EXISTENCE AND NONEXISTENCE OF SOLUTIONS FOR SEMILINEAR EQUATIONS ON EXTERIOR DOMAINS

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Abstract. In this article we study radial solutions of $\Delta u + K(r)f(u) = 0$ on the exterior of the ball of radius $R > 0$ centered at the origin in $\mathbb{R}^N$ where $f$ is odd with $f < 0$ on $(0, \beta)$, $f > 0$ on $(\beta, \delta)$, $f \equiv 0$ for $u > \delta$, and where the function $K(r)$ is assumed to be positive and $K(r) \to 0$ as $r \to \infty$.

The primitive $F(u) = \int_0^u f(t) \, dt$ has a “hilltop” at $u = \delta$. We prove that if $K(r) \sim r^{-\alpha}$ with $\alpha > 2(N - 1)$ and if $R > 0$ is sufficiently small then there are a finite number of solutions of $\Delta u + K(r)f(u) = 0$ on the exterior of the ball of radius $R$ such that $u \to 0$ as $r \to \infty$. We also prove the nonexistence of solutions if $R$ is sufficiently large.

1. Introduction

In this article we study radial solutions of

$$\Delta u + K(r)f(u) = 0 \quad \text{in } \Omega, \quad (1.1)$$
$$u = 0 \quad \text{on } \partial \Omega, \quad (1.2)$$
$$u \to 0 \quad \text{as } |x| \to \infty \quad (1.3)$$

where $x \in \Omega = \mathbb{R}^N \setminus B_R(0)$ is the complement of the ball of radius $R > 0$ centered at the origin.

We assume there exist $\beta, \delta$ with $0 < \beta < \delta$ such that $f(0) = f(\beta) = f(\delta) = 0$ and $F(u) = \int_0^u f(s) \, ds$ where:

(H1) $f$ is odd and locally Lipschitz, $f < 0$ on $(0, \beta)$, $f > 0$ on $(\beta, \delta)$, $f \equiv 0$ on $(\delta, \infty)$, and $F(\delta) > 0$.

We note it follows that $F(u) = \int_0^u f(s) \, ds$ is even and has a unique positive zero, $\gamma$, with $\beta < \gamma < \delta$ such that

(H2) $F < 0$ on $(0, \gamma)$, $F > 0$ on $(\gamma, \infty)$, and $F$ is strictly monotone on $(0, \beta)$ and on $(\beta, \delta)$.

In earlier papers [5–6] we studied (1.1), (1.3) when $\Omega = \mathbb{R}^N$ and $K(r) \equiv 1$. In [7] we studied (1.1), (1.3) with $K(r) \equiv 1$ and $\Omega = \mathbb{R}^N \setminus B_R(0)$. In that paper we proved existence of an infinite number of solutions - one with exactly $n$ zeros for each nonnegative integer $n$ such that $u \to 0$ as $|x| \to \infty$. Interest in the topic for this paper comes from recent papers [4, 11, 13] about solutions of differential equations on exterior domains.

2010 Mathematics Subject Classification. 34B40, 35B05.
Key words and phrases. Exterior domains; semilinear; superlinear; radial.

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When $f$ grows superlinearly at infinity - i.e. $\lim_{u \to \infty} \frac{f(u)}{u^2} = \infty$, and $\Omega = \mathbb{R}^N$ then problem \((1.1)-(1.3)\) has been extensively studied [1,2,10,12,14]. The type of nonlinearity addressed here has not been studied as extensively [5,7].

Since we are interested in radial solutions of \((1.1)-(1.3)\) we assume that $u(x) = u(|x|) = u(r)$ where $x \in \mathbb{R}^N$ and $r = |x| = \sqrt{x_1^2 + \cdots + x_N^2}$ so that $u$ solves

$$u''(r) + \frac{N-1}{r}u'(r) + K(r)f(u(r)) = 0 \quad \text{on} \quad (R, \infty) \quad \text{where} \quad R > 0,$$

$$u(R) = 0, \quad u'(R) = b > 0. \quad (1.4)$$

We assume that there exist constants $c_1 > 0$, $c_2 > 0$, and $\alpha > 0$ such that

 Gaza the case when $0 < \alpha < 2(N-1)$.

**Theorem 1.1.** Let $N \geq 2$ and $\alpha > 2(N-1)$. Assuming (H1)-(H4) then if $R$ is sufficiently large then there are no solutions of \((1.4)-(1.5)\) such that $\lim_{r \to \infty} u(r) = 0$.

**Theorem 1.2.** Let $N > 2$ and $\alpha > 2(N-1)$. Assuming (H1)-(H4) and given a nonnegative integer $n$ then if $R > 0$ is sufficiently small then there are constants $b_i > 0$ and solutions $u_i$ with $0 \leq i \leq n$ of \((1.4)-(1.5)\) with $b = b_i$ such that $\lim_{r \to \infty} u_i(r) = 0$ and $u_i$ has $i$ zeros on $(R, \infty)$.

An important step in proving this result is showing that solutions can be obtained with more and more zeros by choosing $b$ appropriately. Intuitively it can be of help to interpret \((1.4)\) as an equation of motion for a point $u(r)$ moving in a double-well potential $F(u)$ subject to a damping force $-\frac{N-1}{r}u'$. This potential however becomes flat at $u = \pm \delta$. According to \((1.5)\) the system has initial position zero and initial velocity $b > 0$. We will see that if $b > 0$ is sufficiently small then the solution will “fall” into the well at $u = \beta$ and - due to damping - it will be unable to leave the well whereas if $b > 0$ is sufficiently large the solution will reach the top of the hill at $u = \delta$ and will continue to move to the right indefinitely. For an appropriate value of $b$ - which we denote $b^{**}$ - the solution will reach the top of the hill at $u = \delta$ as $r \to \infty$. For values of $b$ slightly less than $b^{**}$ the solutions will not make it to the top of the hill at $u = \delta$ and they will nearly stop moving. Thus the solution “loiters” near the hilltop at $F(\delta)$ on a sufficiently long interval and will usually “fall” into the positive well at $u = \beta$ or the negative well at $u = -\beta$ after passing the origin a finite number of times, say $n$. For the right value of $b$ - which we denote as $b_n$ - the solution comes to rest at the local maximum of the function $F(u)$ at the origin as $r \to \infty$ after passing the origin $n$ times.

In contrast to this is a double-well potential that goes off to infinity as $|u| \to \infty$ - for example $F(u) = u^2(a^2 - 4)$. Here the solutions of \((1.4)-(1.5)\) behave quite differently. As $b$ increases the number of zeros of $u$ increases as $b \to \infty$. Thus the number of times that $u$ reaches the local maximum of $F(u)$ at the origin increases as the parameter $b$ increases. See for example [10,12,14].
2. Preliminaries and Proof of Theorem 1.1

Proof of Theorem 1.1. We observe since \( \alpha > 2(N - 1) \), by (1.4) and (H4)
\[
\left( \frac{1}{2} \frac{u'^2}{K} + F(u) \right)' = - \frac{u'^2}{2rK} \left( 2(N - 1) + \frac{rK'}{K} \right) \geq 0.
\] (2.1)
Hence \( \frac{1}{2} \frac{u'^2}{K} + F(u) \) is nondecreasing. Now suppose there is a solution of (1.4)-(1.5) such that \( \lim_{r \to \infty} u(r) = 0. \) Then \( u \) must have a first local maximum, \( M, \) such that \( u' > 0 \) on \([R, M)\). Then since \( \frac{1}{2} \frac{u'^2}{K} + F(u) \) is nondecreasing we see that
\[
\frac{1}{2} \frac{u'^2}{K} + F(u) \leq F(u(M)) \quad \text{on} \quad (R, M).
\] Rewriting this and using (H3) we see that
\[
\frac{|u'|}{\sqrt{2} \sqrt{F(u(M)) - F(u)}} \leq \sqrt{K} \leq \sqrt{c_2} r^{-\alpha/2} \quad \text{on} \quad (R, M).
\]
Integrating on \((R, M)\) and noting that \( \alpha > 2 \) (since \( \alpha > 2(N - 1) \) and \( N \geq 2 \)) gives
\[
\int_{u(M)}^{u(t)} \frac{dt}{\sqrt{2} \sqrt{F(u(M)) - F(t)}} \leq \frac{\sqrt{c_2}}{\sqrt{2} - 1} (R^{1-\frac{2}{\alpha}} - M^{1-\frac{2}{\alpha}}) \leq \frac{\sqrt{c_2}}{\sqrt{2} - 1} R^{1-\frac{2}{\alpha}}. \tag{2.2}
\]
In addition, since \( \frac{1}{2} \frac{u'^2}{K} + F(u) \) is nondecreasing we see that \( 0 < \frac{1}{2} \frac{u'^2}{K} \leq F(u(M)) \) so \( u(M) > \gamma. \) Further it follows from (H1)-(H2) that \( F(u(M)) \leq F(\delta) \) and \( F(t) \geq -F_0 \) for all \( t \geq 0 \) where \( F_0 > 0 \) and therefore \( F(u(M)) - F(t) \leq F(\delta) + F_0. \) Therefore (2.2) implies
\[
\frac{\gamma}{\sqrt{2} \sqrt{F(\delta) + F_0}} \leq \frac{\sqrt{c_2}}{\sqrt{2} - 1} R^{1-\frac{2}{\alpha}}. \tag{2.3}
\]
We note that the left-hand side of (2.3) is positive and independent of \( R \) but that the right-hand side goes to zero as \( R \to \infty \) since \( \alpha > 2. \) Thus we see that if \( R \) is sufficiently large then (2.3) is violated hence there are no solutions \( u \) of (1.4)-(1.5) such that \( \lim_{r \to \infty} u(r) = 0 \) if \( R \) is sufficiently large. This completes the proof. \( \square \)

For the remainder of this paper we assume \( \alpha > 2(N - 1) \) and \( N > 2. \) Now we make the change of variables
\[
u(r) = w(r^{2-N}).
\] Then (1.4)-(1.5) becomes
\[
w'' + h(t)f(w) = 0, \tag{2.4}
\]
and
\[
w(R^{2-N}) = 0, \quad w'(R^{2-N}) = -bR^{N-1} < 0 \tag{2.5}
\]
where \( h(t) = T(t^{\frac{1}{N-2}}) \) and \( T(r) = \frac{R^{2(N-1)-K(r)}}{(N-2)^2}. \) Then from (H3) and (H4) we see:
\[
h(t) = T(t^{\frac{1}{N-2}}) \sim \frac{t^{q}}{(N-2)^2} \quad \text{for} \quad 0 < t \leq R^{2-N}, \tag{2.6}
\]
where
\[
q = \frac{\alpha - 2(N - 1)}{N - 2} > 0, \quad \lim_{t \to 0^+} \frac{th'(t)}{h(t)} = q.
\]
In addition, it follows from (H3)-(H4) that
\[
\frac{c_1}{(N-2)^2} t^q \leq h(t) \leq \frac{c_2}{(N-2)^2} t^q \quad \text{and} \quad h' > 0 \quad \text{for} \quad 0 < t \leq R^{2-N}. \tag{2.7}
\]

Since we are seeking solutions of (1.4)-(1.5) with \( \lim_{r \to \infty} u(r) = 0 \) we see that this is equivalent to seeking solutions of (2.4)-(2.5) with \( \lim_{t \to 0^+} w(t) = 0 \). Instead we now attempt to solve (2.4) with initial conditions at \( t = 0 \) instead of \( t = R^{2-N} \),
\[
w(0) = 0, \quad w'(0) = a > 0. \tag{2.8}
\]
We attempt now to show that if \( R > 0 \) is sufficiently small and \( n \) is a nonnegative integer then there are \( a_i > 0 \) with \( a_0 < a_1 < \cdots < a_n \) such that \( w(R^{2-N}, a_i) = 0 \) and \( w(t, a) \) has \( i \) zeros on \( (0, R^{2-N}) \).

To proceed we temporarily extend the definition of the function \( h \) so that
\[
h(t) = h(R^{2-N}) + \frac{h'(R^{2-N})}{qR^{2-N} (q-1)} \left[ t^q - R^{2-N} t^q \right] \quad \text{for} \quad t > R^{2-N}.
\]
Note then that (2.7) holds on \( (0, \infty) \).

A useful function in the analysis of (2.4)-(2.5) is
\[
E(t) = \frac{1}{2} \frac{w''(t)}{h(t)} + F(w(t)) \quad \text{for} \quad t > 0.
\tag{2.9}
\]

Using (2.4), we obtain
\[
E'(t) = -\frac{w'^2 h'}{2h^2} \leq 0 \quad \text{since} \quad h' > 0 \quad \text{for} \quad t > 0. \tag{2.10}
\]
Thus \( E \) is nonincreasing. Also note that \( \lim_{t \to 0^+} E(t) = +\infty \). We also observe using (2.4),
\[
\frac{1}{2} w'^2 + h(t) F(w) = \frac{1}{2} a^2 + \int_0^t h'(s) F(w) \, ds. \tag{2.11}
\]

Another useful equation is obtained by integrating (2.4) on \( (0, t) \) and using (2.8) which gives
\[
w'(t) = a - \int_0^t h(x) f(w(x)) \, dx. \tag{2.12}
\]
Integrating again on \( (0, t) \) gives
\[
w(t) = at - \int_0^t \int_0^s h(x) f(w(x)) \, dx \, ds. \tag{2.13}
\]

3. PROOF OF THEOREM 1.2

From the standard theory of ordinary differential equations there exists a unique solution of (2.4), (2.8) on \( [0, 2\epsilon) \) for some \( \epsilon > 0 \). Since \( E \) is nonincreasing then
\[
\frac{1}{2} \frac{w''(t)}{h(t)} + F(w(t)) = E(t) \leq E(\epsilon) \quad \text{for} \quad t > \epsilon \quad \text{from which it follows that} \quad w \quad \text{and} \quad w'
\]

are uniformly bounded on compact subsets of \( [0, \infty) \) and thus the solution \( w(t) \) of (2.4), (2.8) exists on all of \( [0, \infty) \) and varies continuously with respect to \( a \) on compact subsets of \( [0, \infty) \).
Lemma 3.1. Let $\alpha > 2(N-1)$, $N > 2$, and let $w$ satisfy (2.4), (2.8). Suppose (H1)–(H4) hold. Then there exists an $r_a > 0$ such that $w(r_a) = \beta$ and $0 < w < \beta$ on $(0, r_a)$. Also, $r_a \to \infty$ as $a \to 0^+$. In addition, $|w(t, a)| < \delta$ if $a > 0$ is sufficiently small.

Proof. By (2.8) we have $w'(0) = a > 0$ so it follows that $w$ is initially increasing. If $0 < w < \beta$ for all $t > 0$ then $f(w) < 0$ by (H1) and we see from (2.13) that $w(t) > at$. Thus $w(t)$ exceeds $\beta$ for large enough $t$ contradicting that $0 < w < \beta$. Thus there is an $r_a > 0$ such that $w(r_a) = \beta$ and $0 < w < \beta$ on $(0, r_a)$.

For the next part of the lemma we note first that if $|w(t, a)| < \gamma$ for all $t \geq 0$ then there is nothing to prove since $\gamma < \delta$. So suppose now that there exists $s_a > 0$ such that $|w(s_a)| = \gamma$ and $|w| < \gamma$ on $(0, s_a)$. Evaluating (2.11) at $t = s_a$ gives

$$
\frac{1}{2}w'^2(s_a) \leq \frac{1}{2}a^2
$$

(3.1)

since $F(w(s_a)) = F(\gamma) = 0$ and $F(w) \leq 0$ on $(0, s_a)$. Using (3.1) and the fact that $E$ is nonincreasing gives

$$
F(w) \leq \frac{1}{2}w'^2 + F(w) = E(t) \leq E(s_a) = \frac{1}{2}w'^2(s_a) \leq \frac{1}{2}a^2 \text{ for } t \geq s_a.
$$

(3.2)

Thus if $\epsilon > 0$ and $a > 0$ is sufficiently small then we see from (H2) and (3.2) that $|w| < \gamma + \epsilon < \delta$ for $t \geq 0$. This proves the last statement in Lemma 3.1.

Next observe from (H1) that $|f(w)| \leq C_1 |w|$ for all $w$ for some $C_1 > 0$. Using this along with (2.7) in (2.13) and estimating gives

$$
|w(t)| \leq at + \frac{C_1 C_2}{(N-2)^2} t^{q+1} \int_0^t |w(s)| \, ds.
$$

Applying the Gronwall inequality we then obtain

$$
|w(t)| \leq a \left( t + p(t) \int_0^t se^{P(t)-P(s)} \, ds \right)
$$

(3.3)

where:

$$
P(t) = \int_0^t p(s) \, ds \leq \int_0^t C_2 s^{q+1} \frac{(N-2)^2}{(q+1)(N-2)^2} \, ds = \frac{C_1 C_2 t^{q+2}}{(q+1)(N-2)^2}.
$$

Evaluating (3.3) at $t = r_a$ gives

$$
\beta \leq a \left( r_a + p(r_a) \int_0^{r_a} se^{P(r_a)-P(s)} \, ds \right).
$$

(3.4)

It follows from (3.4) and since $p(t), P(t)$ are continuous that $r_a \to \infty$ as $a \to 0^+$. This completes the proof. \qed

Lemma 3.2. Let $\alpha > 2(N-1)$, $N > 2$, and let $w$ satisfy (2.4), (2.8). Suppose (H1)–(H4) hold. If $a > 0$ is sufficiently large then there exists a $t_a > 0$ such that $w(t_a) = \delta$ and $w(t) < \delta$ on $[0, t_a)$.

Proof. It follows from (H1) that $|f(w)| \leq C_2$ for some $C_2 > 0$ so by (2.7) and (2.12):

$$
w' \geq a - \frac{C_2 C_2 t^{q+1}}{(q+1)(N-2)^2} \text{ for } t \geq 0.
$$
Integrating on \((0, t)\) gives
\[
  w(t) \geq at - \frac{C_2c_2t^{q+2}}{(q + 2)(q + 1)(N - 2)} \quad \text{for} \quad t \geq 0.
\]

Thus for large enough \(a\) we have
\[
  w(1) \geq a - \frac{C_2c_2}{(q + 2)(q + 1)(N - 2)} \geq \delta.
\]

Therefore \(w(t)\) exceeds \(\delta\) if \(a > 0\) is sufficiently large. This completes the proof. \(\Box\)

Let

\[
  S = \{ a > 0 : \text{there is a } t_a > 0 \text{ such that } w(t_a, a) = \delta \text{ and } 0 < w < \delta \text{ on } (0, t_a) \}.
\]

By Lemma 3.2 the set \(S\) is nonempty and from Lemma 5.1 the set \(S\) is bounded from below by a positive constant. Now we let:

\[
  0 < a^* = \inf S.
\]

**Lemma 3.3.** Let \(\alpha > 2(N - 1), N > 2, \) and let \(w\) satisfy (2.4), (2.8). Suppose (H1)–(H4) hold. Then \(w(t, a^*) \to \delta\) as \(t \to \infty\) and \(w'(t, a^*) > 0\) on \([0, \infty)\).

**Proof.** We first show \(w(t, a^*) < \delta\) on \([0, \infty)\). If not then there is a \(t_{a^*} > 0\) such that \(w(t_{a^*}, a^*) = \delta\) and \(w(t_{a^*}, a^*) < \delta\) on \([0, t_{a^*}]\). Thus \(w'(t_{a^*}, a^*) \geq 0\). In fact \(w'(t_{a^*}, a^*) > 0\) for \(w'(t_{a^*}, a^*) = 0\) then by uniqueness of solutions of initial value problems \(w(t, a^*) \equiv \delta\) contradicting that \(w(0, a^*) = 0\). So since \(w'(t_{a^*}, a^*) > 0\) and \(w(t_{a^*}, a^*) = \delta\) then there is an \(x_{a^*} > t_{a^*}\) such that \(w(x_{a^*}, a^*) > \delta + \epsilon\) for some \(\epsilon > 0\). Now for \(a < a^*\) but \(a\) close to \(a^*\) then by continuity with respect to initial conditions we have \(w(x_{a^*}, a) > \delta\) contradicting the definition of \(a^*\). Thus \(w(t, a^*) < \delta\) on \([0, \infty)\). Next we show

\[
  E(t, a^*) \geq F(\delta) \quad \text{for all } t > 0.
\]

So suppose not. Then there is a \(t_0 > 0\) such that \(E(t_0, a^*) < F(\delta)\). By continuity with respect to initial conditions \(E(t_0, a) < F(\delta)\) for \(a > a^*\) and \(a\) close to \(a^*\). However, for \(a > a^*\) there is a \(a_0 > 0\) such that \(w(t_0, a_0) = \delta\) and \(w'(t_0, a_0) > 0\) so therefore since \(f(w) = 0\) for \(w > \delta\) (by (H1)) then by (2.4) it follows that \(w(t, a) = w(t_0, a)(t - t_0) + \delta \geq \delta\) for \(t \geq t_0\) and thus \(E(t, a) \geq F(\delta)\) for all \(t > t_0\). Then since \(E\) is nonincreasing (by (2.10)) it follows that \(E(t, a) \geq F(\delta)\) for all \(t > 0\) contradicting that \(E(t_0, a) < F(\delta)\). Thus \(E(t, a^*) \geq F(\delta)\) for \(t > 0\).

Next we show \(w'(t, a^*) > 0\) for \(t \geq 0\). First, since \(w'(0, a) = a > 0\) we see that \(w'(t, a) > 0\) for small \(t > 0\). Suppose then there is an \(M > 0\) such that \(w'(M, a^*) = 0\) and \(w'(t, a^*) > 0\) on \([0, M]\). Then from (2.4) we have \(w''(M, a^*) \leq 0\) and so \(f(w(M, a^*)) \geq 0\). Thus \(w''(M, a^*) \geq \beta\). Also since we showed at the beginning of the proof that \(w(t, a^*) < \delta\) for \(t > 0\) it follows that \(\beta < w(M, a^*) < \delta\) and since \(F\) is increasing on \((\beta, \delta)\) (by (H2)) then \(E(M, a^*) = F(w(M, a^*)) < F(\delta)\). On the other hand it follows from (3.5) that \(E(M, a^*) \geq F(\delta)\) and so we obtain a contradiction. Thus, \(w'(t, a^*) > 0\) on \([0, \infty)\).

It now follows from Lemmas 3.1 and 3.2 that there is an \(L\) with \(\beta < L \leq \delta\) such that \(\lim_{t \to \infty} w(t, a^*) = L\). From (2.4) we see that \(\frac{w''(t, a^*)}{h(t)} \to -f(L)\) as \(t \to \infty\).

If \(f(L) \neq 0\) then \(|w' | \geq c_0h(t) > 0\) for large \(t > 0\) and for some \(c_0 > 0\). Since \(h(t) \sim t^q\) with \(q > 0\) then integrating the inequality \(|w''| \geq c_0h(t) > 0\) twice on \((t_0, t)\) where \(t_0\) is large we see that \(|w| \to \infty\) contradicting that \(w(t, a^*) \to L\).
f(L) = 0 and since \( \beta < L \leq \delta \) it follows from (H1) that \( L = \delta \). This completes the proof. \( \square \)

Next we let
\[
a^\ast = \inf \{ a : w(t, a) > 0 \text{ for } t \geq 0 \text{ and } \lim_{t \to -\infty} w(t, a) = \delta \}.
\]
By Lemma 3.3 we see that
\[
a^\ast \in \{ a : w(t, a) > 0 \text{ for } t \geq 0 \text{ and } \lim_{t \to -\infty} w(t, a) = \delta \}.
\]
Thus the set on the right-hand side of (3.6) is nonempty and by Lemma 3.1 it is bounded from below by a positive constant. Thus \( 0 < a^{**} \leq a^\ast \) and a similar argument as in Lemma 3.3 shows that \( w(t, a^{**}) \to \delta \) as \( t \to \infty \) and \( w'(t, a^{**}) > 0 \) for \( t \geq 0 \).

**Lemma 3.4.** Let \( \alpha > 2(N - 1) \), \( N > 2 \), and let \( w \) satisfy (2.4), (2.8). Suppose (H1)--(H4) hold. If \( 0 < a < a^{**} \) then \( w(t, a) \) has a local maximum, \( M_a > 0 \), and \( M_a \to \infty \) as \( a \to (a^{**})^- \). In addition, \( w(M_a, a) < \delta \) and \( w(M_a, a) \to \delta \) as \( a \to (a^{**})^- \).

**Proof.** If \( a < a^{**} \) and \( w'(t, a) > 0 \) for \( t \geq 0 \) then we see as in Lemma 3.3 that \( w(t, a) \to \delta \) contradicting the definition of \( a^{**} \). Thus there exists \( M_a > 0 \) such that \( w'(t, a) > 0 \) on \( [0, M_a) \) and \( w'(M_a, a) = 0 \). Then \( w''(M_a, a) \leq 0 \) and so \( f(w(M_a, a)) \geq 0 \). Thus \( w(M_a, a) \geq \beta \). Since we know \( w(t, a) \) does not attain the value \( \delta \) because \( a < a^{**} \leq a^\ast \) we therefore have \( \beta \leq w(M_a, a) < \delta \). Now if the \( \{ M_a \} \) were bounded then a subsequence would converge to some \( M_a^{**} \) and so by the Arzela-Ascoli theorem a subsequence of \( w(t, a) \) and \( w'(t, a) \) would converge uniformly to \( w(t, a^{**}) \) and \( w'(t, a^{**}) \) on \( [0, M_a^{**} + 1] \) as \( a \to (a^{**})^- \) and \( w'(M_a^{**}, a^{**}) = 0 \) contradicting \( w'(t, a^{**}) > 0 \) from the remarks after Lemma 3.3.

Thus \( M_a \to \infty \) as \( a \to (a^{**})^- \).

Also, as \( a \to (a^{**})^- \) with \( a < a^{**} \) we know \( w(t, a) \) must get arbitrarily close to \( \delta \) by continuity with respect to initial conditions and so \( w(M_a, a) \to \delta \) as \( a \to (a^{**})^- \). This completes the proof. \( \square \)

**Lemma 3.5.** Let \( \alpha > 2(N - 1) \), \( N > 2 \), and let \( w \) satisfy (2.4), (2.8). Suppose (H1)--(H4) hold. Given a positive integer \( n \) if \( 0 < a < a^{**} \) and \( a \) is sufficiently close to \( a^{**} \) then \( w(t, a) \) has at least \( n \) zeros on \( (0, \infty) \). In addition denoting the \( n \)th zero as \( z_n(a) \) then \( z_n(a) < R^{2^N} \) if \( R \) is sufficiently small and \( a \) is sufficiently close to \( a^{**} \) with \( a < a^{**} \).

**Proof.** From Lemma 3.4 we know that for \( a \) sufficiently close to \( a^{**} \) with \( a < a^{**} \) then \( w \) has a local maximum \( M_a \) and \( w(M_a) > \gamma > \beta \). From (2.4) it follows that \( w'' < 0 \) while \( w > \beta \) and since \( w'(M_a) = 0 \) it follows that there exists \( y_a > M_a \) such that \( w(y_a) = \beta \). Thus there is an \( x_a \) with \( M_a < x_a < y_a \) such that \( w(x_a) = \gamma \).

From (2.10) we have
\[
\frac{1}{2} \frac{w'^2}{h(t)} + F(w) = E(t) \leq E(M_a) = F(w(M_a, a)) \quad \text{for } t \geq M_a.
\]
Rewriting this gives
\[
\frac{|w'|}{\sqrt{h}} \leq \sqrt{2} \sqrt{F(w(M_a, a)) - F(w)}.
\] (3.7)
Now it follows from (2.6) that \(0 < \frac{h'}{h^{3/2}} \leq c_3\) for some \(c_3 > 0\) and \(t > 0\). Then from this and (2.7) we see that
\[
0 < \frac{h'}{h^{3/2}} = \frac{th'}{h^{3/2}} \leq \frac{c_3(N - 2)}{\sqrt{c_1} \ t^{2/3 + 1}}.
\](3.8)

Thus from (2.10), (3.7)-(3.8), and (H3)
\[
-E' = \frac{w''h'}{2h^2} = \frac{|w'|}{2\sqrt h} \frac{h'}{h^{3/2}} |w'|
\leq \frac{c_3(N - 2)}{\sqrt{2c_1}} \sqrt{F(w(M_a, a)) - F(w)} \frac{1}{t^{2/3 + 1}} |w'|.
\](3.9)

Suppose now that \(M_a < s < t\) and that \(w' < 0\) on \((M_a, t)\). Then integrating (3.9) on \((M_a, t)\) and estimating we obtain
\[
E(M_a, a) - E(t, a) \leq \frac{c_3(N - 2)}{\sqrt{2c_1} M_a^{2/3 + 1}} \int_{w(t, a)}^{w(M_a, a)} \sqrt{F(w(M_a, a)) - F(y)} \, dy.
\](3.10)

Let us assume \(w(t, a) > 0\) and \(w'(t, a) < 0\) for \(t > M_a\). Then \([w(t, a), w(M_a, a)] \subset [0, \delta]\) and the integrand in (3.10) is bounded hence the integral in (3.10) is bounded independent of \(a\). Thus the right-hand side of (3.10) goes to 0 as \(a \to (a^{**})^-\) because \(M_a \to \infty\) from Lemma 3.4 and the integral is uniformly bounded. Thus since \(E(M_a, a) = F(w(M_a, a)) - F(\delta)\) as \(a \to (a^{**})^-\) by Lemma 3.4 it follows from (3.10) that \(E(t, a) \to F(\delta)\) as \(a \to (a^{**})^-\). Thus \(E(t, a) \geq \frac{1}{2} F(\delta)\) for \(a\) close to \(a^{**}\) and \(a < a^{**}\). In particular on \((x_a, t)\) where \(0 < w(t, a) < \gamma\) it follows that \(F(w) \leq 0\) so
\[
\frac{1}{2} \frac{w'(t, a)}{h(t)} \geq \frac{1}{2} \frac{w'(t, a)}{h(t)} + F(w(t, a)) = E(t, a) \geq \frac{1}{2} F(\delta) \quad \text{on } (x_a, t)
\](3.11)

hence from (2.6) and (H3)-(H4),
\[
-w'(t, a) \geq \frac{\sqrt{c_1 F(\delta)}}{N - 2} t^{q/2} \quad \text{on } (x_a, t)
\]

and so integrating on \((x_a, t)\) gives
\[
w(t, a) \leq \gamma - \frac{\sqrt{c_1 F(\delta)}}{(N - 2)(\frac{q}{2} + 1)} \left( t^{q/2 + 1} - x_a^{q/2 + 1} \right) \to -\infty \quad \text{as } t \to \infty
\]

which contradicts that \(w > 0\). Thus there exists \(z_a > x_a\) such that \(w(z_a, a) = 0\) and \(w(t, a) > 0\) on \((0, z_a)\). By uniqueness of solutions of initial value problems we have \(w'(z_a, a) < 0\) and so while \(-\beta < w(t, a) < 0\) then \(w'' < 0\) by (2.4) and so we see that there is a \(Y_a > z_a\) such that \(w(Y_a, a) = -\beta\). Now if \(w(t, a)\) does not have a local minimum for \(t > Y_a\) then we can show in a similar way as we did in Lemma 3.3 that \(w \to L\) but now where \(L < -\beta\) and \(f(L) = 0\) implying \(L = -\delta\). But since \(E\) is nonincreasing and \(F\) is even this would imply \(F(\delta) = F(\delta) \leq \lim_{t \to \infty} E(t, a) \leq E(M_a, a) = F(w(M_a, a))\) and hence by (H2) we have \(w(M_a, a) \geq \delta\). But recall from Lemma 3.4 that since \(a < a^{**}\) then \(w(M_a, a) < \delta\) thus we obtain a contradiction. Therefore it must be the case that \(w(t, a)\) has a local minimum, \(m_a > z_a\), and in a similar way as in Lemma 3.4 it is possible to show \(m_a \to \infty\) and \(w(m_a, a) \to -\delta\) as \(a \to (a^{**})^-\). Also as we did at the beginning of this lemma we can show that \(w(t, a)\) has a second zero \(z_{2,a} > z_a\) if \(a\) is sufficiently close to \(a^{**}\) and \(a < a^{**}\). Similarly
we can show that \( w(t, a) \) has any desired (finite) number of zeros by choosing \( a \) sufficiently close to \( a^* \) with \( a < a^* \). This completes the proof. \( \square \)

Thus we see that \( z_k(a) \) the \( k \)th zero of \( w(t, a) \) on \((0, \infty)\) is defined as long as \( a \) is sufficiently close to \( a^* \) with \( a < a^* \). It follows from continuous dependence of solutions on initial conditions that \( z_k(a) \) is a continuous function of \( a \). In addition \( \lim_{a \to (a^*)-} z_k(a) = \infty \). This follows for if the \( z_k(a) \) were bounded then for a subsequence (again labeled \( a \)) we would have \( z_k(a) \to z^* \) and by the Arzela-Ascoli theorem \( w(z^*, a^*) = 0 \) contradicting that \( w(t, a^*) > 0 \) on \((0, \infty)\).

Finally suppose \( R \) is sufficiently small and \( a < a^* \) is sufficiently close to \( a^* \) so that \( z_k(a) < R^{2-N} \). Then since we know \( z_k(a) \) is continuous with \( z_k(a) < R^{2-N} < \infty \) and \( \lim_{a \to (a^*)-} z_k(a) = \infty \) then it follows from the intermediate value theorem that there is a smallest value of \( a \) denoted \( a_k \) such that \( z_k(a_k) = R^{2-N} \).

Thus \( w(t, a_k) \) is a solution of (2.4) with \( k \) zeros on \((0, R^{2-N}] \). Now we let \( b_k = (2-N)R^{1-N} w'(R^{2-N}, a_k) \) and then finally if we let \( u_k(r, b_k) = (-1)^k w(r^{2-N}, a_k+1) \) then \( u_k(r, b_k) \) is a solution of (1.4)-(1.5) with \( b = b_k \), with \( k \) zeros on \((R, \infty)\), and \( \lim_{r \to \infty} u_k(r, b_k) = 0 \). This completes the proof.

**References**


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