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STANDING WAVES FOR DISCRETE NONLINEAR SCHRÖDINGER EQUATIONS

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ABSTRACT. The discrete nonlinear Schrödinger equation is a nonlinear lattice system that appears in many areas of physics such as nonlinear optics, biomolecular chains and Bose-Einstein condensates. By using critical point theory, we establish some new sufficient conditions on the existence results of standing waves for the discrete nonlinear Schrödinger equations. We give an appropriate example to illustrate the conclusion obtained.

1. INTRODUCTION

The discrete nonlinear Schrödinger (DNLS) equation is one of the most important inherently discrete models, having a crucial role in the modeling of a great variety of phenomena, ranging from solid-state and condensed-matter physics to biology [6]. Fundamental states supported by the DNLS equations are standing waves due to their periodic time behavior. This kind of solutions have been found in the experimental observations [10].

In the past decade, the existence of standing waves of the DNLS equations has drawn a great deal of interest [12, 17, 19, 23, 24, 25, 28, 29]. The existence for the periodic DNLS equations with superlinear nonlinearity [19] and with saturable nonlinearity [29] has been studied. And the existence results of standing waves of the DNLS equations without periodicity assumptions were established in [28]. As for the existence of the homoclinic orbits of nonlinear Schrödinger equations, we refer to [5, 26].

We denote by $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{R}$ the sets of all natural numbers, integers and real numbers respectively. For a and b in $\mathbb{Z}$, define $\mathbb{Z}(a,b) = \{a, a+1, \dots, b\}$ when $a \le b$

This article considers the DNLS equation

$$i\dot{\psi}_n = -\Delta\psi_n + v_n\psi_n - \gamma_n f(\psi_n),\ n \in \mathbb{Z}, \tag{1.1}$$

where $\Delta\psi_n = \psi_{n+1} + \psi_{n-1} - 2\psi_n$ is discrete Laplacian operator, v_n and γ_n are real valued for each $n \in \mathbb{Z}$, $f \in C(\mathbb{R},\mathbb{R})$, $f(0) = 0$ and the nonlinearity $f(u)$ is gauge invariant, that is,

$$f(e^{i\theta}u) = e^{i\theta} f(u),\ \theta \in \mathbb{R}. \tag{1.2}$$

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Since standing waves are spatially localized time-periodic solutions and decay to zero at infinity, ψ_n has the form

$$\psi_n = u_n e^{-i\omega t},$$

and

$$\lim_{|n|\to\infty} \psi_n = 0,$$

where ψ_n is real valued for each $n \in \mathbb{Z}$ and $\omega \in \mathbb{R}$ is the temporal frequency. Then (1.1) becomes

$$-\Delta u_n + v_n u_n - \omega u_n = \gamma_n f(u_n), \quad n \in \mathbb{Z}, \tag{1.3}$$

$$\lim_{|n|\to\infty} u_n = 0 \tag{1.4}$$

holds, where $|n|$ is the length of index n. Actually, our methods allow us to consider the more general equation

$$\Delta^k(p_{n-k}\Delta^k u_{n-k}) + (-1)^k q_n u_n = (-1)^k f_n(u_n), \quad k \in \mathbb{Z}(1),\ n \in \mathbb{Z}, \tag{1.5}$$

with the same boundary condition (1.4). Here, Δ is the forward difference operator [16, 22] $\Delta u_n = u_{n+1} - u_n$, $\Delta^k u_n = \Delta(\Delta^{k-1} u_n)$, p_n and q_n are positive real valued for each $n \in \mathbb{Z}$. p_n, q_n and $f_n(x)$ are all T-periodic in n for a given positive integer T. When $k = 1, p_n \equiv 1$ and $q_n \equiv \varepsilon_n - \omega$, we obtain (1.3). Naturally, if we look for standing waves of (1.1), we just need to get the solutions of (1.5) satisfying (1.4).

Peil and Peterson [20] in 1994 studied the asymptotic behavior of solutions of $2k$th-order difference equation

$$\sum_{i=0}^{k} \Delta^i(r_i(n-i)\Delta^i u(n-i)) = 0 \tag{1.6}$$

with $r_i(n) \equiv 0$ for $1 \le i \le k-1$.

In 1998, Anderson [2] considered (1.6) for $n \in \mathbb{Z}(a)$, and obtained a formulation of generalized zeros and (k,k)-disconjugacy for (1.6). Cai and Yu [3] in 2007 obtained some criteria for the existence of periodic solutions of the following difference equation

$$\Delta^k(r_{n-k}\Delta^k u_{n-k}) + f(n, u_n) = 0. \tag{1.7}$$

In 2013, Deng, Liu, Zhang and Shi [9] studied the existence of periodic for the following $2n$th-order difference equation containing both advance and retardation with p-Laplacian

$$\Delta^n(r_{k-n}\varphi_p(\Delta^n u_{k-1})) = (-1)^n f(k, u_{k+1}, u_k, u_{k-1}),\ k \in \mathbb{Z}. \tag{1.8}$$

Recently, Liu, Zhang and Shi [14] established various sets of sufficient conditions of the nonexistence and existence of solutions for mixed boundary value problem and gave some new results to the following $2n$th-order nonlinear difference equation

$$\Delta^n(\gamma_{i-n+1}\Delta^n u_{i-n}) = (-1)^n f(i, u_{i+1}, u_i, u_{i-1}), \quad n \in \mathbb{Z}(1),\ i \in \mathbb{Z}(1,k), \tag{1.9}$$

by using critical point theory.

Using critical point theory, Shi and Zhang [23] in 2016 investigated the more general equation

$$\Delta^k(r_{n-k}\varphi_p(\Delta^k u_{n-1})) + (-1)^k q_n \varphi_p(u_n) = (-1)^k \gamma_n f(u_n), \quad k \in \mathbb{Z}(1),\ n \in \mathbb{Z}, \tag{1.10}$$

and obtained a new result concerning the existence of a standing wave solution.

As it is well known, critical point theory is a powerful tool to deal with the homoclinic solutions of differential equations [11] and is used to study homoclinic solutions of discrete systems in recent years [1, 7, 8, 13, 15]. Our aim in this paper is to obtain the existence results of standing waves for the discrete nonlinear Schrödinger equations by using critical point theory. The main idea is to transfer the problem of solutions in E (defined in Section 2) of (1.5) into that of critical points of the corresponding functional. The motivation for the present work stems from the recent paper [27].

For basic knowledge of variational methods, the reader is referred to [18, 21]. Let $F_n(x) = \int_0^x f_n(t)dt$, $t \in \mathbb{R}$. Our main results are as follows.

Theorem 1.1. *Suppose that the following hypotheses are satisfied:*

- (H1) $F_n(x)$ *is continuously differentiable in* x *for every* $n \in \mathbb{Z}$, $F_n(x) \geq 0$, $F_n(0) = 0$;
- (H2) $\lim_{|x|\to 0} \frac{f_n(x)}{|x|} = 0$ *for* $n \in \mathbb{Z}$;
- (H3) $\lim_{|x|\to\infty} \frac{F_n(x)}{x^2} = \infty$ *for* $n \in \mathbb{Z}$;
- (H4) *for any* $\varrho > 0$, *there exist* $a = a_\varrho > 0$, $b = b_\varrho > 0$ *and* $\alpha < 2$ *such that for all* $n \in \mathbb{Z}$, $|x| > \varrho$,
$$\Big(2 + \frac{1}{a + b|x|^{\alpha/2}}\Big) F_n(x) \leq f_n(x)x.$$

Then (1.5) *has a nontrivial solution satisfying* (1.4).

Theorem 1.2. *Assume that* (H1), (H2) *and the following hypothesis are satisfied:*

- (H5) *there exists* $\gamma > 2$ *such that*
$$0 < \gamma F_n(x) \leq x f_n(x), \quad \forall n \in \mathbb{Z},\ x \in \mathbb{R} \setminus \{0\}.$$

Then (1.5) *has a nontrivial solution satisfying* (1.4).

2. Variational structure

To apply the critical point theory, the corresponding variational framework for equation (1.5) is established. We start by some basic notations for the reader's convenience. Let S be the vector space of all real sequences of the form
$$u = (\dots, u_{-n}, \dots, u_{-1}, u_0, u_1, \dots, u_n, \dots) = \{u_n\}_{n=-\infty}^{+\infty},$$
namely $S = \{\{u_n\} : u_n \in \mathbb{R},\ n \in \mathbb{Z}\}$. Define
$$E = \big\{u \in S : \sum_{n=-\infty}^{+\infty} \big[p_{n-1}(\Delta^k u_{n-1})^2 + q_n u_n^2\big] < +\infty\big\}.$$
The space is a Hilbert space with the inner product
$$\langle u, v\rangle = \sum_{n=-\infty}^{+\infty} (p_{n-1}\Delta^k u_{n-1}\Delta^k v_{n-1} + q_n u_n v_n), \quad \forall u, v \in E, \tag{2.1}$$
and the corresponding norm
$$\|u\| = \Big(\sum_{n=-\infty}^{+\infty} \big[p_{n-1}(\Delta^k u_{n-1})^2 + q_n u_n^2\big]\Big)^{1/2}, \quad \forall u \in E. \tag{2.2}$$

On the other hand, we define the real sequence spaces

$$l^s = \Big\{u \in S : \|u\|_s = \Big(\sum_{n=-\infty}^{+\infty} |u_n|^s\Big)^{1/s} < +\infty\Big\},\ 1 \le s < +\infty, \tag{2.3}$$

with $\|u\|_\infty = \sup_{n\in\mathbb{Z}} |u_n|$ when $s = +\infty$.

Since $u \in E$, it follows that $\lim_{|n|\to\infty} |u_n| = 0$. Hence, there exists $n^* \in \mathbb{Z}$ such that

$$\|u\|_\infty = |u_{n^*}| = \max_{n\in\mathbb{Z}} |u_n|.$$

By (2.2), we have

$$\|u\|^2 \ge \sum_{n\in\mathbb{Z}} q_n u_n^2 \ge \underline{q} \sum_{n\in\mathbb{Z}} u_n^2 \ge \underline{q}\|u\|_\infty^2.$$

Thus,

$$\underline{q}\|u\|_\infty^2 \le \underline{q}\|u\|_2^2 \le \|u\|^2. \tag{2.4}$$

For all $u \in E$, define the functional J on E as follows:

$$\begin{aligned} J(u) :=& \frac{1}{2} \sum_{n=-\infty}^{+\infty} \big[p_{n-1}(\Delta^k u_{n-1})^2 + q_n u_n^2\big] - \sum_{n=-\infty}^{+\infty} F_n(u_n) \\ =& \frac{1}{2}\|u\|^2 - \sum_{n=-\infty}^{+\infty} F_n(u_n), \end{aligned} \tag{2.5}$$

then $J \in C^1(E, \mathbb{R})$. By using

$$\Delta^k u_{n-1} = \sum_{i=0}^{k} (-1)^i \binom{k}{i} u_{n+k-i-1},$$

we can compute the partial derivative as

$$\frac{\partial J(u)}{\partial u_n} = (-1)^k \Delta^k(p_{n-k}\Delta^k u_{n-k}) + q_n u_n - f_n(u_n), \quad k \in \mathbb{Z}(1),\ n \in \mathbb{Z}. \tag{2.6}$$

Thus, the critical points of J in E are solutions of (1.5) satisfying (1.4).

3. Main lemmas

To apply variational methods and critical point theory to study the existence of a nontrivial solution of (1.5) satisfying (1.4), we shall state some lemmas which will be used in the proofs of our main results.

Lemma 3.1 ([4]). *Let E be a real Banach space with its dual space E^* and assume that $J \in C^1(E, \mathbb{R})$ satisfies*

$$\max\{J(0), J(e)\} \le \eta_0 < \eta \le \inf_{\|u\|=\rho} J(u),$$

for some $\eta_0 < \eta$, $\rho > 0$ and $e \in E$ with $\|e\| > \rho$. Let $c \ge \eta$ be characterized by

$$c = \inf_{\gamma\in\Gamma} \max_{0\le t\le 1} J(\gamma(t)),$$

where $\Gamma = \{\gamma \in C([0,1], E) : \gamma(0) = 0,\ \gamma(1) = e\}$ is the set of continuous paths joining 0 to e; then there exists $\{u^{(m)}\}_{m\in\mathbb{N}} \subset E$ such that $J(u^{(m)}) \to c$ and $(1 + \|u^{(m)}\|)\|J'(u^{(m)})\|_{E^*} \to 0$ as $m \to \infty$.

Lemma 3.2. *Assume that* (H1)–(H4) *are satisfied. Then there exists a constant* $c > 0$ *and a sequence* $\{u^{(m)}\}_{m\in\mathbb{Z}}$ *satisfying*

$$J(u^{(m)}) \to c, \quad \|J'(u^{(m)})\|(1+\|u^{(m)}\|) \to 0, \quad m \to \infty. \tag{3.1}$$

Proof. From (H2) there exists $\rho > 0$ such that for any $n \in \mathbb{Z}$ and $|x| \le \rho$,

$$F_n(x) \le \frac{\underline{q}}{4} x^2. \tag{3.2}$$

Define $\|u\| = \underline{q}^{1/2}\rho := \eta$. For any $n \in \mathbb{Z}$, it follows from (2.4) that $|u_n| \le \rho$. Consequently, for $u \in E$, $\|u\| = \rho$, it comes from (2.5) and (3.2) that

$$\begin{aligned} J(u) &= \frac{1}{2}\|u\|^2 - \sum_{n=-\infty}^{+\infty} F_n(u_n) \\ &\ge \frac{1}{2}\eta^2 - \frac{\underline{q}}{4}\sum_{n=-\infty}^{+\infty} u_n^2 \\ &\ge \frac{1}{2}\eta^2 - \frac{\underline{q}}{4}\|u\|_2^2, \\ &\ge \frac{1}{2}\eta^2 - \frac{1}{4}\eta^2 = \frac{1}{4}\eta^2. \end{aligned}$$

Set $u_0^{(0)} = 1$, $u_n^{(0)} = 0$ for $n \ne 0$. Then, by (H1), (H2), (H4) and (2.3), we have

$$\begin{aligned} J(\theta u^{(0)}) &= \frac{\theta^2}{2}\|u^{(0)}\|^2 - \sum_{n=-\infty}^{+\infty} F_n(\theta u_n^{(0)}) \\ &\le \frac{\theta^2}{2}\|u^{(0)}\|^2 - F_0(\theta u_n^{(0)}) \\ &\le \theta^2\Big[\frac{1}{2}\|u^{(0)}\|^2 - \frac{F_0(\theta u_n^{(0)})}{|\theta u_0^{(0)}|^2}\Big] \le 0 \end{aligned}$$

for large enough $\theta > 0$. Thus, we can choose $\bar\theta > 1$ such that $\bar\theta\|u^{(0)}\| > \eta$ and $J(\bar\theta u^{(0)}) \le 0$. Define $e = \bar\theta u^{(0)}$, then $e \in E$, $\|e\| > \eta$ and $J(e) \le 0$. By Lemma 3.1, there exists $c \ge \frac{1}{4}\eta^2$ and a sequence $\{u^{(m)}\}_{m\in\mathbb{Z}} \subset E$ such that (3.1) holds. ☐

Lemma 3.3. *Assume that* (H1)–(H4) *are satisfied. Then any sequence* $\{u^{(m)}\}_{m\in\mathbb{N}}$ *satisfying*

$$J(u^{(m)}) \to c > 0, \quad \|J'(u^{(m)})\|(1+\|u^{(m)}\|) \to 0, \quad m \to \infty \tag{3.3}$$

is bounded in E.

Proof. From (H3) it follows that there exists $0 < \rho < 1$ such that for any $n \in \mathbb{Z}$, $|x| \le \rho$,

$$F_n(x) \le \frac{\underline{q}}{4} x^2. \tag{3.4}$$

For any $n \in \mathbb{Z}$, by (H4), we have

$$f_n(x)x > 2F_n(x) \ge 0 \tag{3.5}$$

and for $|x| > \rho$, we have

$$F_n(x) \le (a + b|x|^{\alpha/2})[f_n(x)x - 2F_n(x)]. \tag{3.6}$$

By (2.5), (2.6) and (3.1), there exist $\tilde{K}$ and $\hat{K}$ such that

$$\begin{aligned} \tilde{K} &\geq 2J(u^{(m)}) - \langle J'(u^{(m)}), u^{(m)}\rangle \\ &= \sum_{n=-\infty}^{+\infty} \big[f_n(u_n^{(m)})u_n^{(m)} - 2F_n(u_n^{(m)})\big] \end{aligned} \tag{3.7}$$

and

$$J(u^{(m)}) \leq \hat{K}. \tag{3.8}$$

It comes from (2.5), (2.6), (3.3), (3.4), (3.5), (3.6), (3.7) and (3.8) that

$$\begin{aligned} &\frac{1}{2}\|u^{(m)}\|^2 \\ &= J(u^{(m)}) + \sum_{n=-\infty}^{+\infty} F_n(u_n^{(m)}) \\ &= J(u^{(m)}) + \sum_{n\in\mathbb{Z}(|u_n^{(m)}|\leq\rho)} F_n(u_n^{(m)}) + \sum_{n\in\mathbb{Z}(|u_n^{(m)}|>\rho)} F_n(u_n^{(m)}) \\ &\leq J(u^{(m)}) + \frac{\underline{q}}{4} \sum_{n\in\mathbb{Z}(|u_n^{(m)}|>\rho)} |u_n^{(m)}|^2 \\ &\quad + \sum_{n\in\mathbb{Z}(|u_n^{(m)}|>\rho)} \{a + b|u_n^{(m)}|^{\alpha/2}\}\big[f_n(u_n^{(m)})u_n^{(m)} - 2F_n(u_n^{(m)})\big] \\ &\leq \hat{K} + \frac{1}{4}\|u^{(m)}\|^2 + \sum_{n\in\mathbb{Z}}\{a + b|u_n^{(m)}|^{\alpha/2}\}\big[f_n(u_n^{(m)})u_n^{(m)} - 2F_n(u_n^{(m)})\big] \\ &\leq \hat{K} + \frac{1}{4}\|u^{(m)}\|^2 + (a + 2b\|u^{(m)}\|_\infty^{\alpha})\big[f_n(u_n^{(m)})u_n^{(m)} - 2F_n(u_n^{(m)})\big] \\ &\leq \hat{K} + \frac{1}{4}\|u^{(m)}\|^2 + \tilde{K}(a + 2b\|u^{(m)}\|_\infty^{\alpha}) \\ &\leq \hat{K} + \frac{1}{4}\|u^{(m)}\|^2 + \tilde{K}(a + 2\underline{q}^{-\frac{\alpha}{2}}b\|u^{(m)}\|^{\alpha}), \quad m \in \mathbb{N}. \end{aligned} \tag{3.9}$$

Combining with $\alpha < 2$, (3.9) imply that $\{u^{(m)}\}_{m\in\mathbb{N}}$ is bounded. Hence, the proof of Lemma 3.3 is complete. □

4. Proof of the main results

In this Section, we shall prove our main results by using the critical point method.

Proof of Theorem 1.1. By Lemma 3.2, there exists a sequence $\{u^{(m)}\}_{m\in\mathbb{N}} \subset E$ satisfying (3.1), and so (3.3). It follows from Lemma 3.3 that $\{u^{(m)}\}_{k\in\mathbb{N}}$ is bounded in E. Therefore, by (2.4), for all $n \in \mathbb{N}$, we have there exists $\bar{K} > 0$ such that

$$\underline{q}^{1/2}\|u^{(m)}\|_\infty \leq \|u^{(m)}\| \leq \bar{K}. \tag{4.1}$$

For any $n \in \mathbb{Z}$, $|x| \leq \underline{q}^{-1/2}\bar{K}$, by (H2), we have

$$\big|\frac{1}{2}f_n(x)x - F_n(x)\big| \leq \frac{c\underline{q}}{4\bar{K}^2}x^2 + \frac{c\underline{q}}{4\bar{K}^2}x^2 = \frac{c\underline{q}}{2\bar{K}^2}x^2. \tag{4.2}$$

By way of contradiction, suppose that $\xi := \limsup_{m\to\infty} \|u^{(m)}\|_\infty = 0$. Then, by (H2), (2.1), (2.3) and (3.2), we have

$$
\begin{aligned}
c &= J(u^{(m)}) - \frac{1}{2}\left\langle J'(u^{(m)}), u^{(m)}\right\rangle + o(1) \\
&= \frac{1}{2}\sum_{n=-\infty}^{+\infty} f_n(u_n^{(m)})u_n^{(m)} - \sum_{n=-\infty}^{+\infty} F_n(u_n^{(m)}) + o(1) \\
&\le \frac{c\underline{q}}{2\bar{K}^2}\sum_{n=-\infty}^{+\infty}(u_n^{(m)})^2 \\
&\le \frac{c\underline{q}}{2\bar{K}^2}\|u^{(k)}\|_2^2 + o(1) \\
&\le \frac{c}{2} + o(1), \quad k \to \infty.
\end{aligned}
$$

This contradiction shows that $\xi > 0$.

First, going to a subsequence if necessary, we can assume that the existence of $n^{(m)} \in \mathbb{Z}$ independent of m such that

$$
|u_{n^{(m)}}^{(m)}| = \|u^{(m)}\|_\infty > \frac{\xi}{2}. \tag{4.3}
$$

Hence, making such shifts, we can assume that $n^{(m)} \in \mathbb{Z}(0, T-1)$ in (4.3). Moreover, passing to a subsequence of ms, we can even assume that $n^{(m)} = n^{(0)}$ is independent of m.

Next, we extract a subsequence, still denote by $u^{(m)}$, such that

$$
u_n^{(m)} \to u_n,\ m \to \infty,\ \forall n \in \mathbb{Z}.
$$

Inequality (4.3) implies that $|u_{n^{(0)}}| \ge \xi$ and, hence, $u = \{u_n\}$ is a nonzero sequence. Moreover,

$$
\begin{aligned}
&\Delta^k(p_{n-k}\Delta^k u_{n-k}) + (-1)^k q_n u_n - (-1)^k f_n(u_n) \\
&= \lim_{m\to\infty}\left[\Delta^k(p_{n-k}\Delta^k u_{n-k}^{(m)}) + (-1)^k q_n u_n^{(m)} - (-1)^k f_n(u_n^{(m)})\right] \quad = \lim_{m\to\infty} 0 = 0.
\end{aligned}
$$

So $u = \{u_n\}$ is a solution of (1.5) satisfying (1.4).

Finally, for any fixed $\varsigma \in \mathbb{Z}$ and m large enough, we have

$$
\sum_{n=-\varsigma}^{\varsigma} |u_n^{(m)}|^2 \le \frac{1}{\underline{q}}\|u^{(m)}\|^2 \le \bar{K}^2.
$$

Since $\bar{K}^2$ is a constant independent of k, passing to the limit, we have

$$
\sum_{n=-\varsigma}^{\varsigma} |u_n|^2 \le \bar{K}^2.
$$

By the arbitrariness of ς, $u \in l^2$. Therefore, u satisfies $u_n \to 0$ as $|n| \to \infty$. The proof is complete. $\square$

Proof of Theorem 1.2. The techniques of the proof of Theorem 1.2 are just the same as those carried out in the proof of [13]. We do not repeat them here. $\square$

5. Example

As an application of Theorem 1.1, we give an example to illustrate our main result. For $n \in \mathbb{Z}$, $k \in \mathbb{Z}(1)$, assume that

$$\begin{aligned} &\Delta^k\Big(\big(3+\sin^2\frac{\pi(n-k)}{T}\big)\Delta^k u_{n-k}\Big) + (-1)^k\big(2+\cos^2\frac{\pi n}{T}\big)u_n \\ &= 2\big(9+\sin^2\frac{\pi n}{T}\big)u_n\ln(1+|u_n|) + \frac{u_n^3}{(1+|u_n|)|u_n|}, \end{aligned} \tag{5.1}$$

where T is a positive integer. We have

$$p_{n-k} = \Big(3+\sin^2\frac{\pi(n-k)}{T}\Big), \quad q_n = \Big(2+\cos^2\frac{\pi n}{T}\Big),$$
$$F_n(x) = \big(9+\sin^2\frac{\pi n}{T}\big)x^2\ln(1+|x|).$$

Then

$$f_n(x)x = 2\big(9+\sin^2\frac{\pi n}{T}\big)x^2\ln(1+|x|) + \frac{x^2|x|}{1+|x|} \ge \big(2+\frac{1}{1+|x|}\big)F_n(x) \ge 0.$$

This shows that (H4) holds with $a = b = \nu = 1$. It is easy to check all the assumptions of Theorem 1.1 are satisfied. Consequently, (5.1) has a nontrivial solution satisfying (1.4).

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