CAUCHY PROBLEM FOR SOME FRACTIONAL NONLINEAR ULTRA-PARABOLIC EQUATIONS

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Abstract. Blowing-up solutions to nonlocal nonlinear ultra-parabolic equations is presented. The obtained results will contribute in the development of ultra-parabolic equations and enrich the existing non-extensive literature on fractional nonlinear ultra-parabolic problems. Our method of proof relies on a suitable choice of a test function and the weak formulation approach of the sought for solutions.

1. Introduction

This article aims to extend recent results by Kerbal and Kirane [10] by considering fractional in time and space nonlinear ultra-parabolic equations instead of classical ones. Indeed, we will present a blow-up result for the nonlocal nonlinear ultra-parabolic 2-times equation

\[ Lu := u_t + D_0^\alpha \left(|u|^q - |u_1|^q \right) + (-\Delta)^{\beta/2}(|u|^m) = |u|^p \]

posed for \((t_1, t_2, x) \in Q = \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^N, N \in \mathbb{N}\) and supplemented with the initial conditions

\[ u(t_1, 0; x) = u_1(t_1; x), \quad u(0, t_2; x) = u_2(t_2; x). \]  

Here \(p > m > 1, p > q > 1\) are real numbers and where for \(0 < \alpha < 1\) and \(D^\alpha\) is the fractional derivative in the sense of Riemann-Liouville. Then, we extend our results to the system of two equations

\[ u_{t_1} + D_0^{\alpha_1} \left(|u|^s - |u_1|^s \right) + (-\Delta)^{\beta_1/2}(|u|^m) = |v|^q, \]

\[ v_{t_2} + D_0^{\alpha_2} \left(|v|^r - |v_1|^r \right) + (-\Delta)^{\beta_2/2}(|v|^n) = |u|^p, \]

posed for \((t_1, t_2, x) \in Q = \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^N, N \in \mathbb{N}\), and supplemented with the initial conditions

\[ u(t_1, 0; x) = u_1(t_1; x), \quad u(0, t_2; x) = u_2(t_2; x), \]

\[ v(t_1, 0; x) = v_1(t_1; x), \quad v(0, t_2; x) = v_2(t_2; x). \]

Here \(p, q, r, s\) are positive real numbers and \(0 < \alpha_1, \alpha_2 < 1, 0 < \beta_1, \beta_2 \leq 2\).
The nonlocal operator $D_{0,t}^{\alpha}$ is defined, for an absolutely continuous function $f : \mathbb{R}^+ \to \mathbb{R}$, by

$$(D_{0,t}^{\alpha})f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_{0}^{t} \frac{f(\sigma)}{(t-\sigma)^\alpha} d\sigma$$

and $\Gamma(\alpha) = \int_{0}^{\infty} r^{\alpha-1} e^{-r} dr$ is the Euler gamma function. The fractional power of the Laplacian $(-\Delta)^{\beta/2}$ ($0 < \beta \leq 2$) stands for diffusion in media with impurities and is defined as

$$(-\Delta)^{\beta/2} v(x) = F^{-1}(\|\xi\|^\beta F(v)(\xi))(x),$$

where $F$ denotes the Fourier transform and $F^{-1}$ denotes its inverse. The operator $D_{0,t}^{\alpha}$ counts for the anomalous diffusion, a recently very much studied topic in probability, physics, chemistry, biology, image processing, etc., see for instance [1, 2, 3, 4, 5, 6, 7, 8, 11, 13, 14, 16] and their references. Classical multi-time or ultraparabolic problems have received a special interest and attention by authors due to their application in real life problems, see for example [9, 10, 12, 17, 19], while the fractional analog are in their preliminary steps.

## 2. Preliminaries

Here, we need the right-hand fractional derivative in the sense of Riemann-Liouville

$$(D_{t,T}^{\alpha})f(t) = -\frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_{t}^{T} \frac{f(\sigma)}{(\sigma-t)^\alpha} d\sigma,$$

for an absolutely continuous function $f : \mathbb{R}^+ \to \mathbb{R}$. Note that for a differentiable function $f$, we have the so-called Caputo’s fractional derivative

$$D_{0,t}^{\alpha}(f-f(0))(t) = \frac{1}{\Gamma(1-\alpha)} \int_{0}^{t} \frac{f'(\sigma)}{(\sigma-t)^\alpha} d\sigma.$$

It is shown in [16] Corollary 2, p.46 that for $f, g$ possessing appropriate regularity, the formula of integration by parts holds true

$$\int_{0}^{T} f(t) D_{0,t}^{\alpha} g(t) dt = \int_{0}^{T} g(t) D_{0,t}^{\alpha} f(t) dt.$$

We also need some preparatory lemmas based on the function $\phi$ defined by

$$\phi(t) = \begin{cases} (1 - \frac{t}{T})^\lambda, & 0 < t \leq T, \\ 0, & t > T, \end{cases} \tag{2.1}$$

where $\lambda \geq 2$.

**Lemma 2.1.** Let $\phi$ be as in (2.1). We have

$$\int_{0}^{T} D_{t,T}^{\alpha}\phi(t) dt = C_{\alpha,\lambda} T^{1-\alpha}, \tag{2.2}$$

where

$$C_{\alpha,\lambda} = \frac{\lambda \Gamma(\lambda - \alpha)}{(\lambda - \alpha + 1) \Gamma(\lambda - 2\alpha + 1)}.$$

For a proof of the above lemma, see [11, 5].
Lemma 2.2. Let $\phi$ be as in (2.1) and $p > 1$. Then for $p < \lambda + 1$,
\[
\int_0^T \phi^{1-p}(t)|\phi'(t)|^p = C_p T^{1-p},
\]
where
\[
C_p = \frac{\lambda^p}{1 + \lambda - p}.
\]
For $\lambda > \alpha p - 1$,
\[
\int_0^T \phi(t)^{1-p}|D_t^{\alpha} \phi(t)|^p dt = C_{p,\alpha} T^{1-\alpha p},
\]
where
\[
C_{p,\alpha} = \frac{\lambda^p (\lambda + 1 - \alpha(\lambda - 2\alpha + 1))}{\Gamma(\lambda - \alpha) \Gamma(\lambda - 2\alpha + 1)}.
\]
For a proof of the above lemma, see [11, 5]. We define the regular function $\psi$:
\[
\psi(\xi) = \begin{cases} 1, & \text{if } 0 \leq \xi \leq 1, \\ \text{decreasing}, & \text{if } 1 \leq \xi \leq 2, \\ 0, & \text{if } \xi \geq 2, \end{cases}
\]
which will be used hereafter.

3. Results

Solutions to (1.1) subject to conditions (1.2) are meant in the following weak sense.

Definition 3.1. A function $u \in L^m(Q) \cap L^p(Q)$ is called a weak solution to (1.1) if
\[
\int_Q |u|^p \varphi dP + \int_S u(0,t_2;x)\varphi(0,t_2;x) dP_2 + \int_Q |u(t_1,0;x)|^q |D_{t_2}^{\alpha} \varphi| T dP
= -\int_Q u \varphi_{t_1} dP + \int_Q |u|^q D_{t_2}^{\alpha} \varphi dP + \int_Q |u|^m (-\Delta)^{\beta/2} \varphi dP
\]
for any test function $\varphi \in C_0^\infty(Q)$; $S = \mathbb{R}^+ \times \mathbb{R}^N$, $P = (t_1,t_2,x)$ and $P_2 = (t_2,x)$, such that $\varphi(T,t_2;x) = \varphi(t_1,T;x) = 0$.

Note that every weak solution is a classical solution near the points $(t_1,t_2,x)$ where $u(t_1,t_2,x)$ is positive.

Our main result dealing with equation (1.1) subject to (1.2) is given by the following theorem.

Theorem 3.2. Assume that
\[
\int_S u(0,t_2;x)\varphi(0,t_2;x) dP_2 > 0, \quad \int_Q |u(t_1,0;x)|^q |D_{t_2}^{\alpha} \varphi| T dP > 0.
\]
If $1 < p \leq \min \left( 1 + \frac{1}{N+1}, q \left( 1 + \frac{\alpha}{N+2-\alpha} \right), m \left( 1 + \frac{\beta}{N+2-\beta} \right) \right)$, then Problem (1.1)-(1.2) does not admit global weak solutions.

For the proof, we need to recall the following proposition from [8, proposition 3.3].
Proposition 3.3 ([8]). Suppose that $\delta \in [0, 2]$, $\beta + 1 \geq 0$, and $\theta \in C_0^{\infty}(R^N)$. Then, the following point-wise inequality holds:

$$|\theta(x)|^\beta \theta(x)(-\Delta)^{\delta/2}\theta(x) \geq \frac{1}{\beta + 2}(-\Delta)^{\delta/2}|\theta(x)|^{\beta + 2}.$$  

Proof of Theorem 3.2. Our strategy of proof is to use the weak formulation of the solution with a suitable choice of the test function (see for example [15]). We assume that the solution is nontrivial and global. We choose the test function $\varphi(t_1, t_2, x)$ in the form

$$\varphi(t_1, t_2; x) = \varphi_1(t_1)\varphi_2(t_2)\varphi_3(x)$$  

where $\varphi_1(t_1) = \psi(t_1/T)$, $\varphi_2(t_2) = (1 - t_2/T)^\lambda$ and $\varphi_3(x) = \psi(|x|^2/T^2)$.

Now, replacing $\varphi$ by $\varphi^\mu$ in (3.1), we estimate $\int_{Q_T} u\varphi_1^\mu dP$ using the $\varepsilon$-Young inequality as follows

$$\int_Q |u||\varphi_1^\mu| dP \leq \varepsilon \int_Q |u|^p \varphi^\mu dP + C_\varepsilon \int_Q \varphi^\mu - \frac{\varepsilon}{\varepsilon^*} |\varphi_1|^{\varepsilon^*/\varepsilon} dP.$$  

Similarly, we have

$$\int_Q |u|^q D_{t_2}^{\alpha}|\varphi^\mu| dP \leq \varepsilon \int_Q |u|^p \varphi^\mu dP + C_\varepsilon \int_Q |D_{t_2}^{\alpha}|\varphi^\mu|^{\frac{\alpha}{\alpha-1}} \varphi^{-\frac{\alpha}{\alpha-1}} dP,$$  

where $p > q$. Observe that

$$\int_Q |u(t_1, 0; x)|^q D_{t_2}^{\alpha}|\varphi^\mu| dP$$

$$= \left(\int_0^T D_{t_2}^{\alpha}|\varphi_2^\mu(t_2)| dt_2\right) \int_S |u(t_1, 0; x)|^q \varphi_3^\mu(x) \varphi_1^\mu(t_1) dP_1$$  

with the help of Lemma 2.1 one can rewrite the equation (3.5) as

$$\int_Q |u(t_1, 0; x)|^q D_{t_2}^{\alpha}|\varphi^\mu| dP = C_{\alpha, \lambda, \mu} T^{1-\alpha} \int_S |u(t_1, 0; x)|^q \varphi_3^\mu(x) \varphi_1^\mu(t_1) dP_1,$$  

where $P_1 = (t_1, x)$. Using the convexity inequality in proposition 3.3 and the $\varepsilon$-Young inequality, the last term in the right hand side of equation (3.6) can be estimated by

$$\int_Q |u|^m (-\Delta)^{\delta/2}\varphi^\mu dP$$

$$\leq \int_Q \mu \varphi^\mu |u|^m(-\Delta)^{\delta/2}\varphi dP$$

$$\leq \varepsilon \int Q \varphi^\mu |u|^p dP + C(\varepsilon) \int Q |(-\Delta)^{\delta/2}\varphi|^{\frac{m}{p}} \varphi^{\left(\mu-\frac{m}{p}\right)} dP.$$  

Now, using (3.3), (3.4), (3.5), and (3.7), we obtain
\[
\int_Q |u|^p \varphi^\mu dP + \int_S u(0, t_2; x) \varphi^\mu(0, t_2; x) dP_2 \\
+ C_{\alpha, \lambda} T^{1-\alpha} \int_S |u(t_1, 0; x)|^p \varphi^\mu_3(x) \varphi_1^{\alpha}(t_1) dP_1 \\
\leq 3 \varepsilon \int_Q |u|^p \varphi^\mu dP + C_{\varepsilon} \left( \int_{Q_T} \varphi^\mu - \frac{\mu}{\varepsilon} \right) |\varphi_{t_1}| \frac{\mu}{\varepsilon} dP \\
+ \int_Q |D_{t_2}^\alpha |T \varphi^\mu| \frac{\mu}{\varepsilon} \varphi^{-\frac{\mu}{\varepsilon}} dP \\
+ \int_Q \left| (-\Delta)^{\beta/2} \varphi \right| \frac{\mu}{\varepsilon} \varphi^{p(\mu - 1) - m \mu} \frac{\mu}{\varepsilon} dP.
\]
(3.8)

If we choose \( \varepsilon = 1/6 \) (for example), then we obtain the estimate
\[
\int_Q |u|^p \varphi^\mu dP + 2 \int_S u(0, t_2; x) \varphi^\mu(0, t_2; x) dP_2 \\
+ C_{\alpha, \lambda} T^{1-\alpha} \int_S |u(t_1, 0; x)|^p \varphi^\mu_3(x) \varphi_1^{\alpha}(t_1) dP_1 \\
\leq C \left( \int_Q \varphi^\mu - \frac{\mu}{\varepsilon} \right) |\varphi_{t_1}| \frac{\mu}{\varepsilon} dP + \int_Q |D_{t_2}^\alpha |T \varphi^\mu| \frac{\mu}{\varepsilon} \varphi^{-\frac{\mu}{\varepsilon}} dP \\
+ \int_Q \left| (-\Delta)^{\beta/2} \varphi \right| \frac{\mu}{\varepsilon} \varphi^{p(\mu - 1) - m \mu} \frac{\mu}{\varepsilon} dP
\]
for some positive constant \( C \). The right hand side of (3.9) is now free of the unknown function \( u \). Let us now pass to the new variables
\[
\tau_1 = T^{-1} t_1, \quad \tau_2 = T^{-1} t_2, \quad y = T^{-1} x.
\]
(3.10)

We have
\[
\int_Q \varphi^\mu - \frac{\mu}{\varepsilon} |\varphi_{t_1}| \frac{\mu}{\varepsilon} dP = \left( \int_{Q_2} \varphi_2^\mu \varphi_3^\mu dP_2 \right) \left( \int_{Q_{\Omega_2}} \varphi_1^{\mu} \varphi_{t_1} |\varphi_{t_1}| \frac{\mu}{\varepsilon} d\tau_1 \right)
\]
(3.11)

where
\[
C_1 = \left( \int_{Q_2} \varphi_2^\mu \varphi_3^\mu dP_2 \right) \left( \int_{Q_{\Omega_2}} \varphi_1^{\mu} \varphi_{t_1} |\varphi_{t_1}| \frac{\mu}{\varepsilon} d\tau_1 \right) < \infty
\]
with \( \mu > \frac{\mu}{\varepsilon} \) and \( P_{\tau_2} = (\tau_2, y), \Omega_2 = \{1 \leq \tau_2 + |y| \leq 2\} \). Similarly, we obtain
\[
\int_Q |D_{t_2}^\alpha |T \varphi^\mu| \frac{\mu}{\varepsilon} \varphi^{-\frac{\mu}{\varepsilon}} dP
\]
(3.12)

where
\[
C_2 = \left( \int_{Q_2} \varphi_2^\mu \varphi_3^\mu dP_2 \right) \left( \int_{Q_{\Omega_2}} \varphi_2^{\mu} \varphi_{t_2} |D_{t_2}^\alpha \varphi_{t_2}| \frac{\mu}{\varepsilon} d\tau_2 \right) < \infty
\]
and $P_{r_1} = (\tau, y)$, $\Omega_1 = \{1 \leq \tau + |y| \leq 2\}$, and
\[
\int_Q |(-\Delta)^{\beta/2} \varphi|^{\frac{p}{\mu}} \varphi^{(p-1) - \mu} \frac{1}{\varphi^{\mu}} dP
= \left( \int_{\mathbb{R}^N} |(-\Delta)^{\beta/2} \varphi_3|^{\frac{p}{\mu}} \varphi_3^{(p-1) - \mu} \frac{1}{\varphi_3^{\mu}} dx \right) \left( \int_{Q_T} \varphi_1^{\mu} \varphi_2^{\mu} dt_1 dt_2 \right)
= C_3 T^{2+N-\frac{\beta p}{2N-p}}
\]
where
\[
C_3 = \int_{\text{support } \psi} |(-\Delta)^{\beta/2} \psi| \frac{1}{\varphi_3^{\mu}} \varphi_3^{(p-1) - \mu} \frac{1}{\varphi_3^{\mu}} dy \int_{Q_T} \varphi_1^{\mu} \varphi_2^{\mu} d\tau_1 d\tau_2 < \infty
\]
with $\mu > \frac{p}{p-\beta}$ and $Q_T = [0, T] \times [0, T]$. By (3.11) and (3.13), we obtain for (3.9) the following estimate
\[
\int_Q |u| P \varphi^{\mu} dP + 2 \int_S u(0, t_2; x) \varphi^{\mu}(0, t_2; x) dP_2
+ C_{\alpha, \lambda_0} T^{1-\alpha} \int_S |u(t_1, 0; x)|^q \varphi_3^{\mu}(x) \varphi_1^{\mu}(t_1) dP_1
\leq C_1 T^{2+N-\frac{\beta p}{2N-p}} + C_2 T^{2+N-\frac{\beta p}{2N-p}} + C_3 T^{2+N-\frac{\beta p}{2N-p}},
\]
then
\[
\int_Q |u| P \varphi^{\mu} dP + 2 \int_S u(0, t_2; x) \varphi^{\mu}(0, t_2; x) dP_2
+ C_{\alpha, \lambda_0} T^{1-\alpha} \int_S |u(t_1, 0; x)|^q \varphi_3^{\mu}(x) \varphi_1^{\mu}(t_1) dP_1
\leq \tilde{C} \left( T^{2+N-\frac{\beta p}{2N-p}} + T^{2+N-\frac{\beta p}{2N-p}} + T^{2+N-\frac{\beta p}{2N-p}} \right)
\]
where $\tilde{C} = \max\{C_1, C_2, C_3\}$. Now, for the first case, we require:
(a) $2 + N - \frac{\beta p}{p-\beta} < 0$ or $1 < p \leq 1 + \frac{1}{N+\alpha}$, for $p > q$ and $m > 1$.
(b) $2 + N - \frac{\beta p}{p-q} < 0$ or $1 < p < q(1 + \frac{\alpha}{N+2-\alpha})$, for $p > m > 1$.
(c) $2 + N - \frac{\beta p}{p-m} < 0$ or $1 < p < m(1 + \frac{\beta}{N+2-\beta})$.
Letting $T$ aproach infinity in (3.15), we obtain a contradiction as the left hand side is positive while the right hand side goes to zero.
For the second case, we assume the exponents of $T$ in (3.15) are zeros. Applying Hölder’s inequality to the right hand side of inequality (3.9), we obtain
\[
\int_Q |u| P \varphi^{\mu} dP + 2 \int_S u(0, t_2; x) \varphi^{\mu}(0, t_2; x) dP_2
+ C_{\alpha, \lambda_0} T^{1-\alpha} \int_S |u(t_1, 0; x)|^q \varphi_3^{\mu}(x) \varphi_1^{\mu}(t_1) dP_1
\leq \left( \int_{Q_T} |u| P \varphi^{\mu} dP \right)^{1/p} C(\varphi)
\]
where
\[
C(\varphi) = C \left( \int_Q \varphi^{-\frac{\beta p}{2N-p}} |\varphi_1^{\mu}|^{\frac{p}{2N-p}} dP + \int_{Q_T} |D^{\alpha}_{t_2} \varphi^{\mu} |^{\frac{p}{2N-p}} \varphi^{\frac{\beta p}{2N-p}} dP + \int_Q |(-\Delta)^{\beta/2} \varphi| \frac{p}{\mu} \varphi^{(p-1) - \mu} \frac{1}{\varphi^{\mu}} dP \right).
\]
Whereupon, using Lebesgue’s dominated convergence theorem we have
\[
\int_Q |u|^p \varphi \, dP \leq \bar{C} \implies \lim_{T \to \infty} \int_{C_T} |u|^p \, dP = 0,
\]
where \( C_T = \{(t_1, t_2, x) \mid T \leq t_1 + t_2 + |x| \leq 2T\} \).
Then, letting \( T \) approach infinity in (3.16), the right-hand side approaches zero, which is again contradiction. \( \square \)

4. A 2 × 2 system with a 2-dimensional fractional time

We consider
\[
u_{t_1} + D_{0+}^{\alpha_1}(|u|^s - |u_1|^s) + (-\Delta)^{\beta_1/2}(|u|^m) = |v|^q, \tag{4.1} \\
v_{t_2} + D_{0+}^{\alpha_2}(|v|^r - |v_1|^r) + (-\Delta)^{\beta_2/2}(|v|^n) = |u|^p, \tag{4.2}
\]
posed for \((t_1, t_2, x) \in Q = \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^N, N \in \mathbb{N},\) and supplemented with the initial conditions
\[
u(t_1, 0; x) = u_1(t_1; x), \quad u(0, t_2; x) = u_2(t_2; x), \tag{4.3} \\
v(t_1, 0; x) = v_1(t_1; x), \quad v(0, t_2; x) = v_2(t_2; x). \tag{4.4}
\]
Here \(p, q, r, s,\) are positive real numbers and \(0 < \alpha_1, \alpha_2 < 1, 0 < \beta_1, \beta_2 \leq 2.\) Let us set
\[
I_0 = \int_S u_2(0, t_2, x) \varphi(0, t_2, x) \, dP_2 + \int_Q |u_1|^s D_{t_2}^{\alpha_1} |\varphi| \, dP, \\
J_0 = \int_S v_2(0, t_2, x) \varphi(0, t_2, x) \, dP_2 + \int_Q |v_1|^r D_{t_2}^{\alpha_2} |\varphi| \, dP
\]

Definition 4.1. We say that \((u, v) \in (L^p \cap L^m) \times (L^q \cap L^n)\) is a weak formulation to system (4.1)-(4.2) if
\[
\int_{Q} |v|^q \varphi \, dP + I_0 = -\int_{Q} u \varphi_{t_1} \, dP + \int_{Q} |u|^s D_{t_2}^{\alpha_1} \varphi \, dP + \int_{Q} |u|^m (-\Delta)^{\beta_1/2} \varphi \, dP \\
\int_{Q} |u|^p \varphi \, dP + J_0 = -\int_{Q} v \varphi_{t_1} \, dP + \int_{Q} |v|^r D_{t_2}^{\alpha_2} \varphi \, dP + \int_{Q} |v|^n (-\Delta)^{\beta_2/2} \varphi \, dP
\]
for any test function \(\varphi \in C_0^\infty.\) Now, set
\[
\sigma_1 = -\frac{q[1 - p(N + 1)] + N + 2}{pq - 1}, \\
\sigma_2 = -\frac{q[\alpha_1 - p(N + 1)] + r(N + 2)}{pq - r}, \\
\sigma_3 = -\frac{q[\beta_1 - p(N + 1)] + n(N + 2)}{pq - n}, \\
\sigma_4 = -\frac{q[s - p(N + 2 - \alpha_1)] + s(N + 2)}{pq - s}, \\
\sigma_5 = -\frac{q[s\alpha_2 - p(N + 2 - \alpha_1)] + sr(N + 2)}{pq - sr}, \\
\sigma_6 = -\frac{q[s\beta_2 - p(N + 2 - \alpha_1)] + sn(N + 2)}{pq - sn},
\]
\[ \sigma_7 = -\frac{q[m - p(N + 2 - \beta_1)] + m(N + 2)}{pq - m}, \]
\[ \sigma_8 = -\frac{q[m\alpha_2 - p(N + 2 - \beta_1)] + rm(N + 2)}{pq - rm}, \]
\[ \sigma_9 = -\frac{q[m\beta_2 - p(N + 2 - \beta_1)] + nm(N + 2)}{pq - nm}. \]

**Theorem 4.2.** Let \( p > 1, \ q > 1, \ p > m, \ p > s, \ q > n, \ q > r \) and assume that
\[
\int_S u_2(0,t_2,x)\varphi^\mu(0,t_2,x)dP > 0, \quad \int_Q |u_1|^sD^\alpha_{t_2}|T\varphi^\mu dP > 0,
\]
\[
\int_S v_2(0,t_2,x)\varphi^\mu(0,t_2,x)dP > 0, \quad \int_Q |v_1|^rD^\alpha_{t_2}|T\varphi^\mu dP > 0,
\]
then solutions to system (4.1)-(4.2) blow-up whenever
\[
\max\{\sigma_1, \ldots, \sigma_9; \delta_1, \ldots, \delta_9\} \leq 0.
\]

**Proof of theorem 4.2.** Assume that the solution is nontrivial and global. Next, replacing \( \varphi \) by \( \varphi^\mu \) in (4.5) and then using Hölder’s inequality to estimate the RHS, we obtain the following estimates:

- For \( p > 1, \)
\[
-\int_Q u_\varphi^\mu dP \leq \mu \left( \int_Q |u|^p \varphi^\mu dP \right)^{1/p} \left( \int_Q \varphi^\mu |\varphi^\mu|^{\frac{\mu}{r-m}} dP \right)^{\frac{r-m}{\mu}}. \quad (4.6)
\]

- For \( p > s, \)
\[
\int_Q |u|^sD^\alpha_{t_2}|T\varphi^\mu dP \leq \left( \int_Q |u|^p \varphi^\mu dP \right)^{s/p} \left( \int_Q \varphi^\mu |\varphi^\mu|^{\frac{\mu}{r-m}} dP \right)^{\frac{r-m}{\mu}}. \quad (4.7)
\]

- For \( p > m, \)
\[
\int_Q |u|^m(\Delta)^\frac{\mu}{2}\varphi^\mu \leq \mu \left( \int_Q |u|^p \varphi^\mu dP \right)^{\frac{m}{p}} \left( \int_Q \varphi^\mu |(-\Delta)^\frac{\mu}{2}\varphi|^{\frac{r-m}{\mu}} dP \right)^{\frac{r-m}{\mu}}. \quad (4.8)
\]

Similarly, we have

- For \( q > 1, \)
\[
-\int_Q v_\varphi^\mu dP \leq \mu \left( \int_Q |v|^q \varphi^\mu dP \right)^{1/q} \left( \int_Q \varphi^\mu |\varphi^\mu|^{\frac{\mu}{r-m}} dP \right)^{\frac{r-m}{\mu}}. \quad (4.9)
\]

- For \( q > r, \)
\[
\int_Q |v|^rD^\alpha_{t_2}|T\varphi^\mu dP \leq \left( \int_Q |v|^q \varphi^\mu dP \right)^{\frac{r}{q}} \left( \int_Q \varphi^\mu |D^\alpha_{t_2}|T\varphi^\mu|^{\frac{\mu}{r-m}} dP \right)^{\frac{r-m}{\mu}}. \quad (4.10)
\]

- For \( q > n, \)
\[
\int_Q |v|^n(\Delta)^\frac{\mu}{2}\varphi^\mu \leq \mu \left( \int_Q |v|^q \varphi^\mu dP \right)^{\frac{n}{q}} \left( \int_Q \varphi^\mu |(-\Delta)^\frac{\mu}{2}\varphi|^{\frac{r-m}{\mu}} dP \right)^{\frac{r-m}{\mu}}. \quad (4.11)
\]

If we set
\[
I_u := \int_Q |u|^p \varphi^\mu dP, \quad I_v := \int_Q |v|^q \varphi^\mu dP, \quad A(p) = \mu \left( \int_Q \varphi^\mu |\varphi^\mu|^{\frac{r-m}{\mu}} dP \right)^{\frac{r-m}{\mu}},
\]
then, using estimates (4.6)-(4.11), we can write (4.5) as

\[ A(q) = \mu \left( \int_Q \varphi^{p-q} |\varphi_t|^q \, dP \right)^{\frac{p-1}{q}}, \]

\[ B(p, s) = \left( \int_Q \varphi^{-\frac{m}{m-\alpha}} |D_{t_1}^{\alpha} \varphi|^\frac{p}{m-\alpha} \, dP \right)^{\frac{m-\alpha}{p}}, \]

\[ B(q, r) = \left( \int_Q \varphi^{-\frac{m}{m-\alpha}} |D_{t_2}^{\alpha} \varphi|^\frac{q}{m-\alpha} \, dP \right)^{\frac{m-\alpha}{q}}, \]

\[ C(p, m) = \mu \left( \int_Q \varphi^{p-q} (-\Delta)^{\frac{\alpha}{q}} \varphi |\varphi|^q \, dP \right)^{\frac{q-1}{q}}, \]

\[ C(q, n) = \mu \left( \int_Q \varphi^{p-q} (-\Delta)^{\frac{\alpha}{q}} \varphi |\varphi|^q \, dP \right)^{\frac{q-1}{q}}, \]

\[ I_0 = \int_s u_2(0, t_2, x)\varphi^\alpha (0, t_2, x)\, dP_2 + \int_Q |u_1|^\delta D_{t_2}^{\alpha} \varphi \, dP, \]

\[ J_0 = \int_s v_2(0, t_2, x)\varphi^\alpha (0, t_2, x)\, dP_2 + \int_Q |v_1|^\gamma D_{t_2}^{\alpha} \varphi \, dP, \]

Then Young’s inequality implies

\[ I_v + I_0 \leq I_u^{1/p} A(p) + I_u^{s/p} B(p, s) + I_u^{\frac{m}{n}} C(p, m), \]

\[ I_u + J_0 \leq I_u^{\frac{1}{2}} A(q) + I_u^{\frac{r}{2}} B(q, r) + I_u^{\frac{m}{n}} C(q, n). \]

Since \( I_u^{\frac{m}{n}}, J_0^{\frac{m}{n}} > 0 \), we have

\[ I_v \leq I_u^{1/p} A(p) + I_u^{s/p} B(p, s) + I_u^{\frac{m}{n}} C(p, m), \]

\[ I_u \leq I_u^{\frac{1}{2}} A(q) + I_u^{\frac{r}{2}} B(q, r) + I_u^{\frac{m}{n}} C(q, n). \]

Now, from (4.12) and (4.13), we have

\[ I_v + I_0 \leq \left( I_v^{\frac{m}{n}} A^{1/p}(q) + I_v^{\frac{m}{n}} B^{1/p}(q, r) + I_v^{\frac{m}{n}} C^{1/p}(q, n) \right) A(p) \]

\[ + \left( I_v^{\frac{m}{n}} A^{s/p}(q) + I_v^{\frac{m}{n}} B^{s/p}(q, r) + I_v^{\frac{m}{n}} C^{s/p}(q, n) \right) B(p, s) \]

\[ + \left( I_v^{\frac{m}{n}} A^{\frac{r}{2}}(q) + I_v^{\frac{m}{n}} B^{\frac{r}{2}}(q, r) + I_v^{\frac{m}{n}} C^{\frac{r}{2}}(q, n) \right) C(p, m). \]

Then Young’s inequality implies

\[ I_v + I_0 \leq K \left\{ \left( A^{1/p}(q) A(p) \right)^{\frac{m}{n}p} + \left( B^{1/p}(q, r) A(p) \right)^{\frac{m}{n}p} \right\} \]

\[ + \left( C^{1/p}(q, n) A(p) \right)^{\frac{m}{n}p} + \left( A^{s/p}(q) B(p, s) \right)^{\frac{m}{n}p} \]

\[ + \left( B^{s/p}(q, r) B(p, s) \right)^{\frac{m}{n}p} + \left( C^{s/p}(q, n) B(p, s) \right)^{\frac{m}{n}p} \]

\[ + \left( A^{\frac{r}{2}}(q) C(p, m) \right)^{\frac{m}{n}p} + \left( B^{\frac{r}{2}}(q, r) C(p, m) \right)^{\frac{m}{n}p} \]

\[ + \left( C^{\frac{r}{2}}(q, n) C(p, m) \right)^{\frac{m}{n}p} \]

for some positive constant \( K \). Using the scaled variables (3.2) we obtain

\[ A(p) = CT^{-1+(N+2)(1-1/p)}, \quad A(q) = CT^{-1+(N+2)(1-1/q)}, \]
\[ B(p, s) = CT^{-\alpha_1 + (N+2)(1-s/p)}, \quad B(q, r) = CT^{-\alpha_2 + (N+2)(1-r/q)}, \]
\[ C(p, m) = CT^{-\beta_1 + (N+2)(1-m/p)}, \quad C(q, n) = CT^{-\beta_2 + (N+2)(1-n/q)}, \]
for some positive constant \( C \). Hence, we obtain
\[
I_\nu + I_\mu^0 \leq K \{ T^\sigma_1 + T^\sigma_2 + \cdots + T^\sigma_9 \}. \tag{4.14}
\]
Similarly, we obtain for \( I_\nu \) the estimate
\[
I_\nu + J_0^\mu \leq K \{ T^\delta_1 + T^\delta_2 + \cdots + T^\delta_9 \}. \tag{4.15}
\]
Finally, passing to the limit as \( T \to \infty \), we observe that:

Either \( \max\{\sigma_1, \ldots, \sigma_9; \delta_1, \ldots, \delta_9\} < 0 \) and in this case, the right hand side tends to zero while the left hand side is strictly positive. Hence, we obtain a contradiction.

Or \( \max\{\sigma_1, \ldots, \sigma_9; \delta_1, \ldots, \delta_9\} = 0 \) and in this case, following the analysis similar as in one equation, we prove a contradiction. \( \Box \)

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