EXISTENCE OF NONNEGATIVE SOLUTIONS FOR SINGULAR ELLIPTIC PROBLEMS

TOMAS GODOY, ALFREDO J. GUERIN

Abstract. We prove the existence of nonnegative nontrivial weak solutions to the problem
\[-\Delta u = au - \alpha \chi_{\{u>0\}} - bu^p \quad \text{in } \Omega,
\]
\[u = 0 \quad \text{on } \partial \Omega,
\]
where \(\Omega\) is a bounded domain in \(\mathbb{R}^n\). A sufficient condition for the existence of a continuous and strictly positive weak solution is also given, and the uniqueness of such a solution is proved. We also prove a maximality property for solutions that are positive a.e. in \(\Omega\).

1. Introduction and statement of the problem

Let \(\Omega\) be a bounded domain in \(\mathbb{R}^n\) with \(C^{1,1}\) boundary, let \(a\) and \(b\) be nonnegative functions on \(\Omega\), and let \(\alpha\) and \(p\) be positive real numbers. Consider the following singular elliptic problem
\[-\Delta u = au - \alpha - bu^p \quad \text{in } \Omega,
\]
\[u = 0 \quad \text{on } \partial \Omega,
\]
\[u > 0 \quad \text{in } \Omega.
\]
Problems like (1.1) appear in chemical catalysts process, non-Newtonian fluids, and in models for the temperature of electrical devices (see e.g., \[10, 7, 16, 19\]).

Several works can be found concerning the existence of positive solutions to (1.1) for the case \(b = 0\), i.e., for the problem \(-\Delta u = au - \alpha\) in \(\Omega\), \(u = 0\) on \(\partial \Omega\), \(u > 0\) in \(\Omega\); let us mention a few: Classical solutions \(u \in C^2(\Omega) \cap C(\Omega)\) satisfying \(u(x) > 0\) for all \(x \in \Omega\) were obtained by Crandall, Rabinowitz and Tartar \[11\] under the following hypothesis: \(a \in C^1(\Omega)\) and \(\min_{\Omega} a > 0\). Lazer and McKenna \[24\] proved the existence of positive weak solutions \(u \in H_0^1(\Omega)\) to (1.1) assuming that \(a \in C^1(\Omega)\), \(\gamma \in (0,1)\), and, again, \(a\) strictly positive on \(\Omega\). The case \(0 \leq a \in L^\infty(\Omega), a \neq 0\) (that is: \(|\{x \in \Omega : a(x) > 0\}| > 0\) was studied by Del Pino \[12\]. Situations where \(a\) is singular on the boundary \(\partial \Omega\) were considered by Bougerara, Giacomoni and Hernández \[5\].

The existence of classical solutions to problem (1.1) was proved by Coclite and Palmieri \[9\] for \(a\) and \(b\) in \(C^1(\Omega)\), \(0 < p < 1\), and \(a\) strictly positive on \(\Omega\) (see \[9\].

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Theorem 1}). Related singular elliptic problems were treated by Shi and Yao [29], and by Aranda and Godoy [3, 2]. Elliptic problems with singular terms and free boundaries were considered by Dávila and Montenegro [13, 14].

Ghergu and Rădulescu [22] studied multi-parameter singular bifurcation problems of the form $-\Delta u = g(u) + \lambda |\nabla u|^p + \mu f(\cdot, u)$ in $\Omega$, $u = 0$ on $\partial \Omega$, $u > 0$ in $\Omega$, where $\Omega$ is a smooth bounded domain in $\mathbb{R}^n$, $\lambda, \mu \geq 0$, $0 < p \leq 2$, $f : \overline{\Omega} \times (0, \infty) \to [0, \infty)$ is a Hölder continuous function such that $f(\cdot, s)$ is nondecreasing with respect to $s$, and $g : (0, \infty) \to (0, \infty)$ is a nonincreasing Hölder continuous function such that $\lim_{s \to 0^+} g(s) = \infty$. When $g(s)$ behaves like $s^{-\alpha}$ near the origin, with $0 < \alpha < 1$, the asymptotic behavior of the solution around the bifurcation point is established.

Dupaigne, Ghergu and Rădulescu [18] obtained various existence and nonexistence results for Lane–Emden–Fowler equations with convection and singular potential of the form $-\Delta u \pm p(d_1(x))g(u) = \lambda f(x, u) + \mu |\nabla u|^\beta$ in $\Omega$, $u = 0$ on $\partial \Omega$, $u > 0$ in $\Omega$, where $\Omega$ is a smooth bounded domain in $\mathbb{R}^n$, $d_1(x) = \text{dist}(x, \partial \Omega)$, $\lambda > 0$, $\mu \in \mathbb{R}$, $0 < \beta \leq 2$, $p(d_1(x))$ is a positive weight possibly singular at $\partial \Omega$. $g \in C^1((0, \infty)$ is a positive decreasing function such that $\lim_{s \to 0^+} g(s) = \infty$, $f : \overline{\Omega} \times (0, \infty) \to (0, \infty)$ is a Hölder continuous function which is positive on $\Omega \times (0, \infty)$ and satisfies that $s \to f(x, s)$ is nondecreasing and also that $f(x, s)$ is either linear or sublinear with respect to $s$.


Existence and nonexistence results for solutions to the inequality $Lu \geq K(x)u^p$ in $\Omega$, $u > 0$ in $\Omega$ were obtained by Ghergu, Liskevich and Sobol [20] for the case where $\Omega$ is a punctured ball $B_R(0) \setminus \{0\}$, $p \in \mathbb{R}$, $K \in L^\infty_{\text{ess}}(B_R(0) \setminus \{0\})$, $\text{ess inf} K > 0$, and $Lu := \sum_{1 \leq i, j \leq n} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{1 \leq j \leq n} b_j(x) \frac{\partial u}{\partial x_j}$, where the matrix $a = \{a_{ij}(x)\}_{1 \leq i, j \leq n}$ is symmetric, uniformly elliptic on $\Omega$, with each $a_{ij} \in L^\infty(B_R(0))$, and each $b_j$ is a measurable function and satisfies $\text{ess sup}_{x \in B_R(0) \setminus \{0\}} |x| b_j(x) < \infty$.

Existence and uniqueness results were obtained by Bouguera and Giacomoni [4] for mild solutions to singular initial value parabolic problems involving the p-Laplacian operator of the form $u_t - \Delta_p u = u^{-\alpha} + f(x, u)$ in $Q_T := (0, T) \times \Omega$, $u = 0$ on $(0, T) \times \partial \Omega$, $u > 0$ in $Q_T$, $u(0, x) = u_0(x)$ in $\Omega$ where $\Omega$ is a regular bounded domain in $\mathbb{R}^n$ : $\Omega : \mathbb{R} \to \mathbb{R}$ is a bounded below Carathéodory function and nonincreasing with respect to the second variable, $\Delta_p u := \text{div}(|\nabla u|^{p-2} \nabla u)$, $1 < p < \infty$, $\alpha > 0$, $T > 0$, and $u_0$ in a suitable functional space.

Singularly perturbed elliptic problems on an annulus whose solutions concentrate in a circle were studied by Manna and Srikanth [27].

Let us mention also that Loc and Schmitt [26, 25], extended the method of sub and supersolutions to deal with singular elliptic problems. A comprehensive treatment of the subject can be found in Ghergu and Rădulescu’s book [21] (see also [28]), and in the survey article [15], by Díaz and Hernández.

Let us state the problem that we will consider from now on: Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ with $C^{1,1}$ boundary, $\alpha \in (0, 1)$, and $p \in (0, 2^* - 1)$, where $2^*$ is defined by $\frac{1}{2^*} = \frac{1}{2} - \frac{1}{n}$ if $n > 2$ and $2^* = \infty$ if $n \leq 2$. Let $a$ and $b$ be nonnegative functions such that $a$ belongs to $L^\infty(\Omega)$, $a \neq 0$, and $b$ is in $L^1(\Omega)$, with $r = \frac{1}{1-p}$ if $p < 1$, and $r = \infty$ otherwise.
We are concerned with weak solutions to the problem
\begin{equation}
-\Delta u = a u^{-\alpha} \chi_{\{u > 0\}} - bu^p \quad \text{in } \Omega, \\
u = 0 \quad \text{on } \partial \Omega, \\
u \geq 0 \quad \text{in } \Omega
\end{equation}
where \(a u^{-\alpha} \chi_{\{u > 0\}}\) stands for the function defined by \(a u^{-\alpha} \chi_{\{u > 0\}}(x) = a(x)u(x)^{-\alpha}\) if \(u(x) \neq 0\), and \(a u^{-\alpha} \chi_{\{u > 0\}}(x) = 0\) if \(u(x) = 0\).

By a weak solution to (1.2) we mean a nonnegative function \(u \in H^1_0(\Omega)\) such that, for all \(\varphi \in H^1_0(\Omega) \cap L^\infty(\Omega)\), \((a u^{-\alpha} \chi_{\{u > 0\}} - b u^p) \varphi \in L^1(\Omega)\), and the following holds
\begin{equation}
\int_{\Omega} \langle \nabla u, \nabla \varphi \rangle = \int_{\Omega} (a u^{-\alpha} \chi_{\{u > 0\}} - b u^p) \varphi.
\end{equation}

The main aim of this work is to prove the existence of at least one nonnegative weak solution \(u \neq 0\) to the stated problem (see Theorem 3.1). Additionally, we give a condition on \(a, b\) that guarantees the existence of a strictly positive weak solution to (1.2) (see Theorem 3.5). In Theorem 3.8 we prove that there is at most one solution that is positive a.e. in \(\Omega\), and give a maximality property for such a solution. Examples of non-existence of strictly positive solutions, and of non-uniqueness of the nonnegative solutions, are also provided.

To prove Theorem 3.1 we show that the energy functional \(J\) associated with (1.2) attains its minimum at some nonnegative nontrivial \(u \in H^1_0(\Omega) \cap L^\infty(\Omega)\). Note that \(J\) may fail to be Gateaux differentiable at \(u\); despite this fact, we manage to prove that the said minimizer is indeed a weak solution of problem (1.2). Theorem 3.5 is proved using the sub and supersolutions method for singular elliptic problems developed in [20].

2. Preliminary lemmas

Let \(J : H^1_0(\Omega) \to \mathbb{R}\) be the energy functional associated with (1.2),
\begin{equation}
J(u) := \frac{1}{2} \int_{\Omega} |\nabla u|^2 - \frac{1}{1-\alpha} \int_{\Omega} a |u|^{1-\alpha} + \frac{1}{1+p} \int_{\Omega} b |u|^{1+p}.
\end{equation}

Let us start with the following lemma.

Lemma 2.1. The following statements hold:

(i) \(J\) is coercive on \(H^1_0(\Omega)\).

(ii) \(\inf_{u \in H^1_0(\Omega)} J(u) > -\infty\).

(iii) \(\inf_{u \in H^1_0(\Omega)} J(u)\) is achieved at some \(u \in H^1_0(\Omega)\).

Proof. Let \(u \in H^1_0(\Omega)\). Since \(0 < 1-\alpha < 1\), the Hölder’s and Poincare’s inequalities give
\[
\frac{1}{1-\alpha} \int_{\Omega} a |u|^{1-\alpha} \leq c \|\nabla u\|^{-\alpha}_2
\]
for some positive constant \(c\) independent of \(u\), and so \(J(u) \geq \frac{1}{2} \|\nabla u\|^{-\alpha}_2 - c \|\nabla u\|^{-\alpha}_2\), which clearly implies (i) and (ii).

To prove (iii), let \(\beta = \inf_{u \in H^1_0(\Omega)} J(u)\), and consider a sequence \(\{u_j\}_{j \in \mathbb{N}} \subset H^1_0(\Omega)\) such that \(\lim_{j \to \infty} J(u_j) = \beta\). Then, by (i), \(\{u_j\}_{j \in \mathbb{N}}\) is bounded in \(H^1_0(\Omega)\). Let \(q\) be in \((p+1, 2^*)\). Since the inclusion \(H^1_0(\Omega) \hookrightarrow L^q(\Omega)\) is a compact map, we can assume (taking a subsequence if necessary) that \(\{u_j\}_{j \in \mathbb{N}}\) converges strongly to some \(u \in L^q(\Omega)\). Since \(\{u_j\}_{k \in \mathbb{N}}\) is bounded in \(H^1_0(\Omega)\), there exists \(v \in H^1_0(\Omega)\),
and a subsequence \( \{ u_{j_k} \}_{k \in \mathbb{N}} \), such that the subsequence converges strongly to \( v \) in \( L^2(\Omega) \), and \( \{ \nabla u_{j_k} \}_{k \in \mathbb{N}} \) converges weakly to \( \nabla v \) in \( L^2(\Omega, \mathbb{R}^n) \). Thus \( v = u \), \( \{ u_{j_k} \}_{k \in \mathbb{N}} \) converges to \( u \) in \( L^q(\Omega) \), and

\[
\| \nabla u \|_2 \leq \lim \inf_{k \to \infty} \| \nabla u_{j_k} \|_2. \tag{2.2}
\]

On the other hand, the Nemytskii operators \( f(u) := |u|^{1-\alpha} \) and \( g(u) := |u|^{1+p} \) are continuous from \( L^2(\Omega) \) into \( L^{\frac{2}{1-\alpha}}(\Omega) \), and from \( L^q(\Omega) \) into \( L^{\frac{q}{1+p}}(\Omega) \), respectively [1 Theorem 1.2.1] and so, since \( a \in L^\infty(\Omega) \) and \( b \in L^r(\Omega) \),

\[
\lim_{j \to \infty} \int_\Omega \left( \frac{1}{1-\alpha} a |u_{j_k}|^{1-\alpha} - \frac{1}{1+p} b |u_{j_k}|^{1+p} \right) = \int_\Omega \left( \frac{1}{1-\alpha} a |u|^{1-\alpha} - \frac{1}{1+p} b |u|^{1+p} \right) \tag{2.3}
\]

which, combined with \( \text{(2.2)} \), gives \( J(u) \leq \lim \inf_{k \to \infty} J(u_{j_k}) = \beta \), therefore (iii) holds (since \( \beta \leq J(u) \)).

**Corollary 2.2.** \( \inf_{u \in H^1_0(\Omega)} J(u) \) is achieved at some nonnegative \( u \in H^1_0(\Omega) \).

**Proof.** Lemma 2.1 states that \( J \) attains its minimum at some \( u \in H^1_0(\Omega) \). Since \( J(u) = J(|u|) \), a nonnegative minimizer exists. \( \square \)

For the rest of this article, we fix a nonnegative minimizer for \( J \) on \( H^1_0(\Omega) \), and denote it by \( u \).

**Lemma 2.3.** The equality

\[
\int_\Omega \langle \nabla u, \nabla (u \varphi) \rangle = \int_\Omega (a u^{1-\alpha} - b u^{1+p}) \varphi \tag{2.4}
\]

holds for any \( \varphi \in H^1(\Omega) \cap L^\infty(\Omega) \) such that \( \varphi u \in H^1_0(\Omega) \).

**Proof.** Let \( \varphi \in H^1(\Omega) \cap L^\infty(\Omega) \) be such that \( \varphi u \in H^1_0(\Omega) \); satisfying, in addition, \( \| \varphi \|_\infty \leq \frac{1}{2} \). Let \( \tau \in \mathbb{R} \) such that \( |\tau| < 1 \). Then \( u + \tau u \varphi \geq 0 \), and \( J(u) \leq J(u + \tau u \varphi) \).

A computation shows that this inequality can be written as

\[
\tau \int_\Omega \langle \nabla u, \nabla (u \varphi) \rangle \\
\geq \frac{1}{1-\alpha} \int_\Omega a u^{1-\alpha} ((1 + \tau \varphi)^{1-\alpha} - 1) - \frac{1}{1+p} \int_\Omega b u^{1+p} ((1 + \tau \varphi)^{1+p} - 1) \tag{2.5}
\]

\[
- \frac{\tau^2}{2} \int_\Omega |u|^2 |\nabla \varphi|^2 - \frac{\tau^2}{2} \int_\Omega |\nabla u|^2 - \tau^2 \int_\Omega u \varphi \langle \nabla u, \nabla \varphi \rangle.
\]

Note that, for \( \gamma > 0 \), the second-order Taylor expansion of the function \( h(t) = (1 + t)^\gamma - 1 \) gives

\[
(1 + \tau \varphi)^\gamma - 1 = \gamma \tau \varphi - \frac{\tau^2}{2} \gamma (\gamma - 1)(1 + \zeta_{\tau, \gamma}) \varphi^2 \tag{2.6}
\]
for some measurable function $\zeta_{\tau, \gamma} : \Omega \to \mathbb{R}$ satisfying $|\zeta_{\tau, \gamma}| \leq |\tau \varphi| \leq \frac{1}{2}$. Inserting (2.6) (used with $\gamma = 1 - \alpha$ and $\gamma = 1 + p$) in (2.5), we obtain

$$
\tau \int_{\Omega} \langle \nabla u, \nabla (u \varphi) \rangle \\
\geq \tau \int_{\Omega} a u^{1-\alpha} \varphi - \frac{\tau^2}{2} \int_{\Omega} a u^{1-\alpha} (1 + \zeta_{\tau,1 - \alpha})^{-\alpha - 1} \varphi^2 \\
- (\tau \int_{\Omega} b u^{1+p} \varphi + \frac{\tau^2}{2} \int_{\Omega} b u^{1+p} (1 + \zeta_{\tau,1 + p})^{-p - 1} \varphi^2) \\
- \frac{\tau^2}{2} \int_{\Omega} u^2 |\nabla \varphi|^2 - \frac{\tau^2}{2} \int_{\Omega} \varphi^2 |\nabla u|^2 - \frac{\tau^2}{2} \int_{\Omega} u \varphi (\nabla u, \nabla \varphi). 
$$

(2.7)

Also, $1 + \zeta_{\tau, 1 - \alpha} \geq \frac{1}{2}$ and $1 + \zeta_{\tau, 1 + p} \geq \frac{1}{2}$, and thus

$$
| \int_{\Omega} a u^{1-\alpha} (1 + \zeta_{\tau, 1 - \alpha})^{-\alpha - 1} \varphi^2 | \leq c,
$$

$$
| \int_{\Omega} b u^{1+p} (1 + \zeta_{\tau,1 + p})^{-p - 1} \varphi^2 | \leq c
$$

for some positive constant $c$ independent of $\tau$. Dividing by $\tau$, and then letting $\tau \to 0^+$, from (2.7) we obtain

$$
\int_{\Omega} \langle \nabla u, \nabla (u \varphi) \rangle \geq \int_{\Omega} a u^{1-\alpha} \varphi - \int_{\Omega} b u^{1+p} \varphi.
$$

We note that this inequality holds if we put $-\varphi$ instead of $\varphi$; therefore we obtain also the reverse inequality, and we conclude that (2.4) is valid for $\|\varphi\|_{\infty} \leq \frac{1}{2}$. Finally, since both sides in (2.4) are linear on $\varphi$, the assumption $\|\varphi\|_{\infty} \leq \frac{1}{2}$ can be removed.

**Lemma 2.4.** There exists $v \in H^1_0(\Omega)$ such that $J(v) < 0$.

**Proof.** It is sufficient to show that there exists a function $\Phi \in H^1_0(\Omega)$ such that $\int_{\Omega} a|\Phi|^{1-\alpha} > 0$. Indeed, if such a $\Phi$ exists, then, for $t > 0$, we have

$$
J(t\Phi) = \frac{t^2}{2} \|\nabla \Phi\|_2^2 - \frac{t^{1-\alpha}}{1-\alpha} \int_{\Omega} a|\Phi|^{1-\alpha} + \frac{t^{1+p}}{1+p} \int_{\Omega} b|\Phi|^{1+p}
$$

$$
= t^{1-\alpha} \left( \frac{t^{1+\alpha}}{2} \|\nabla \Phi\|_2^2 - \frac{1}{1-\alpha} \int_{\Omega} a|\Phi|^{1-\alpha} + \frac{t^{p+\alpha}}{1+p} \int_{\Omega} b|\Phi|^{1+p} \right)
$$

which gives that $J(t\Phi)$ is positive for $t$ positive and small enough. Such a $\Phi$ can be constructed as follows: Let $h \in C^\infty_c(\mathbb{R}^n)$ be a nonnegative radial function with support in the unit ball $B = \{x \in \mathbb{R}^n : |x| < 1\}$, and such that $\int_B h = 1$. For $\varepsilon > 0$ let $h_\varepsilon(x) := \frac{1}{\varepsilon^n} h(\frac{x}{\varepsilon})$. For $\delta > 0$ let $\Omega_\delta := \{x \in \Omega : \text{dist}(x, \partial \Omega) > \delta\}$. Since $\{|x \in \Omega : a(x) > 0| > 0$, we have $\{|x \in \Omega : a(x) > 0\} \cap \Omega_\delta > 0$ for $\delta$ positive and small enough. We fix such a $\delta$, and set $E = \{x \in \Omega : a(x) > 0\} \cap \Omega_\delta$. For $\varepsilon > 0$ we define $\Phi_\varepsilon := h_\varepsilon * \chi_E$. Then $\Phi_\varepsilon \in C^\infty(\mathbb{R}^n)$ and $\text{supp}(\Phi_\varepsilon) \subset \Omega$ for $\varepsilon < \delta$. Thus $\Phi_\varepsilon \in C^\infty_c(\Omega)$ for $\varepsilon < \delta$. Also, $\lim_{\varepsilon \to 0^+} \Phi_\varepsilon = \chi_E$ with convergence in $L^1(\Omega)$ (see [10 Theorem 4.22]). Then $\lim_{\varepsilon \to 0^+} a\Phi_\varepsilon^{1-\alpha} = a\chi_E$ with convergence in $L^1(\Omega)$ (see [11 Theorem 1.2.1]), therefore

$$
\lim_{\varepsilon \to 0^+} \int_{\Omega} a\Phi_\varepsilon^{1-\alpha} = \int_{\Omega} a(\chi_E)^{1-\alpha} = \int_{\Omega} a\chi_E > 0.
$$
Then \( \int_\Omega a |\Phi_\varepsilon|^{1-\alpha} > 0 \) for \( \varepsilon \) small enough. \( \square \)

**Corollary 2.5.** \( u \neq 0 \).

**Remark 2.6.** Let us observe that \( \nabla (u^2) = 2u \nabla (u) \) for any (possibly unbounded) \( v \in H^1(\Omega) \). Indeed, for \( k \in \mathbb{N} \), let \( v_k \) be the truncation of \( v \), defined by \( v_k(x) = v(x) \) if \( |v(x)| \leq k \), and by \( v_k(x) = k \text{ sign}(v(x)) \) otherwise. Then \( \{v_k\}_{k \in \mathbb{N}} \) converges to \( v \) in \( H^1(\Omega) \) as \( k \) tends to \( \infty \), and, since each \( v_k \) is bounded, it follows from the chain rule (as stated e.g. in [23 Lemma 7.5]) that \( \frac{\partial}{\partial x_i} (v_k^2) = 2v_k \frac{\partial v_k}{\partial x_i} \), \( i = 1, 2, \ldots, n \).

Since \( \{v_k\}_{k \in \mathbb{N}} \) converges to \( v \) in \( L^2(\Omega) \), we have that \( \{v_k^2\}_{k \in \mathbb{N}} \) converges to \( v^2 \) in \( L^1(\Omega) \), and so also in \( D'(\Omega) \). Then \( \{\frac{\partial}{\partial x_i} (v_k^2)\}_{k \in \mathbb{N}} \) converges to \( \frac{\partial}{\partial x_i} (v^2) \) in \( D'(\Omega) \).

Since \( \{2v_k \frac{\partial v_k}{\partial x_i}\}_{k \in \mathbb{N}} \) converges to \( 2v \frac{\partial v}{\partial x_i} \) in \( L^1(\Omega) \), and therefore in \( D'(\Omega) \), we obtain that, for each \( i \), \( \frac{\partial}{\partial x_i} (v^2) = 2v \frac{\partial v}{\partial x_i} \).

**Lemma 2.7.** \( u \in L^\infty(\Omega) \).

**Proof.** Let \( \Omega' \) be a bounded \( C^{0,1} \) domain such that \( \overline{\Omega} \subset \Omega' \), and let \( \tilde{u}, \tilde{a} : \mathbb{R}^n \to \mathbb{R} \) be the extensions by zero of \( u \) and \( a \) respectively. We consider first the case \( n \geq 2 \).

Let \( r = \frac{1-\alpha}{2} \), \( \eta = \frac{2}{r-n} \). Then \( 0 < r < 1 \), \( \eta > 1 \), and \( au^2r \in L^\eta(\Omega) \). Let \( z \in W^{2,\eta}(\Omega') \cap W_0^{1,\eta}(\Omega') \) be the solution of

\[ -\Delta z = 2\tilde{a}u^2r \quad \text{in} \quad \Omega', \]
\[ z = 0 \quad \text{on} \quad \partial \Omega'. \quad (2.8) \]

Let \( \tilde{z} : \mathbb{R}^n \to \mathbb{R} \) be the extension by zero of \( z \) and let \( \varphi \) be a nonnegative function in \( C_c^\infty(\Omega') \). By Remark 2.6 and Lemma 2.3, we have

\[ \int_{\Omega'} \langle \nabla (\tilde{u}^2), \nabla \varphi \rangle = \int_{\Omega'} \langle 2\tilde{u} \nabla \tilde{u}, \nabla \varphi \rangle \]
\[ = \int_{\Omega} 2u (\nabla u, \nabla \varphi) \leq \int_{\Omega} 2 (\nabla (u \varphi), \nabla u) \]
\[ = 2 \int_{\Omega} (au^{1-\alpha} - bu^{p+1}) \varphi \]
\[ \leq 2 \int_{\Omega'} \tilde{a} \tilde{u}^2 r \varphi = \int_{\Omega'} \langle \nabla z, \nabla \varphi \rangle \quad (2.9) \]

For \( \varepsilon > 0 \) let \( h_\varepsilon \) be the mollifiers defined as in the proof of Lemma 2.3. For \( \varepsilon \) small enough we have \( 0 \leq \varphi * h_\varepsilon \in C_c^\infty(\Omega') \), and so, by (2.9),

\[ \int_{\Omega'} \langle \nabla (h_\varepsilon * \tilde{u}^2), \nabla \varphi \rangle = \int_{\Omega'} \langle \nabla (\tilde{u}^2), h_\varepsilon * \nabla \varphi \rangle \]
\[ = \int_{\Omega'} \langle \nabla (\tilde{u}^2), h_\varepsilon * \varphi \rangle \]
\[ \leq \int_{\Omega'} \langle \nabla z, \nabla (h_\varepsilon * \varphi) \rangle \]

where we have used that, since \( h_\varepsilon \) is an even function, the convolution operator with kernel \( h_\varepsilon \) is self-adjoint in \( L^2(\mathbb{R}^n) \). Recall that \( \tilde{z} \in W^{1,\eta}(\mathbb{R}^n) \) and \( \text{supp}(\tilde{z}) \subset \Omega' \).

Also, \( \nabla \tilde{z} = \nabla z \) a.e. in \( \Omega' \), and \( \nabla z = 0 \) a.e. in \( \mathbb{R}^n - \Omega' \). Thus

\[ \int_{\Omega'} \langle \nabla z, \nabla (h_\varepsilon * \varphi) \rangle = \int_{\mathbb{R}^n} \langle \nabla \tilde{z}, \nabla (h_\varepsilon * \varphi) \rangle \]
Let $\psi$ defined as above, we have $u \in L^p(\Omega)$, and so $\lim_{\epsilon \to 0^+} (\Delta u) = \Delta u$. Then $\lim_{\epsilon \to 0^+} (\Delta u) = \Delta u$, and in each case with convergence in $L^p(\Omega)$. Since $u \in L^p(\Omega)$, we obtain $u \in L^p(\Omega)$, and as before, $u^2 \leq z \in \Omega$. Then $u \in L^p(\Omega)$ also in this case.

**Lemma 2.8.**

$$
\int_{\Omega} \langle \nabla u, \nabla (u \varphi) \rangle = \int_{\Omega} \left( a u^{1-\alpha} - b u^{1+p} \right) \varphi
$$

for all $\varphi \in H^1(\Omega) \cap L^\infty(\Omega)$.

**Proof.** Let $\varphi \in H^1(\Omega) \cap L^\infty(\Omega)$. By Lemma 2.7 we have $u \in L^\infty(\Omega)$ and so $u \varphi \in H^1_0(\Omega)$. Thus Lemma 2.3 gives (2.10). $\square$

3. Main results

**Theorem 3.1.** There exists a nonnegative weak solution $0 \neq u \in H^1_0(\Omega) \cap L^\infty(\Omega)$ of problem (1.2).

**Proof.** Let $u$ be the nonnegative minimizer of $J$ considered in the previous section. Let $\psi$ be a nonnegative function in $H^1_0(\Omega) \cap L^\infty(\Omega)$, and let $\epsilon > 0$. Note that $\frac{\psi}{u+\epsilon} \in H^1(\Omega) \cap L^\infty(\Omega)$, and that $\nabla (\frac{\psi}{u+\epsilon}) = \frac{\psi}{(u+\epsilon)^2} \nabla u + \frac{u}{u+\epsilon} \nabla \psi$, and so Lemma 2.8 gives

$$
\epsilon \int_{\Omega} \frac{|\nabla u|^2}{(u+\epsilon)^2} + \int_{\Omega} \frac{u}{u+\epsilon} \langle \nabla u, \nabla \psi \rangle = \int_{\Omega} \left( a u^{1-\alpha} - b u^{1+p} \right) \frac{1}{u+\epsilon} \psi.
$$

(3.1)
Since $\nabla u = 0$ a.e. on the set $\{x \in \Omega : u(x) = 0\}$, and since $au^{1-\alpha} = bu^{1+p} = 0$ on the same set, \(3.1\) can be written as

$$\varepsilon \int_{\{u > 0\}} \frac{|\nabla u|^2}{(u + \varepsilon)^2} + \int_{\{u > 0\}} \frac{u}{u + \varepsilon} \langle \nabla u, \nabla \psi \rangle + \int_{\{u > 0\}} bu^{p} \frac{u}{u + \varepsilon} \psi = \int_{\{u > 0\}} au^{-\alpha} \frac{u}{u + \varepsilon} \psi. \quad (3.2)$$

Also

$$\lim_{\varepsilon \to 0^{+}} \left( \frac{u}{u + \varepsilon} \langle \nabla u, \nabla \psi \rangle \right) = \chi_{\{u > 0\}} \langle \nabla u, \nabla \psi \rangle = \langle \nabla u, \nabla \psi \rangle$$
a.e. in $\Omega$, and $|u/(u + \varepsilon) \langle \nabla u, \nabla \psi \rangle| \leq |\langle \nabla u, \nabla \psi \rangle| \in L^1(\Omega)$, and so Lebesgue’s dominated convergence theorem gives

$$\lim_{\varepsilon \to 0^{+}} \int_{\{u > 0\}} \frac{u}{u + \varepsilon} \langle \nabla u, \nabla \psi \rangle = \int_{\Omega} \langle \nabla u, \nabla \psi \rangle. \quad (3.3)$$

On the other hand, $\lim_{\varepsilon \to 0^{+}} au^{-\alpha} \frac{u}{u + \varepsilon} \psi = au^{-\alpha} \psi$ on the set $\{x \in \Omega : u(x) > 0\}$ and, since $au^{-\alpha} \frac{u}{u + \varepsilon} \psi$ is non-increasing in $\varepsilon$, the monotone convergence theorem gives

$$\lim_{\varepsilon \to 0^{+}} \int_{\{u > 0\}} au^{-\alpha} \frac{u}{u + \varepsilon} \psi = \int_{\{u > 0\}} au^{-\alpha} \psi = \int_{\Omega} au^{-\alpha} \chi_{\{u > 0\}} \psi. \quad (3.4)$$

Also

$$\lim_{\varepsilon \to 0^{+}} \int_{\{u > 0\}} bu^{p} \frac{u}{u + \varepsilon} \psi = \int_{\Omega} bu^{p} \psi \quad (3.5)$$

Then, from (3.2), (3.3), (3.4) and (3.5), we obtain

$$\int_{\Omega} \langle \nabla u, \nabla \psi \rangle + \int_{\Omega} bu^{p} \psi = \lim_{\varepsilon \to 0^{+}} \left( \int_{\{u > 0\}} \frac{u}{u + \varepsilon} \langle \nabla u, \nabla \psi \rangle + \int_{\{u > 0\}} bu^{p} \frac{u}{u + \varepsilon} \right)$$

$$\leq \limsup_{\varepsilon \to 0^{+}} \left( \int_{\{u > 0\}} \frac{\varepsilon |\nabla u|^2}{(u + \varepsilon)^2} + \int_{\{u > 0\}} \frac{u}{u + \varepsilon} \langle \nabla u, \nabla \psi \rangle + \int_{\{u > 0\}} bu^{p} \frac{u}{u + \varepsilon} \right)$$

$$= \limsup_{\varepsilon \to 0^{+}} \int_{\{u > 0\}} au^{-\alpha} \frac{u}{u + \varepsilon} \psi$$

$$= \int_{\Omega} au^{-\alpha} \chi_{\{u > 0\}} \psi.$$
and so
\[
\frac{1}{(1-\alpha)t} \int_{\Omega} a((u + t\psi)^{1-\alpha} - u^{1-\alpha}) 
\leq \int_{\Omega} \langle \nabla u, \nabla \psi \rangle + \frac{1}{(1+p)t} \int_{\Omega} b((u + t\psi)^{1+p} - u^{1+p}) + \frac{t}{2} \int_{\Omega} |\nabla \psi|^2. \tag{3.7}
\]

The mean value theorem gives \((u + t\psi)^{1-\alpha} - u^{1-\alpha} = (1-\alpha)(u + \sigma_t)^{-\alpha}t\psi\) for some measurable function \(\sigma_t\) such that \(0 < \sigma_t < t\psi\). Thus
\[
\frac{1}{(1-\alpha)t} \int_{\Omega} a((u + t\psi)^{1-\alpha} - u^{1-\alpha}) =\]
\[
= \frac{1}{(1-\alpha)t} \int_{\{a>0\} \cap \{\psi>0\}} a((u + t\psi)^{1-\alpha} - u^{1-\alpha}) = \int_{\{a>0\} \cap \{\psi>0\}} a(u + \sigma_t)^{-\alpha} \psi.
\]

Now, \(\lim_{t \to 0^+} a(u + \sigma_t)^{-\alpha} \psi = a u^{-\alpha} \psi\ a.e\ on\ the\ set\ \{a > 0\} \cap \{\psi > 0\}\). Then, by Fatou’s Lemma,
\[
\liminf_{t \to 0^+} \frac{1}{(1-\alpha)t} \int_{\Omega} a((u + t\psi)^{1-\alpha} - u^{1-\alpha}) = \liminf_{t \to 0^+} \int_{\{a>0\} \cap \{\psi>0\}} a(u + \sigma_t)^{-\alpha} \psi 
\geq \int_{\{a>0\} \cap \{\psi>0\}} a u^{-\alpha} \psi \geq \int_{\Omega} a u^{-\alpha} \chi_{\{u>0\}} \psi. \tag{3.8}
\]

Again by the mean value theorem, we have
\[
\frac{1}{(1+p)t} \int_{\Omega} b((u + t\psi)^{1+p} - u^{1+p}) = \int_{\Omega} b(u + \sigma_t)^{p} \psi.
\]

Note that, for \(0 < t < 1\), we have \(0 \leq b(u + \sigma_t) \leq b(u + \psi) = b(u + \psi)^{p+1} \in L^1(\Omega)\). Also, \(\lim_{t \to 0^+} b(u + \sigma_t)^{p} \psi = bu^p \psi\ a.e\ in\ \Omega\). Thus, by Lebesgue’s dominated convergence theorem, we have
\[
\lim_{t \to 0^+} \frac{1}{(1+p)t} \int_{\Omega} b((u + t\psi)^{1+p} - u^{1+p}) = \int_{\Omega} bu^p \psi. \tag{3.9}
\]

Now, from (3.7), (3.8), and (3.9), we obtain
\[
\int_{\Omega} \langle \nabla u, \nabla \psi \rangle + \int_{\Omega} bu^p \psi \geq \int_{\Omega} a u^{-\alpha} \chi_{\{u>0\}} \psi \tag{3.10}
\]

Since \(bu^p \psi \in L^1(\Omega)\), (3.10) implies that \(a u^{-\alpha} \chi_{\{u>0\}} \psi \in L^1(\Omega)\). We apply (3.10), combined with (3.6), to complete the proof. \(\square\)

**Remark 3.2.** It is well known (see e.g., [17]) that, for \(m \in L^\infty(\Omega)\) such that \(|\{x \in \Omega : m(x) > 0\}| > 0\), there exists a unique \(\lambda = \lambda_1(-\Delta, \Omega, m)\) such that the problem
\[
-\Delta \varphi_1 = \lambda m \varphi_1 \ in\ \Omega,
\]
\[
\varphi_1 = 0 \ on\ \partial\Omega,
\]
\[
\varphi_1 > 0 \ in\ \Omega
\]
has a solution \( \varphi_1 \in H^1_0(\Omega) \). This solution is unique up to a multiplicative constant, belongs to \( C^{1,\gamma}((\Omega) \) for some \( 0 < \gamma < 1 \), satisfies that \( |\nabla \varphi| > 0 \) for all \( x \in \partial \Omega \), and there are positive constants \( c_1, c_2 \) such that \( c_1 d_\Omega \leq \varphi \leq c_2 d_\Omega \) in \( \Omega \), where \( d_\Omega : \Omega \to \mathbb{R} \) is the function defined by
\[
d_\Omega(x) = \text{dist}(x, \partial \Omega).
\]
\( \lambda_1 \) and \( \varphi_1 \) are called, respectively, the principal eigenvalue and a positive principal eigenfunction for \( -\Delta \) in \( \Omega \), with Dirichlet boundary condition and weight \( m \).

**Remark 3.3.** It is well known that, under our assumptions on \( \Omega, \alpha, \) and \( a \), the problem
\[
-\Delta \theta = a\theta^{-\alpha} \quad \text{in} \; \Omega, \\
\theta = 0 \quad \text{on} \; \partial \Omega, \\
\theta > 0 \quad \text{in} \; \Omega
\]
has a unique weak solution \( \theta \in H^1_0(\Omega) \). Moreover, \( \theta \) is in \( C(\overline{\Omega}) \), and \( \theta \geq c'd_\Omega \) for some positive constant \( c' \) (see \([12, 3]\)). A computation shows that (in weak sense)
\[
-\Delta(\theta^{p+1}) = -(\alpha + 1) \theta^\alpha \Delta \theta - (\alpha + 1)\alpha \theta^{\alpha - 2} |\nabla \theta|^2 \leq (\alpha + 1)\|a\|_\infty \; \text{in} \; \Omega,
\]
and so we have \( \theta \leq c'd_\Omega^{\frac{1}{p+1}} \) in \( \Omega \), for some constant \( c'' > 0 \).

**Remark 3.4.** Following \([26]\), we say that \( w \in W^{1,2}_{\text{loc}}(\Omega) \) is a subsolution (supersolution) to the problem
\[
-\Delta z = az^{-\alpha} - bz^p \quad \text{in} \; \Omega \tag{3.11}
\]
in the sense of distributions, if, and only if: \( w > 0 \) a.e. in \( \Omega \), \( aw^{-\alpha} - bw^p \in L^1_{\text{loc}}(\Omega) \), and for all nonnegative \( \varphi \in C^\infty_c(\Omega) \), it holds that
\[
\int_\Omega (\nabla w, \nabla \varphi) \leq (\geq) \int_\Omega (aw^{-\alpha} - bw^p) \varphi.
\]
We also say that \( z \in W^{1,2}_{\text{loc}}(\Omega) \) is a solution, in the sense of distributions, of \((3.11)\) if, and only if, \( z > 0 \) a.e. in \( \Omega \), and, for all \( \varphi \in C^\infty_c(\Omega) \) it holds that
\[
\int_\Omega (\nabla z, \nabla \varphi) = \int_\Omega (az^{-\alpha} - bz^p) \varphi.
\]
For subsolutions, supersolutions and solutions defined in the above sense, \([26]\), Theorem 2.4] says that, if \((3.11)\) has a subsolution \( \underline{z} \) and a supersolution \( \overline{z} \) (in the sense of distributions), both in \( L^\infty(\Omega) \), and such that \( 0 < \underline{z}(x) \leq \overline{z}(x) \) a.e. \( x \in \Omega \), and if there exists \( k \in L^\infty(\Omega) \) such that \( |as^{-\alpha} - bs^p| \leq k(x) \) a.e. \( x \in \Omega \) for all \( s \in [\underline{z}(x), \overline{z}(x)] \); then \((3.11)\) has a solution \( z \) in the sense of distributions, and \( z \) satisfies \( \underline{z} \leq z \leq \overline{z} \) a.e. in \( \Omega \).

**Theorem 3.5.** Suppose that \( a \geq \varepsilon b \) for some \( \varepsilon > 0 \). Then there exists a weak solution \( v \in H^1_0(\Omega) \cap L^\infty(\Omega) \) of \((1.2)\) such that \( v \geq cd_\Omega \) in \( \Omega \) for some \( c > 0 \), and \( v \in C^1_{\text{loc}}(\Omega) \cap C(\overline{\Omega}) \).

**Proof.** Suppose that \( a \geq \varepsilon b \) for some \( \varepsilon > 0 \). Let \( \varphi_1 \in H^1_0(\Omega) \) be the positive principal eigenfunction associated to the weight function \( a \), normalized by \( \|\varphi_1\|_\infty = 1 \) (see Remark 3.2). Note that (in weak sense), for \( t \) positive and small enough,
\[
-\Delta(t\varphi_1) \leq a(t\varphi_1)^{-\alpha} - b(t\varphi_1)^p \quad \text{in} \; \Omega. \tag{3.12}
\]
Indeed, \( -\Delta(t\varphi_1) = \lambda_1 t\varphi_1 \), and so \((3.12)\) is equivalent to \((1 - \lambda_1(t\varphi_1)^{1+\alpha})a \geq (t\varphi_1)^{p+\alpha}b \) in \( \Omega \). But, for \( t \) small enough, we have \( b(t\varphi_1)^{p+\alpha} \leq b\varepsilon^{p+\alpha} \leq \frac{1}{2} \varepsilon b \leq \varepsilon b \).
\( \frac{1}{2} a \leq (1 - \lambda_1(t\varphi_1)^{1+\alpha}) a \) in \( \Omega \). Since \( t\varphi_1 > 0 \) in \( \Omega \), it follows that, for such a \( t \), \( t\varphi_1 \) is a subsolution of \( (1.2) \), in the sense of Remark 3.4. On the other hand, let \( \theta \in H^1_0(\Omega) \cap C(\overline{\Omega}) \) be the solution of the problem \( -\Delta \theta = a\theta^{-\alpha} \) in \( \Omega \), \( \theta = 0 \) on \( \partial \Omega \). Since \( \theta \geq c'd\Omega \) in \( \Omega \) for some \( c' > 0 \), we have that \( \theta \) is strictly positive in \( \Omega \), and, by diminishing \( t \) if necessary, we can assume, that \( t\varphi_1 \leq \theta \). Clearly (in weak sense) \( -\Delta \theta \geq a\theta^{-\alpha} - b\theta^p \) in \( \Omega \), and so \( \theta \) is a supersolution of \( (1.2) \), again in the sense of Remark 3.4. Since \( t\varphi_1 \geq c_1 t\delta_{\Omega} \) in \( \Omega \) for some \( c_1 > 0 \), and since \( \theta \leq c''d_{\Omega}^{-\frac{1}{\alpha}} \) in \( \Omega \) for some \( c'' > 0 \), we have \([t\varphi_1(x), \theta(x)] \in [c_1 t\delta_{\Omega}(x), c''d_{\Omega}^{-\frac{1}{\alpha}}(x)]\) for \( x \in \Omega \).

Therefore a.e. \( x \in \Omega \), for all \( s \in [t\varphi_1(x), \theta(x)] \), the following holds

\[ |as^{-\alpha} - bs^p| \leq \|a\|_{\infty}(c_1 t)^{-\alpha}d_{\Omega}^{-\alpha} + \|b\|_{\infty}(c'')^p d_{\Omega}^{-\frac{1}{\alpha}}(x) := k(x). \]

Since \( k \in L_{loc}^\infty(\Omega) \), 

\[ \text{Remark 3.4}, \]  

(see Remark 3.4), says that there exists \( v \in W^{1,2}_{loc}(\Omega) \) such that \( t\varphi_1 \leq v \leq \theta \) in \( \Omega \), and such that, for any \( \varphi \in C^\infty_c(\Omega) \),

\[ \int_{\Omega} (\nabla v \cdot \nabla \varphi) = \int_{\Omega} (av^{-\alpha} - bv^p)\varphi. \quad (3.13) \]

Note that \( v \in H^1_0(\Omega) \): Indeed, let \( \Omega' \) be a subdomain of \( \Omega \) such that \( \overline{\Omega'} \subset \Omega \). Since \( v \geq c'd\Omega \) in \( \Omega \) for some \( c'' > 0 \), we have \( av^{-\alpha} - bv^p \in L^\infty(\Omega') \). Therefore, from (3.13), a density argument, and Lebesgue’s dominated convergence theorem give that, for any \( \varphi \in H^1_0(\Omega') \), it holds

\[ \int_{\Omega'} (\nabla v \cdot \nabla \varphi) = \int_{\Omega'} (av^{-\alpha} - bv^p)\varphi. \quad (3.14) \]

Let \( \varepsilon > 0 \). Since \( v \leq \theta \leq c''d_{\Omega}^{-\frac{1}{\alpha}} \) for some \( c'' > 0 \), we have that \( \text{supp}(v - \varepsilon)^+ \subset \Omega' \) for some subdomain \( \Omega' \) such that \( \overline{\Omega'} \subset \Omega \). Also \( (v - \varepsilon)^+ \in H^1(\Omega) \) and so \( (v - \varepsilon)^+ \in H^1_0(\Omega) \). Thus, from (3.14), we obtain

\[ \int_{\Omega} \chi_{\{v > \varepsilon\}} \nabla v \cdot \nabla v = \int_{\Omega'} \nabla v \cdot \nabla (v - \varepsilon)^+ \]

\[ = \int_{\Omega'} (av^{-\alpha} - bv^p)(v - \varepsilon)^+ \]

\[ = \int_{\Omega} (av^{-\alpha} - bv^p)(v - \varepsilon) \chi_{\{v > \varepsilon\}}. \]

The monotone convergence theorem gives

\[ \lim_{\varepsilon \to 0^+} \int_{\Omega} \chi_{\{v > \varepsilon\}} \nabla v \cdot \nabla v = \int_{\Omega} \nabla v \cdot \nabla v \]

and, since \( av^{-\alpha} - bv^p \in L^1(\Omega) \), and \( v \in L^\infty(\Omega) \), Lebesgue’s dominated convergence theorem gives

\[ \lim_{\varepsilon \to 0^+} \int_{\Omega} (av^{-\alpha} - bv^p)(v - \varepsilon) \chi_{\{v > \varepsilon\}} = \int_{\Omega} (av^{1-\alpha} - bv^{1+p}). \]

Taking limits in (3.15), we obtain

\[ \int_{\Omega} \nabla v \cdot \nabla v = \int_{\Omega} (av^{1-\alpha} - bv^{1+p}) < \infty. \]

Thus \( v \in H^1(\Omega) \) and, since \( t\varphi_1 \leq v \leq \theta \), we have \( v \in H^1_0(\Omega) \). Note also that \( av^{-\alpha} - bv^p \in L^1(\Omega) \) and so, again by a density argument, and applying Lebesgue’s
dominated convergence theorem, we conclude that (3.13) holds for all $\varphi$ in $H^{1}_0(\Omega) \cap L^\infty(\Omega)$.

Let $\Omega'$ be an arbitrary subdomain of $\Omega$ such that $\overline{\Omega'} \subset \Omega$, and let $\Omega''$ be such that $\overline{\Omega''} \subset \overline{\Omega'} \subset \Omega$. Since $v \in L^\infty(\Omega'')$ and $(a_1v - b_1v)|_{\Omega''} \in L^\infty(\Omega'')$, we have $v|_{\Omega''} \in W^{2,s}(\Omega'')$ for all $s \in [1, \infty)$ (see, e.g., Proposition 4.1.2 in [8]) and so $v|_{\Omega''} \in C^1(\Omega'')$. Thus $v \in C^1_{loc}(\Omega)$ and, since $1/\varphi_1 \leq v \leq \varphi$, $v$ is continuous on $\partial \Omega$. \hfill $\Box$

**Example 3.6.** Let $\Omega = (0, 2\pi)$, $\alpha = 1/3$, and $p \in (0, 1/5)$. Let $a$ and $b$ be the functions defined on $\Omega$ by $a = 2(1 - \cos(2x))/\sqrt{\sin^2(x)}$, $b(x) = 2|\sin^2(x)|^{-p}$. Then $a \geq 0$, $b \geq 0$, $0 \neq a \in L^\infty(\Omega)$ and $b \in L^{2/p}(\Omega)$. Consider now the following three functions in $C^1(\Omega)$: $u(x) = \sin^2(x)\chi_{(0, \pi)}$, $v(x) = \sin^2(x)\chi_{(0, 2\pi)}$, and $w(x) = \sin^2(x)\chi_{(\pi, 2\pi)}$. A computation shows that $u$, $v$, and $w$ are all weak solutions of (1.2) (v is in fact a classical solution). Therefore (without additional assumptions on $a$ and $b$) uniqueness is not to be expected for nonnegative nontrivial weak solutions of (1.2). Notice that $w \equiv 0$ on $(0, \pi)$. Note also that $v(x) > 0$ for $x \in \Omega - \{\pi\}$ and $v(\pi) = 0$, therefore, by Theorem 3.8 below, there is no continuous and strictly positive solution to (1.2).

**Example 3.7.** Let $\Omega = (0, 2)$, let $\alpha \in (0, 1)$, $p \in (0, 1)$, let $b := \chi_{(0, 1)}$ and let $a := \chi_{(1, 1 + \delta)}$, with

$$0 < \delta \leq \left(\frac{1 - \alpha}{2}\right)^{1/\alpha} \left(\frac{2}{p + 1}\right)^{1/(p + 1)} \frac{1}{\Gamma(1 + \frac{1}{p})}.$$

Let us show that the problem

$$-u'' = au^{-\alpha} - bu^p \quad \text{in} \quad \Omega,$$

$$u = 0 \quad \text{on} \quad \partial \Omega$$

has no weak solution $u \in H^1_0(\Omega)$ such that $u > 0$ a.e. in $\Omega$. Let us suppose, for the sake of contradiction, that $u$ is a weak solution such that $u > 0$ a.e. in $\Omega$. Since $H^1_0(\Omega) \subset C^\gamma(\Omega)$ for some $\gamma \in (0, 1)$, we have $u \in C^\gamma(\Omega)$ for such a $\gamma$. Throughout this example, unless there is risk of confusion, the restrictions of $u$ to $(0, 1)$, $(1, 1 + \delta)$, and $(1 + \delta, 2)$, will be still denoted by $u$. Since $u$ belongs to $C^\gamma([0, 1])$, and $|u(x) - u(y)| \leq |x - y|^\gamma$ for any $x, y \in [0, 1]$, we have $u^p \in C^{\gamma p}([0, 1])$. Let $A = u(1)$. Since

$$-u'' = -u^p \quad \text{in} \quad (0, 1),$$

$$u(0) = 0,$$  \hspace{1cm} (3.17)

$$u(1) = A$$

we have that $u$ is a classical solution of (3.17) that belongs to $C^2([0, 1]) \cap C([0, 1])$ and so $-u'' = -u^p$ in $[0, 1]$. (see, e.g., [23] Theorem 6.14). Note also that

$$u(x) \geq \left(\frac{1 - p}{2}\right)^{2/p} \left(\frac{2}{p + 1}\right)^{1/(p + 1)} x^{-\frac{p}{p + 1}} \quad \text{for all} \quad x \in [0, 1].$$  \hspace{1cm} (3.18)

Indeed, multiplying (3.17) by $u'$ we obtain $\frac{1}{2}((u')^2)' = \frac{1}{p + 1}(u^{p + 1})'$ on $[0, 1]$, and so

$$\frac{1}{2}(u'(x))^2 - \frac{1}{p + 1}u(x)^{p + 1} = \frac{1}{2}(u'(0))^2 \geq 0 \quad \text{for all} \quad x \in [0, 1].$$

Thus

$$u'(x)^2 \geq \frac{2}{p + 1}u^{p + 1} \quad \text{in} \quad [0, 1].$$  \hspace{1cm} (3.19)
As \( u \geq 0 \) on \([0,1]\) and \( u(0) = 0 \), we have \( u'(0) \geq 0 \). Observe also that (3.17) implies \( u'' \geq 0 \) on \([0,1]\), and so \( u \) is a convex function on \([0,1]\). Thus \( u' \) is nondecreasing on \([0,1]\) and, since \( u'(0) \geq 0 \), we have \( u' \geq 0 \) in \([0,1]\), and so, from (3.19), we conclude

\[
u' \geq \left( \frac{2}{p+1} \right)^{1/2} u^\frac{p+1}{p} \quad \text{in } [0,1].
\] (3.20)

If \( u(x) = 0 \) for some \( x \in (0,1) \) we would have \( u(x) = 0 \) for all \( x \in (0,x) \), which contradicts the assumption that \( u > 0 \) a.e. in \( \Omega \). Thus \( u(x) > 0 \) for all \( x \in [0,1] \), therefore (3.20) can be rewritten as

\[
\left( \frac{2}{p+1} \right)^{1/2} u' \geq \left( \frac{2}{p+1} \right)^{1/2} \text{ on } [0,1].
\]

By integrating this inequality over \((0,x)\) we obtain

\[
\frac{2}{1-p} (u(x))^{\frac{1-p}{p}} \geq \left( \frac{2}{p+1} \right)^{1/2} \text{ for all } x \in [0,1],
\]

and so (3.18) holds. In particular we have

\[
u(1) \geq \left( \frac{1-p}{2} \right)^{\frac{2}{p}} \left( \frac{2}{p+1} \right)^{\frac{p}{1-p}} x^{\frac{2}{p}} \quad \text{(3.21)}\]

and then, by (3.20),

\[
u'(1) \geq \left( \frac{2}{p+1} \right)^{\frac{1-p}{p}} \left( \frac{1-p}{2} \right)^{\frac{p}{1-p}} \quad \text{(3.22)}\]

Consider now the restriction of \( u \) to \((1,1+\delta)\): \( u \in H^1(1,1+\delta) \subset C([1,1+\delta]) \), and solves

\[-u'' = u^{-\alpha} \quad \text{in } (1,1+\delta) \]

\[u(1) \geq 0, \quad u(1+\delta) \geq 0.
\]

Let \( \zeta \in H^1_0(1,1+\delta) \subset C([1,1+\delta]) \) be the solution to the problem

\[-\zeta'' = \zeta^{-\alpha} \quad \text{in } (1,1+\delta) \]

\[\zeta > 0 \quad \text{in } (1,1+\delta) \]

\[\zeta(1) = 0, \quad \zeta(1+\delta) = 0.
\]

Observe that \( u \geq \zeta \) on \((1,1+\delta)\). To prove this, suppose, for the sake of contradiction, that \( \{x \in (1,1+\delta): u(x) < \zeta(x)\} \neq \emptyset \), and let \( U \) be one of its connected components. Note that \( U \) is an open interval, since \( u \) and \( \zeta \) are continuous on \((1,1+\delta)\). Since \(-\zeta'' = \zeta^{-\alpha} \leq u^{-\alpha} = -u'' \) on \( U \), and \( \zeta = u \) on \( \partial U \), the maximum principle gives \( \zeta \leq u \) on \( U \), which is a contradiction. Thus \( u \geq \zeta \) on \((1,1+\delta)\) as claimed.

Recall that there exists \( c > 0 \) such that \( \zeta \geq cd \) on \((1,1+\delta)\), where \( d(x) = \text{dist}(x, \partial(1,1+\delta)) \) for all \( x \in (1,1+\delta) \) (see Remark 3.3); therefore \( u \geq cd \) on \((1,1+\delta)\). Note also that \( u(1+\delta) > 0 \). If not, since \( u(2) = 0 \) and \( u'' = 0 \) in \((1,2)\), we would have \( u = 0 \) in \((1,2)\); which would contradict \( u > 0 \) a.e. in \( \Omega \). Since \( u(1) > 0 \), \( u(1+\delta) > 0 \), and \( u \geq cd \) on \((1,1+\delta)\), it follows that \( u(x) > 0 \) for any \( x \in [1,1+\delta] \), and, since \( u \) is continuous on \([1,1+\delta]\), we have \( u \geq \text{const} > 0 \) on \([1,1+\delta]\). Now

\[
|u^{-\alpha}(x) - u^{-\alpha}(y)| = (u(x)u(y))^{-\alpha}|u(x)^\alpha - u(y)^\alpha|
\]

\[
\leq (u(x)u(y))^{-\alpha}|u(x) - u(y)|^\alpha
\]

and so, since \( u \in C^\gamma(\overline{\Omega}) \), we have \(-u'' \in C^{\alpha\gamma}([1,1+\delta])\). Let \( A = u(1) \), \( B = u(1+\delta) \).

Since \( u \) solves

\[-u'' = u^{-\alpha} \quad \text{in } (1,1+\delta) \]

\[u(1) = A, \quad u(1+\delta) = B, \tag{3.23}
\]
it follows that $u$ is a classical solution of \eqref{3.23} that belongs to $C^2([1,1 + \delta]) \cap C([1,1 + \delta])$ (see [23, Theorem 6.14]).

On the other hand, since $u'' = 0$ on $(1 + \delta, 2)$ and $u(2) = 0$, we have
\begin{equation}
  u(x) = \frac{u(1 + \delta)}{1 - \delta} (2 - x) \quad \text{for all } x \in (1 + \delta, 2) \tag{3.24}
\end{equation}

Since $u^{-\alpha} \in C^\infty([1,1 + \delta])$ and $u \in H^1(\Omega) \subset C(\overline{\Omega})$, we have $au^{-\alpha} - bu^p \in L^2(\Omega)$, and thus, from \eqref{3.16}, it follows that $u \in W^{2,2}(\Omega) \subset C^1(\overline{\Omega})$. Multiplying \eqref{3.23} by $u'$ we obtain
\begin{equation}
  \left( \frac{1}{2} (u')^2 \right)' = -\frac{1}{1 - \alpha} (u^{-\alpha})' \quad \text{on } (1,1 + \delta) \tag{3.25}
\end{equation}

and so $\frac{1}{2} (u')^2 + \frac{1}{1 - \alpha} u^{-\alpha} = \text{const} = \frac{1}{2} (u(1'))^2 + \frac{1}{1 - \alpha} u(1)^{-\alpha}$. Therefore, for $x \in (1,1 + \delta)$: $u'(x) = 0$ if, and only if, $\frac{1}{1 - \alpha} u^{-\alpha}(x) = \frac{1}{2} (u'(1))^2 + \frac{1}{1 - \alpha} u(1)^{-\alpha}$. If there were no $x$ in $(1,1 + \delta)$ such that $\frac{1}{1 - \alpha} u^{-\alpha}(x) = \frac{1}{2} (u'(1))^2 + \frac{1}{1 - \alpha} u(1)^{-\alpha}$, we would have $u'(x) \neq 0$ for all $x \in (1,1 + \delta)$, which would imply that $u'(x) > 0$ for all $x \in (1,1 + \delta)$ (since $u'$ is continuous on $[1,1 + \delta]$, and since $u'(1) > 0$). Thus $u'(1 + \delta) \geq 0$, but, by \eqref{3.24}, $u'(1 + \delta) = -\frac{(u(1 + \delta))'}{u(1 + \delta)} < 0$, which is a contradiction. Therefore $\{ x \in (1,1 + \delta) : \frac{1}{1 - \alpha} u^{-\alpha}(x) = \frac{1}{2} (u'(1))^2 + \frac{1}{1 - \alpha} u(1)^{-\alpha} \} \neq \emptyset$; let $x_1$ be its infimum. Since $u$ is continuous, $x_1$ is a minimum, therefore we have $u(x_1) = (\frac{1}{1 - \alpha} (u'(1))^2 + u(1)^{-\alpha})^{\frac{1}{1 - \alpha}}$. Note that $u'(x) > 0$ for all $x \in [1, x_1)$. Moreover, \eqref{3.23} gives that $u$ is concave on $[1,1 + \delta]$, and so $\frac{u(x_1) - u(1)}{x_1 - 1} \leq u'(1)$. Then, recalling \eqref{3.22},

\begin{align*}
  x_1 - 1 & \geq \frac{u(x_1) - u(1)}{u'(1)} = \frac{(\frac{1}{1 - \alpha} (u'(1))^2 + u(1)^{-\alpha})^{\frac{1}{1 - \alpha}} - u(1)}{u'(1)} \\
  & \geq \left( \frac{1 - \alpha}{2} (u'(1))^2 \right)^{\frac{1}{1 - \alpha}} + (u(1)^{1 - \alpha})^{\frac{1}{1 - \alpha}} - u(1) \quad \frac{u'(1)}{u'(1)} \tag{3.26}
\end{align*}

\begin{align*}
  & = \left( \frac{1 - \alpha}{2} (u'(1))^2 \right)^{\frac{1}{1 - \alpha}} \left( u'(1) \right)^{\frac{1 + \alpha}{1 - \alpha}} \\
  & \geq \left( \frac{1 - \alpha}{2} \right)^{\frac{1}{1 - \alpha}} \left( \frac{2}{p + 1} \right)^{\frac{1}{p}} \left( \frac{1 - p}{2} \right)^{\frac{1}{1 - p}} \geq \delta,
\end{align*}

which contradicts $x_1 < 1 + \delta$.

**Theorem 3.8.** There is at most one weak solution $v \in H^1(\Omega) \cap L^\infty(\Omega)$ of \eqref{1.2} such that $v(x) > 0$ a.e. in $\Omega$; and, if it exists, it satisfies $v \geq u$ for any other nonnegative weak solution $u \in H^1(\Omega) \cap L^\infty(\Omega)$ of \eqref{1.2}.

**Proof.** Since $s \to f(s) := as^{-\alpha} - bs^p$ is nondecreasing, the uniqueness assertion of the theorem follows from a standard argument: If $w$ is another solution which is positive a.e. in $\Omega$, take $\varphi := v - w$ as a test function in the weak form of the equation
\begin{align*}
  -\Delta(v - w) &= f(v) - f(w) \quad \text{in } \Omega, \\
  v - w &= 0 \quad \text{on } \partial \Omega
\end{align*}

to obtain $\int_\Omega |\nabla(v - w)|^2 = \int_\Omega (f(v) - f(w))(v - w) \leq 0$, which implies $v = w$. 

Let $u \in H^1_0(\Omega) \cap L^\infty(\Omega)$ be a nonnegative solution of 1.2. Therefore, for any $\varphi \in H^1_0(\Omega) \cap L^\infty(\Omega)$, we have

$$
\int_\Omega (\nabla (u - v), \nabla \varphi) = \int_\Omega (au - \alpha \chi_{\{u>0\}} - bu^p - (av^{-\alpha} - bv^p)) \varphi
\quad (3.26)
$$

$$
= \int_{\{u>0\}} (f(u) - f(v)) \varphi + \int_{\{u=0\}} (-av^{-\alpha} + bv^p) \varphi.
$$

Now, we take $\varphi = (u - v)^+$. Since $v > 0$ a.e. in $\Omega$, we have

$$
\int_{\{u=0\}} (-av^{-\alpha} + bv^p)(u - v)^+ = 0.
$$

Thus, from (3.26), we obtain $\int_\Omega |\nabla (u - v)^+|^2 \leq 0$, and so $u \leq v$ in $\Omega$. □

References


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