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LOGARITHMICALLY IMPROVED REGULARITY CRITERIA FOR SUPERCRITICAL QUASI-GEOSTROPHIC EQUATIONS IN ORLICZ-MORREY SPACES

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Abstract. This article provides a regularity criterion for the surface quasi-geostrophic equation with supercritical dissipation. This criterion is in terms of the norm of the solution in a Orlicz-Morrey space. The result shows that, if a weak solutions θ satisfies

$$\int_0^T \frac{\|\nabla\theta(\cdot,s)\|_{\mathcal{M}^{2/r}_{L^2\log^P L}}^{\frac{\alpha}{\alpha-r}}}{1+\ln(e+\|\nabla^{\perp}\theta(\cdot,s)\|_{L^{2/r}})}ds<\infty,$$

for some $0<r<\alpha$ and $0<\alpha<1$, then θ is regular at $t=T$. In view of the embedding $L^{2/r}\subset \mathcal{M}_p{}^{2/r}\subset \mathcal{M}^{2/r}_{L^2\log^P L}$ with $2<p<2/r$ and $P>1$, our result extends the results due to Xiang [29] and Jia-Dong [15].

1. Introduction

The surface quasi-geostrophic equation is

$$\begin{gathered}\partial_t\theta+u\cdot\nabla\theta+\Lambda^{\alpha}\theta=0,\\ \theta(x,0)=\theta_0(x),\end{gathered}\tag{1.1}$$

where $\theta=\theta(x,t)$ is a scalar real-valued function of $(x,t)\in\mathbb{R}^2\times\mathbb{R}^+$ and $u=(u^1,u_2)$ is the associated incompressible velocity field of the fluid with $\nabla\cdot u=0$, and determined from θ by

$$u:=(-\partial_2\Lambda^{-1}\theta,\partial^1\Lambda^{-1}\theta)=\mathcal{R}^{\perp}\theta=(-\mathcal{R}_2\theta,\mathcal{R}^1\theta),$$

where $\Lambda=(-\Delta)^{1/2}$ is the Zygmund operator and $\mathcal{R}_i$, $i=1,2$ are the Riesz transforms.

The surface quasi-geostrophic equation with subcritical ($1<\alpha\le 2$) or critical dissipation ($\alpha=1$) have been shown to possess global classical solutions whenever the initial data is sufficiently smooth. However, the global regularity problem remains open for the supercritical case ($0<\alpha<1$). Various regularity (or blow-up) criteria have been produced to shed light on this difficult global regularity problem (see e.g. [1, 2, 4, 5, 7, 8, 16, 17, 28, 30] and the references therein). The difficulties in understanding this problem are similar to those in solving the three-dimensional Navier-Stokes equations.

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For the study of the regularity criterion to (1.1) for the critical and supercritical case, Constantin, Majda and Tabak [6] obtained the following conditions

$$\limsup_{t\nearrow T}\|\theta(t)\|_{H^m}<\infty \quad \text{if and only if} \quad \int_0^T \|\nabla^{\perp}\theta(\cdot,t)\|_{L^\infty}dt<\infty, \tag{1.2}$$

with $m\geq 3$ and $\nabla^{\perp}=(-\partial_{x_2},\partial_{x^1})$. Later on, Chae [2] (see also [3]) generalizes (1.2) to obtain the regularity criterion of the supercritical quasi-geostrophic equation (1.1) under the assumption

$$\int_0^T \|\nabla^{\perp}\theta(\cdot,t)\|_{L^p}^r dt<\infty \quad \text{with } \frac{2}{p}+\frac{\alpha}{r}\leq\alpha \text{ and } \frac{2}{\alpha}<p<\infty.$$

Recently, Xiang [29] improved the Chae's result and obtained another logarithmically improved regularity criterion in terms of the Lebesgue space subject to the assumption

$$\int_0^T \frac{\|\nabla\theta(\cdot,t)\|_{L^p}^r}{1+\ln(e+\|\nabla\theta(\cdot,t)\|_{L^\infty})}dt<\infty \quad \text{with } \frac{2}{p}+\frac{\alpha}{r}\leq\alpha \text{ and } \frac{2}{\alpha}<p<\infty. \tag{1.3}$$

Very recently, Jia and Dong [15] improves the above regularity criterion (1.3) from Lebesgue space framework to Morrey-Campanato space framework. More precisely, they show the regularity of weak solution when the temperature function θ satisfies the growth condition

$$\int_0^T \frac{\|\nabla\theta(\cdot,t)\|_{\mathcal{M}_q^p}^r}{1+\ln(e+\|\nabla\theta(\cdot,t)\|_{L^p})}dt<\infty \quad \text{with } \frac{2}{p}+\frac{\alpha}{r}=\alpha \text{ and } \frac{2}{\alpha}<p<\infty. \tag{1.4}$$

The regularity criterion presented in this article states that, if a weak solution of (1.1) satisfies

$$\int_0^T \frac{\|\nabla\theta(\cdot,s)\|_{\mathcal{M}^{2/r}_{L^2\log^P L}}^{\frac{\alpha}{\alpha-r}}}{1+\ln(e+\|\nabla^{\perp}\theta(\cdot,s)\|_{L^{2/r}})}ds<\infty \quad \text{with } 0<r<\alpha,$$

for some $0<r<\alpha$ and $0<\alpha<1$, then θ is actually regular in H^2 on $[0,T]$, where $\mathcal{M}^{2/r}_{L^2\log^P L}$ denotes the Orlicz-Morrey space. Since the embedding relation $L^{2/r}\subset\mathcal{M}^{2/r}_{L^2\log^P L}$ with $P>1$ holds, our regularity criterion can be understood as an extension of the regularity results of Xiang [29] and Jia-Dong [15]. Main tools used in this paper are a weighted norm inequality for the Riesz potential and the Gagliardo-Nirenberg inequality.

2. Orlicz-Morrey spaces and statement of the main result

Before stating our result, let us recall some definitions and properties of the spaces that we are going to use (see e.g. [10, 11, 12, 13, 26] and references therein).

Definition 2.1. For $P\in\mathbb{R}$ and $1<w<v<\infty$, the Orlicz-Morrey space $\mathcal{M}^v_{L^u\log^P L}$ is defined by

$$\|f\|^v_{\mathcal{M}_{L^u\log^P L}} := \sup\big\{r^{2/v}\|f\|_{B(x,r),L^u\log^P L} : x\in\mathbb{R}^2,\, r>0\big\}, \tag{2.1}$$

where $\|f\|_{B(x,r),L^u\log^P L}$ denotes the $t^w\log^P(3+t)$ average given by

$$\|f\|_{B(x,R),L^u\log^P L}$$

$$:= \inf\Big\{\lambda > 0 : \frac{1}{|B(x,R)|}\int_{B(x,R)} \Big(\frac{|f(x)|}{\lambda}\Big)^w \log\Big(3 + \frac{|f(x)|}{\lambda}\Big)^P\,dx \le 1\Big\}.$$

Our main result now reads as follows.

Theorem 2.2. *Let θ be a Leray-Hopf weak solutions of* (1.1) *with $0 < \alpha < 1$, namely*

$$\theta \in L^\infty(0,T;L^2(\mathbb{R}^2)) \cap L^2(0,T;\dot{H}^\alpha(\mathbb{R}^2)).$$

and satisfies the condition

$$\int_0^T \frac{\|\nabla\theta(\cdot,s)\|_{\mathcal{M}^{2/r}_{L^2\log^P L}}^{\frac{\alpha}{\alpha-r}}}{1+\ln(e+\|\nabla^\perp\theta(\cdot,s)\|_{L^{2/r}})}ds < \infty \quad \textit{with } 0 < r < \alpha. \tag{2.2}$$

Then, the solution $\theta(x,t)$ is regular on $(0,T]$.

Remark 2.3. This criterion is in terms of the norm of the solution in a Orlicz-Morrey space. It is clear that Theorem 2.2 gives a logarithmic improvement of Xiang's regularity criteria (1.3) (see also 1.4). As a consequence, this result extends several previous works.

Meanwhile, the definition of classical Morrey-Campanato spaces is as follows (see e.g. [18]):

Definition 2.4. For $1 < p \le q \le +\infty$, the Morrey-Campanato space is defined by

$$\mathcal{M}^p_q = \Big\{f \in L^p_{\rm loc}(\mathbb{R}^2) : \|f\|_{\mathcal{M}^p_q} = \sup_{x\in\mathbb{R}^2}\sup_{R>0}|B|^{1/q-1/p}\|f\|_{L^p(B(x,R))} < \infty\Big\}, \tag{2.3}$$

where $B(x,R)$ denotes the closed ball in $\mathbb{R}^2$ with center x and radius R.

In view of (2.1) and (2.3), the definition (2.1) covers (2.3) as a special case when $P = 0$. Here and below we write $(\log a)^P =: \log^P a$.

Recall the following crucial result established in [10, 13, 12] (see also [22, 23, 24, 25]).

Theorem 2.5. *Let $0 < \alpha < 1$ and fractional integral operator I_α be defined by*

$$I_\alpha f(x) = \int_{\mathbb{R}^2}\frac{f(y)}{|x-y|^{2-\alpha}}\,dy. \tag{2.4}$$

If $P > 1$, then

$$\|g\cdot I_\alpha f\|_{L^2} \le C\,\|g\|_{\mathcal{M}^{3/\alpha}_{L^2\log^P L}}\|f\|_{L^2}. \tag{2.5}$$

Additionally, we have the following embeddings: for $P > 0$ and $0 < u < \tilde{u} < v$,

$$L^v \hookrightarrow L^{v,\infty} \hookrightarrow \mathcal{M}^v_{\tilde{u}} \hookrightarrow \mathcal{M}^v_{L^u\log^P L} \hookrightarrow \mathcal{M}^v_u \tag{2.6}$$

in the sense of continuous embedding and the inclusion is proper, where $L^{p,\infty}$ denotes the usual Lorentz (weak-L^p) space. For more details see [10, 13, 12]. We shall use as well the following useful Sobolev inequality.

Lemma 2.6. *Suppose that $s > 1$ and $p \in [2,+\infty]$. Then, there is a constant $C \ge 0$ such that*

$$\|f\|_{L^p(\mathbb{R}^2)} \le C\|f\|_{H^s(\mathbb{R}^2)}.$$

In particular,

$$\|f\|_{L^{2/r}(\mathbb{R}^2)} \le C\|f\|_{H^2(\mathbb{R}^2)} \quad \textit{with } 0 \le r \le 1.$$

The above lemma can be proved using the well-known boundedness property of the Riesz potential operator (see, e.g., Stein [27]). In the proof of the main result, we employ the following Gagliardo-Nirenberg inequality having fractional derivatives contained in [14].

Lemma 2.7. *Let $1 < p$, $p_0, p^1 \le \infty$, $s, \gamma \in \mathbb{R}_+$, $0 \le \beta \le 1$. Then, there exists a constant C such that*

$$\|f\|_{\dot{H}^s_p} \le C\|f\|^{1-\beta}_{L_{p_0}} \|f\|^{\beta}_{\dot{H}^{\gamma}_{p^1}},$$

where

$$\frac{1}{p} - \frac{s}{2} = \frac{1-\beta}{p_0} + \beta\big(\frac{1}{p^1} - \frac{\gamma}{2}\big), \quad s \le \beta\gamma.$$

In particular,

$$\|f\|_{\dot{H}^s} \le C\|f\|^{1-\frac{s}{\gamma}}_{L^2} \|f\|^{s/\gamma}_{\dot{H}^{\gamma}}. \tag{2.7}$$

Now we are in a position to prove our regularity criterion.

Proof of Theorem 2.2. It suffices to prove that (2.2) ensures the a priori estimate

$$\int_0^T \|\nabla^{\perp}\theta(\cdot,t)\|_{L^\infty}\, dt < \infty,$$

hence guaranteeing the desired regularity until T by (1.2).

For this, applying Λ^2 to (1.1) and taking the L^2 inner product of the resulting equation with $\Lambda^2\theta$ and integrating by parts, we obtain

$$\begin{aligned}
&\frac{d}{dt}\|\Lambda^2\theta(\cdot,t)\|^2_{L^2} + 2\|\Lambda^{2+\frac{\alpha}{2}}\theta(\cdot,t)\|^2_{L^2}\\
&= -2\int_{\mathbb{R}^2} (u\cdot\nabla\theta)\Lambda^4\theta\, dx\\
&= -2\int_{\mathbb{R}^2} \Lambda^2(u\cdot\nabla\theta)\Lambda^2\theta\, dx\\
&= -2\int_{\mathbb{R}^2} (\Lambda^2 u\cdot\nabla\theta)\Lambda^2\theta\, dx - 2\int_{\mathbb{R}^2} (\Lambda u\cdot\nabla\Lambda\theta)\Lambda^2\theta\, dx - 2\int_{\mathbb{R}^2} (u\cdot\nabla\Lambda^2\theta)\Lambda^2\theta\, dx\\
&= -2\int_{\mathbb{R}^2} (\Lambda^2 u\cdot\nabla\theta)\Lambda^2\theta\, dx - 2\int_{\mathbb{R}^2} (\Lambda u\cdot\nabla\Lambda\theta)\Lambda^2\theta\, dx,
\end{aligned}$$

where we have used the following cancelation property

$$\int_{\mathbb{R}^2} (u\cdot\nabla\Lambda^2\theta)\Lambda^2\theta\, dx = \frac{1}{2}\int_{\mathbb{R}^2} u\cdot\nabla|\Lambda^2\theta|^2 dx = -\frac{1}{2}\int_{\mathbb{R}^2} (\nabla\cdot u).|\Lambda^2\theta|^2 dx = 0.$$

Notice that Λ^s and ∇ commute. Hence, by Hölder inequality, first we estimate J. By using the Schwarz inequality, the fact that $\dot{B}^r_{2.2} = \dot{H}^r$ and the interpolation inequality, we have

$$\begin{aligned}
&\frac{d}{dt}\|\Lambda^2\theta(\cdot,t)\|^2_{L^2} + 2\|\Lambda^{2+\frac{\alpha}{2}}\theta(\cdot,t)\|^2_{L^2}\\
&= -2\int_{\mathbb{R}^2} (\Lambda^2 u\cdot\nabla\theta) I_r(-\Delta)^{\frac{r}{2}}\Lambda^2\theta\, dx - 2\int_{\mathbb{R}^2} (\Lambda u\cdot\nabla\Lambda\theta) I_r(-\Delta)^{\frac{r}{2}}\Lambda^2\theta\, dx\\
&\le 2\|\mathcal{R}^{\perp}\Lambda^2\theta(\cdot,t)\|_{L^2}\|(\nabla\theta I_r(-\Delta)^{\frac{r}{2}}\Lambda^2\theta)(\cdot,t)\|_{L^2}\\
&\quad + 2\|\nabla\Lambda\theta(\cdot,t)\|_{L^2}\|(\mathcal{R}^{\perp}\Lambda\theta I_r(-\Delta)^{\frac{r}{2}}\Lambda^2\theta)(\cdot,t)\|_{L^2}.
\end{aligned}$$

If we invoke Theorem 2.5 and (2.7), then we have by Young inequality and the boundedness of $\mathcal{R}^{\perp}$ in the space L^2 and $\mathcal{M}^{2/r}_{L^2\log^P L}$

$$\begin{aligned}
&\frac{d}{dt}\|\Lambda^2\theta(\cdot,t)\|_{L^2}^2 + 2\|\Lambda^{2+\frac{\alpha}{2}}\theta(\cdot,t)\|_{L^2}^2\\
&\le C\|\Lambda^2\theta(\cdot,t)\|_{L^2}\|\Lambda^2\theta(\cdot,t)\|_{\dot H^r}\|\nabla\theta(\cdot,t)\|^{2/r}_{\mathcal{M}_{L^2\log^P L}}\\
&\quad + C\|\Lambda^2\theta(\cdot,t)\|_{L^2}\|\Lambda^2\theta(\cdot,t)\|_{\dot H^r}\|\mathcal{R}^{\perp}\Lambda\theta(\cdot,t)\|\\
&\le C\|\Lambda^2\theta(\cdot,t)\|_{L^2}^{2-\frac{2r}{\alpha}}\|\Lambda^2\theta(\cdot,t)\|_{\dot H^{\frac{\alpha}{2}}}^{\frac{2r}{\alpha}}\|\nabla\theta(\cdot,t)\|^{2/r}_{\mathcal{M}_{L^2\log^P L}}\\
&= \Big(C\|\Lambda^2\theta(\cdot,t)\|_{L^2}^2\|\nabla\theta(\cdot,t)\|^{\frac{\alpha}{\alpha-r}}_{\mathcal{M}^{2/r}_{L^2\log^P L}}\Big)^{1-\frac{r}{\alpha}}\Big(\|\Lambda^2\theta(\cdot,t)\|^2_{\dot H^{\frac{\alpha}{2}}}\Big)^{r/\alpha}\\
&\le \frac12\|\Lambda^2\theta(\cdot,t)\|^2_{\dot H^{\alpha/2}} + C\|\Lambda^2\theta(\cdot,t)\|_{L^2}^2\|\nabla\theta(\cdot,t)\|^{\frac{\alpha}{\alpha-r}}_{\mathcal{M}^{2/r}_{L^2\log^P L}}.
\end{aligned}$$

Consequently, by absorbing the diffusion term into the left hand side, we obtain

$$\begin{aligned}
&\frac{d}{dt}\|\Lambda^2\theta(\cdot,t)\|_{L^2}^2 + \|\Lambda^{\frac{\alpha}{2}+2}\theta(\cdot,t)\|_{L^2}^2\\
&\le C\|\nabla\theta(\cdot,t)\|^{\frac{\alpha}{\alpha-r}}_{\mathcal{M}^{2/r}_{\mathrm{L^2\log^P L}}}\|\Lambda^2\theta(\cdot,t)\|_{L^2}^2\\
&\le C\frac{\|\nabla\theta(\cdot,t)\|^{\frac{\alpha}{\alpha-r}}_{\mathcal{M}^{2/r}_{L^2\log^P L}}}{1+\ln(e+\|\nabla^{\perp}\theta(\cdot,t)\|_{L^{2/r}})}[1+\ln(e+\|\nabla^{\perp}\theta(\cdot,t)\|_{L^{\frac{2}{r}}})]\|\Lambda^2\theta(\cdot,t)\|_{L^2}^2\\
&\le C\frac{\|\nabla\theta(\cdot,t)\|^{\frac{\alpha}{\alpha-r}}_{\mathcal{M}^{2/r}_{L^2\log^P L}}}{1+\ln(e+\|\nabla^{\perp}\theta(\cdot,t)\|_{L^{2/r}})}[1+\ln(e+\|\Lambda^2\theta(\cdot,t)\|_{L^2})]\|\Lambda^2\theta(\cdot,t)\|_{L^2}^2,
\end{aligned}$$

where we have used the Sobolev embedding (see Lemma 2.6)

$$\|\nabla^{\perp}\theta(\cdot,t)\|_{L^{2/r}} \le C\|\Lambda^2\theta(\cdot,t)\|_{L^2}\quad \text{for } 0<r<1.$$

It follows that

$$\frac{d}{dt}\ln(e+\|\Lambda^2\theta(\cdot,t)\|_{L^2}^2) \le C\frac{\|\nabla\theta(\cdot,t)\|^{\frac{\alpha}{\alpha-r}}_{\mathcal{M}^{2/r}_{L^2\log^P L}}}{1+\ln(e+\|\nabla^{\perp}\theta(\cdot,t)\|_{L^{2/r}})}[1+\ln(e+\|\Lambda^2\theta(\cdot,t)\|_{L^2}^2)]$$

and thus by Gronwall's inequality,

$$\begin{aligned}
&\ln(e+\|\Lambda^2\theta(\cdot,t)\|_{L^2}^2)\\
&\le \ln(e+\|\Lambda^2\theta_0(\cdot,t)\|_{L^2}^2)\exp\Big(C\int_0^T\frac{\|\nabla\theta(\cdot,t)\|^{\frac{\alpha}{\alpha-r}}_{\mathcal{M}^{2/r}_{L^2\log^P L}}}{1+\ln(e+\|\nabla^{\perp}\theta(\cdot,t)\|_{L^{2/r}})}dt\Big).
\end{aligned}$$

This gives the uniform boundedness of $\|\Lambda^2\theta(\cdot,t)\|_{L^2}^2$ in the time interval $[0,T]$. Recall that

$$\begin{aligned}
&\frac{d}{dt}\|\Lambda^2\theta(\cdot,t)\|_{L^2}^2 + \|\Lambda^{\frac{\alpha}{2}+2}\theta(\cdot,t)\|_{L^2}^2\\
&\le C\frac{\|\nabla\theta(\cdot,t)\|^{\frac{\alpha}{\alpha-r}}_{\mathcal{M}^{2/r}_{L^2\log^P L}}}{1+\ln(e+\|\nabla^{\perp}\theta(\cdot,t)\|_{L^{2/r}})}[1+\ln(e+\|\Lambda^2\theta(\cdot,t)\|_{L^2})]\|\Lambda^2\theta(\cdot,t)\|_{L^2}^2.
\end{aligned}\tag{2.8}$$

Integrating (2.8) over $[0,T]$, we have

$$
\begin{aligned}
&\|\Lambda^2\theta(\cdot,t)\|_{L^2}^2+\int_0^T\|\Lambda^{\frac{\alpha}{2}+2}\theta(\cdot,t)\|_{L^2}^2dt\\
&\le C\int_0^T\frac{\|\nabla\theta(\cdot,t)\|_{\mathcal{M}^{2/r}_{L^2\log^P L}}^{\frac{\alpha}{\alpha-r}}}{1+\ln(e+\|\nabla^\perp\theta(\cdot,t)\|_{L^{2/r}})}dt\\
&\quad\sup_{0\le t\le T}\big\{[1+\ln(e+\|\Lambda^2\theta(\cdot,t)\|_{L^2})]\|\Lambda^2\theta(\cdot,t)\|_{L^2}^2\big\}+\|\Lambda^2\theta_0\|_{L^2}^2,
\end{aligned}
$$

which implies

$$
\int_0^T\|\Lambda^{\frac{\alpha}{2}+2}\theta(\cdot,t)\|_{L^2}^2dt<\infty.
$$

On the other hand, by the Gagliardo-Nirenberg inequality in $\mathbb{R}^2$, it follows that

$$
\begin{aligned}
\|\nabla^\perp\theta\|_{L^\infty}&\le C\|\theta\|_{L^2}^{\frac{\alpha}{\alpha+4}}\|\nabla^\perp\theta\|_{\dot H^{1+\frac{\alpha}{2}}}^{\frac{4}{\alpha+4}}\\
&\le C\|\theta\|_{L^2}^{\frac{\alpha}{\alpha+4}}\|\Lambda^{\frac{\alpha}{2}+2}\theta\|_{L^2}^{\frac{4}{\alpha+4}}.
\end{aligned}
$$

Noting that $\|\theta\|_{L^2}\le\|\theta_0\|_{L^2}$, implies

$$
\begin{aligned}
\int_0^T\|\nabla^\perp\theta(\cdot,t)\|_{L^\infty}dt&\le C\int_0^T\|\theta(\cdot,t)\|_{L^2}^{\frac{\alpha}{\alpha+4}}\|\Lambda^{\frac{\alpha}{2}+2}\theta(\cdot,t)\|_{L^2}^{\frac{4}{\alpha+4}}dt\\
&\le C\|\theta_0\|_{L^2}^{\frac{\alpha}{\alpha+4}}\int_0^T\|\Lambda^{\frac{\alpha}{2}+2}\theta(\cdot,t)\|_{L^2}^{\frac{4}{\alpha+4}}dt\\
&\le C\|\theta_0\|_{L^2}^{\frac{\alpha}{\alpha+4}}T^{\frac{\alpha+2}{\alpha+4}}\Big(\int_0^T\|\Lambda^{\frac{\alpha}{2}+2}\theta(\cdot,t)\|_{L^2}^2dt\Big)^{\frac{2}{\alpha+4}}<\infty.
\end{aligned}
$$

By the blow-up criterion (2.2) of smooth solutions to (1.1), we complete the proof. $\square$

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References

[1] L. Caffarelli, A. Vasseur; *Drift diffusion equations with fractional diffusion and the quasi-geostrophic equation*, Ann. Math. 171 (2010), 1903-1930.

[2] D. Chae; *On the regularity conditions for the dissipative quasi-geostrophic equations*, SIAM J. Math. Anal. 37 (2006), 1649-1656.

[3] D. Chae, J. Lee; *Global well-posedness in the supercritical dissipative quasi-geostrophic equations*, Commun Math. Phys, 233 (2003), 297-311.

[4] A. Cordoba, D. Cordoba; *A maximum principle applied to quasi-geostrophic equations*, Commum. Math. Phys. 249 (2004), 511-528.

[5] P. Constantin, D. Cordoba, J. Wu; *On the critical dissipative quasi-geostrophic equation*, Indiana Univ. Math. J. 50 (2001), 97-107.

[6] P. Constantin, A. Majda, E. Tabak; *Formation of strong fronts in the 2D quasi-geostrophic thermal active scalar*, Nonlinearity 7 (1994), 1495-1533.

[7] P. Constantin, J. Wu; *Behavior of solutions of 2D quasi-geostrophic equations*, SIAM J. Math. Anal. 30 (1999), 937-948.

[8] P. Constantin, J. Wu; *Regularity of Hölder continuous solutions of the super-critical quasi-geostrophic equation*, Ann. Inst. H. Poincaré Anal. NonLineaire 25 (2008), 1103-1110.

[9] S. Gala; *Regularity criteria for the 3D magneto-micropolar fluid equations in the Morrey-Campanato space*, Nonlinear Differential Equations and Applications 17 (2010), 181-194.
[10] S. Gala, Y. Sawano, H. Tanaka; *A new Beale-Kato-Majda criteria for the 3D magneto-micropolar fluid equations in Orlicz-Morrey spaces*, Math. Meth. Appl. Sci. 35 (2012), 1321-1334.
[11] S. Gala, Y. Sawano, H. Tanaka; *A remark on two generalized Orlicz–Morrey spaces*, J. Approx. Theory 198 (2015), 1-9.
[12] S. Gala, Y. Sawano, H. Tanaka; *On the uniqueness of weak solutions of the 3D MHD equations in the Orlicz-Morrey space*, Applicable Analysis 92 (2013), 776-783.
[13] S. Gala, Y. Sawano, H. Tanaka, M. A. Ragusa; *On the uniqueness of weak solutions of the 3D MHD equations in the Orlicz–Morrey space*, Applicable Analysis 93 (2014), 356-368.
[14] D. Gilbarg, N. S. Trudinger; *Elliptic Partial Difrential Equations of Second Order*, Springer-Verlag, Berlin, New York, 1983.
[15] Y. Jia, B.-Dong; *Remarks on the logarithmical regularity criterion of the supercritical surface quasi-geostrophic equation in Morrey spaces*, Appl. Math. Lett. 43 (2015), 80-84.
[16] N. Ju; *Existence and uniqueness of the solution to the dissipative 2D quasi-geostrophic equation in the Sobolev space*, Comm. Math. Phys. 251 (2004), 365-376.
[17] A. Kiselev, F. Nazarov, A. Volberg; *Global well-posedness for the critical 2D dissipative quasi-geostrophic equation*, Invent.Math. 167 (2007), 445-453.
[18] P. G. Lemarié-Rieusset; *Recent Developments in the Navier-Stokes Problem*, London: Chapman & Hall/CRC, 2002.
[19] F. Marchand; *Weak-strong uniqueness criteria for the critical quasi-geostrophic equation*, Physica D 237 (2008), 1346-1351.
[20] J. Pedlosky; *Geophysical Fluid Dynamics*, Springer, New York (1987).
[21] C. Pérez; *Sharp Lp-weighted Sobolev inequalities*, Ann. Inst. Fourier (Grenoble) 45 (1995), 809-824.
[22] Y. Sawano, S. Sugano, H. Tanaka; *Generalized fractional integral operators and fractional maximal operators in the framework of Morrey spaces*, Trans. Amer. Math. Soc. 363 (2011), 6481–6503.
[23] Y. Sawano, S. Sugano, H. Tanaka; *A note on Generalized Fractional Integral Operators on Generalized Morrey Space*, Boundary Value Problems, 2009 (2009), Article ID 835865, 18pp.
[24] Y. Sawano, S. Sugano, H. Tanaka; *Orlicz-Morrey spaces and fractional operators*, Potential Analysis 36 (2012), 517-556.
[25] Y. Sawano, S. Sugano, H. Tanaka; *Olsen's inequality and its applications to Schrödinger equations.* Harmonic analysis and nonlinear partial differential equations, 51–80, RIMS Kôkyûroku Bessatsu, B26, Res. Inst. Math. Sci. (x RIMS), Kyoto, 2011.
[26] Y. Sawano, S. Sugano, H. Tanaka, S. Gala; *Olsen's inequality and its applications to the MHD equations* (Regularity and Singularity for Geometric Partial Differential Equations and Conservation Laws), RIMS Kokyuroku 1845 (2013), 121-144.
[27] E. M. Stein; *Harmonic Analysis. Princeton University Press*, Princeton, 1993.
[28] J. Wu; *The 2D dissipative quasi-geostrophic equation*, Appl. Math. Lett. 15 (2002), 925-930.
[29] Z. Xiang; *A regularity criterion for the critical and supercritical dissipative quasi-geostrophic equations*, Appl. Math. Lett. 23 (2010), 1286-1290.
[30] B. Yuan; *Regularity condition of solutions to the quasi-geostrophic equations in Besov spaces with negative indices*, Acta Mathematicae Applicatae Sinica, English Series 26 (2010), 381-386.

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