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ANALYSIS OF A NONLINEAR SURFACE WIND WAVES MODEL VIA LIE GROUP METHOD

BEN GAO

ABSTRACT. This article focuses on two aspects. Firstly, symmetry analysis is performed for a nonlinear equation which can model surface wind wave patterns in nature. As a byproduct, the similarity reductions and exact solutions of the equation are constructed based on the optimal systems. Secondly, the explicit solutions are considered by the power series method. Moreover, the convergence of the power series solutions are shown.

1. INTRODUCTION

Manna [6] studied the surface wind waves qualitative behavior, represented by the nonlinear equation

$$u_{xt} = -\frac{3g}{hc_0}u - uu_{xx} + (u_x)^2, \tag{1.1}$$

where $u(x,t)$ is considered as an unidirectional surface wave propagating in the x-direction on a fluid medium involved in a large scale flow. h is the unperturbed initial depth, g is the acceleration of gravity, c_0 is the wind velocity and subscripts denote partial derivatives. For the sake of providing more information to understand (1.1), some works have been devoted to study (1.1) [6, 12]. The author in [6] provided some peakon solutions with amplitude, velocity, and width in interrelation and static compacton solutions with amplitude and width in interrelation for (1.1). The exact explicit traveling wave solutions of (1.1) are given by using the method of dynamical systems in [12]. However, as the authors known, the Lie group analysis and explicit power series solutions of (1.1) are left as open problems.

The application of Lie transformation group theory for the construction of solutions of nonlinear partial differential equations (PDEs) is one of the most active fields of research in the theory of nonlinear PDEs and applications [3, 4, 5, 7, 11]. The main idea of Lie group method is to transform solutions of a system of differential equations to other solutions. Once the symmetry group of a system of differential equations has be determined, one can directly use the defining property of such a group and construct new solutions to the system from known ones.

The rest of article is arranged as follows: Section 2 concentrates on symmetries of (1.1); in Section 3, the similarity reductions for (1.1) are dealt with and exact solutions are provided by using Lie group method; in Section 4, the explicit solutions

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for the reduced equations are obtained by using the power series method; the last section contains a conclusion of our work.

2. Lie point symmetry

In this section, we apply Lie point symmetry method for (1.1), and obtain its infinitesimal generators, commutation table of Lie algebra.

First of all, let us consider a one-parameter Lie symmetry group admitted by (1.1) with an infinitesimal operator of the form

$$X = \xi\partial_x + \tau\partial_t + \phi\partial_u. \tag{2.1}$$

where ξ, τ, ϕ are functions of x, t, u respectively and are called infinitesimals of the symmetry group.

The classical infinitesimal Lie invariance criterion for (1.1) with respect to the operator (2.1) reads as in [2, 7],

$$\operatorname{pr} X^{(2)}\Big[u_{xt} + \frac{3g}{hc_0}u + uu_{xx} - (u_x)^2\Big] = 0$$

for any u solves (1.1). Here, the symbol $\operatorname{pr} X^{(2)}$ is the usual 2 th-order prolongation of the operator [2, 7], in this situation,

$$\operatorname{pr} X^{(2)} = X + \phi_x^{(1)}\frac{\partial}{\partial u_x} + \phi_{xx}^{(2)}\frac{\partial}{\partial u_{xx}} + \phi_{xt}^{(2)}\frac{\partial}{\partial u_{xt}},$$

where

$$\begin{gathered}
\phi_x^{(1)} = D_x\phi - u_xD_x\xi - u_tD_x\tau,\\
\phi_{xx}^{(2)} = D_x^2(\phi - \xi u_x - \tau u_t) + \xi u_{xxx} + \tau u_{xxt},\\
\phi_{xt}^{(2)} = D_xD_t(\phi - \xi u_x - \tau u_t) + \xi u_{xxt} + \tau u_{xtt},
\end{gathered}$$

and D_x, D_t stand for the total derivative operators, for example,

$$D_t = \frac{\partial}{\partial t} + u_t\frac{\partial}{\partial u} + u_{tx}\frac{\partial}{\partial u_x} + u_{tt}\frac{\partial}{\partial u_t} + \dots .$$

Substituting $\operatorname{pr} X^{(2)}$ into (1.1) and splitting it with respect to the different order derivatives of u, one obtains a system of linear over-determining equations for the unknown functions ξ, τ and ϕ. We can find the following equations for the symmetry group of (1.1)

$$\begin{gathered}
\xi_x = -\tau_t, \quad \xi_t = \xi_u = 0,\\
\tau_x = \tau_u = 0, \quad \tau_{tt} = 0,\\
\phi = -2u\tau_t.
\end{gathered} \tag{2.2}$$

Solving above (2.2), we obtain

$$\xi = -c_1x + c_3, \quad \tau = c_1t + c_2, \quad \phi = -2c_1u,$$

where c_1, c_2, c_3 and c_4 are arbitrary constants, we find that (1.1) admits the operators

$$X_1 = -x\partial_x + t\partial_t - 2u\partial_u, \quad X_2 = \partial_t, \quad X_3 = \partial_x.$$

It is easy to check that $\{X_1, X_2, X_3\}$ is closed under the Lie bracket. In fact, we have

$$\begin{gathered}
[X_1, X_1] = [X_2, X_2] = [X_3, X_3] = 0,\\
[X_1, X_2] = -[X_2, X_1] = -X_2, [X_1, X_3] = -[X_3, X_1] = X_3,
\end{gathered}$$

$$[X_2, X_3] = -[X_3, X_2] = 0.$$

Furthermore, to obtain their adjoint representation, employing the following Lie series

$$\mathrm{Ad}(\exp(\epsilon X_i))X_j = X_j - \epsilon[X_i, X_j] + \frac{1}{2}\epsilon^2[X_i, [X_i, X_j]] - \dots,$$

we can compute the following results

$$\begin{gathered}
\mathrm{Ad}(\exp(\epsilon X_i))X_i = X_i, \quad i = 1,2,3, \quad \mathrm{Ad}(\exp(\epsilon X_1))X_2 = e^{\epsilon}X_2, \\
\mathrm{Ad}(\exp(\epsilon X_2))X_1 = X_1 - \epsilon X_2, \quad \mathrm{Ad}(\exp(\epsilon X_1))X_3 = e^{-\epsilon}X_3, \\
\mathrm{Ad}(\exp(\epsilon X_3))X_1 = X_1 + \epsilon X_3, \quad \mathrm{Ad}(\exp(\epsilon X_2))X_3 = X_3, \\
\mathrm{Ad}(\exp(\epsilon X_3))X_2 = X_2,
\end{gathered}$$

where ϵ is an arbitrary constant.

Based on the adjoint representation of the infinitesimal operators, we obtain the optimal systems of (1.1) as follows,

$$\{X_1, X_2, X_3, X_3 + aX_2\},$$

where a is an arbitrary constant.

Remark 2.1. The optimal systems of (1.1) can also be obtained using the results of the paper [9].

3. Similarity reductions and exact solutions

The symmetry group properties are very useful for the construction of invariant solutions of the differential equation under study. Next, We will consider the following similarity reductions and group-invariant solutions for (1.1) based on the optimal system.

Case 1: Reduction by X_1. Integrating the characteristic equation for X_1, we get similarity variables

$$z = tx, \quad p = \frac{u}{x^2},$$

and the group-invariant solution is $p = f(z)$, that is,

$$u = x^2 f(tx). \tag{3.1}$$

Substituting this expression into (1.1), we obtain

$$\frac{3g}{hc_0}f - 2f^2 + 3f' - z^2 f'^2 + zf'' + z^2 ff'' = 0 \tag{3.2}$$

where $f' = \frac{df}{dz}$.

Case 2: Reduction by X_2. Similarly, we have $u = f(z)$ in which $z = x$. Substituting it into (1.1), we obtain

$$\frac{3g}{hc_0}f - f'^2 + ff'' = 0 \tag{3.3}$$

where $f' = \frac{df}{dz}$.

Case 3: For generator X_3, we have $z = t, u = f(z)$. The corresponding reduction equation is

$$\frac{3g}{hc_0}f = 0. \tag{3.4}$$

Therefore, (1.1) has a solution $u = 0$. Obviously, the solution is not meaningful.

Case 4: For generator X_3+aX_2, we have $z=-ax+t, u=f(z)$. The corresponding reduction equation is

$$\frac{3g}{hc_0}f-a^2f'^2-af''+a^2ff''=0 \tag{3.5}$$

where $f'=\frac{df}{dz}$.

4. Explicit power series solutions

In Section 4, we obtained the reduced equations by using symmetry analysis. The power series can be used to treat differential equations, including many complicated differential equations with nonconstant coefficients [1]. In this section, we solve the nonlinear ODEs (3.2), (3.3), and (3.5) by the power series method.

4.1. Explicit solutions to (3.2).

Now, we seek a solution of (3.2) in a power series of the form

$$f(z)=\sum_{n=0}^{\infty}p_nz^n, \tag{4.1}$$

where the coefficients p_n $(n=0,1,2,\dots)$ are constants to be determined. Substituting (4.1) into (3.2), we have

$$\begin{aligned}
&\frac{3g}{hc_0}\sum_{n=0}^{\infty}p_nz^n-2\sum_{n=0}^{\infty}\sum_{k=0}^{n}p_kp_{n-k}z^n+3\sum_{n=0}^{\infty}(n+1)p_{n+1}z^n\\
&-z^2\sum_{n=0}^{\infty}\sum_{k=0}^{n}(k+1)(n+1-k)p_{k+1}p_{n+1-k}z^n+z\sum_{n=0}^{\infty}(n+1)(n+2)p_{n+2}z^n\\
&+z^2\sum_{n=0}^{\infty}\sum_{k=0}^{n}(k+1)(k+2)p_{n-k}p_{k+2}z^n=0.
\end{aligned} \tag{4.2}$$

From this equality, comparing coefficients, we have

$$\begin{aligned}
p_1&=\frac{1}{3}p_0(2p_0-\frac{3g}{hc_0}),\\
p_2&=\frac{1}{8}p_1(4p_0-\frac{3g}{hc_0}).
\end{aligned} \tag{4.3}$$

Generally, for $n\geq 2$, we have

$$\begin{aligned}
p_{n+1}=\frac{1}{(n+1)(n+3)}\Big\{&2p_{n-1}p_1-(\frac{3g}{hc_0}-2p_0)p_n\\
&+\sum_{k=0}^{n-2}[2p_kp_{n-k}+(k+1)(n-1-k)p_{k+1}p_{n-1-k}\\
&-(k+1)(k+2)p_{n-2-k}p_{k+2}]\Big\}.
\end{aligned} \tag{4.4}$$

In view of this equality, we can obtain all the coefficients $p_i(i\geq 3)$ of the power series (4.1), e.g.,

$$p_3=\frac{1}{15}(3p_1^2-\frac{3g}{hc_0}p_2+2p_0p_2). \tag{4.5}$$

Therefore, for arbitrary chosen constant number p_0, the other terms of the sequence $\{p_n\}_{n=0}^{\infty}$

can be determined by (4.3) and (4.4). This implies that for (3.2), there is a power series solution (4.1) with the coefficients constructed by (4.3) and (4.4).

Now we show that the convergence of the power series solution (4.1) of (3.2). In fact, from (4.4), we have

$$|p_{n+1}|\leq M[|p_{n-1}|+|p_n|+\sum_{k=0}^{n-2}(|p_k||p_{n-k}|+|p_{k+1}||p_{n-1-k}|+|p_{n-2-k}||p_{k+2}|)],$$

for $n=2,3,\dots$, where $M=\max\{|p_1|,|\frac{3g}{hc_0}-2p_0|,1\}$.

Now, we define a power series $R=R(z)=\sum_{n=0}^{\infty}r_nz^n$ by

$$r_i=|p_i|,\quad i=0,1,2,$$

and

$$r_{n+1}=M[r_{n-1}+r_n+\sum_{k=0}^{n-2}(r_kr_{n-k}+r_{k+1}r_{n-1-k}+r_{n-2-k}r_{k+2})]$$

where $n=2,3,\dots$. Then, it is easily seen that

$$|p_n|\leq r_n,\quad n=0,1,2,\dots.$$

Thus, the series $R=R(z)=\sum_{n=0}^{\infty}r_nz^n$ is a majorant series of (4.1). Next, we show that the series $R=R(z)$ has a positive radius of convergence.

$$\begin{aligned}
R(z)&=r_0+r_1z+r_2z^2+\sum_{n=2}^{\infty}r_{n+1}z^{n+1}\\
&=r_0+r_1z+r_2z^2+M\Big(\sum_{n=2}^{\infty}r_{n-1}z^{n+1}+\sum_{n=2}^{\infty}r_nz^{n+1}\\
&\quad+\sum_{n=2}^{\infty}\sum_{k=0}^{n-2}r_kr_{n-k}z^{n+1}+\sum_{n=2}^{\infty}\sum_{k=0}^{n-2}r_{k+1}r_{n-1-k}z^{n+1}\\
&\quad+\sum_{n=2}^{\infty}\sum_{k=0}^{n-2}r_{n-2-k}r_{k+2}z^{n+1}\Big)\\
&=r_0+r_1z+r_2z^2+M[z^2(R-r_0)+z(R-r_0-r_1z)\\
&\quad+zR(R-r_0-r_1z)+z(R-r_0)^2+zR(R-r_0-r_1z)].
\end{aligned}$$

Consider now the implicit functional equation with respect to the independent variable z,

$$\begin{aligned}
F(z,R)&=R-r_0-r_1z-r_2z^2-M[z^2(R-r_0)+z(R-r_0-r_1z)\\
&\quad+zR(R-r_0-r_1z)+z(R-r_0)^2+zR(R-r_0-r_1z)].
\end{aligned}$$

Since F is analytic in the neighborhood of $(0,r_0)$ and $F(0,r_0)=0, F_R(0,r_0)=1\neq 0$. if we choose the parameter $r_0=|p_0|$ properly. By the implicit function theorem [10], we see that $R=R(z)$ is analytic in a neighborhood of the point $(0,r_0)$ and with the positive radius. This implies that the power series (4.1) converges in a neighborhood of the point $(0,r_0)$. This completes the proof.

Hence, the power series solution (4.1) for (3.2) is an analytic solution. The power series solution of (3.2) can be written as

$$f(z)=p_0+p_1z+p_2z^2+\sum_{n=2}^{\infty}p_{n+1}z^{n+1}$$

$$
\begin{aligned}
&= p_0 + \frac{1}{3}p_0(2p_0 - \frac{3g}{hc_0})z + \frac{1}{8}p_1(4p_0 - \frac{3g}{hc_0})z^2 + \sum_{n=2}^{\infty} \frac{1}{(n+1)(n+3)}\{2p_{n-1}p_1 \\
&\quad - (\frac{3g}{hc_0} - 2p_0)p_n + \sum_{k=0}^{n-2}[2p_k p_{n-k} + (k+1)(n-1-k)p_{k+1}p_{n-1-k} \\
&\quad - (k+1)(k+2)p_{n-2-k}p_{k+2}]\}z^{n+1}.
\end{aligned}
$$

Furthermore, the explicit power series solution of (1.1) is

$$
\begin{aligned}
u(x,t) &= p_0x^2 + p_1tx^3 + p_2t^2x^4 + \sum_{n=2}^{\infty} p_{n+1}t^{n+1}x^{n+3} \\
&= p_0x^2 + \frac{1}{3}p_0(2p_0 - \frac{3g}{hc_0})tx^3 + \frac{1}{8}p_1(4p_0 - \frac{3g}{hc_0})t^2x^4 \\
&\quad + \sum_{n=2}^{\infty} \frac{1}{(n+1)(n+3)}\{2p_{n-1}p_1 \\
&\quad - (\frac{3g}{hc_0} - 2p_0)p_n + \sum_{k=0}^{n-2}[2p_k p_{n-k} + (k+1)(n-1-k)p_{k+1}p_{n-1-k} \\
&\quad - (k+1)(k+2)p_{n-2-k}p_{k+2}]\}t^{n+1}x^{n+3},
\end{aligned}
$$

where p_0 is an arbitrary constant, the other coefficients $p_n (n \geq 1)$ depend on (4.3) and (4.4) completely.

4.2. **Explicit solutions to** (3.3)**.** Similarly, we seek a solution of (3.3) in a power series of the form (4.1). Substituting it into (3.3), and comparing coefficients, we have

$$
p_2 = \frac{1}{2p_0}(p_1^2 - \frac{3g}{hc_0}p_0). \tag{4.6}
$$

Generally, for $n \geq 1$, we have

$$
\begin{aligned}
p_{n+2} = \frac{1}{(n+1)(n+2)p_0}\Big\{ &\sum_{k=0}^{n-1}(k+1)[(n+1-k)p_{k+1}p_{n+1-k} \\
&- (k+2)p_{n-k}p_{k+2}] + (n+1)p_1p_{n+1} - \frac{3g}{hc_0}p_n\Big\}.
\end{aligned} \tag{4.7}
$$

In view of (4.6) and (4.7), we can obtain all the coefficients $p_i (i \geq 3)$ of the power series (4.1), e.g.,

$$
p_3 = \frac{p_1}{6p_0}(2p_2 - \frac{3g}{hc_0}), \quad p_4 = \frac{p_2}{12p_0}(2p_2 - \frac{3g}{hc_0}).
$$

Therefore, for arbitrary chosen constant numbers $p_0 \neq 0$ and p_1, the other terms of the sequence $\{p_n\}_{n=0}^{\infty}$ can be determined by (4.6) and (4.7). This implies that for (3.3), there is a power series solution (4.1) with the coefficients constructed by (4.6) and (4.7).

4.3. **Explicit solutions to** (3.5)**.** Now, we seek a solution of (3.5) in a power series of the form (4.1). Substituting (4.1) into (3.5), and comparing coefficients, we have

$$
p_2 = \frac{1}{2a(ap_0 - 1)}(a^2p_1^2 - \frac{3g}{hc_0}p_0). \tag{4.8}
$$

Generally, for $n \geq 1$, we have

$$\begin{aligned} p_{n+2} = &\frac{1}{(n+1)(n+2)a(ap_0-1)} \\ &\times \Big\{a^2 \sum_{k=0}^{n-1}(k+1)[(n+1-k)p_{k+1}p_{n+1-k} - (k+2)p_{n-k}p_{k+2}] \\ &+ a^2(n+1)p_1p_{n+1} - \frac{3g}{hc_0}p_n\Big\}. \end{aligned} \tag{4.9}$$

In view of (4.8) and (4.9), we can obtain all the coefficients $p_n (n \geq 3)$ of the power series (4.1), e.g.,

$$p_3 = \frac{p_1}{6a(ap_0-1)}(2a^2p_2 - \frac{3g}{hc_0}), \quad p_4 = \frac{p_2}{12a(ap_0-1)}(2a^2p_2 - \frac{3g}{hc_0}).$$

Thus, for arbitrary chosen constant numbers $a \neq 0, p_0 \neq \frac{1}{a}$ and p_1, the other terms of the sequence $\{p_n\}_{n=0}^{\infty}$ can be constructed by (4.8) and (4.9). This implies that for (3.5), there is a power series solution (4.1) with the coefficients determined by (4.8) and (4.9).

Furthermore, the explicit power series solution of (1.1) is

$$\begin{aligned} u(x,t) &= p_0 + p_1(-ax+t) + p_2(-ax+t)^2 + \sum_{n=1}^{\infty} p_{n+2}(-ax+t)^{n+2} \\ &= p_0 + p_1(-ax+t) + \frac{1}{2a(ap_0-1)}(a^2p_1^2 - \frac{3g}{hc_0}p_0)(-ax+t)^2 \\ &\quad + \sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)a(ap_0-1)}\Big\{a^2\sum_{k=0}^{n-1}(k+1)[(n+1-k)p_{k+1}p_{n+1-k} \\ &\quad - (k+2)p_{n-k}p_{k+2}] + a^2(n+1)p_1p_{n+1} - \frac{3g}{hc_0}p_n\Big\}(-ax+t)^{n+2}, \end{aligned}$$

where $a \neq 0, p_0 \neq \frac{1}{a}$ and p_1 are arbitrary constants, the other coefficients $p_n (n \geq 2)$ depend on (4.8) and (4.9) completely.

Remark 4.1. The proofs of convergence of the power series solutions to (3.3), (3.5) are similar to the one of (3.2). The details are omitted here. Furthermore, such power series solutions can greatly enrich the solutions of the (1.1) and converge quickly, so it is convenient for computations in both theory and applications.

Remark 4.2. The solutions of equations (3.3) and (3.5) (obviously, it is sufficiently to consider only the case $a = 1$) in the explicit form can also be found without using the power series method (see the handbook [8]) and read as

$$f(z) = f(x) = be^{cx} - \frac{d}{c^2} + \frac{d^2}{4bc^4}e^{-cx},$$

and

$$f(z) = be^{cz} - (\frac{d}{c^2} - 1) + \frac{d}{4bc^2}(\frac{d}{c^2} - 1)e^{-cz} (\text{for} a = 1),$$

where $d = \frac{3g}{hc_0}$, $b \neq 0$, $c \neq 0$.

Conclusions. In this article, We present Lie approach for a nonlinear surface wind waves model. As a byproduct, new invariant solutions are constructed based on the optimal systems which may exert potential applications about the problems of physical phenomena. Moreover, we apply the power series method to obtain the explicit solutions of (1.1). It can be seen that the Lie symmetry analysis and the power series method are very efficient to research the explicit solutions of PDEs.

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Ben Gao
College of Mathematics, Taiyuan University of Technology, Taiyuan 030024, China
E-mail address: benzi0116@163.com