Electronic Journal of Differential Equations, Vol. 2016 (2016), No. 156, pp. 1–11.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

INVARIANT REGIONS AND EXISTENCE OF GLOBAL SOLUTIONS TO REACTION-DIFFUSION SYSTEMS WITHOUT CONDITIONS ON THE GROWTH OF NONLINEARITIES

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Abstract. This article concerns the existence of global solutions for a coupled 2-component reaction diffusion system with a full matrix diffusion and exponential nonlinearities. We show that some results of global and bounded solutions are established via invariant regions and the Lyapunov functional. A numerical example is used to illustrate our results.

1. Introduction

Reaction-diffusion systems have received considerable attention from mathematicians and other scientists and engineers alike because of their ability to model real-life phenomena in a wide variety of fields. The study of these systems has allowed for a deeper understanding of the dynamics and characteristics of the phenomena. In this article, we study the generic reaction-diffusion system with a full diffusion-matrix,

$$\begin{aligned} \frac{\partial u}{\partial t} - a\Delta u - b\Delta v = f(u,v) \quad \text{in } \mathbb{R}^+ \times \Omega \\ \frac{\partial v}{\partial t} - c\Delta u - d\Delta v = g(u,v) \quad \text{in } \mathbb{R}^+ \times \Omega, \end{aligned} \tag{1.1}$$

with the boundary conditions

$$\frac{\partial u}{\partial \eta} = \frac{\partial v}{\partial \eta} = 0, \tag{1.2}$$

and initial data

$$u(0,x) = u_0(x), \quad v(0,x) = v_0(x), \quad \text{in } \Omega. \tag{1.3}$$

Here Ω is an open bounded domain of class $\mathbb{C}^1$ in $\mathbb{R}^N$, with boundary $\partial\Omega$ and $\frac{\partial}{\partial\eta}$ denotes the outward normal derivative on $\partial\Omega$. We will assume that the nonlinearities f and g are continuously differentiable functions on $\mathbb{R}^+$. The constants a, b, c and d are positive and satisfying the condition

$$(b+c)^2 < 4ad, \tag{1.4}$$

2010 *Mathematics Subject Classification.* 35K45, 35K57.
Key words and phrases. Reaction-diffusion system; global solution, Lyapunov functional.

Submitted May 2, 2016. Published June 21, 2016.

which reflects the parabolicity of the system and implies at the same time that the diffusion matrix:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{1.5}$$

is positive definite. That means the eigenvalues λ_1 and λ_2 ($\lambda_1 < \lambda_2$) of A^T are positive.

One of the earliest works on this subject is that of Kanel and Kirane [4], where they proved the existence of global solutions given $g(u;v) = -f(u;v) = -uv^n$ with n being an odd integer subject to

$$\|b - c\| < C_p,$$

where C_p contains a constant from Solonnikov's estimate. They further improved their work in [6] and relaxed the conditions on the diffusion matrix. They showed the existence of global solutions subject to

(H1) $d < a + c$,
(H2) $b < \varepsilon_0 \equiv (\frac{ad(a+c-d)}{ad+(a+c-d)})$ if $a \le d < a + c$,
(H3) $b < \min\{\frac{1}{2}(a + c), \varepsilon_0\}$, and

$$\|F(v)\| \le C_F(1 + \|v\|^{1+\varepsilon}),$$

where ε and C_F are positive constants with $\varepsilon < 1$ being sufficiently small and $g(u;v) = -f(u;v) = uF(v)$.

Kouachi [10] considered the case where the nonlinearities $f(u,v) \ge 0$ and $g(u,v) \ge 0$ are continuously differentiable polynomially bounded functions satisfying

$$\mu_2 g(\mu_2 s, s) \le f(\mu_2 s, s), \quad \text{and} \quad f(\mu_1 s, s) \le \mu_1 g(\mu_1 s, s),$$

for all $s \ge 0$, and

$$f(u,v) + Cg(u,v) \le C_1(u + \alpha v + 1),$$

for positive $C, \alpha > -\mu_2$ and C_1 a positive constant and with

$$\mu_1 = \frac{\min\{a,d\} - \lambda_1}{c} > 0 > \mu_2 = \frac{\min\{a,d\} - \lambda_2}{c}. \tag{1.6}$$

The author was able to determine the invariant regions of the system and establish the existence of global solutions through an appropriate Lyapunov functional.

Kouachi [9] again considered the case that $f(u,v) = -\frac{\rho}{\sigma} g(u,v)$ with $g(u,v) \ge 0$, and a full diffusion-matrix with a balance law. The study established the invariant regions of the system as well as the existence of global solutions. The reaction term was assumed to be of polynomial or sub-exponential growth. This work was later extended by Rebiai and Benachour [13], where the authors relaxed the conditions on the nonlinearities.

In our work, we will distinguish between the following two main cases for the diffusion matrix A and the corresponding invariant regions.

Case 1: $c = 0$**.** We assume that $a < d$ and that the initial data is in the region

$$\Sigma_1 = \{(u_0, v_0) \in \mathbb{R}^2 \text{ such that } 0 \le \frac{b}{d-a} v_0 \le u_0\}. \tag{1.7}$$

If $w = u - \frac{b}{d-a} v$ is uniformly bounded, we can suppose that for all positive constants M, the nonlinearity g is controlled for v being sufficiently large

$$0 \le g(u,v) \le H(v), \quad \text{for } 0 \le u - \frac{b}{d-a} v \le M, \tag{1.8}$$

where H is a continuously differentiable function satisfying

$$\lim_{v\to\infty} \frac{H'(v)}{f(u,v)-\frac{b}{d-a}g(u,v)} = 0, \quad \text{for all } (u,v)\in\Sigma_1, \tag{1.9}$$

$$f(u,v) \le \frac{b}{d-a}g(u,v) \text{ for all } (u,v)\in\Sigma_1, \tag{1.10}$$

$$f(\frac{b}{d-a}v,v) \le \frac{b}{d-a}g(\frac{b}{d-a}v,v), \tag{1.11}$$

with $g(u,0)\ge 0$ for all $u\ge 0$ and $v\ge 0$.

Case 2: $c\neq 0$**.** The initial data are assumed to be in one of the following regions:

$$\begin{aligned}
\Sigma_2 &= \{(u_0,v_0)\in\mathbb{R}^2 \text{ such that } \mu_2 v_0 \le u_0 \le \mu_1 v_0\}\\
\Sigma_3 &= \{(u_0,v_0)\in\mathbb{R}^2 \text{ such that } \mu_1 v_0 \le u_0 \le \mu_2 v_0\}\\
\Sigma_4 &= \{(u_0,v_0)\in\mathbb{R}^2 \text{ such that } \min(\mu_2 v_0,\mu_1 v_0) \ge u_0\}\\
\Sigma_5 &= \{(u_0,v_0)\in\mathbb{R}^2 \text{ such that } u_0 \ge \max(\mu_2 v_0,\mu_1 v_0)\},
\end{aligned} \tag{1.12}$$

with μ_1 and μ_2 as defined in (1.6).

In our work we will only deal with the first case (Σ_2). Generalization to the remaining regions is trivial and can be looked up in the appendix. We suppose that the reaction terms f and g satisfy:

$$\mu_1 g(\mu_2 s,s) \ge f(\mu_2 s,s) \quad \text{for all } s\ge 0, \tag{1.13}$$

$$f(r,\frac{1}{\mu_1}r) \ge \mu_2 g(r,\frac{1}{\mu_1}r), \quad \text{for all } r\ge 0, \tag{1.14}$$

$$f(r,s) \le \mu_2 g(r,s), \quad \text{for all } (r,s)\in\Sigma_2\,. \tag{1.15}$$

If $w=u-\mu_2 v$ is uniformly bounded, we know that for all positive constants M, the nonlinearity g is controlled for v being sufficiently large, i.e. for $0\le(\mu_1-\mu_2)p-s\le M$,

$$\mu_1 g(\mu_1 p - s, s) - f(\mu_1 p - s, p) \le H(s), \tag{1.16}$$

where H is a continuously differentiable function satisfying

$$\lim_{s\to\infty} \frac{H'(s)}{f(\mu_1 p - s,p)-\mu_2 g(\mu_1 p - s,p)} = 0. \tag{1.17}$$

This class of systems motivates us to construct the type of functionals considered in this paper with the aim of proving the existence of global solutions.

2. Invariant regions

In this section, we are concerned with the invariant regions of the proposed system. We will prove that if the pair (f,g) points into one of the previously defined regions Σ (either Σ_1, Σ_2, Σ_3, Σ_4, or Σ_5) on $\partial\Sigma$, then Σ is an invariant region for problem (1.1)–(1.3), i.e the solution remains in Σ for any initial data in Σ. Once the invariant regions are constructed, one can apply the Lyapunov technique in order to establish the global existence of unique solutions for the proposed problem (1.1)–(1.3) as will be shown later on in Section 4 (see for related examples the work of Kirane and Kouachi in [7] and [8]).

Proposition 2.1. *Suppose that the functions f and g point into the region Σ on $\partial\Sigma$, then for any (u_0,v_0) in Σ, the solution $(u(t,\cdot),v(t,\cdot))$ of problem* (1.1)–(1.3) *remains in Σ for any time $t\in[0,T^*]$.*

Proof. We will approach the two cases discussed in Section 1 separately. In the first case, $c = 0$. Multiplying the second equation in (1.1) by $\frac{b}{a-d}$ and adding the result to the first equation in (1.1) yields the equivalent system

$$\begin{gathered}
\frac{\partial w}{\partial t} - a\Delta w = F(w,v) = f(u,v) + \frac{b}{a-d}g(u,v) \\
\frac{\partial v}{\partial t} - d\Delta v = g(u,v),
\end{gathered} \tag{2.1}$$

where

$$w = u + \frac{b}{a-d}v,$$

and the initial data

$$w(0,x) = w_0(x) \geq 0, \quad v(0,x) = v_0(x) \geq 0 \quad \text{in } \Omega,$$

with the Neuman boundary conditions

$$\frac{\partial w}{\partial \eta} = \frac{\partial v}{\partial \eta} = 0.$$

Using (1.11), the first property is assured by the quasi-positivity of the nonlinearities; that is:

$$F(0,v) \geq 0 \text{ and } g(u,0) \geq 0 \quad \text{for all } w \geq 0 \text{ and } v \geq 0. \tag{2.2}$$

In the second case, $c \neq 0$. It suffices to show that region Σ_2 is invariant. The proof can be trivially extended to the other regions. We construct a new system, which is equivalent to (1.1). The first equation is formed by multiplying the second equation in (1.1) by μ_1 and subtracting the second equation from it. The second is obtained by multiplying the second equation in (1.1) by $-\mu_2$ and adding it to the first one. Then, assuming without loss of generality that $a < d$ and with the fact that λ_1 and λ_2 are the eingenvalues of A, we can write

$$\begin{gathered}
\frac{\partial w}{\partial t} - \lambda_2\Delta w = k(w,z), \\
\frac{\partial z}{\partial t} - \lambda_1\Delta z = h(w,z),
\end{gathered} \tag{2.3}$$

where

$$\begin{gathered}
w = -\mu_2 v + u \geq 0 \\
z = \mu_1 v - u \geq 0,
\end{gathered} \tag{2.4}$$

the eigenvalues of A are given by

$$\begin{gathered}
\lambda_1 = \frac{1}{2}(a + d - \sqrt{(a-d)^2 + 4bc}) \\
\lambda_2 = \frac{1}{2}(a + d + \sqrt{(a-d)^2 + 4bc}),
\end{gathered} \tag{2.5}$$

and

$$\begin{gathered}
k(w,z) = -\mu_2 g(u,v) + f(u,v) \\
h(w,z) = \mu_1 g(u,v) - f(u,v).
\end{gathered} \tag{2.6}$$

Using (1.13) and (1.14), the first property is assured by the quasi-positivity of the nonlinearities; that is:

$$k(0,z) \geq 0, \ h(w,0) \geq 0 \text{ for all } w \geq 0 \text{ and } z \geq 0. \tag{2.7}$$

Now using (1.16), we can suppose that for all positive constants M, the nonlinearity g is controlled for v being sufficiently large

$$h(w,z) \le H(z), \quad \text{for } 0 \le w \le M, \tag{2.8}$$

where H is a continuously differentiable function satisfying

$$\lim_{z\to\infty} \frac{H'(z)}{k(w,z)} = 0, \quad \text{for } 0 \le w \le M. \tag{2.9}$$

This concludes the proof of the proposition for regions Σ_1 and Σ_2. □

3. Existence of local solutions

In this section, we prove the existence of local solutions using basic existence theory. First of all, let us define the usual norms in spaces $L^p(\Omega)$, $L^\infty(\Omega)$, and $C(\overline{\Omega})$ as

$$\|u\|_p^p = \frac{1}{\|\Omega\|}\int_\Omega \|u(x)\|^p dx, \quad \|u\|_\infty = \operatorname*{esssup}_{x\in\Omega}\|u(x)\|,$$
$$\|u\|_{C(\overline{\Omega})} = \max_{x\in\overline{\Omega}} \|u(x)\|,$$

respectively.

Now, for any initial data in $C(\overline{\Omega}) \times C(\overline{\Omega})$ or in $L^p(\Omega)\times L^p(\Omega)$ with $p \in (1,+\infty)$, the existence and uniqueness of local solutions to (1.1)–(1.3) follow from the basic existence theory for abstract semilinear differential equations (see Henry [2]). Also note that the solutions are classical on $[0, T_{\max})$ where $T_{\max}$ denotes the eventual blowing-up time in $L^\infty(\Omega)$.

4. Existence of global solutions

In this section, we prove the existence of global solutions for the diagonalized system (2.3)-(2.5). Our main results are summarized in the following theorem and corollary.

Theorem 4.1. *Let $(w(t,\cdot), z(t,\cdot))$ be any positive solution of the diagonal problem (2.3)-(2.5) on the interval $[0,T]$ f or some $T < T^*$. Then, assuming Neumann boundary conditions, the functional*

$$t \to L(t) = \int_\Omega (M-w)^{-\gamma} H^p(z) dx, \tag{4.1}$$

is uniformly bounded on $[0,T)$ for any positive constants γ, M and p satisfying

$$0 < \gamma < \frac{4ab}{(\lambda_2-\lambda_1)^2}, \quad \|w(t,x)\|_\infty < M \quad \text{for all } 0 < t \le T, \tag{4.2}$$

with

$$p > \frac{4(\gamma+1)\lambda_1\lambda_2}{4\lambda_1\lambda_2 - \gamma(\lambda_2-\lambda_1)^2}. \tag{4.3}$$

Proof. Differentiating the functional L with respect to t yields:

$$\begin{aligned} L'(t) &= \frac{d}{dt}\int_\Omega (M-w)^{-\gamma} H^p(z)\, dx \\ &= \int_\Omega \big[H^p(z)\frac{d}{dt}(M-w)^{-\gamma} + (M-w)^{-\gamma}\frac{d}{dt}H^p(z)\big]\, dx \end{aligned}$$

$$\begin{aligned}
&= \int_\Omega \Big\{\gamma(M-w)^{-\gamma-1}H^p(z)[\lambda_2\Delta w + k(w,z)] \\
&\quad + p(M-w)^{-\gamma}H^{p-1}H'[\lambda_1\Delta z + h(w,z)]\Big\}\,dx \\
&= I + J,
\end{aligned}$$

where

$$\begin{gathered}
I = \int_\Omega \big\{\lambda_2\gamma(M-w)^{-\gamma-1}H^p(z)\Delta w + \lambda_1 p(M-w)^{-\gamma}H^{p-1}H'\Delta z\big\}\,dx, \\
J = \int_\Omega (M-w)^{-\gamma-1}H^p(z)\big\{\gamma k(w,z) + p(M-w)\frac{H'}{H}h(w,z)\big\}\,dx.
\end{gathered}$$

A simple application of Green's formula in (1.2) to I yields

$$I := I_1 + I_2.$$

Simplifying the first part of I leads to

$$\begin{aligned}
I_1 &= -\int_\Omega \nabla[\lambda_2\gamma(M-w)^{-\gamma-1}H^p(z)]\nabla w\,dx \\
&= -\lambda_2\gamma\int_\Omega \Big\{(\gamma+1)(M-w)^{-\gamma-2}H^p(z)\|\nabla w\|^2 \\
&\quad + p(M-w)^{-\gamma-1}H^{p-1}(z)H'\nabla z\nabla w\Big\}\,dx,
\end{aligned}$$

and similarly, the second part can be simplified to produce

$$\begin{aligned}
I_2 &= -\int_\Omega \nabla[\lambda_1 p(M-w)^{-\gamma}H^{p-1}H']\nabla z\,dx \\
&= -\lambda_1 p\int_\Omega \Big\{\gamma(M-w)^{-\gamma-1}H^{p-1}H'\nabla w\nabla z + (M-w)^{-\gamma}H^{p-1}H''\|\nabla z\|^2 \\
&\quad + (p-1)(M-w)^{-\gamma}H^{p-2}(H')^2\|\nabla z\|^2\Big\}\,dx.
\end{aligned}$$

Hence, we can write

$$\begin{aligned}
I = -\int_\Omega \Big\{&\lambda_2\gamma(\gamma+1)\|\nabla w\|^2 + \big(\lambda_2\gamma p(M-w)\frac{H'}{H} + \lambda_1 p\gamma(M-w)\frac{H'}{H}\big)\nabla z\nabla w \\
&+ \lambda_1 p\Big((p-1)(\frac{H'}{H})^2 + \frac{H''}{H}\Big)(M-w)^2\|\nabla z\|^2\}(M-w)^{-\gamma-2}H^p\,dx.
\end{aligned}$$

The formula can be rearranged in the form

$$I = -\int_\Omega (T(\nabla w,\nabla z))H^p(M-w)^{-\gamma-2}dx, \tag{4.4}$$

where

$$\begin{aligned}
T(\nabla w,\nabla z) &= \gamma(\gamma+1)\lambda_2\|\nabla w\|^2 + (\lambda_2+\lambda_1)p\gamma(M-w)\frac{H'}{H}\nabla w\nabla z \\
&\quad + \lambda_1\big[p\frac{H''}{H} + (p^2-p)(\frac{H'}{H})^2\big](M-w)^2\|\nabla z\|^2.
\end{aligned}$$

The discriminant D of T is given by

$$\begin{aligned}
\frac{D}{(M-w)^2} &= \gamma p[((\lambda_2-\lambda_1)^2\gamma - 4\lambda_1\lambda_2)p + 4\lambda_2\lambda_1(\gamma+1)]\big(\frac{H'}{H}\big)^2 \\
&\quad - 4\lambda_2\gamma(\gamma+1)p\frac{H''}{H}.
\end{aligned}$$

If γ and p are chosen so as to satisfy conditions (4.2) and (4.3), then $D < 0$ and consequently

$$I \leq 0. \tag{4.5}$$

Now, let us examine the second part of the derivative. We have

$$\begin{aligned} J &= \int_\Omega (M-w)^{-\gamma-1} H^p(z)\big\{\gamma k(w,z) + p(M-w)\frac{H'}{H}h(w,z)\big\}\,dx \\ &= \int_\Omega \big[p(M-w)(-\frac{H'}{k})\frac{h}{H} - \gamma\big](-k(w,z))H^p(M-w)^{-\gamma-1}dx. \end{aligned}$$

This is along to (2.9), and (1.15) yields

$$\lim_{z\to\infty}(-\frac{H'}{k}) < \frac{2\gamma}{nM}.$$

Hence, there exists $\tilde{z} > 0$ such that

$$[(M-w)(-p\frac{H'}{k}) - \gamma]H^p(M-w)^{-\gamma-1} \leq 0,$$

for all $z \geq \overline{z}$ and $0 \leq w \leq M$. Rearranging the inequality and simplifying yields

$$(-\frac{H'}{k}) \leq \frac{2\gamma}{nM} \quad (0 \leq w \leq M).$$

Since the function in J is continuous, it is uniformly bounded for $z \geq 0$ and $0 \leq w \leq M$. Therefore, there exists $C_1 > 0$ such that

$$J \leq C_1, \tag{4.6}$$

and the proof is complete. □

Corollary 4.2. *For any initial data (u_0, v_0) in $L^\infty(\Omega) \times L^\infty(\Omega)$ and any functions f and g pointing into the region Σ_2 on $\partial\Sigma_2$ and satisfying either* (1.7)–(1.11) *or* (1.12)–(1.14)*, the solutions of the problem* (1.1)-(1.2) *are global in time and uniformly bounded on* $(0, +\infty) \times \Omega$.

Proof. From (2.8), we easily deduce that $h(w) \in L^\infty([0, T^*),\ L^p(\Omega))$ for all $p \geq 1$ and consequently $w \in L^\infty([0, T^*),\ L^\infty(\Omega))$ (see [2] and [1]). It follows that the solutions of the system (2.1)-(2.2) are global in time and uniformly bounded on $(0, +\infty) \times \Omega$. □

5. Numerical Example

This section will present numerical solutions for an example drawn from the proposed model and satisfying the conditions for the existence of global solutions. The results are obtained through the finite difference method with appropriately chosen descritization intervals in space and time. Let us consider the case where $c = 0$ and

$$\begin{aligned} f(u,v) &= -\big(u + \frac{b}{a-d}v\big)^k e^{(\epsilon v + e^v)} \\ g(u,v) &= \big(u + \frac{b}{a-d}v\big)^l e^{e^v}, \end{aligned} \tag{5.1}$$

with $\epsilon > 1$ and $k, l > 0$. Note that taking $b = 0$, we obtain

$$\begin{aligned} f(u,v) &= -u^k e^{(\epsilon v + e^v)} \\ g(u,v) &= g(u,v) = u^l e^{e^v}, \end{aligned}$$

which is the same example considered by Kouachi in [11]. However, let us consider the non-diagonal case with $b \neq 0$. Consider the diffusion matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 4 & 6 \\ 0 & 5 \end{pmatrix},$$

with eigenvalues $\lambda_1 = 4$ and $\lambda_2 = 5$. Since $c = 0$, we have the invariant region Σ_1 defined as

$$\Sigma_1 = \{(u_0, v_0) \in \mathbb{R}^2 \text{ such that } 6v_0 \leq u_0\}.$$

Let us choose $(u_0, v_0) = (1.5, 0.15)$ and $\epsilon = 3 > 1$, $(k, l) = (0.6, 0.8)$. The system can now be written as

$$\begin{gathered} \frac{\partial w}{\partial t} - 4\Delta w = F(u, v) = f(u, v) - 6g(u, v) \\ \frac{\partial v}{\partial t} - 5\Delta v = g(u, v), \end{gathered} \tag{5.2}$$

where $w = u - 6v \geq 0$. We can easily verify that the resulting system satisfies conditions (1.10) and (1.11). Solving the resulting system (5.2) numerically yields the solutions shown in Figure 1 for the one-dimensional diffusion case.

6. Appendix

In the main result of this article, we proved the existence of global solutions for the proposed system (1.1)–(1.3) in the region Σ_2 using an appropriate Lyapunov functional. The work can be triavially extended to the remaining regions. Recall that we started by finding an equivalent system in the region in question, then we used this equivalent system to prove the existence of global solutions. The proof of Theorem 4.1 in the region Σ_2 can be extended to the remaining regions, i.e. Σ_1, Σ_3, Σ_4, and Σ_5, in a similar fashion using the following equivalent systems.

Case 3. For Σ_3, a new system of two equations is formed. The first equation is the result of multiplying the second equation of (1.1) by μ_2 and subtracting the first from it. The second equation is formed by multiplying the second equation of (1.1) by $(-\mu_1)$ and adding the first. Hence,

$$\begin{gathered} \frac{\partial(\mu_2 v - u)}{\partial t} - \Delta[(c\mu_2 - a)u + (d\mu_2 - b)v] = \mu_2 g(u, v) - f(u, v) \\ \frac{\partial(-\mu_1 v + u)}{\partial t} - \Delta[(a - c\mu_1)u + (-d\mu_1 + b)v] = -\mu_1 g(u, v) + f(u, v). \end{gathered} \tag{6.1}$$

Assuming without loss that $a < d$ and with the fact that λ_1 and λ_2 are the eingenvalues of A, we obtain

$$\begin{gathered} \frac{\partial w}{\partial t} - \lambda_2 \Delta w = k(w, z) \\ \frac{\partial z}{\partial t} - \lambda_1 \Delta z = h(w, z), \end{gathered} \tag{6.2}$$

where

$$\begin{gathered} w = \mu_2 v - u \geq 0 \\ z = -\mu_1 v + u \geq 0, \end{gathered} \tag{6.3}$$

and

$$\begin{gathered} k(w, z) = \mu_2 g(u, v) - f(u, v) \\ h(w, z) = -\mu_1 g(u, v) + f(u, v). \end{gathered} \tag{6.4}$$

The rest is trivial.

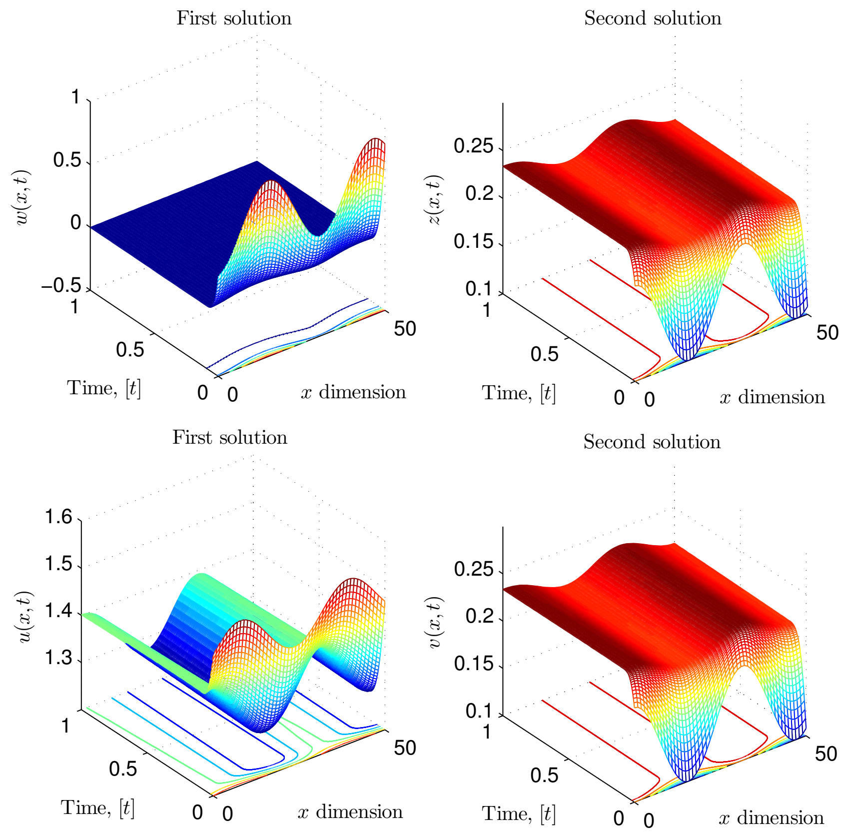

FIGURE 1. Solutions of (5.1) in the one-dimensional case. The top half depicts solutions of the equivalent diagonal system (w, v) and the bottom half depicts the raw solutions (u, v).

Case 4. For Σ_4, we form a new system of equations based on (1.1). The second equation is multiplied by μ_1 and the first is subtracted from it to yield the first equation. The second is obtained by multiplying the second equation of (1.1) by μ_2 and subtracting the first from it. Thus we obtain

$$\begin{aligned} &\frac{\partial(\mu_2 v - u)}{\partial t} + (a - c\mu_2)\Delta u + (b - d\mu_2)\Delta v = \mu_2 g(u, v) - f(u, v) \\ &\frac{\partial(\mu_1 v - u)}{\partial t} + (a - c\mu_1)\Delta u + (b - d\mu_1)\Delta v = \mu_1 g(u, v) - f(u, v). \end{aligned} \tag{6.5}$$

Then, if we assume without loss that $a < d$ and with the fact that λ_1 and λ_2 are the eingenvalues of A, we have

$$\begin{aligned} &\frac{\partial w}{\partial t} - \lambda_2 \Delta w = k(w, z) \\ &\frac{\partial z}{\partial t} - \lambda_1 \Delta z = h(w, z), \end{aligned} \tag{6.6}$$

where

$$\begin{aligned} w &= \mu_2 v - u \geq 0 \\ z &= \mu_1 v - u \geq 0, \end{aligned} \tag{6.7}$$

and

$$\begin{aligned} k(w,z) &= \mu_2 g(u,v) - f(u,v) \\ h(w,z) &= \mu_1 g(u,v) - f(u,v). \end{aligned} \tag{6.8}$$

Again, the remainder of the proof is trivial.

Case 5. For Σ_5, we multiply the second equation of (1.1) by $(-\mu_1)$ and add the first to it, and separately multiply it by $(-\mu_2)$ and add the first to it. This yields a new system of equations:

$$\begin{aligned} \frac{\partial(-\mu_2 v + u)}{\partial t} + (c\mu_2 - a)\Delta u + (d\mu_2 - b)\Delta v &= f(u,v) - \mu_2 g(u,v) \\ \frac{\partial(-\mu_1 v + u)}{\partial t} + (c\mu_1 - a)\Delta u + (d\mu_1 - b)\Delta v &= f(u,v) - \mu_1 g(u,v). \end{aligned} \tag{6.9}$$

Then, assuming without loss that $a < d$ and with the fact that λ_1 and λ_2 are the eingenvalues of A, we can write:

$$\begin{aligned} \frac{\partial w}{\partial t} - \lambda_2 \Delta w &= k(w,z) \\ \frac{\partial z}{\partial t} - \lambda_1 \Delta z &= h(w,z), \end{aligned} \tag{6.10}$$

where

$$\begin{aligned} w &= -\mu_2 v + u \geq 0 \\ z &= -\mu_1 v + u \geq 0, \end{aligned} \tag{6.11}$$

and

$$\begin{aligned} k(w,z) &= f(u,v) - \mu_2 g(u,v) \\ h(w,z) &= f(u,v) - \mu_1 g(u,v). \end{aligned} \tag{6.12}$$

The rest follows in the same way as the proof for region Σ_2.

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