Electronic Journal of Differential Equations, Vol. 2016 (2016), No. 151, pp. 1–12.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

MULTIPLE SOLUTIONS FOR A FRACTIONAL p-LAPLACIAN EQUATION WITH SIGN-CHANGING POTENTIAL

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Abstract. We use a variant of the fountain Theorem to prove the existence of infinitely many weak solutions for the fractional p-Laplace equation

$$(-\Delta)_p^s u + V(x)|u|^{p-2}u = f(x,u) \quad \text{in } \mathbb{R}^N,$$

where $s \in (0,1)$, $p \geq 2$, $N \geq 2$, $(-\Delta)_p^s$ is the fractional p-Laplace operator, the nonlinearity f is p-superlinear at infinity and the potential $V(x)$ is allowed to be sign-changing.

1. Introduction

In this article we are interested in the study of the nonlinear fractional p-Laplacian equation

$$(-\Delta)_p^s u + V(x)|u|^{p-2}u = f(x,u) \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

where $s \in (0,1)$, $p \geq 2$ and $N \geq 2$. Here $(-\Delta)_p^s$ is the fractional p-Laplace operator defined, for u smooth enough, by setting

$$(-\Delta)_p^s u(x) = 2 \lim_{\epsilon \to 0} \int_{\mathbb{R}^N \setminus B_\epsilon(x)} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+sp}} dy, \quad x \in \mathbb{R}^N,$$

up to some normalization constant depending upon N and s.

When $p = 2$, equation (1.1) arises in the study of the nonlinear Fractional Schrödinger equation

$$\imath \frac{\partial \psi}{\partial t} + (-\Delta)^s \psi = H(x,\psi) \quad \text{in } \mathbb{R}^N \times \mathbb{R}$$

when looking for standing wave functions $\psi(x,t) = u(x)e^{-\imath ct}$, where c is a constant. This equation was introduced by Laskin [12, 13] and comes from an extension of the Feynman path integral from the Brownian-like to the Levy-like quantum mechanical paths.

Nowadays there are many articles related to the nonlinear fractional Schrödinger equation: see for instance [1, 4, 5, 7, 8, 15, 16] and references therein. More recently, a new nonlocal and nonlinear operator was considered, namely the fractional p-Laplacian. In the works of Franzina and Palatucci [9] and of Lindgren and Linqvist [14], the eigenvalue problem associated with $(-\Delta)_p^s$ is studied, in particular

2010 *Mathematics Subject Classification.* 35A15, 35R11, 35J92.
Key words and phrases. Fractional p-Laplacian; sign-changing potential; fountain theorem.

Submitted May 10, 2016. Published June 20, 2016.

some properties of the first eigenvalue and of the higher order eigenvalues are obtained. Torres [17] established an existence result for the problem (1.1) when f is p-superlinear and subcritical. Iannizzotto et al. [11] investigated existence and multiplicity of solutions for a class of quasi-linear nonlocal problems involving the p-Laplacian operator.

Motivated by the above papers, we aim to study the multiplicity of nontrivial weak solutions to (1.1), when f is p-superlinear and $V(x)$ can change sign. More precisely, we require that the potential $V(x)$ satisfies the following assumptions:

(A1) $V \in C(\mathbb{R}^N)$ is bounded from below;

(A2) There exists $r > 0$ such that

$$\lim_{|y|\to\infty} |\{x \in \mathbb{R}^N : |x-y| \le r, V(x) \le M\}| = 0$$

for any $M > 0$,

while the nonlinearity $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ and its primitive $F(x,t) = \int_0^t f(x,z)\,dz$ are such that

(A3) $f \in C(\mathbb{R}^N \times \mathbb{R})$, and there exist $c_1 > 0$ and $p < \nu < p_s^*$ such that

$$|f(x,t)| \le c_1(|t|^{p-1} + |t|^{\nu-1}) \quad \forall (x,t) \in \mathbb{R}^N \times \mathbb{R},$$

where $p_s^* = \frac{Np}{N-sp}$ if $sp < N$ and $p_s^* = \infty$ for $sp \ge N$.

(A4) $F(x,0) \equiv 0, F(x,t) \ge 0$ for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$ and

$$\lim_{|t|\to\infty} \frac{F(x,t)}{|t|^p} = +\infty \text{ uniformly in } x \in \mathbb{R}^N.$$

(A5) There exists $\theta \ge 1$ such that

$$\theta\mathcal{F}(x,t) \ge \mathcal{F}(x,\tau t) \quad \forall (x,t) \in \mathbb{R}^N \times \mathbb{R} \text{ and } \tau \in [0,1]$$

where $\mathcal{F}(x,t) = tf(x,t) - pF(x,t)$.

(A6) $f(x,-t) = -f(x,t)$ for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$.

We recall that the conditions (A1) and (A2) on the potential V and (A3)–(A6) with $p = 2$ and $s = 1$, have been used in [18] to extend the well-known multiplicity result due to Bartsch and Wang [2]. Examples of V and f satisfying the above assumptions are

$$V(x) = \begin{cases} 2n|x| - 2n(n-1) + c_0 & \text{if } n-1 \le |x| \le (2n-1)/2 \\ -2n|x| + 2n^2 + c_0 & \text{if } (2n-1)/2 \le |x| \le n. \end{cases},$$

for $n \in \mathbb{N}$ and $c_0 \in \mathbb{R}$; and

$$f(x,t) = a(x)|t|^{p-2}t\ln(1+|t|) \quad \forall (x,t) \in \mathbb{R}^N \times \mathbb{R},$$

where $a(x)$ is a continuous bounded function with positive lower bound. Our main result can be stated as follows.

Theorem 1.1. *Assume that* (A1)–(A6) *are satisfied. Then the problem* (1.1) *has infinitely many nontrivial weak solutions.*

To prove Theorem 1.1, we will consider the family of functionals

$$\mathcal{J}_\lambda(u) = \frac{1}{p}\Big[\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,dx\,dy + \int_{\mathbb{R}^N} V(x)|u(x)|^p\,dx\Big] - \lambda\int_{\mathbb{R}^N} F(x,u)\,dx,$$

with $\lambda \in [1,2]$ and $u \in E$, where E is the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\|_E^p = \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,dx\,dy + \int_{\mathbb{R}^N} V(x)|u(x)|^p\,dx,$$

and we will show that $\mathcal{J}_\lambda$ satisfies the assumptions of the following variant of fountain Theorem due to Zou [19].

Theorem 1.2 ([19]). *Let $(E,\|\cdot\|)$ be a Banach space, $E = \overline{\oplus_{j\in\mathbb{N}} X_j}$, with $\dim X_j < \infty$ for any $j \in \mathbb{N}$. Set $Y_k = \oplus_{j=1}^k X_j$ and $Z_k = \overline{\oplus_{j=k}^\infty X_j}$. Let $\mathcal{J}_\lambda : E \to \mathbb{R}$ a family of $C^1(E,\mathbb{R})$ functionals defined by*

$$\mathcal{J}_\lambda(u) = A(u) - \lambda B(u), \quad \lambda \in [1,2].$$

Assume that $\mathcal{J}_\lambda$ satisfies the following assumptions:

(i) *$\mathcal{J}_\lambda$ maps bounded sets to bounded sets uniformly for $\lambda \in [1,2]$, $\mathcal{J}_\lambda(-u) = \mathcal{J}_\lambda(u)$ for all $(\lambda,u) \in [1,2]\times E$.*
(ii) *$B(u) \geq 0$ for all $u \in E$, and $A(u) \to \infty$ or $B(u) \to \infty$ as $\|u\| \to \infty$.*
(iii) *There exists $r_k > \rho_k$ such that*

$$\beta_k(\lambda) = \max_{u\in Y_k,\|u\|=r_k} \mathcal{J}_\lambda(u) < \alpha_k(\lambda) = \inf_{u\in Z_k,\|u\|=\rho_k} \mathcal{J}_\lambda(u), \quad \forall \lambda \in [1,2].$$

Then

$$\alpha_k(\lambda) \leq \xi_k(\lambda) = \inf_{\gamma\in\Gamma_k} \max_{u\in B_k} \mathcal{J}_\lambda(\gamma(u)) \quad \forall \lambda \in [1,2],$$

where

$$B_k = \{u \in Y_k : \|u\| \leq r_k\} \text{ and } \Gamma_k = \{\gamma \in C(B_k,X) : \gamma \text{ is odd }, \gamma = Id \text{ on } \partial B_k\}.$$

Moreover, for a.e. $\lambda \in [1,2]$, there exists a sequence $\{u_m^k(\lambda)\}_{m\in\mathbb{N}} \subset E$ such that

$$\sup_{m\in\mathbb{N}} \|u_m^k(\lambda)\| < \infty, \mathcal{J}'_\lambda(u_m^k(\lambda)) \to 0, \quad \mathcal{J}_\lambda(u_m^k(\lambda)) \to \xi_k(\lambda) \quad \text{as } m \to \infty.$$

Remark 1.3. By using (A1) we know that there exists $V_0 > 0$ such that $V_1(x) = V(x) + V_0 \geq 1$ for any $x \in \mathbb{R}^N$. Let $f_1(x,t) = f(x,t) + V_0|t|^{p-2}t$ for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$. Then it is easy to verify that the study of (1.1) is equivalent to investigate the problem

$$(-\Delta)_p^s u + V_1(x)|u|^{p-2}u = f_1(x,u) \quad \text{in } \mathbb{R}^N.$$

Hence, from now on, we assume that $V(x) \geq 1$ for any $x \in \mathbb{R}^N$ in (A1).

2. Preliminaries and functional setting

In this preliminary Section, for the reader's convenience, we recall some basic results related to the fractional Sobolev spaces. For more details about this topic we refer to [6].

Let $u : \mathbb{R}^N \to \mathbb{R}$ be a measurable function. We say that u belongs to the space $W^{s,p}(\mathbb{R}^N)$ if $u \in L^p(\mathbb{R}^N)$ and

$$[u]^p_{W^{s,p}(\mathbb{R}^N)} := \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,dx\,dy < \infty.$$

Then $W^{s,p}(\mathbb{R}^N)$ is a Banach space with respect to the norm

$$\|u\|_{W^{s,p}(\mathbb{R}^N)} = \left[[u]^p_{W^{s,p}(\mathbb{R}^N)} + |u|^p_{L^p(\mathbb{R}^N)}\right]^{1/p}.$$

We recall the main embeddings results for this class of fractional Sobolev spaces:

Theorem 2.1 ([6]). *Let $s \in (0,1)$ and $p \in [1,\infty)$ be such that $sp < N$. Then there exists $C = C(N,p,s) > 0$ such that*

$$|u|_{L^{p_s^*}(\mathbb{R}^N)} \leq C\|u\|_{W^{s,p}(\mathbb{R}^N)}.$$

for any $u \in W^{s,p}(\mathbb{R}^N)$. Moreover the embedding $W^{s,p}(\mathbb{R}^N) \subset L^q(\mathbb{R}^N)$ is locally compact whenever $q < p_s^*$.

- *If $sp = N$ then $W^{s,p}(\mathbb{R}^N) \subset L^q(\mathbb{R}^N)$ for any $q \in [p,\infty)$.*
- *If $sp > N$ then $W^{s,p}(\mathbb{R}^N) \subset C^{0,\frac{sp-N}{p}}_{\rm loc}(\mathbb{R}^N)$.*

Now we give the definition of weak solution for the problem (1.1). Taking into account the presence of the potential $V(x)$, we denote by E the closure of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\| := \Big([u]^p_{W^{s,p}(\mathbb{R}^N)} + |u|^p_V\Big)^{1/p}, \quad |u|^p_V = \int_{\mathbb{R}^N} V(x)|u(x)|^p\,dx.$$

Equivalently

$$E = \big\{u \in L^{p_s^*}(\mathbb{R}^N) : [u]_{W^{s,p}(\mathbb{R}^N)}, |u|_V < \infty\big\}.$$

Let us denote by $(E^*, \|\cdot\|_*)$ the dual space of $(E, \|\cdot\|)$. We define the nonlinear operator $A : E \to E^*$ by setting

$$\begin{aligned}\langle A(u), v\rangle = &\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+sp}}(v(x)-v(y))\,dxdy\\ &+ \int_{\mathbb{R}^N} V(x)|u|^{p-2}uv\,dx,\end{aligned}$$

for $u, v \in E$. Here $\langle\cdot,\cdot\rangle$ denotes the duality pairing between E and E^*. Let

$$B(u) = \int_{\mathbb{R}^N} F(x,u)\,dx$$

for $u \in E$, and we set $\mathcal{J}(u) = \frac{1}{p}\langle A(u), u\rangle - B(u)$ for $u \in E$.

Definition 2.2. We say that $u \in E$ is a weak solution to (1.1) if u satisfies

$$\langle A(u), v\rangle = \langle B'(u), v\rangle$$

for all $v \in E$.

Now we show a compactness result.

Lemma 2.3. *Under the assumption* (A1) *and* (A2)*, the embedding $E \subset L^q(\mathbb{R}^N)$ is compact for any $q \in [p, p_s^*)$.*

Proof. Let $\{u_n\} \subset E$ such that $u_n \rightharpoonup 0$ in E. We have to show that $u_n \to 0$ in $L^q(\mathbb{R}^N)$ for $q \in [p, p_s^*)$. By the interpolation inequality we only need to consider $q = p$. By using the Theorem 2.1 we know that $u_n \to 0$ in $L^p_{\rm loc}(\mathbb{R}^N)$. Thus it suffices to show that, for any $\epsilon > 0$, there exists $R > 0$ such that

$$\int_{B^c_R(0)} |u_n|^p\,dx < \epsilon;$$

here $B^c_R(0) = \mathbb{R}^N \setminus B_R(0)$. Let $\{y_i\}_{i\in\mathbb{N}}$ be a sequence of points in $\mathbb{R}^N$ satisfying $\mathbb{R}^N \subset \bigcup_{i\in\mathbb{N}} B_r(y_i)$ and such that each point x is contained in at most 2^N such balls $B_r(y_i)$. We recall that we are assuming $V(x) \geq 1$ for any $x \in \mathbb{R}^N$. Let

$$A_{R,M} = \{x \in B^c_R : V(x) \geq M\}, \quad B_{R,M} = \{x \in B^c_R : V(x) < M\}.$$

Then

$$\int_{A_{R,M}} |u_n|^p\,dx \le \frac{1}{M}\int_{\mathbb{R}^N} V(x)|u_n|^p\,dx$$

and this can be made arbitrarily small by choosing M large.

Take $\gamma > 1$ such that $p\gamma \le p_s^*$ and let $\gamma' = \frac{\gamma}{\gamma-1}$ be the dual exponent of γ. Then for fixed $M > 0$ we have

$$\begin{aligned}
\int_{B_{R,M}} |u_n|^p\,dx &\le \sum_{i\in\mathbb{N}} \int_{B_{R,M}\cap B_r(y_i)} |u_n|^p\,dx \\
&\le \sum_{i\in\mathbb{N}} \Big(\int_{B_{R,M}\cap B_r(y_i)} |u_n|^{p\gamma}\,dx\Big)^{1/\gamma} |B_{R,M}\cap B_r(y_i)|^{1/\gamma'} \\
&\le \epsilon_R \sum_{i\in\mathbb{N}} \Big(\int_{B_{R,M}\cap B_r(y_i)} |u_n|^{p\gamma}\,dx\Big)^{1/\gamma} \\
&\le C\epsilon_R \sum_{i\in\mathbb{N}} \|u_n\|^p_{W^{s,p}(B_r(y_i))} \\
&\le C\epsilon_R 2^N \|u_n\|^p_{W^{s,p}(\mathbb{R}^N)}
\end{aligned}$$

where $\epsilon_R = \sup_{y_i} |B_{R,M}\cap B_r(y_i)|^{1/\gamma'}$. By assumption (A1) we can infer that $\epsilon_R \to 0$ as $R \to \infty$. Then we may make this term small by choosing R large. $\square$

Next we prove the following result which will be fundamental later.

Lemma 2.4. *If $u_n \rightharpoonup u$ in E and $\langle A(u_n), u_n - u\rangle \to 0$ then $u_n \to u$ in E.*

Proof. Firstly, let us observe that for any $u, v \in E$

$$|\langle A(u), v\rangle| \le [u]^{p-1}_{W^{s,p}(\mathbb{R}^N)}[v]_{W^{s,p}(\mathbb{R}^N)} + |v|_V^{p-1}|v|_V \le \|u\|^{p-1}\|v\|.$$

Then, elementary calculations yield

$$\begin{aligned}
0 &\le (\|u_n\|^{p-1} - \|u\|^{p-1})(\|u_n\| - \|u\|) \\
&\le \|u_n\|^p - \langle A(u_n), u\rangle - \langle A(u), u_n\rangle + \|u\|^p \\
&= \langle A(u_n), u_n\rangle - \langle A(u_n), u\rangle - \langle A(u), u_n\rangle + \langle A(u), u\rangle \\
&= \langle A(u_n), u_n - u\rangle - \langle A(u), u_n - u\rangle =: I_n + II_n.
\end{aligned} \tag{2.1}$$

By the hypotheses of the lemma, it follows that $I_n, II_n \to 0$ as $n \to \infty$ so, in view of (2.1), we have $\|u_n\| \to \|u\|$ as $n \to \infty$. Since it is well known that the weak convergence and the norm convergence in a uniformly convex space imply the strong convergence, to conclude the proof it will be sufficient to prove that E is uniformly convex.

Fix $\varepsilon \in (0, 2)$ and let $u, v \in E$ such that $\|u\|, \|v\| \le 1$ and $\|u - v\| > \varepsilon$. By using that

$$|\frac{a+b}{2}|^p + |\frac{a-b}{2}|^p \le \frac{1}{2}(|a|^p + |b|^p) \quad \text{for any } a, b \in \mathbb{R},$$

it follows that

$$\begin{aligned}
&\|\frac{u+v}{2}\|^p + \|\frac{u-v}{2}\|^p \\
&= \Big\{ [\frac{u+v}{2}]^p_{W^{s,p}(\mathbb{R}^N)} + [\frac{u-v}{2}]^p_{W^{s,p}(\mathbb{R}^N)} + |\frac{u+v}{2}|^p_V + |\frac{u-v}{2}|^p_V \Big\} \\
&\le \frac{1}{2}\Big([u]^p_{W^{s,p}(\mathbb{R}^N)} + [v]^p_{W^{s,p}(\mathbb{R}^N)} + |u|^p_V + |v|^p_V \Big)
\end{aligned}$$

$$= \frac{1}{2}\big(\|u\|^p + \|v\|^p\big) = 1,$$

which gives $\|\frac{u+v}{2}\|^p < 1 - \frac{\varepsilon^p}{2^p}$. Choosing

$$\delta = 1 - \big[1 - \Big(\frac{\varepsilon}{2}\Big)^p\big]^{1/p} > 0,$$

we can infer that $\|\frac{u+v}{2}\| < 1 - \delta$. Then E is uniformly convex. $\square$

Let us introduce a family of functionals $\mathcal{J}_\lambda : E \to \mathbb{R}$ defined by

$$\mathcal{J}_\lambda(u) = \frac{1}{p}\langle A(u), u\rangle - \lambda B(u), \quad \lambda \in [1,2].$$

After integrating, from (A3), we obtain that for any $(x,t) \in \mathbb{R}^N \times \mathbb{R}$,

$$|F(x,t)| \le \frac{c_1}{p}|t|^p + \frac{c_1}{\nu}|t|^\nu \le c_1(|t|^p + |t|^\nu). \tag{2.2}$$

By using (A1), (A2), (2.2) and Lemma 2.3 follows that $\mathcal{J}_\lambda$ is well defined on E. Moreover $\mathcal{J}_\lambda \in C^1(E,\mathbb{R})$, and

$$\mathcal{J}_\lambda'(u)v = \langle A(u), v\rangle - \lambda\langle B'(u), v\rangle \tag{2.3}$$

where

$$\langle B'(u), v\rangle = \int_{\mathbb{R}^N} f(x,u)v\,dx.$$

Then the critical points of $\mathcal{J}_1 = \mathcal{J}$ are weak solutions to (1.1).

To apply the Theorem 1.2, we can observe that E is a separable ($C_0^\infty(\mathbb{R}^N)$ is separable and dense in $W^{s,p}(\mathbb{R}^N)$) and reflexive Banach space, so there exist $(\phi_n) \subset E$ and $(\phi_n^*) \subset E^*$ such that $E = \overline{\operatorname{span}\{\phi_n : n \in \mathbb{N}\}}$, $E^* = \overline{\operatorname{span}\{\phi_n^* : n \in \mathbb{N}\}}$ and $\langle \phi_n^*, \phi_m\rangle = 1$ if $n = m$ and zero otherwise. Then, for any $n \in \mathbb{N}$, we set $X_n = \operatorname{span}\{\phi_n\}$, $Y_n = \oplus_{j=1}^n X_j$ and $Z_n = \overline{\oplus_{j=n}^\infty X_j}$.

3. Proof of Theorem 1.1

In this section we give the proof of the main result of this paper. Firstly we prove the following Lemmas:

Lemma 3.1. *Assume that* (A1)–(A3) *are satisfied. Then there exists* $k_1 \in \mathbb{N}$ *and a sequence* $\rho_k \to \infty$ *as* $k \to \infty$ *such that*

$$\alpha_k(\lambda) = \inf_{u \in Z_k, \|u\| = \rho_k} \mathcal{J}_\lambda(u), \quad \forall k \ge k_1.$$

Proof. Let us define

$$b_p(k) = \sup_{u \in Z_k, \|u\|=1} |u|_{L^p(\mathbb{R}^N)}, \quad b_\nu(k) = \sup_{u \in Z_k, \|u\|=1} |u|_{L^\nu(\mathbb{R}^N)}.$$

We aim to prove that

$$b_p(k) \to 0, b_\nu(k) \to 0 \quad \text{as } k \to \infty. \tag{3.1}$$

It is clear that $b_p(k)$ and $b_\nu(k)$ are decreasing with respect to k so there exist $b_p, b_\nu \ge 0$ such that $b_p(k) \to b_p$ and $b_\nu(k) \to b_\nu$ as $k \to \infty$. For any $k \ge 0$, there exists $u_k \in Z_k$ such that $\|u_k\| = 1$ and $|u_k|_p \ge \frac{b_p(k)}{2}$.

Taking into account that E is reflexive, we can assume that $u_k \rightharpoonup u$ in E. Now, for any $\phi_n^* \in \{\phi_j^*\}_{j\in\mathbb{N}}$, we can see that $\langle \phi_n^*, u_k\rangle = 0$ for $k > n$, so $\langle \phi_n^*, u\rangle = \lim_{k\to\infty}\langle \phi_n^*, u_k\rangle = 0$. Then $\langle \phi_n^*, u\rangle = 0$ for any $\phi_n^* \in \{\phi_j^*\}_{j\in\mathbb{N}}$, which gives $u = 0$.

Since E is compactly embedded in $L^p(\mathbb{R}^N)$ by Lemma 2.3, we have $u_k \to 0$ in $L^p(\mathbb{R}^N)$, which implies that $b_p = 0$. Similarly we can prove $b_\nu = 0$. Then, for any $u \in Z_k$ and $\lambda \in [1,2]$, we can see that

$$\begin{aligned}
\mathcal{J}_\lambda(u) &= \frac{1}{p}\langle A(u), u\rangle - \lambda B(u)\\
&\geq \frac{\|u\|^p}{p} - 2\int_{\mathbb{R}^N} F(x,u)\,dx\\
&\geq \frac{\|u\|^p}{p} - 2c_1(|u|^p_{L^p(\mathbb{R}^N)} + |u|^\nu_{L^\nu(\mathbb{R}^N)})\\
&\geq \frac{\|u\|^p}{p} - 2c_1(b_p^p(k)\|u\|^p + b_\nu^\nu(k)\|u\|^\nu).
\end{aligned}$$

By using (3.1), we can find $k_1 \in \mathbb{N}$ such that

$$2c_1 b_p^p(k) \leq \frac{1}{2p} \quad \forall k \geq k_1.$$

For each $k \geq k_1$, we choose

$$\rho_k := (8pc_1 b_\nu^\nu(k))^{\frac{1}{p-\nu}}.$$

Let us note that

$$\rho_k \to \infty \quad \text{as } k \to \infty, \tag{3.2}$$

since $\nu > p$. Then we deduce that

$$\alpha_k(\lambda) := \inf_{u\in Z_k, \|u\|=\rho_k} \mathcal{J}_\lambda(u) \geq \frac{1}{4p}\rho_k^p > 0$$

for any $k \geq k_1$. $\square$

Lemma 3.2. *Assume that* (A1)–(A4) *hold. Then for the positive integer k_1 and the sequence ρ_k obtained in Lemma 3.1, there exists $r_k > \rho_k$ for any $k \geq k_1$ such that*

$$\beta_k(\lambda) = \max_{u\in Y_k, \|u\|=r_k} \mathcal{J}_\lambda(u) < 0.$$

Proof. Firstly we prove that for any finite dimensional subspace $F \subset E$ there exists a constant $\delta > 0$ such that

$$|\{x \in \mathbb{R}^N : |u(x)| \geq \delta\|u\|\}| \geq \delta, \quad \forall u \in F \setminus \{0\}. \tag{3.3}$$

We argue by contradiction and we suppose that for any $n \in \mathbb{N}$ there exists $0 \neq u_n \in F$ such that

$$\big|\big\{x \in \mathbb{R}^N : |u_n(x)| \geq \frac{1}{n}\|u\|\big\}\big| < \frac{1}{n}, \quad \forall n \in \mathbb{N}.$$

Let $v_n := \frac{u_n}{\|u_n\|} \in F$ for all $n \in \mathbb{N}$. Then $\|v_n\| = 1$ for all $n \in \mathbb{N}$ and

$$\big|\big\{x \in \mathbb{R}^N : |v_n(x)| \geq \frac{1}{n}\big\}\big| < \frac{1}{n}, \quad \forall n \in \mathbb{N}. \tag{3.4}$$

Up to a subsequence, we may assume that $v_n \to v$ in E for some $v \in F$ since F is a finite dimensional space. Clearly $\|v\| = 1$. By using Lemma 2.3 and the fact that all norms are equivalent on F, we deduce that

$$|v_n - v|_{L^p(\mathbb{R}^N)} \to 0 \quad \text{as } n \to \infty. \tag{3.5}$$

Since $v \neq 0$, there exists $\delta_0 > 0$ such that

$$|\{x \in \mathbb{R}^N : |v(x)| \geq \delta_0\}| \geq \delta_0. \tag{3.6}$$

Set

$$\Lambda_0 := \{x \in \mathbb{R}^N : |v(x)| \geq \delta_0\}$$

and for all $n \in \mathbb{N}$,

$$\Lambda_n := \big\{x \in \mathbb{R}^N : |v_n(x)| \geq \frac{1}{n}\big\}, \quad \Lambda_n^c := \mathbb{R}^N \setminus \Lambda_n.$$

Taking into account (3.4) and (3.6), we obtain

$$|\Lambda_n \cap \Lambda_0| \geq |\Lambda_0| - |\Lambda_n^c| \geq \delta_0 - \frac{1}{n} \geq \frac{\delta_0}{2}.$$

for n large enough. Therefore,

$$\begin{aligned}
\int_{\mathbb{R}^N} |v_n - v|^p \, dx &\geq \int_{\Lambda_n \cap \Lambda_0} |v_n - v|^p \, dx \\
&\geq \int_{\Lambda_n \cap \Lambda_0} (|v|^p - |v_n|^p) \, dx \\
&\geq \Big(\delta_0 - \frac{1}{n}\Big)^p |\Lambda_n \cap \Lambda_0| \\
&\geq \Big(\frac{\delta_0}{2}\Big)^{p+1} > 0
\end{aligned}$$

which contradicts (3.5).

Now, by using that Y_k is finite dimensional and (3.3), we can find $\delta_k > 0$ such that

$$|\{x \in \mathbb{R}^N : |u(x)| \geq \delta_k \|u\|\}| \geq \delta_k, \quad \forall u \in Y_k \setminus \{0\}. \tag{3.7}$$

By (A4), for any $k \in \mathbb{N}$ there exists a constant $R_k > 0$ such that

$$F(x,u) \geq \frac{|u|^p}{\delta_k^{p+1}} \quad \forall x \in \mathbb{R}^N \text{ and } |u| \geq R_k.$$

Set

$$A_u^k = \{x \in \mathbb{R}^N : |u(x)| \geq \delta_k \|u\|\}$$

and let us observe that, by (3.7), $|A_u^k| \geq \delta_k$ for any $u \in Y_k \setminus \{0\}$. Then for any $u \in Y_k$ such that $\|u\| \geq \frac{R_k}{\delta_k}$, we have

$$\begin{aligned}
\mathcal{J}_\lambda(u) &\leq \frac{1}{p}\|u\|^p - \int_{\mathbb{R}^N} F(x,u) \, dx \\
&\leq \frac{1}{p}\|u\|^p - \int_{A_u^k} \frac{|u|^p}{\delta_k^{p+1}} \, dx \\
&\leq \frac{1}{p}\|u\|^p - \|u\|^p = -\big(\frac{p-1}{p}\big)\|u\|^p.
\end{aligned}$$

Choosing $r_k > \max\{\rho_k, \frac{R_k}{\delta_k}\}$ for all $k \geq k_1$, follows that

$$\beta_k(\lambda) = \max_{u \in Y_k, \|u\| = r_k} \mathcal{J}_\lambda(u) \leq -\Big(\frac{p-1}{p}\Big) r_k^p < 0, \quad \forall k \geq k_1.$$

$\square$

By using (2.2) and Lemma 2.3 we can see that $\mathcal{J}_\lambda$ maps bounded sets to bounded sets uniformly for $\lambda \in [1,2]$. Moreover, by (A6), $\mathcal{J}_\lambda$ is even. Then condition (i) in Theorem 1.2 is satisfied. Condition (ii) is clearly true, while (iii) follows by Lemma 3.1 and Lemma 3.2. Then, by Theorem 1.2, for any $k \geq k_1$ and $\lambda \in [1,2]$ there exists a sequence $\{u_m^k(\lambda)\} \subset E$ such that

$$\sup_{m\in\mathbb{N}} \|u_m^k(\lambda)\| < \infty, \quad \mathcal{J}'_\lambda(u_m^k(\lambda)) \to 0, \quad \mathcal{J}_\lambda(u_m^k(\lambda)) \to \xi_k(\lambda) \quad \text{as } m \to \infty$$

where

$$\xi_k(\lambda) = \inf_{\gamma\in\Gamma_k} \max_{u\in B_k} \mathcal{J}_\lambda(\gamma(u))$$

with

$$B_k = \{u \in Y_k : \|u\| \leq r_k\}, \quad \Gamma_k = \{\gamma \in C(B_k, X) : \gamma \text{ is odd }, \gamma = Id \text{ on } \partial B_k\}.$$

In particular, from the proof of Lemma 3.1, we deduce that for any $k \geq k_1$ and $\lambda \in [1,2]$

$$\frac{1}{4p}\rho_k^p =: c_k \leq \xi_k(\lambda) \leq d_k := \max_{u\in B_k} \mathcal{J}_1(u), \tag{3.8}$$

and $c_k \to \infty$ as $k \to \infty$ by (3.2). As a consequence, for any $k \geq k_1$, we can choose $\lambda_n \to 1$ (depending on k) and get the corresponding sequences satisfying

$$\sup_{m\in\mathbb{N}} \|u_m^k(\lambda_n)\| < \infty, \quad \mathcal{J}'_{\lambda_n}(u_m^k(\lambda_n)) \to 0 \quad \text{as } m \to \infty. \tag{3.9}$$

Now, we prove that for any $k \geq k_1$, $\{u_m^k(\lambda_n)\}_{m\in\mathbb{N}}$ admits a strongly convergent subsequence $\{u_n^k\}_{n\in\mathbb{N}}$, and that such subsequence is bounded.

Lemma 3.3. *For each λ_n given above, the sequence $\{u_m^k(\lambda_n)\}_{m\in\mathbb{N}}$ has a strong convergent subsequence.*

Proof. By (3.9) we may assume, without loss of generality, that as $m \to \infty$,

$$u_m^k(\lambda_n) \rightharpoonup u_n^k \text{ in } E$$

for some $u_n^k \in E$. By Lemma 2.3 we have

$$u_m^k(\lambda_n) \to u_n^k \quad \text{in } L^p(\mathbb{R}^N) \cap L^\nu(\mathbb{R}^N). \tag{3.10}$$

By (A3) and Hölder inequality it follows that

$$\begin{aligned}
&\Big|\int_{\mathbb{R}^N} f(x, u_m^k(\lambda_n))(u_m^k(\lambda_n) - u_n^k)\,dx\Big| \\
&\leq c_1|u_m^k(\lambda_n)|_p^{p-1}|u_m^k(\lambda_n) - u_n^k|_p + c_1|u_m^k(\lambda_n)|_\nu^{\nu-1}|u_m^k(\lambda_n) - u_n^k|_\nu
\end{aligned}$$

so, by using (3.10), we obtain

$$\lim_{m\to\infty} \int_{\mathbb{R}^N} f(x, u_m^k(\lambda_n))(u_m^k(\lambda_n) - u_n^k)\,dx = 0.$$

Since $\mathcal{J}'_{\lambda_n}(u_m^k(\lambda_n)) \to 0$ as $m \to \infty$, and

$$\langle \mathcal{J}'_\lambda(u), v\rangle = \langle A(u), v\rangle - \lambda\langle B'(u), v\rangle,$$

we deduce that

$$\langle A(u_m^k(\lambda_n)), u_m^k(\lambda_n) - u_n^k\rangle \to 0 \quad \text{as } m \to \infty.$$

Then, by using Lemma 2.4, we infer that

$$u_m^k(\lambda_n) \to u_n^k \quad \text{in } E \text{ as } m \to \infty.$$

□

Therefore, without loss of generality, we may assume that

$$\lim_{m\to\infty} u_m^k(\lambda_n) = u_n^k, \quad \forall n \in \mathbb{N}, k \geq k_1.$$

As a consequence, we obtain

$$\mathcal{J}'_{\lambda_n}(u_n^k) = 0, \mathcal{J}_{\lambda_n}(u_n^k) \in [c_k, d_k], \quad \forall n \in \mathbb{N}, k \geq k_1. \tag{3.11}$$

Lemma 3.4. *For any $k \geq k_1$, the sequence $\{u_n^k\}_{n\in\mathbb{N}}$ is bounded.*

Proof. For simplicity we set $u_n = u_n^k$. We suppose by contradiction that, up to a subsequence,

$$\|u_n\| \to \infty \quad \text{as } n \to \infty. \tag{3.12}$$

Let $w_n = u_n/\|u_n\|$ for any $n \in \mathbb{N}$. Then, up to subsequence, we may assume that

$$\begin{gathered} w_n \rightharpoonup w \quad \text{in } E \\ w_n \to w \quad \text{in } L^p(\mathbb{R}^N) \cap L^\nu(\mathbb{R}^N) \\ w_n \to w \quad \text{a.e. in } \mathbb{R}^N. \end{gathered} \tag{3.13}$$

Now we distinguish two cases.

Case $w = 0$. As in [10], we can say that for any $n \in \mathbb{N}$ there exists $t_n \in [0, 1]$ such that

$$\mathcal{J}_{\lambda_n}(t_n u_n) = \max_{t\in[0,1]} \mathcal{J}_{\lambda_n}(t u_n). \tag{3.14}$$

Since (3.12) holds, for any $j \in \mathbb{N}$, we can choose $r_j = (2jp)^{1/p} w_n$ such that

$$r_j \|u_n\|^{-1} \in (0, 1) \tag{3.15}$$

provided n is large enough. By (3.13), $F(\cdot, 0) = 0$ and the continuity of F, we can see that

$$F(x, r_j w_n) \to F(x, r_j w) = 0 \quad \text{a.e. } x \in \mathbb{R}^N \tag{3.16}$$

as $n \to \infty$ for any $j \in \mathbb{N}$. Then, taking into account (2.2), (3.13), (3.16), $(A4)$ and by using the Dominated Convergence Theorem we deduce that

$$F(x, r_j w_n) \to 0 \quad \text{in } L^1(\mathbb{R}^N) \tag{3.17}$$

as $n \to \infty$ for any $j \in \mathbb{N}$. Then (3.14), (3.15) and (3.17) yield

$$\mathcal{J}_{\lambda_n}(t_n u_n) \geq \mathcal{J}_{\lambda_n}(r_j w_n) \geq 2j - \lambda_n \int_{\mathbb{R}^N} F(x, r_j w_n)\, dx \geq j$$

provided n is large enough and for any $j \in \mathbb{N}$. As a consequence

$$\mathcal{J}_{\lambda_n}(t_n u_n) \to \infty \quad \text{as } n \to \infty. \tag{3.18}$$

Since $\mathcal{J}_{\lambda_n}(0) = 0$ and $\mathcal{J}_{\lambda_n}(u_n) \in [c_k, d_k]$, we deduce that $t_n \in (0, 1)$ for n large enough. Thus, by (3.14) we have

$$\langle \mathcal{J}'_{\lambda_n}(t_n u_n), t_n u_n\rangle = t_n \frac{d}{dt}\Big|_{t=t_n} \mathcal{J}_{\lambda_n}(t u_n) = 0. \tag{3.19}$$

Taking into account (A5), (3.19) and (2.3) we obtain

$$\begin{aligned} \frac{1}{\theta}\mathcal{J}_{\lambda_n}(t_n u_n) &= \frac{1}{\theta}\Big(\mathcal{J}_{\lambda_n}(t_n u_n) - \frac{1}{p}\langle \mathcal{J}'_{\lambda_n}(t_n u_n), t_n u_n\rangle\Big) \\ &= \frac{\lambda_n}{\theta p} \int_{\mathbb{R}^N} \mathcal{F}(x, t_n u_n)\, dx \end{aligned}$$

$$\begin{aligned}&\le \frac{\lambda_n}{p}\int_{\mathbb{R}^N}\mathcal{F}(x,u_n)\,dx\\&=\mathcal{J}_{\lambda_n}(u_n)-\frac{1}{p}\langle \mathcal{J}'_{\lambda_n}(u_n),u_n\rangle=\mathcal{J}_{\lambda_n}(u_n)\end{aligned}$$

which contradicts (3.11) and (3.18).

Case $w \not\equiv 0$. Thus the set $\Omega := \{x \in \mathbb{R}^N : w(x) \neq 0\}$ has positive Lebesgue measure. By using (3.12) and that $w \not\equiv 0$, we have

$$|u_n(x)| \to \infty \quad \text{a.e. } x \in \Omega \text{ as } n \to \infty. \tag{3.20}$$

Putting together (3.13), (3.20), and ($A4$), and by applying Fatou's Lemma, we can easily deduce that

$$\begin{aligned}\frac{1}{p}-\frac{\mathcal{J}_{\lambda_n}(u_n)}{\|u_n\|^p}&=\lambda_n\int_{\mathbb{R}^N}\frac{F(x,u_n(x))}{\|u_n\|^p}\,dx\\&\ge \lambda_n\int_{\Omega}|w_n|^p\frac{F(x,u_n(x))}{|u_n|^p}\,dx \to \infty \text{ as } n\to\infty\end{aligned}$$

which gives a contradiction because of (3.11).

Then, we have proved that the sequence $\{u_n\}$ is bounded in E. □

Proof of Theorem 1.1. Taking into account Lemma 3.4 and (3.11), for each $k \ge k_1$, we can use similar arguments to those in the proof of Lemma 3.3, to show that the sequence $\{u_n^k\}$ admits a strong convergent subsequence with the limit u^k being just a critical point of $\mathcal{J}_1 = \mathcal{J}$. Clearly, $\mathcal{J}(u^k) \in [c_k, d_k]$ for all $k \ge k_1$. Since $c_k \to \infty$ as $k \to \infty$ in (3.8), we deduce the existence of infinitely many nontrivial critical points of $\mathcal{J}$. As a consequence, we have that (1.1) possesses infinitely many nontrivial weak solutions. □

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