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OPTIMAL HARVESTING IN DIFFUSIVE POPULATION MODELS WITH SIZE RANDOM GROWTH AND DISTRIBUTED RECRUITMENT

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Abstract. In this article, we consider an optimal harvesting control problem for a spatial diffusion population system, which incorporates individual's random growth of size and distributed style of recruitment. The existence and uniqueness of nonnegative solutions to this practical model is established by means of Banach's fixed point theorem. The continuous dependence of population density on the harvesting effort is analyzed. The optimal harvesting strategies are discussed through normal cone and adjoint techniques. Some conditions are presented to assure that there is only one optimal policy.

1. Introduction

Structured population models provide the connection between the population level dynamics and individual level vital rates. It has attracted a lot of attention from a rather diverse group of scientific researchers in biology and mathematics [8, 19]. Dynamic analyses on the size-structured and age-structured population models are presented in [1, 10]. Optimal control and optimization analyses have also been considered extensively from the economical and ecological points of view [1, 2, 3, 9, 16, 20]. To the optimal harvesting problems, there are quite many meaningful results on the age-structured population systems with or without spatial diffusion [1, 2, 17, 18, 22] and the references therein.

For more realistic biological significance of modeling, Hadeler [15] proposed structured population models with diffusion in the size-space. The biological motivation is that the diffusion allows for "stochastic noise" to be incorporated in the models, namely, the stochastic fluctuations around the tendency to growth. Faugeras and Maury [14] established an advection-diffusion-reaction model of fish with length (i.e. size structure) and plane position (i.e. spatial structure) distributions. The diffusion-convection process with respect to size is also called the random growth process [7]. For these models, the existence and asymptotic behaviors of solutions were shown by semigroup theories in [13] and Hopf bifurcation properties with the modified Ricker type birth function were studied in [7]. Recently, some numerical approximate solutions by the method of lines were investigated in [6]. Up to

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present, it seems that very few results on the optimal harvesting control problems is presented for these biological models with the size random growth.

Inspired by the above results, we are concerned with the optimal harvesting for the following diffusive population model with the size random growth (Let $Q := \Omega \times (S_0, S_1) \times (0, T)$, and $\Sigma := \partial\Omega \times (S_0, S_1) \times (0, T)$):

$$\begin{aligned} \partial_t p - k\Delta p = \partial_s(d(s)\partial_s p - g(s)p) - \mu p - up \\ + \int_{S_0}^{S_1} \beta(x, s, t, \hat{s}) p(x, \hat{s}, t)\,\mathrm{d}\hat{s}, \quad \text{in } Q, \end{aligned} \tag{1.1}$$

$$\frac{\partial p}{\partial s}(x, S_0, t) = \frac{\partial p}{\partial s}(x, S_1, t) = 0, \quad \forall (x, t) \in \Omega \times (0, T), \tag{1.2}$$

$$\frac{\partial p}{\partial n}(x, s, t) = 0, \quad \text{on } \Sigma, \tag{1.3}$$

$$p(x, s, 0) = p^0(x, s), \quad \forall (x, s) \in \Omega \times (S_0, S_1), \tag{1.4}$$

where Δ stands for the Laplace operator with respect to the spatial variable x, and $\Omega \subset R^N (N \le 3)$ is a bounded open domain with a boundary $\partial\Omega$ smooth enough, $k > 0$ is the spatial diffusion coefficient, $\mu := \mu(x, s, t)$ denotes the death rate, and $u := u(x, s, t)$ is the harvesting effect which can be controlled by the outside force. The constants S_0 and S_1 stand for the minimal and maximal sizes of individuals, respectively. $T > 0$ is the finite horizon of control, independent of any initial-boundary conditions. $p(x, s, t)$ denotes the population density of individuals of size $s \in [S_0, S_1]$ at time $t \in [0, T]$ at location $x \in \Omega$. The homogeneous Neumann boundary conditions are introduced with respect to the N-dimension spatial variable x and 1-dimension size variable s.

The individuals' size random growth process is described here by the term $\partial_s(d(s)\partial_s p - g(s)p)$ in the Eq. (1). Here, $d(s) > 0$ on $[S_0, S_1]$ stands for the size-specific diffusion coefficient, and $g(s)$ is the growth modulus. Similar to [13], we choose the non-local integral term in (1) as the recruitment process. The distributed recruitment means that individuals may be recruited into the population at different sizes with the rate $\beta(x, s, t, \hat{s})$. This choice is different from the one given in [14].

The aim of this article is to study the optimal harvesting control problem maximize

$$J(u) := \int_0^T \int_{S_0}^{S_1} \int_\Omega [wp^u u - \frac{1}{2}\rho u^2]\,\mathrm{d}x\,\mathrm{d}s\,\mathrm{d}t, \tag{1.5}$$

subject to

$$u \in \mathcal{U} = \{v \in L^2(Q) : 0 \le \zeta_1(x, s, t) \le v(x, s, t) \le \zeta_2(x, s, t) \text{ a.e. in } Q\},$$

where $w := w(x, s, t)$ denotes the economic value of an individual of size $s \in [S_0, S_1]$ at time $t \in [0, T]$ at $x \in \Omega$. $\rho > 0$ is a cost factor for implementing the harvesting policy u, and $p^u(x, s, t)$ is the solution of the system (1.1)–(1.4) corresponding to u.

We assume that the following conditions hold throughout this article:

- (H1) $g \in C^1[S_0, S_1]$, $g(S_0) > 0$;
- (H2) $\mu \in L^\infty_{loc}(\overline{\Omega} \times [S_0, S_1) \times [0, T])$, $\mu(x, s, t) \ge 0$, a.e. in Q;
- (H3) $\beta(x, s, t, \hat{s}) \ge 0$ a.e. in $\Omega \times (S_0, S_1) \times (0, T) \times (S_0, S_1)$, $\beta \in L^\infty$ and let $\overline{\beta} := \|\beta\|_\infty$;
- (H4) $d(s) \ge d_1 > 0$ a.e. in (S_0, S_1), $d \in L^\infty(S_0, S_1)$ and let $\overline{d} := \|d\|_\infty$;

(H5) $p^0(x,s) \geq 0$ a.e. in $\Omega \times (S_0, S_1)$, $p_0 \in L^2(\Omega \times (S_0, S_1))$;
(H6) $w(s,t,x) > 0$ a.e. in Q from (1.5), $w \in L^\infty(Q)$ and let $W := \|w\|_\infty$.

The rest of this article is organized as follows. In Section 2, we deal with the existence and uniqueness of solutions of the state system (1.1)–(1.4) with the given parameters. Then we display the optimal strategies by feedback laws in Section 3, and establish the existence of optimal harvesting control and a unique optimal policy in Section 4. A short conclusion is given in Section 5.

2. Existence and uniqueness of solution of the state system

In this section, we establish the existence and uniqueness of a positive weak solution to the state system (1.1)–(1.4).

Let $\mathcal{Q} = \Omega \times (S_0, S_1)$ be an open subset of R^{N+1}. Then $Q = \mathcal{Q} \times (0,T)$. We regard $p(x,s,\cdot)$ as an element of the functional space $H := L^2(\mathcal{Q})$. For any $t \in [0,T]$ we have

$$\int_{S_0}^{S_1} \int_\Omega |p(x,s,t)|^2 \,\mathrm{d}x\,\mathrm{d}s < \infty. \tag{2.1}$$

Denote by $H^1(\mathcal{Q})$ the Sobolev space $W^{1,2}(\mathcal{Q})$ endowed with the norm

$$\|p\|_{H^1(\mathcal{Q})} = \Big(\int_{S_0}^{S_1} \int_\Omega (p^2 + |\nabla_x p|^2 + |\partial_s p|^2) \,\mathrm{d}x\,\mathrm{d}s \Big)^{1/2}. \tag{2.2}$$

Let $H^1(\mathcal{Q})^*$ denote the dual of $H^1(\mathcal{Q})$. Then we have the chain of dense and continuous embeddings

$$H^1(\mathcal{Q}) \hookrightarrow H \hookrightarrow H^1(\mathcal{Q})^*, \tag{2.3}$$

and any $F \in H^1(\mathcal{Q})^*$ can be continuously extended to H if and only if there is some $f \in H$ such that

$$F(p) = \int_{S_0}^{S_1} \int_\Omega f \cdot p \,\mathrm{d}x\,\mathrm{d}s = (f,p)_H, \quad \forall p \in H^1(\mathcal{Q}). \tag{2.4}$$

Definition 2.1. We denote by $W(0,T)$ the linear space of all $p \in L^2(0,T;H^1(\mathcal{Q}))$ which has a distributional derivative $p' \in L^2(0,T;H^1(\mathcal{Q})^*)$, equipped with the norm

$$\|p\|_{W(0,T)} = \Big(\int_0^T \Big(\|p\|^2_{H^1(\mathcal{Q})} + \|p'(t)\|^2_{H^1(\mathcal{Q})^*} \Big) \,\mathrm{d}t \Big)^{1/2}. \tag{2.5}$$

The space $W(0,T) = \{p \in L^2(0,T;H^1(\mathcal{Q})) : \frac{dp}{dt} \in L^2(0,T;H^1(\mathcal{Q})^*)\}$ is a Hilbert space with the inner product

$$(p,q)_{W(0,T)} = \int_0^T (p,q)_{H^1(\mathcal{Q})} \,\mathrm{d}t + \int_0^T (p'(t), q'(t))_{H^1(\mathcal{Q})^*} \,\mathrm{d}t. \tag{2.6}$$

From [20], we have

$$W(0,T) \hookrightarrow C([0,T];H). \tag{2.7}$$

For the sake of convenience, we change the unknown function p in the equation (1.1)) by $\hat{p} = e^{-\theta t} p$ (θ is to be determined latter). Then we have the following proposition.

Proposition 2.2. *The function p satisfies the state system* (1.1)–(1.4) *if and only if $\hat{p}$ is a solution to the equation*

$$\partial_t\hat{p}-k\Delta\hat{p}=\partial_s(d(s)\partial_s\hat{p}-g(s)\hat{p})-\mu\hat{p}-u\hat{p}-\theta\hat{p}+\int_{S_0}^{S_1}\beta(x,s,t,\hat{s})\hat{p}(x,\hat{s},t)\,\mathrm{d}\hat{s}, \tag{2.8}$$

endowed with the analogical initial-boundary conditions:

$$\frac{\partial\hat{p}}{\partial s}(x,S_0,t)=\frac{\partial\hat{p}}{\partial s}(x,S_1,t)=0,\quad \forall(x,t)\in\Omega\times(0,T), \tag{2.9}$$

$$\frac{\partial\hat{p}}{\partial n}(x,s,t)=0,\quad \text{on } \Sigma, \tag{2.10}$$

$$\hat{p}(x,s,0)=p^0(x,s),\quad \forall(x,s)\in\Omega\times(S_0,S_1). \tag{2.11}$$

Multiplying the equation (2.8) by a function q and using integration by parts on $\mathcal{Q}$, we arrive at the following definition.

Definition 2.3. The bilinear mapping $a(t;\cdot,\cdot):H^1(\mathcal{Q}))\times H^1(\mathcal{Q}))\to\mathbb{R}$ for $t\in[0,T]$, is defined as

$$a(t;\hat{p},q)=\int_{\mathcal{Q}}(k\nabla\hat{p}\cdot\nabla q+d(\partial_s\hat{p})(\partial_s q)+g(\partial_s\hat{p})q+(\mu+u+\theta+g_s)\hat{p}q)\,\mathrm{d}x\,\mathrm{d}s. \tag{2.12}$$

According to classical discussions (see, e.g. [21]), we cite the following result and omit the proof.

Lemma 2.4. *For almost every $t\in(0,T)$, $a(t;\hat{p},q)$ is continuous on $H^1(\mathcal{Q})\times H^1(\mathcal{Q})$, and for θ large enough, $a(t;\hat{p},q)$ is coercive on $H^1(\mathcal{Q})$. There exist two constants $M>0$ and $\delta>0$, depending on k, $\overline{d}$, $\|\mu\|_\infty$, $|g_s|_{\max}$, d_1, and θ, such that*

$$|a(t;\hat{p},q)|\le M\|\hat{p}\|_{H^1(\mathcal{Q})}\|q\|_{H^1(\mathcal{Q})},\quad \forall\hat{p},q\in H^1(\mathcal{Q}), \tag{2.13}$$

$$a(t;\hat{p},\hat{p})\ge\delta\|\hat{p}\|^2_{H^1(\mathcal{Q})},\quad \forall\hat{p}\in H^1(\mathcal{Q}). \tag{2.14}$$

Now we are ready to define the weak solutions $\hat{p}$ to (2.8)–(2.11).

Definition 2.5. A function $\hat{p}\in W(0,T)$ is said to be a solution of (2.8)–(2.11) if the following variational equation holds for all $q\in L^2(0,T;H^1(\mathcal{Q}))$:

$$\int_0^T\Big(\frac{d\hat{p}}{dt},q\Big)_H dt+\int_0^T a(t;\hat{p},q)dt=\int_0^T(\mathcal{I}\hat{p},q)_H dt, \tag{2.15}$$

and

$$\hat{p}(x,s,0)=p^0(x,s)\text{ in }\Omega\times(S_0,S_1),$$

where $\mathcal{I}\hat{p}:=\int_{S_0}^{S_1}\beta(x,s,t,\hat{s})\hat{p}(x,\hat{s},t)\,\mathrm{d}\hat{s}$.

Lemma 2.6. *System* (2.8)–(2.11) *has a unique non-negative bounded solution $\hat{p}\in W(0,T)$.*

Proof. Firstly, we define an operator $\mathcal{A}$ by freezing the integral term $\mathcal{I}\hat{p}$, and then apply the Banach fixed-point theorem to $\mathcal{A}$. So it is clear to see that the fixed point is our desired solution.

Let $\hat{p}^*$ be fixed in $W(0,T)$ and replace $(\mathcal{I}\hat{p},q)_H$ by $(\mathcal{I}\hat{p}^*,q)_H$ in (2.15). For all $q\in L^2(0,T;H^1(\mathcal{Q}))$ and some appropriate T, the problem reduces to the following standard linear problem in the sense of distribution:

$$\begin{gathered}\Big(\frac{d\hat p}{dt},q\Big)_H + a(t;\hat p,q) = (\mathcal{I}\hat p^*,q)_H,\\ \hat p(x,s,0) = p^0(x,s).\end{gathered} \tag{2.16}$$

We get a unique solution $\hat p \in W(0,T)$ of the problem (2.16) by the classical discussion. So this solution defines an operator $\mathcal{A}$ on $W(0,T)$ and $\mathcal{A}\hat p^* = \hat p$.

Taking $q = \hat p$ in (2.16), integrating it on $[0,t]$, using the coerciveness of $a(t;p,q)$ in (15) and Cauchy-Schwarz inequality, we have

$$\int_0^t \Big(\frac12\frac{d}{dt}\|\hat p(\tau)\|_H^2 + \delta\|\hat p(\tau)\|_{H_1}^2\Big)d\tau \le \int_0^t \|\mathcal{I}\hat p^*(\tau)\|_H\cdot\|\hat p(\tau)\|_H d\tau. \tag{2.17}$$

By using Young's inequality, for all $\alpha > 0$ we obtain

$$\frac12\left(\|\hat p(t)\|_H^2 - \|p^0\|_H^2\right) + \delta\int_0^t \|\hat p(\tau)\|_{H_1}^2 d\tau \le \int_0^t \frac{1}{\alpha}\|\mathcal{I}\hat p^*(\tau)\|_H^2\, d\tau + \int_0^t \alpha\|\hat p(\tau)\|_H^2\, d\tau. \tag{2.18}$$

Choosing $\alpha = \delta$, by the norm definition of H_1 in (2.2) and the assumption (H3), we derive that

$$\begin{aligned}\|\hat p(t)\|_H^2 &\le \frac{2}{\delta}\int_0^t \|\mathcal{I}\hat p^*(\tau)\|_H^2 d\tau + 2\|p^0\|_H^2\\ &\le \frac{2}{\delta}\int_0^T\int_{\mathcal{Q}}\Big(\int_{S_0}^{S_1}\beta(x,s,\tau,\hat s)\hat p^*(x,\hat s,\tau)d\hat s\Big)^2 dx\,ds\,d\tau + 2\|p^0\|_H^2\\ &\le \frac{2\overline{\beta}^2(S_1-S_0)^2T}{\delta}\|\hat p^*\|_H^2 + 2\|p^0\|_H^2.\end{aligned} \tag{2.19}$$

Thus, we have

$$\|\hat p(t)\|_{L^\infty(0,T;H)}^2 \le \frac{2\overline{\beta}^2(S_1-S_0)^2T}{\delta}\|\hat p^*\|_{L^\infty(0,T;H)}^2 + 2\|p^0\|_H^2. \tag{2.20}$$

Define a ball domain

$$B_r := \Big\{p \in W(0,T) : \|p\|_{L^\infty(0,T;H)} \le r, r \ge \frac{\|p^0\|_H}{\sqrt{\frac12 - \frac{\overline{\beta}^2(S_1-S_0)^2T}{\delta}}}\Big\}, \tag{2.21}$$

where $T < \delta/(2\overline{\beta}^2(S_1-S_0)^2)$. Then we have $\mathcal{A}B_r \subset B_r$ by (2.20), because if $\|\hat p^*\|_{L^\infty(0,T;H)} \le r$, it gives $\|\hat p\|_{L^\infty(0,T;H)} \le r$ for $\frac{2\overline{\beta}^2(S_1-S_0)^2T}{\delta}r^2 + 2\|p^0\|_H^2 \le r^2$ from (2.21).

Furthermore, we claim that $\mathcal{A}$ is a strict contraction on B_r. In fact, Let $\mathcal{A}\hat p_i^* = \hat p_i$, $\hat p_i, {\hat p_i}^* \in B_r, i = 1,2$. By using a similar deduction from (2.16) to (2.20) to $\hat p_1 - \hat p_2$, we have

$$\|\hat p_1 - \hat p_2|_{L^\infty(0,T;H)}^2 \le \frac{2\overline{\beta}^2(S_1-S_0)^2T}{\delta}\|\hat p_1^* - \hat p_2^*\|_{L^\infty(0,T;H)}^2. \tag{2.22}$$

For $T < \frac{\delta}{2\overline{\beta}^2(S_1-S_0)^2}$, namely, $\frac{2\overline{\beta}^2(S_1-S_0)^2T}{\delta} < 1$, $\mathcal{A}$ is a strict contraction. Banach fixed-point theorem allows us to conclude that there exists a unique $\hat p \in B_r$ such that $\mathcal{A}\hat p = \hat p$. This point is the desired unique bounded solution $\hat p \in W(0,T)$.

Since T does not depend on p^0, we can apply the same procedure as the above on $(T, 2T)$, $(2T, 3T)$, ... and so on. So we deduce that a solution of (2.8)–(2.11) can be found on the desired time interval.

We now prove the non-negativity. Let $\hat{p}_1 \geq 0$ be given in $W(0,T)$ and define the sequence $\{\hat{p}_n\}_{n\geq 1}$ by $\mathcal{A}\hat{p}_n = \hat{p}_{n+1}$.

Taking $\hat{p}_2^- = \max\{0, -\hat{p}_2\}$ as a test function in (2.16) leads to

$$\Big(\frac{d\hat{p}_2}{dt}, \hat{p}_2^-\Big)_H + a(t; \hat{p}_2, \hat{p}_2^-) = (\mathcal{I}\hat{p}_1, \hat{p}_2^-)_H. \tag{2.23}$$

If we let $\hat{p}_2^+ = \max\{0, \hat{p}_2\}$, then $\hat{p}_2 = \hat{p}_2^+ - \hat{p}_2^-$ and $\hat{p}_2^+ \cdot \hat{p}_2^- = 0$, and it gives

$$\frac{1}{2}\frac{d}{dt}\|\hat{p}_2^-\|_H^2 \leq \frac{1}{2}\frac{d}{dt}\|\hat{p}_2^-\|_H^2 + a(t; \hat{p}_2^-, \hat{p}_2^-) = -(\mathcal{I}\hat{p}_1, \hat{p}_2^-)_H. \tag{2.24}$$

Since $\hat{p}_1 \geq 0$, it leads to $\mathcal{I}\hat{p}_1 \geq 0$ and $-(\mathcal{I}\hat{p}_1, \hat{p}_2^-)_H \leq 0$. Then we find $\frac{d}{dt}\|\hat{p}_2^-\|_H^2 \leq 0$, and

$$\|\hat{p}(t)_2^-\|_H^2 \leq \|\hat{p}(0)_2^-\|_H^2 = \|p^{0-}\|_H^2 = 0, \tag{2.25}$$

which means that $\hat{p}_2 \geq 0$. By induction, we can further show that $\hat{p}_n \geq 0$, for all $n \geq 1$. The unique solution $\hat{p} \in W(0,T)$ (the limiting point of the sequence) is non-negative. □

From Lemma 2.6 and Proposition 2.2, we have the following result.

Theorem 2.7. *Assume that the hypotheses* (H1)–(H5) *hold. For any $u \in \mathcal{U}$, the system* (1.1)–(1.4) *has a unique nonnegative solution $p^u(x,s,t) \in W(0,T)$ in Q and*

$$0 \leq p^u(x,s,t) \leq M_1, \quad \text{a.e. in } Q, \tag{2.26}$$

where $M_1 > 0$ is a constant independent of p^u and u.

3. Optimal strategies

In this section, we derive the first-order necessary optimality conditions for the optimal harvesting control problem (1.5).

We present an auxiliary result for the continuous dependence of the population density with the harvesting effort u reads as follows.

Lemma 3.1. *Let p^{u_1}, p^{u_2} be the solutions of* (1.1)–(1.4) *corresponding to the controls $u_1, u_2 \in \mathcal{U}$, respectively. Then we have*

$$|p^{u_1} - p^{u_2}| \leq TC_1\|u_1 - u_2\|_{L^\infty(Q)}, \tag{3.1}$$

where C_1 is a positive constant independent of u_1 and u_2.

Proof. Let $y = p^{u_1} - p^{u_2}$. Then y is the solution of the system

$$\begin{gathered}
\partial_t y - k\Delta y = \partial_s(d(s)\partial_s y - g(s)y) - \mu y - u_1 y + (u_2 - u_1)p^{u_2} \\
+ \int_{S_0}^{S_1} \beta(x,s,t,\hat{s})y(x,\hat{s},t)\,\mathrm{d}\hat{s}, \\
\frac{\partial y}{\partial s}(x,S_0,t) = \frac{\partial y}{\partial s}(x,S_1,t) = 0, \quad \forall (x,t) \in \Omega \times (0,T), \\
\frac{\partial y}{\partial n}(x,s,t) = 0, \quad \text{on } \Sigma, \\
y(x,s,0) = 0, \quad \forall (x,s) \in \Omega \times (S_0,S_1).
\end{gathered} \tag{3.2}$$

Multiplying the first equation of (3.2) by y and integrating it on $Q_t := \Omega\times(S_0,S_1)\times(0,t)$, we deduce that

$$
\begin{aligned}
&\|y(\cdot,t,\cdot)\|_H^2\\
&\le \int_0^t\int_\Omega g(S_0)y(x,S_0,\tau)^2dxd\tau - \int_{Q_t} g'(s)y^2d\sigma\\
&\quad + 2\int_{Q_t}(u_2-u_1)p^{u_2}yd\sigma + 2\int_{Q_t}\Big(\int_{S_0}^{S_1}\beta(x,s,\tau,\hat{s})y(x,\hat{s},\tau)d\hat{s}\Big)yd\sigma\\
&\le |g'(s)|_{\max}\int_0^t\|y(\cdot,\tau,\cdot)\|_H^2d\tau + 2\int_{Q_t}|u_1-u_2||M_1y|d\sigma\\
&\quad + 2\overline{\beta}(S_1-S_0)\int_0^t\|y(\cdot,\tau,\cdot)\|_H^2d\tau\\
&\le |g'(s)|_{\max}\int_0^t\|y(\cdot,\tau,\cdot)\|_H^2d\tau + \int_{Q_t}\big(|M_1(u_1-u_2)|^2+|y|^2\big)\,d\sigma\\
&\quad + 2\overline{\beta}(S_1-S_0)\int_0^t\|y(\cdot,\tau,\cdot)\|_H^2d\tau\\
&= M_1^2\int_0^t\|u_1-u_2\|_H^2d\tau + \Big(|g'(s)|_{\max}+2\\
&\quad + 2\overline{\beta}(S_1-S_0)\Big)\int_0^t\|y(\cdot,\tau,\cdot)\|_H^2d\tau.
\end{aligned}\tag{3.3}
$$

It follows from Bellman's lemma that

$$
\|y(\cdot,t,\cdot)\|_H^2 \le M_1^2e^{\left(|g'(s)|_{\max}+1+2\overline{\beta}(S_1-S_0)\right)T}\|u_1-u_2\|_H^2. \tag{3.4}
$$

Integrating it on $(0,T)$ yields

$$
\|y\|_{L^2(0,T;H)}^2 \le TM_1^2e^{\left(|g'(s)|_{\max}+1+2\overline{\beta}(S_1-S_0)\right)T}\|u_1-u_2\|_{L^2(0,T;H)}^2. \tag{3.5}
$$

Thus, by the fundamental embedding inequality, we know that (2.25) holds for some constant $C_1>0$. □

To characterize the structure of the optimal controller, we need to define the following dual problem associated with (1.1)–(1.4):

$$
\begin{gathered}
\partial_tq + k\Delta q = -\partial_s(d(s)\partial_sq) - g(s)\partial_sq + (\mu+u^*)q + wu^*\\
- \int_{S_0}^{S_1}\beta(x,\hat{s},t,s)q(x,\hat{s},t)\,\mathrm{d}\hat{s},\\
d(s)\partial_sq + g(s)q|_{s=S_0} = d(s)\partial_sq + g(s)q|_{s=S_1} = 0,\quad \forall(x,t)\in\Omega\times(0,T),\\
\frac{\partial q}{\partial n}(x,s,t) = 0,\quad \text{on } \Sigma,\\
q(x,s,T) = 0,\quad \forall(x,s)\in\Omega\times(S_0,S_1).
\end{gathered}\tag{3.6}
$$

Under the changes $\tau = T - t$ and $\tilde{q}(x,s,\tau) := q(x,s,T-\tau)$, the above problem becomes

$$\begin{gathered}
\partial_\tau \tilde{q} - k\Delta\tilde{q} = \partial_s(d(s)\partial_s\tilde{q}) + g(s)\partial_s\tilde{q} - (\mu+u^*)\tilde{q} - wu^* \\
+ \int_{S_0}^{S_1} \beta(x,\hat{s},\tau,s)\tilde{q}(x,\hat{s},\tau)\,\mathrm{d}\hat{s}, \\
d(s)\partial_s\tilde{q} + g(s)\tilde{q}|_{s=S_0} = d(s)\partial_s\tilde{q} + g(s)\tilde{q}|_{s=S_1} = 0, \quad \forall (x,\tau) \in \Omega\times(0,T), \\
\frac{\partial\tilde{q}}{\partial n}(x,s,\tau) = 0, \quad \text{on } \Sigma, \\
\tilde{q}(x,s,0) = 0, \quad \forall (x,s) \in \Omega\times(S_0,S_1).
\end{gathered} \tag{3.7}$$

Using classical results for parabolic equations associated with (3.7), and discussing in the same manner as that in Lemmas 2.6 and 3.1, we can derive the following lemma.

Lemma 3.2. *Problem* (3.6) *has a unique solution* $q^u \in L^2(0,T;H^1(\mathcal{Q}))$ *and*

$$|q^u(x,s,t)| \le M_2, \quad \text{a.e. in } Q, \tag{3.8}$$

where M_2 *is a positive constant independent of* q^u *and* u.

Furthermore, let q^{u_1}, q^{u_2} *be the solutions of* (3.6) *corresponding to* $u_1, u_2 \in \mathcal{U}$, *respectively. Then there exists a positive constant* C_2, *which is independent of* u_1, u_2, *such that*

$$|q^{u_1} - q^{u_2}| \le TC_2\|u_1 - u_2\|_{L^\infty(Q)}. \tag{3.9}$$

We now describe the structure of optimal controllers as follows.

Theorem 3.3. *Let* $u^*(x,s,t) \in \mathcal{U}$ *be an optimal control for the problem* (1.1)–(1.5), *and* p^{u^*} *and* q^{u^*} *be the corresponding solutions of system* (1.1)–(1.4) *and* (3.6), *respectively. Then we have*

$$u^*(x,s,t) = \mathcal{F}\big\{\frac{[w+q^{u^*}]p^{u^*}}{\rho}\big\}(x,s,t), \tag{3.10}$$

in which the mapping $\mathcal{F}$ *is defined as*

$$(\mathcal{F}h)(x,s,t) = \begin{cases} \zeta_1(x,s,t), & h(s,t,x) < \zeta_1(x,s,t), \\ h(x,s,t), & \zeta_1(x,s,t) \le h(x,s,t) \le \zeta_2(x,s,t), \\ \zeta_2(x,s,t), & h(x,s,t) > \zeta_2(x,s,t). \end{cases} \tag{3.11}$$

Proof. Let $\mathcal{T}_{\mathcal{U}}(u^*)$ be the tangent cone to $\mathcal{U}$ at u^*(see [4]). For any $v \in \mathcal{T}_{\mathcal{U}}(u^*)$, we know that $u^* + \varepsilon v \in \mathcal{U}$ for the sufficient small $\varepsilon > 0$. Since u^* is optimal, it follows that

$$\begin{aligned}
&\int_Q \big(wu^*p^{u^*} - \frac{1}{2}\rho u^{*2}\big)\,dx\,ds\,dt \\
&\ge \int_Q \big(w(u^*+\varepsilon v)p^{u^*+\varepsilon v} - \frac{1}{2}\rho(u^*+\varepsilon v)^2\big)\,dx\,ds\,dt,
\end{aligned} \tag{3.12}$$

which implies

$$\int_Q \Big(wu^*\frac{p^{u^*+\varepsilon v} - p^{u^*}}{\varepsilon} + wvp^{u^*+\varepsilon v} - \frac{1}{2}\rho v(2u^*+\varepsilon v)\Big)\,dx\,ds\,dt \le 0. \tag{3.13}$$

Let $z(x,s,t)$ be the solution of

$$\begin{gathered}
\partial_t z - k\Delta z = \partial_s(d(s)\partial_s z - g(s)z) - (\mu + u^*)z - vp^{u^*} \\
+ \int_{S_0}^{S_1} \beta(x,s,t,\hat{s})z(x,\hat{s},t)\,\mathrm{d}\hat{s}, \\
\frac{\partial z}{\partial s}(x,S_0,t) = \frac{\partial z}{\partial s}(x,S_1,t) = 0, \quad \forall (x,t) \in \Omega \times (0,T), \\
\frac{\partial z}{\partial n}(x,s,t) = 0, \quad \text{on } \Sigma, \\
z(x,s,0) = 0, \quad \forall (x,s) \in \Omega \times (S_0,S_1).
\end{gathered} \tag{3.14}$$

The existence and uniqueness of solutions to (3.14) follows from the theory of nonhomogeneous parabolic equations (see, e.g. [12]).

Let

$$w_\varepsilon(x,s,t) = \frac{p^{u^*+\varepsilon v} - p^{u^*}}{\varepsilon} - z(x,s,t), \quad (x,s,t) \in Q. \tag{3.15}$$

It is not hard to deduce that $w_\varepsilon(x,s,t)$ is the solution of

$$\begin{gathered}
\partial_t w - k\Delta w = \partial_s(d(s)\partial_s w - g(s)w) - \mu w - u^* w \\
- v(p^{u^*+\varepsilon v} - p^{u^*}) + \int_{S_0}^{S_1} \beta(x,s,t,\hat{s})w(x,\hat{s},t)\,\mathrm{d}\hat{s}, \\
\frac{\partial w}{\partial s}(x,S_0,t) = \frac{\partial w}{\partial s}(x,S_1,t) = 0, \quad \forall (x,t) \in \Omega \times (0,T), \\
\frac{\partial w}{\partial n}(x,s,t) = 0, \quad \text{on } \Sigma, \\
w(x,s,0) = 0, \quad \forall (x,s) \in \Omega \times (S_0,S_1).
\end{gathered} \tag{3.16}$$

In what follows, we show $w_\varepsilon \to 0$ as $\varepsilon \to 0+$. By estimating (2.25), we may infer

$$p^{u^*+\varepsilon v} - p^{u^*} \to 0, \quad \text{in } L^2(0,T;H) \text{ as } \varepsilon \to 0^+. \tag{3.17}$$

We now consider the limit system

$$\begin{gathered}
\partial_t w - k\Delta w = \partial_s(d(s)\partial_s w - g(s)w) - \mu w - u^* w + \int_{S_0}^{S_1} \beta(x,s,t,\hat{s})w(x,\hat{s},t)\,\mathrm{d}\hat{s}, \\
\frac{\partial w}{\partial s}(x,S_0,t) = \frac{\partial w}{\partial s}(x,S_1,t) = 0, \quad \forall (x,t) \in \Omega \times (0,T), \\
\frac{\partial w}{\partial n}(x,s,t) = 0, \quad \text{on } \Sigma, \\
w(x,s,0) = 0, \quad \forall (x,s) \in \Omega \times (S_0,S_1),
\end{gathered} \tag{3.18}$$

which is a homogenous linear parabolic system and has a unique solution $w(x,s,t) = 0$ a.e. in Q. Hence, we have

$$\frac{p^{u^*+\varepsilon v} - p^{u^*}}{\varepsilon} \to z, \quad \text{in } L^2(0,T;H) \text{ as} \varepsilon \to 0^+. \tag{3.19}$$

Passing to the limit in (3.13) we find

$$\int_Q \big(wu^* z + (wp^{u^*} - \rho u^*)v\big)\,dx\,ds\,dt \le 0. \tag{3.20}$$

Multiplying (3.6) by $z(x,s,t)$ and integrating it over Q (denoting u by u^*), we deduce that

$$\int_Q wu^* z\,dx\,ds\,dt = \int_Q vp^{u^*}q^{u^*}\,dx\,ds\,dt. \tag{3.21}$$

Then it follows from (3.20) and (3.21) that

$$\int_Q \Big\{\Big[(w+q^{u^*})p^{u^*} - \rho u^*\Big]v\Big\}(s,t,x)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}s \le 0, \quad \forall v \in \mathcal{T}_{\mathcal{U}}(u^*). \tag{3.22}$$

According to the properties of normal cone (see [22]), the expression in the square brackets of (3.22) satisfies $(w+q^{u^*})p^{u^*} - \rho u^* \in \mathcal{N}_{\mathcal{U}}(u^*)$, the normal cone to $\mathcal{U}$ at u^*. Consequently the conclusion follows. □

4. Existence and uniqueness of optimal solutions

In this section, we show that there is one and only one solution for optimal harvesting control problem (1.5). We need the following lemma which can be proven by the definition of normal cones (see, e.g. [5]).

Lemma 4.1. *Suppose that $\eta(x,s,t) \in L^1(Q)$ satisfies*

$$\int_Q [\eta(x,s,t)v(x,s,t) + \alpha|v(x,s,t)|]\,dx\,ds\,dt \ge 0, \quad \forall v \in \mathcal{T}_{\mathcal{U}}(u), \tag{4.1}$$

where α is some small positive constant. Then there exists some $\theta \in L^\infty(Q)$ such that $|\theta|_\infty \le 1$ and $u + \alpha\theta \in \mathcal{N}_{\mathcal{U}}(u)$.

The following result guarantees the existence and uniqueness of the optimal strategies.

Theorem 4.2. *Assume that* (H1)–(H6) *hold. If*

$$T(C_1(W+M_2) + C_2M_1)\rho^{-1} < 1, \tag{4.2}$$

where W is the same as in (H6)*, and M_i, C_i $(i=1,2)$ are given in Theorem 2.7, Lemmas 3.1 and 3.2, then the optimal control problem* (1.1)–(1.5) *has a unique solution.*

Proof. Define a functional $\Phi : L^1(Q) \to (-\infty, +\infty]$ by

$$\Phi(u) = \begin{cases} -J(u) = \int_Q \left(\frac{1}{2}\rho u^2 - wp^u u\right)\,dx\,ds\,dt, & \text{if } u \in \mathcal{U}, \\ +\infty, & \text{if } u \notin \mathcal{U}, \end{cases} \tag{4.3}$$

where $J(\cdot)$ is of the form (1.5). By Lemma 3.1, it is easily seen that Φ is lower semi-continuous. According to the Ekeland variational principle [11], for each $\varepsilon > 0$ there exists $u_\varepsilon \in \mathcal{U}$ such that

$$\Phi(u_\varepsilon) \le \inf_{u\in\mathcal{U}} \Phi(u) + \varepsilon, \tag{4.4}$$

$$\Phi(u_\varepsilon) \le \inf_{u\in\mathcal{U}} \{\Phi(u) + \sqrt{\varepsilon}|u-u_\varepsilon|_1\}, \tag{4.5}$$

where $|\cdot|_1$ denotes the norm in $L^1(Q)$.

Note that the perturbed functional $\Phi_\varepsilon(u) := \Phi(u) + \sqrt{\varepsilon}|u-u_\varepsilon|_1$ attains its infimum at u_ε. By the same argument as in the previous section, we obtain the condition

$$\int_Q \big(\rho u_\varepsilon - (w+q^{u_\varepsilon})p^{u_\varepsilon}\big)v\,dx\,ds\,dt + \sqrt{\varepsilon}\int_Q |v(x,s,t)|\,dx\,ds\,dt \ge 0, \quad \forall v \in \mathcal{T}_{\mathcal{U}}(u_\varepsilon). \tag{4.6}$$

By Lemma 4.1, we see that there exists $\theta_\varepsilon \in L^\infty(Q)$, and $|\theta|_\infty \le 1$, such that

$$\rho u_\varepsilon - (w + q^{u_\varepsilon}) p^{u_\varepsilon} - \sqrt{\varepsilon}\theta_\varepsilon \in \mathcal{N}_{\mathcal{U}}(u_\varepsilon), \tag{4.7}$$

and consequently,

$$u_\varepsilon(x,s,t) = \mathcal{F}[(1/\rho)\big((w + q^{u_\varepsilon})p^{u_\varepsilon} + \sqrt{\varepsilon}\theta_\varepsilon\big)], \quad \text{a.e. in } Q. \tag{4.8}$$

To show the uniqueness of the optimal controller u, we define $\mathcal{J} : \mathcal{U} \subset L^\infty(Q) \to \mathcal{U}$ by

$$(\mathcal{J}(u))(x,s,t) = \mathcal{F}\Big(\frac{w(x,s,t) + q^u(x,s,t)}{\rho} p^u(x,s,t)\Big), \quad \text{a.e. in } Q. \tag{4.9}$$

For $(x,s,t) \in Q$ we have

$$\begin{aligned}
&|(\mathcal{J}(u))(x,s,t) - (\mathcal{J}(v))(x,s,t)| \\
&= |\mathcal{F}((1/\rho)(w + q^u)p^u) - \mathcal{F}((1/\rho)(w + q^v)p^v)| \\
&\le \frac{W}{\rho}|p^u - p^v| + \frac{1}{\rho}|q^u||p^u - p^v| + \frac{1}{\rho}|p^v||q^u - q^v|.
\end{aligned} \tag{4.10}$$

By the estimates (2.24), (2.25), (3.8) and (3.9), we obtain

$$\begin{aligned}
&\|(\mathcal{J}(u))(x,s,t) - (\mathcal{J}(v))(x,s,t)\|_{L^\infty(Q)} \\
&\le T(C_1(W + M_2) + C_2M_1)\rho^{-1}\|u - v\|_{L^\infty(Q)}.
\end{aligned} \tag{4.11}$$

So $\mathcal{J}$ is a contraction if $T(C_1(W + M_2) + C_2M_1)\rho^{-1} < 1$, and $\mathcal{J}$ has one and only one fixed point $\overline{u} \in \mathcal{U}$. Theorem 3.3 implies that an optimal controller u^*, if it exists, must coincide with this fixed point. Hence, the uniqueness of optimal controls is proved.

Next, we prove the existence of optimal controls. In fact, we only need to show that the fixed point $\overline{u}$ minimizes $\Phi(\cdot)$. By (4.8) and (4.9), we have

$$\begin{aligned}
&\|\mathcal{J}(u_\varepsilon) - u_\varepsilon\|_{L^\infty(Q)} \\
&= \|\mathcal{F}(\rho^{-1}(w + q^u)p^u) - \mathcal{F}[\rho^{-1}((w + q^{u_\varepsilon})p^{u_\varepsilon} + \sqrt{\varepsilon}\theta_\varepsilon)]\|_{L^\infty(Q)} \\
&\le \rho^{-1}\sqrt{\varepsilon}.
\end{aligned} \tag{4.12}$$

This leads to

$$\begin{aligned}
\|\overline{u} - u_\varepsilon\|_{L^\infty(Q)} &\le \|\mathcal{J}(\overline{u}) - \mathcal{J}(u_\varepsilon)\|_{L^\infty(Q)} + \rho^{-1}\sqrt{\varepsilon} \\
&\le T(C_1(W + M_2) + C_2M_1)\rho^{-1}\|\overline{u} - u_\varepsilon\|_{L^\infty(Q)} + \rho^{-1}\sqrt{\varepsilon};
\end{aligned} \tag{4.13}$$

that is,

$$\|\overline{u} - u_\varepsilon\|_{L^\infty(Q)} \le [1 - T(C_1(W + M_2) + C_2M_1)\rho^{-1}]^{-1}\rho^{-1}\sqrt{\varepsilon}. \tag{4.14}$$

So, we see that $u_\varepsilon \to \overline{u}$ in $L^\infty(Q)$, and by (4.4) we have $\Phi(\overline{u}) = \inf_{u\in\mathcal{U}} \Phi(u)$ which completes the proof. □

Conclusions and comments. In this article, we introduced a linear structured population model with spatial diffusion, size random growth (namely, diffusion in the size space) and the distributed recruitment. Application of the size diffusion is natural and significant in the biological phenomena [15, 13], since individuals that have the same size initially, may disperse as time progresses. We equipped our model with the homogeneous Neumann boundary condition with respect to the N-dimension spatial variable x and 1-dimension size variable s, and presented the results of the existence and uniqueness of solution of the state system, which laid a sufficient foundation for the optimal harvesting control problems.

By applying the Ekeland variational principle [11] and the properties of normal cone and adjoint techniques [5], we developed the optimal harvesting strategies and deduced the conditions to assure only one optimal policy. Theorem 4.2 tells us that, under the given conditions, our optimal control problem admits one and only one solution. Furthermore, Theorem 3.3 described the structure, other than a specific analytical expression, of optimal strategy. Unfortunately, one could not derive the explicit formula for the optimal strategy since the strategy, the state and the costate are coupled into a complex system. The results at this stage may be regarded as a middle step to real world applications and serve as a starting point for numerical computations.

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