LOGARITHMICALLY IMPROVED REGULARITY CRITERIA FOR SUPERCRITICAL QUASI-GEOSTROPHIC EQUATIONS IN ORLICZ-MORREY SPACES

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Abstract. This article provides a regularity criterion for the surface quasi-geostrophic equation with supercritical dissipation. This criterion is in terms of the norm of the solution in a Orlicz-Morrey space. The result shows that, if a weak solutions $\theta$ satisfies
\[
\int_0^T \frac{\|\nabla \theta(\cdot,s)\|_{L^{2/r}}^\alpha}{1 + \ln(e + \|\nabla \perp \theta(\cdot,s)\|_{L^{2/r}})} \, ds < \infty,
\]
for some $0 < r < \alpha$ and $0 < \alpha < 1$, then $\theta$ is regular at $t = T$. In view of the embedding $L^{2/r} \subset M_p^{2/\alpha} \subset M_\alpha^{2/r} \subset L^2 \log^P L$ with $2 < p < 2/r$ and $P > 1$, our result extends the results due to Xiang [29] and Jia-Dong [15].

1. Introduction

The surface quasi-geostrophic equation is
\[
\partial_t \theta + u \cdot \nabla \theta + \Lambda^\alpha \theta = 0,
\]
where $\theta = \theta(x,t)$ is a scalar real-valued function of $(x,t) \in \mathbb{R}^2 \times \mathbb{R}^+$ and $u = (u_1, u_2)$ is the associated incompressible velocity field of the fluid with $\nabla \cdot u = 0$, and determined from $\theta$ by
\[
u := (-\partial_2 \Lambda^{-1} \theta, \partial_1 \Lambda^{-1} \theta) = \mathcal{R} \perp \theta = (-\mathcal{R}_2 \theta, \mathcal{R}_1 \theta),
\]
where $\Lambda = (-\Delta)^{1/2}$ is the Zygmund operator and $\mathcal{R}_i$, $i = 1, 2$ are the Riesz transforms.

The surface quasi-geostrophic equation with subcritical ($1 < \alpha \leq 2$) or critical dissipation ($\alpha = 1$) have been shown to possess global classical solutions whenever the initial data is sufficiently smooth. However, the global regularity problem remains open for the supercritical case ($0 < \alpha < 1$). Various regularity (or blow-up) criteria have been produced to shed light on this difficult global regularity problem (see e.g. [1, 2, 4, 5, 7, 8, 16, 17, 28, 30] and the references therein). The difficulties in understanding this problem are similar to those in solving the three-dimensional Navier-Stokes equations.

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For the study of the regularity criterion to (1.1) for the critical and supercritical case, Constantin, Majda and Tabak [6] obtained the following conditions
\[
\limsup_{t \nearrow T} \| \theta(t) \|_{H^m} < \infty \quad \text{if and only if} \quad \int_0^T \| \nabla \perp \theta(\cdot, t) \|_{L^\infty} dt < \infty, \quad (1.2)
\]
with \( m \geq 3 \) and \( \nabla \perp = (-\partial_{x_2}, \partial_{x_1}) \). Later on, Chae [2] (see also [3]) generalizes (1.2) to obtain the regularity criterion of the supercritical quasi-geostrophic equation (1.1) under the assumption
\[
\int_0^T \| \nabla \perp \theta(\cdot, t) \|_{L^p} dt < \infty \quad \text{with} \quad \frac{2}{p} + \frac{\alpha}{r} \leq \alpha \quad \text{and} \quad \frac{2}{\alpha} < p < \infty. \quad (1.3)
\]
Recently, Xiang [29] improved the Chae’s result and obtained another logarithmically improved regularity criterion in terms of the Lebesgue space subject to the assumption
\[
\int_0^T \| \nabla \theta(\cdot, t) \|_{L^p}^{\frac{\alpha}{r}} \frac{1}{1 + \ln(e + \| \nabla \theta(\cdot, t) \|_{L^\infty})} dt < \infty \quad \text{with} \quad \frac{2}{p} + \frac{\alpha}{r} = \alpha \quad \text{and} \quad \frac{2}{\alpha} < p < \infty. \quad (1.4)
\]
Very recently, Jia and Dong [15] improves the above regularity criterion (1.3) from Lebesgue space framework to Morrey-Campanato space framework. More precisely, they show the regularity of weak solution when the temperature function \( \theta \) satisfies the growth condition
\[
\int_0^T \| \nabla \theta(\cdot, t) \|_{M^{\alpha}_{L^p, L^2/\log P L}}^{\alpha/r} \frac{1}{1 + \ln(e + \| \nabla \theta(\cdot, t) \|_{L^\infty})} ds < \infty \quad \text{with} \quad 0 < r < \alpha < 1, \quad (1.4)
\]
for some \( 0 < r < \alpha \) and \( 0 < \alpha < 1 \), then \( \theta \) is actually regular in \( H^2 \) on \([0, T]\), where \( M^{\alpha}_{L^p, L^2/\log P L} \) denotes the Orlicz-Morrey space. Since the embedding relation \( L^{2/r} \subset M^{\alpha}_{L^p, L^2/\log P L} \) with \( P > 1 \) holds, our regularity criterion can be understood as an extension of the regularity results of Xiang [29] and Jia-Dong [15]. Main tools used in this paper are a weighted norm inequality for the Riesz potential and the Gagliardo-Nirenberg inequality.

2. Orlicz-Morrey spaces and statement of the main result

Before stating our result, let us recall some definitions and properties of the spaces that we are going to use (see e.g. [10, 11, 12, 13, 26] and references therein).

**Definition 2.1.** For \( P \in \mathbb{R} \) and \( 1 < w < v < \infty \), the Orlicz-Morrey space \( M_w^{v, \log P L} \) is defined by
\[
\| f \|_{M_w^{v, \log P L}} := \sup \left\{ r^{2/v} \| f \|_{B(x, r), L^v \log P L} : x \in \mathbb{R}^2, \ r > 0 \right\}, \quad (2.1)
\]
where \( \| f \|_{B(x, r), L^v \log P L} \) denotes the \( t^w \log P(3 + t) \) average given by
\[
\| f \|_{B(x, r), L^v \log P L} := \frac{\int_{B(x, r)} | f(t) |^w dt}{r^w \log P(3 + r)}.
\]
\[ := \inf \left\{ \lambda > 0 : \frac{1}{|B(x,R)|} \int_{B(x,R)} \left( \frac{|f(x)|}{\lambda} \right)^w \log \left( 3 + \frac{|f(x)|}{\lambda} \right)^P dx \leq 1 \right\}. \]

Our main result now reads as follows.

**Theorem 2.2.** Let \( \theta \) be a Leray-Hopf weak solutions of (1.1) with \( 0 < \alpha < 1 \), namely
\[
\theta \in L^\infty(0,T; L^2(\mathbb{R}^2)) \cap L^2(0,T; \dot{H}^\alpha(\mathbb{R}^2)).
\]
and satisfies the condition
\[
\int_0^T \frac{\|\nabla \theta(\cdot,s)\|^\alpha_{\mathcal{M}^{2/r}_{L^2 \log r \rightarrow L^2}}}{1 + \ln(e + \|\nabla \theta(\cdot,s)\|_{L^2})} ds < \infty \quad \text{with} \quad 0 < r < \alpha. \tag{2.2}
\]
Then, the solution \( \theta(x,t) \) is regular on \((0,T].\)

**Remark 2.3.** This criterion is in terms of the norm of the solution in a Orlicz-Morrey space. It is clear that Theorem 2.2 gives a logarithmic improvement of Xiang’s regularity criteria (1.3) (see also 1.4). As a consequence, this result extends several previous works.

Meanwhile, the definition of classical Morrey-Campanato spaces is as follows (see e.g. [18]):

**Definition 2.4.** For \( 1 < p \leq q \leq +\infty \), the Morrey-Campanato space is defined by
\[
\mathcal{M}^p_q = \left\{ f \in L^p_{\text{loc}}(\mathbb{R}^2) : \|f\|_{\mathcal{M}^p_q} = \sup_{x \in \mathbb{R}^2} \sup_{R > 0} |B|^{1/p - 1/q} \|f\|_{L^p(B(x,R))} < \infty \right\}, \tag{2.3}
\]
where \( B(x,R) \) denotes the closed ball in \( \mathbb{R}^2 \) with center \( x \) and radius \( R \).

In view of (2.1) and (2.3), the definition (2.1) covers (2.3) as a special case when \( P = 0 \). Here and below we write \((\log a)^P := \log^P a\).

Recall the following crucial result established in [10, 13, 12] (see also [22, 23, 24, 25]).

**Theorem 2.5.** Let \( 0 < \alpha < 1 \) and fractional integral operator \( I_\alpha \) be defined by
\[
I_\alpha f(x) = \int_{\mathbb{R}^2} \frac{f(y)}{|x-y|^{2-\alpha}} dy. \tag{2.4}
\]
If \( P > 1 \), then
\[
\|g \cdot I_\alpha f\|_{L^2} \leq C \|g\|_{\mathcal{M}^{3/2}_{L^2 \log r \rightarrow L^2}} \|f\|_{L^2}. \tag{2.5}
\]
Additionally, we have the following embeddings: for \( P > 0 \) and \( 0 < u < \tilde{u} < v \),
\[
L^u \hookrightarrow L^{v, \infty} \hookrightarrow \mathcal{M}^v_{\tilde{u}} \hookrightarrow \mathcal{M}^v_{L^n \log^P \rightarrow L} \hookrightarrow \mathcal{M}^v_u \tag{2.6}
\]
in the sense of continuous embedding and the inclusion is proper, where \( L^{p, \infty} \) denotes the usual Lorentz (weak-\(L^p\)) space. For more details see [10, 13, 12]. We shall use as well the following useful Sobolev inequality.

**Lemma 2.6.** Suppose that \( s > 1 \) and \( p \in [2, +\infty] \). Then, there is a constant \( C \geq 0 \) such that
\[
\|f\|_{L^p(\mathbb{R}^2)} \leq C \|f\|_{H^s(\mathbb{R}^2)}. \]
In particular,
\[
\|f\|_{L^{2/r}(\mathbb{R}^2)} \leq C \|f\|_{H^{s/2}(\mathbb{R}^2)} \quad \text{with} \quad 0 \leq r \leq 1.
\]
The above lemma can be proved using the well-known boundedness property of the Riesz potential operator (see, e.g., Stein [27]). In the proof of the main result, we employ the following Gagliardo-Nirenberg inequality having fractional derivatives contained in [13].

**Lemma 2.7.** Let $1 < p, p_0, p^1 \leq \infty$, $s, \gamma \in \mathbb{R}_+$, $0 \leq \beta \leq 1$. Then, there exists a constant $C$ such that

$$\|f\|_{\dot{H}_p^s} \leq C\|f\|_{L_{p_0}^\gamma}^{1-\beta}\|f\|_{\dot{H}_{p^1}^\gamma}^\beta,$$

where $\frac{1}{p} - \frac{s}{2} = \frac{1-\beta}{p_0} + \beta(\frac{1}{p^1} - \frac{\gamma}{2})$, $s \leq \beta\gamma$.

In particular,

$$\|f\|_{H^s} \leq C\|f\|_{L_2}^{1-\frac{s}{2}}\|f\|_{H_{\gamma}}^{s/\gamma}. \quad (2.7)$$

Now we are in a position to prove our regularity criterion.

**Proof of Theorem 2.2.** It suffices to prove that (2.2) ensures the a priori estimate

$$\int_0^T \|\nabla^\perp \theta(\cdot, t)\|_{L^\infty} dt < \infty,$$

hence guaranteeing the desired regularity until $T$ by (1.2).

For this, applying $\Lambda^2$ to (1.1) and taking the $L^2$ inner product of the resulting equation with $\Lambda^2 \theta$ and integrating by parts, we obtain

$$\frac{d}{dt}\|\Lambda^2 \theta(\cdot, t)\|_{L_2}^2 + 2\|\Lambda^{2+\frac{s}{2}} \theta(\cdot, t)\|_{L_2}^2$$

$$= -2\int_{\mathbb{R}^2} (u \cdot \nabla \theta) \Lambda^4 \theta \, dx$$

$$= -2\int_{\mathbb{R}^2} \Lambda^2 (u \cdot \nabla \theta) \Lambda^2 \theta \, dx$$

$$= -2\int_{\mathbb{R}^2} \Lambda^2 (u \cdot \nabla \theta) \Lambda^2 \theta \, dx - 2\int_{\mathbb{R}^2} (\Lambda u \cdot \nabla \Lambda \theta) \Lambda^2 \theta \, dx$$

$$= -2\int_{\mathbb{R}^2} \Lambda^2 (u \cdot \nabla \theta) \Lambda^2 \theta \, dx - 2\int_{\mathbb{R}^2} (\Lambda u \cdot \nabla \Lambda \theta) \Lambda^2 \theta \, dx,$$

where we have used the following cancelation property

$$\int_{\mathbb{R}^2} (u \cdot \nabla \Lambda^2 \theta) \Lambda^2 \theta \, dx = \frac{1}{2} \int_{\mathbb{R}^2} u \cdot \nabla |\Lambda^2 \theta|^2 \, dx = -\frac{1}{2} \int_{\mathbb{R}^2} (\nabla \cdot u) |\Lambda^2 \theta|^2 \, dx = 0.$$

Notice that $\Lambda^s$ and $\nabla$ commute. Hence, by Hölder inequality, first we estimate $J$. By using the Schwarz inequality, the fact that $B^s_{2,2} = H^s$ and the interpolation inequality, we have

$$\frac{d}{dt}\|\Lambda^2 \theta(\cdot, t)\|_{L_2}^2 + 2\|\Lambda^{2+\frac{s}{2}} \theta(\cdot, t)\|_{L_2}^2$$

$$= -2\int_{\mathbb{R}^2} (\Lambda^2 u \cdot \nabla \theta) I_r(-\Delta)^{\frac{s}{2}} \Lambda^2 \theta \, dx - 2\int_{\mathbb{R}^2} (\Lambda u \cdot \nabla \Lambda \theta) I_r(-\Delta)^{\frac{s}{2}} \Lambda^2 \theta \, dx$$

$$\leq 2\|\mathcal{R}^s \Lambda^2 \theta(\cdot, t)\|_{L_2} \|\nabla \theta I_r(-\Delta)^{\frac{s}{2}} \Lambda^2 \theta(\cdot, t)\|_{L_2}$$

$$+ 2\|\nabla \Lambda \theta(\cdot, t)\|_{L_2} \|\mathcal{R}^s \Lambda^2 \theta(\cdot, t)\|_{L_2}.$$
If we invoke Theorem 2.5 and (2.7), then we have by Young inequality and the boundedness of $R^\perp$ in the space $L^2$ and $M_{L^2 \log^P L}^{2/r}$

$$\frac{d}{dt} \|A^2 \theta(\cdot, t)\|_{L^2}^2 + 2 \|A^{2+\frac{\alpha}{2}} \theta(\cdot, t)\|_{L^2}^2 \leq C \|A^2 \theta(\cdot, t)\|_{L^2} \|A^3 \theta(\cdot, t)\|_{H^{r/2} M_{L^2 \log^P L}^{2/r}}$$

$$+ C \|A^2 \theta(\cdot, t)\|_{L^2} \|A^3 \theta(\cdot, t)\|_{H^{r/2} R^\perp A^2 \theta(\cdot, t)}$$

$$\leq C \|A^2 \theta(\cdot, t)\|_{L^2}^2 \|\nabla \theta(\cdot, t)\|_{H^{r/2} M_{L^2 \log^P L}^{2/r}}$$

$$= \left( C \|A^2 \theta(\cdot, t)\|_{L^2}^2 \|\nabla \theta(\cdot, t)\|_{H^{r/2} M_{L^2 \log^P L}^{2/r}} \right)^{1-\frac{r}{2}} \left( \|A^2 \theta(\cdot, t)\|_{H^{r/2} M_{L^2 \log^P L}^{2/r}} \right)^{r/2}$$

$$\leq \frac{1}{2} \|A^2 \theta(\cdot, t)\|_{H^{r/2} M_{L^2 \log^P L}^{2/r}}^2 + C \|A^2 \theta(\cdot, t)\|_{L^2} \|\nabla \theta(\cdot, t)\|_{H^{r/2} M_{L^2 \log^P L}^{2/r}}^2.$$ 

Consequently, by absorbing the diffusion term into the left hand side, we obtain

$$\frac{d}{dt} \|A^2 \theta(\cdot, t)\|_{L^2}^2 + 2 \|A^{2+\frac{\alpha}{2}} \theta(\cdot, t)\|_{L^2}^2 \leq C \|\nabla \theta(\cdot, t)\|_{H^{r/2} M_{L^2 \log^P L}^{2/r}} \|A^2 \theta(\cdot, t)\|_{L^2}^2$$

$$\leq C \frac{\|\nabla \theta(\cdot, t)\|_{H^{r/2} M_{L^2 \log^P L}^{2/r}} \|A^2 \theta(\cdot, t)\|_{L^2}^2}{1 + \ln(e + \|\nabla \theta(\cdot, t)\|_{L^2}^2)} \left[ 1 + \ln(e + \|\nabla \theta(\cdot, t)\|_{L^2}^2) \right]$$

$$\|A^2 \theta(\cdot, t)\|_{L^2}^2 \leq C \|\nabla \theta(\cdot, t)\|_{H^{r/2} M_{L^2 \log^P L}^{2/r}} \left[ 1 + \ln(e + \|\nabla \theta(\cdot, t)\|_{L^2}^2) \right] \|A^2 \theta(\cdot, t)\|_{L^2}^2,$$

where we have used the Sobolev embedding (see Lemma 2.6)

$$\|\nabla \theta(\cdot, t)\|_{L^2} \leq C \|A^2 \theta(\cdot, t)\|_{L^2} \quad \text{for } 0 < r < 1.$$ 

It follows that

$$\frac{d}{dt} \ln(e + \|A^2 \theta(\cdot, t)\|_{L^2})_2 \leq C \frac{\|\nabla \theta(\cdot, t)\|_{H^{r/2} M_{L^2 \log^P L}^{2/r}}}{1 + \ln(e + \|\nabla \theta(\cdot, t)\|_{L^2})_2} \left[ 1 + \ln(e + \|\nabla \theta(\cdot, t)\|_{L^2})_2 \right]$$

and thus by Gronwall’s inequality,

$$\ln(e + \|A^2 \theta(\cdot, t)\|_{L^2})_2 \leq C \int_0^T \frac{\|\nabla \theta(\cdot, t)\|_{H^{r/2} M_{L^2 \log^P L}^{2/r}}}{1 + \ln(e + \|\nabla \theta(\cdot, t)\|_{L^2})_2} \, dt \right).$$

This gives the uniform boundedness of $\|A^2 \theta(\cdot, t)\|_{L^2}^2$ in the time interval $[0, T]$. Recall that

$$\frac{d}{dt} \|A^2 \theta(\cdot, t)\|_{L^2}^2 + 2 \|A^{2+\frac{\alpha}{2}} \theta(\cdot, t)\|_{L^2}^2 \leq C \frac{\|\nabla \theta(\cdot, t)\|_{H^{r/2} M_{L^2 \log^P L}^{2/r}}}{1 + \ln(e + \|\nabla \theta(\cdot, t)\|_{L^2})_2} \left[ 1 + \ln(e + \|A^2 \theta(\cdot, t)\|_{L^2})_2 \right] \|A^2 \theta(\cdot, t)\|_{L^2}^2.$$

2.8
Integrating (2.8) over $[0, T]$, we have
\[
\|\Lambda^{\frac{5}{2}}\theta(\cdot, t)\|_{L^2}^2 + \int_0^T \|\Lambda^{\frac{5}{2} + 2}\theta(\cdot, t)\|_{L^2}^2 dt
\leq C \int_0^T \frac{\|\nabla\theta(\cdot, t)\|_{L^{\frac{4}{3}}(\mathbb{R}^2)}^2}{1 + \ln(e + \|\nabla^\perp\theta(\cdot, t)\|_{L^2(\mathbb{R}^2)})} dt
\]
\[
\sup_{0 \leq t \leq T} \left\{ [1 + \ln(e + \|\Lambda^2\theta(\cdot, t)\|_{L^2})] \|\Lambda^2\theta(\cdot, t)\|_{L^2}^2 \right\} + \|\Lambda^2\theta_0\|_{L^2}^2,
\]
which implies
\[
\int_0^T \|\Lambda^{\frac{5}{2} + 2}\theta(\cdot, t)\|_{L^2}^2 dt < \infty.
\]
On the other hand, by the Gagliardo-Nirenberg inequality in $\mathbb{R}^2$, it follows that
\[
\|\nabla^\perp\theta\|_{L^{\infty}} \leq C\|\theta\|_{L^{\frac{4}{3}}(\mathbb{R}^2)} \|\nabla^\perp\theta\|_{L^\infty(\mathbb{R}^2)}^{\frac{1}{2}} \|\theta\|_{L^2(\mathbb{R}^2)} \|\Lambda^{\frac{3}{2} + 2}\theta\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}}.
\]
Noting that $\|\theta\|_{L^2} \leq \|\theta_0\|_{L^2}$, implies
\[
\int_0^T \|\nabla^\perp\theta(\cdot, t)\|_{L^{\infty}} dt \leq C \int_0^T \|\theta(\cdot, t)\|_{L^{\frac{4}{3}}(\mathbb{R}^2)} \|\Lambda^{\frac{3}{2} + 2}\theta(\cdot, t)\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}} dt
\leq C\|\theta_0\|_{L^2} \int_0^T \|\Lambda^{\frac{3}{2} + 2}\theta(\cdot, t)\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}} dt
\leq C\|\theta_0\|_{L^2} \frac{T^{\frac{3}{2} + 2}}{2}\|\Lambda^{\frac{3}{2} + 2}\theta(\cdot, t)\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}} < \infty.
\]
By the blow-up criterion (2.2) of smooth solutions to (1.1), we complete the proof. 

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