau$, then solving the ODE (3.6) yields

$$w^{1-\gamma}(t)-w^{1-\gamma}(\tau)\le -(1-\gamma)K(\tau)(t-\tau),\quad \forall t>\tau. \tag{3.7}$$

Inequality (3.7) holds for any $t>\tau$, so it deduces a contradiction as t is large enough.

Finally, we shall show that the vanishing time (i.e. the quenching time) of $u(t)$ can be estimated by a constant only depending on $\|u_0\|_{L^1(\Omega)}$ and N, β, $|\Omega|$. In fact, by the basic semigroup estimate (see [9, 4]), we have

$$w(\tau)^{1/2}=\|u(\tau)\|_{L^2(\Omega)}\le C\tau^{-\frac{N}{4}}\|u_0\|_{L^1(\Omega)}.$$

Combining this fact, and (3.7) yields

$$w^{1-\gamma}(t)+(1-\gamma)K(\tau)(t-\tau)\le \Big(C\tau^{-\frac{N}{4}}\|u_0\|_{L^1(\Omega)}\Big)^{2(1-\gamma)},\quad \text{for } t>\tau. \tag{3.8}$$

Let $T_{\min}$ be a minimum vanishing time of $u(t)$. According to (3.8), we have for any $\tau>0$,

$$T_{\min}\le T(\tau)=\tau+C_1\tau^{-\frac{N}{2}(1-\gamma)}K(\tau)^{-1}\|u_0\|_{L^1(\Omega)}^{2(1-\gamma)},$$

with $C_1=C_1(N,|\Omega|)$. By a computation based on the definition of $K(\tau)$ and $M(\tau)$, we obtain

$$T(\tau)=\tau+C_2\tau^{-\left(\frac{N(1-\gamma)}{2}+N\beta(1-\theta)\gamma\right)}\|u_0\|_{L^1(\Omega)}^{2\beta(1-\theta)\gamma+2(1-\gamma)}:=\tau+C_2\tau^{-\alpha_1}\|u_0\|_{L^1(\Omega)}^{\alpha_2}.$$

However,

$$\min_{\tau>0}\{\tau+C_2\tau^{-\alpha_1}\|u_0\|_{L^1(\Omega)}^{\alpha_2}\}=\tau_0+C_2\tau_0^{-\alpha_1}\|u_0\|_{L^1(\Omega)}^{\alpha_2},$$

with $\tau_0^{\alpha_1+1}=\alpha_1C_2\|u_0\|_{L^1(\Omega)}^{\alpha_2}$. Then the previous equality gives us

$$\min_{\tau>0}\{\tau+C_2\tau^{-\alpha_1}\|u_0\|_{L^1(\Omega)}^{\alpha_2}\}=C_3\|u_0\|_{L^1(\Omega)}^{\frac{\alpha_2}{\alpha_1+1}}=C_3\|u_0\|_{L^1(\Omega)}^{\frac{2(1+\beta)}{3+\beta}}:=T^*,$$

with $C_3 = C_3(N, \gamma, |\Omega|)$. Then, $T_{\min} \le T^*$, which completes the proof. $\square$

Remark 3.2. The quenching property was established in the literature (see e.g., [21, 30]) only for the special case of bounded initial data or $u_0 \in L^2(\Omega)$, so the obtained quenching time T^* always depends on $\|u_0\|_{L^\infty(\Omega)}$ or $\|u_0\|_{L^2(\Omega)}$. Thus, our result is sharper in the sense that we merely require that $u_0 \in L^1(\Omega)$.

Next, we show that the uniqueness result holds for a class of weak solutions satisfying some conditions. Let $\mathcal{A}$ be the set of weak solution of equation (1.1) such that any solution $v \in \mathcal{A}$, $v(x,t) > 0$ in $\Omega \times (0, T_0)$. In other words, the set $\mathcal{A}$ contains the weak solutions such that they have the same quenching time T_0 as the maximal solution u.

Theorem 3.3. *Assume $\beta \in (0,1)$. Then* (1.1) *has at most one solution in the set $\mathcal{A}$.*

Remark 3.4. To obtain a solution, which stays positive for some time, we refer to [11, Lemma 1.9].

Proof of Theorem 3.3. Let $v \in \mathcal{A}$. Thanks to Theorem 1.2, we obtain

$$v \le u, \quad \text{in } \Omega \times (0, T^*). \tag{3.9}$$

Since u is a weak solution of (1.1), we have that for any $s \in (0, T^*)$,

$$\begin{aligned}
&\int_\Omega u(T^*)\phi(T^*)dx + \int_s^{T^*}\int_\Omega \nabla u \nabla \phi \,dx\,d\sigma + \int_s^{T^*}\int_\Omega \chi_{\{u>0\}} u^{-\beta}\phi\,dx\,d\sigma \\
&= \int_\Omega u(s)\phi(s)dx,
\end{aligned}$$

for any test function $\phi \in L^\infty_{loc}((0,\infty); L^\infty(\Omega)) \cap L^2_{loc}((0,\infty); H^1_0(\Omega))$. The fact that $u \in \mathcal{A}$ implies

$$\int_s^{T^*}\int_\Omega \nabla u \nabla \phi \,dx\,d\sigma + \int_s^{T^*}\int_\Omega u^{-\beta}\phi\,dx\,d\sigma = \int_\Omega u(s)\phi(s)dx, \tag{3.10}$$

By choosing $\phi = v$ as a test function in (3.10), we obtain

$$\int_s^{T^*}\int_\Omega \nabla u \nabla v \,dx\,d\sigma + \int_s^{T^*}\int_\Omega u^{-\beta} v \quad dx\,d\sigma = \int_\Omega u(s)v(s)dx, \tag{3.11}$$

Similarly, we also get the following equation by changing the roles of u and v,

$$\int_s^{T^*}\int_\Omega \nabla u \nabla v \,dx\,d\sigma + \int_s^{T^*}\int_\Omega v^{-\beta} u \quad dx\,d\sigma = \int_\Omega u(s)v(s)dx, \tag{3.12}$$

Combining (3.11) and (3.12), we obtain

$$\int_s^{T^*}\int_\Omega v^{-\beta} u \quad dx\,d\sigma = \int_s^{T^*}\int_\Omega u^{-\beta} v \quad dx\,d\sigma.$$

The above equation and (3.9) imply $u = v$ in $\Omega \times (s, T^*)$. This conclusion holds for any $s > 0$, thus we complete the proof. $\square$

4. Appendix

4.1. **Proof of Theorem 2.1.** Let us regularize the initial condition u_0 by considering a nonnegative sequence $\{u_{0,k}\}_k \subset \mathcal{C}_c^\infty(\Omega)$ such that $u_{0,k} \to u_0$ in $L^1(\Omega)$ as $k \to \infty$, and consider the problem

$$\begin{gathered}\partial_t v_k - \Delta v_k + g_\varepsilon(v_k) = 0, \quad \text{in } \Omega \times (0,T), \\ v_k = 0, \quad \text{on } \partial\Omega \times (0,T), \\ v_k(\cdot,0) = u_{0,k}(\cdot) \quad \text{on } \Omega.\end{gathered} \tag{4.1}$$

Since g_ε is a global Lipschitz-continuous function, the classical result ensures the existence and the uniqueness of a classical solution v_k. Moreover, v_k fulfils that for any $t > 0$,

$$v_k(t) = S(t)u_{0,k} - \int_0^t S(t-s)g_\varepsilon(v_k(s))ds. \tag{4.2}$$

Next, we claim that for any $T > 0$, $v_k \geq 0$ in $\Omega \times (0,T)$. Indeed, it is sufficient to show that

$$\min_{(x,t)\in\Omega\times(0,T)} v_k(x,t) \geq 0.$$

We can assume by contradiction that there is a point $(x_0,t_0) \in \Omega \times (0,T)$ such that

$$\min_{\Omega\times(0,T)} v_k(x,t) = v_k(x_0,t_0) < 0.$$

Let $\overline{v}_k(x,t) := v_k(x,t) + \delta t$, with $\delta > 0$ small enough such that $\overline{v}_k(x_0,t_0) = v_k(x_0,t_0) + \delta t_0 < 0$. This implies that $\overline{v}_k$ attains its minimum at a point inside of $\Omega \times (0,T)$, say $(x_1,t_1) \in \Omega \times (0,T)$, and $\overline{v}_k(x_1,t_1) \leq \overline{v}_k(x_0,t_0) < 0$. Then, we have $\partial_t\overline{v}_k(x_1,t_1) = 0$ and $\Delta\overline{v}_k(x_1,t_1) \geq 0$, so

$$0 = \partial_t v_k(x_1,t_1) - \Delta v_k(x_1,t_1) + g_\varepsilon(v_k(x_1,t_1)) = (\partial_t\overline{v}_k(x_1,t_1) - \delta) - \Delta\overline{v}_k(x_1,t_1) + 0.$$

This leads to a contradiction. Thus, we obtain the claim.

Next, we proceed as in the proof of Theorem 2.3 to obtain $v_k \to u_\varepsilon$, in the space $L^r(0,T;W_0^{1,r}(\Omega))$, as $k \to +\infty$ (up to a subsequence if necessary), and that $u_\varepsilon \in \mathcal{C}([0,T];L^1(\Omega))$. Then, it suffices to pass to the limit in (4.2) as $k \to +\infty$ in obtain (2.2).

It remains to show now that $u_\varepsilon \in \mathcal{C}_{x,t}^{2+\alpha,1+\frac{\alpha}{2}}(\overline{\Omega}\times[\tau,T])$ for any $0 < \tau < T < +\infty$, with some $\alpha \in (0,1)$. Indeed, applying the result of [27] to v_k we obtain that

$$\partial_t v_k,\ \nabla v_k,\ D^2_{x_ix_j}v_k \in L^p(\Omega \times (\tau,T)),$$

for any $p > 1$.

When p is large enough (such as $p>N+2$), we have $v_k \in \mathcal{C}_{x,t}^{\gamma,\frac{\gamma}{2}}(\overline{\Omega} \times [\tau,T])$, for some $\gamma \in (0,1)$. Note that v_k is bounded in $\mathcal{C}_{x,t}^{\gamma,\frac{\gamma}{2}}(\overline{\Omega} \times [\tau,T])$ by a constant independent of k. Therefore, Ascoli's theorem implies that, there is a subsequence (still denoted as $\{v_k\}_k$) such that

$$v_k \to u_\varepsilon, \quad \text{in } \mathcal{C}_{x,t}^{\gamma,\frac{\gamma}{2}}(\overline{\Omega} \times [\tau,T]).$$

On the other hand, u_ε satisfies

$$\partial_t u_\varepsilon - \Delta u_\varepsilon = -g_\varepsilon(u_\varepsilon).$$

But, since g_ε is Lipschitz-continuous, we have that $g_\varepsilon(u_\varepsilon) \in \mathcal{C}^{\gamma,\frac{\gamma}{2}}_{x,t}(\overline{\Omega}\times(\tau,T))$. Then, the conclusion $u_\varepsilon \in \mathcal{C}^{2+\gamma,1+\frac{\gamma}{2}}_{x,t}(\overline{\Omega}\times(\tau,T))$ follows from the α-Holder regularity of parabolic equations.

Proof of claim (2.42)**.** Let v be a mild solution of (1.1) and let us consider the problem

$$
\begin{gathered}
\partial_t \overline{v} - \Delta \overline{v} + f = 0, \quad \text{in } \Omega\times(0,T),\\
\overline{v} = 0, \quad \text{on } \partial\Omega\times(0,T),\\
\overline{v}(\cdot,0) = u_0(\cdot) \quad \text{on } \Omega.
\end{gathered} \tag{4.3}
$$

where $f := v^{-\beta}\chi_{\{v>0\}} \in L^1(\Omega\times(0,T))$ and $0<T<+\infty$. Then, a classical result (see for example [3, Lemma 3.3]) ensures that there is a unique mild (or weak) solution $\overline{v}$ of (4.3). Moreover, [3, Lemma 3.4] asserts that $v=\overline{v}$ in $\Omega\times(0,T)$. To prove (2.42), it is enough to show that, for any $0<\tau<T<+\infty$, $\overline{v}\in L^2(\tau,T;W^{1,2}_0(\Omega))$. Indeed, let $[f_n]_n \subset \mathcal{C}^\infty_c(\Omega\times(0,+\infty))$ be a sequence converging to f in $L^1(\Omega\times(0,+\infty))$ as $n\to+\infty$. Then, there exists a unique classical solution of the equation

$$
\begin{gathered}
\partial_t \overline{v}_n - \Delta \overline{v}_n + f_n = 0, \quad \text{in } \Omega\times(0,T),\\
\overline{v}_n = 0, \quad \text{on } \partial\Omega\times(0,T),\\
\overline{v}_n(\cdot,0) = u_0(\cdot) \quad \text{on } \Omega.
\end{gathered}
$$

Consider the equation satisfied by the difference between two solutions $\overline{v}_n$ and $\overline{v}_m$:

$$
\partial_t(\overline{v}_n - \overline{v}_m) - \Delta(\overline{v}_n - \overline{v}_m) + f_n - f_m = 0,
$$

Multiplying the above equation with $\overline{v}_{n,m} := \overline{v}_n - \overline{v}_m$ and integrating by parts we obtain

$$
\begin{aligned}
&\frac{1}{2}\int_\Omega (\overline{v}_{n,m})^2(T)dx + \int_\tau^T\int_\Omega |\nabla \overline{v}_{n,m}|^2\,dx\,ds\\
&= \int_\tau^T\int_\Omega (f_m - f_n)\overline{v}_{n,m}\,dx\,ds + \frac{1}{2}\int_\Omega (\overline{v}_{n,m})^2(\tau)dx.
\end{aligned}
$$

This implies

$$
\int_\tau^T\int_\Omega |\nabla \overline{v}_{n,m}|^2\,dx\,ds \le \int_\tau^T\int_\Omega |f_m - f_n||\overline{v}_{n,m}|\,dx\,ds + \frac{1}{2}\int_\Omega (\overline{v}_{n,m})^2(\tau)dx.
$$

The fact that $(f_n - f_m)$ converges to 0 in $L^1(\Omega\times(0,T))$ as $n,m\to+\infty$, and that $\{v_n\}_n$ is bounded by (2.6) assert that

$$
\lim_{n,m\to+\infty}\int_\tau^T\int_\Omega |f_m - f_n||\overline{v}_{n,m}|\,dx\,ds = 0.
$$

Moreover, using the same compactness argument as in the proof of Theorem 2.3, we obtain

$$
\lim_{n,m\to+\infty}\int_\Omega (\overline{v}_{n,m})^2(\tau)dx = 0.
$$

Finally, combining the previous three inequalities, we deduce

$$
\lim_{n,m\to+\infty}\int_\tau^T\int_\Omega |\nabla \overline{v}_{n,m}|^2\,dx\,ds = 0.
$$

Then, the uniqueness result implies that $[\nabla \overline{v}_n]_n$ converges to ∇v in $L^2(\Omega \times (\tau, T))$ and we reach the conclusion.

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