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COMBINED EFFECTS OF CHANGING-SIGN POTENTIAL AND CRITICAL NONLINEARITIES IN KIRCHHOFF TYPE PROBLEMS

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Abstract. In this article, we study the existence and multiplicity of positive solutions for a class of Kirchhoff type problems involving changing-sign potential and critical growth terms. Using the concentration compactness principle and Nehari manifold, we obtain the existence and multiplicity of nonzero non-negative solutions.

1. Introduction and statement of main result

In this article, we consider the multiplicity of non-negative solutions of the Kirchhoff type equation

$$\begin{gathered} -\Big(a+b\int_\Omega |\nabla u|^2dx\Big)\Delta u = |u|^4u+\mu|x|^{\alpha-2}u+\lambda f(x)|u|^{q-2}u \quad \text{in } \Omega, \\ u=0, \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^3$, $a,b>0$, $0<\alpha<1$, $1<q<2$, $\lambda>0$ is a positive real number, and $0<\mu<a\mu_1$ (μ_1 is the first eigenvalue of $-\Delta u=\mu|x|^{\alpha-2}u$, under Dirichlet boundary condition). The weight functions $f\in C(\overline{\Omega})$ is changing-sign potential, satisfying $f^+=\max\{f,0\}\neq 0$.

In recent years, the existence and multiplicity of solutions to the nonlocal Kirchhoff type problem

$$\begin{gathered} -\Big(a+b\int_\Omega |\nabla u|^2dx\Big)\Delta u = g(x,u) \quad \text{in } \Omega, \\ u=0, \quad \text{on } \partial\Omega \end{gathered} \tag{1.2}$$

has been the focus of a great deal of research and some results can be found. For instance, in [1, 2, 13, 16, 17, 21, 28]. In particular, when $g(x,u)$ is involved in critical nonlinearities terms, readers can be referred to [10, 12, 15, 25, 29] for details. The authors in [7, 8, 11, 18] have investigated Kirchhoff type equation with concave and convex nonlinear. In addition, there are some results for $g(x,u)$ being changing-sign potential, see for example [19, 22, 31]. Especially, Chen et al. [7] considered the

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following nonlocal Kirchhoff type problem

$$\begin{gathered} -\Big(a+b\int_\Omega |\nabla u|^2dx\Big)\Delta u = f(x)u^{p-2}u+\lambda g(x)|u|^{q-2}u \quad \text{in } \Omega, \\ u=0, \quad \text{on } \partial\Omega, \end{gathered} \tag{1.3}$$

the authors assumed that $1<q<2<p<6$, the sign-changing weight functions $f,g\in C(\overline{\Omega})$ and $f^+=\max\{f,0\}\neq 0$ and $g^+=\max\{g,0\}\neq 0$ hold. Then there exists a positive constant $\lambda_0(a)>0$ such that for each $a>0$ and $\lambda\in(0,\lambda_0(a))$, problem (1.3) has at least two positive solutions. In equation (1.3), assume $1<q<2$, $p=6$, $f(x)\equiv 1$ and add a term of $\mu|x|^{\alpha-2}u$, then an interesting question is put forward if the existence and multiplicity of solutions can be established for Kirchhoff type problems with critical and changing-sign terms.

Throughout this paper, we use the following notation:

- The space $H_0^1(\Omega)$ is equipped with the norm $\|u\|=(\int_\Omega|\nabla u|^2dx)^{1/2}$, the norm in $L^p(\Omega)$ is represented by $|u|_p=(\int_\Omega |u|^pdx)^{\frac{1}{p}}$;
- Let S be the best Sobolev constant, namely

$$S:=\inf_{u\in D^{1,2}(\mathbb{R}^3)\setminus\{0\}}\frac{\int_{\mathbb{R}^3}|\nabla u|^2dx}{\big(\int_{\mathbb{R}^3}|u|^6dx\big)^{1/3}}. \tag{1.4}$$

The energy functional $I_\lambda(u)$: $H_0^1(\Omega)\to\mathbb{R}$ corresponding to (1.1) is defined by

$$I_\lambda(u)=\frac{a}{2}\|u\|^2+\frac{b}{4}\|u\|^4-\frac{\mu}{2}\int_\Omega |x|^{\alpha-2}|u|^2dx-\frac16\int_\Omega|u|^6dx-\frac{\lambda}{q}\int_\Omega f|u|^qdx.$$

Generally speaking, a function u is called a weak solution of (1.1) if $u\in H_0^1(\Omega)$ and for all $\varphi\in H_0^1(\Omega)$ it holds

$$(a+b\|u\|^2)\int_\Omega(\nabla u\cdot\nabla\varphi)dx=\mu\int_\Omega |x|^{\alpha-2}u\varphi dx+\int_\Omega|u|^4u\varphi dx+\lambda\int_\Omega f|u|^{q-2}u\varphi dx.$$

Our main result is as follows:

Theorem 1.1. *Assume that $1<q<2$, $0<\alpha<1$, and $f\in L^\infty(\Omega)$ changes sign, then there exists $\lambda_*>0$ such that for every $\lambda\in(0,\lambda_*)$, problem (1.1) has at least two nonzero non-negative solutions, and one of the solutions is a ground state solution.*

Remark 1.2. It is well known that the difficulty lies in the lack of compactness of the embedding: $H_0^1\hookrightarrow L^6(\Omega)$, then we overcome the difficulty by the concentration compactness principle. The nonlocal Kirchhoff problem becomes difficult when $b>0$ for estimating the critical value level, however, by adding a particular term $\mu|x|^{\alpha-2}u$, we could get over the trouble.

In section 2 we present some preliminary results, while in section 3 we present the proof of Theorem 1.1.

2. Preliminary results

Since I_λ is not bounded below on $H_0^1(\Omega)$, we will work on the Nehari manifold

$$\mathcal{N}_\lambda=\{u\in H_0^1(\Omega)\setminus\{0\}:\langle I_\lambda'(u),u\rangle=0\},$$

which implies that $\mathcal{N}_\lambda$ holds all nonzero solutions of (1.1). In addition, $u \in \mathcal{N}_\lambda$ if and only if

$$a\|u\|^2 + b\|u\|^4 - \int_\Omega |u|^6 dx - \mu \int_\Omega |x|^{\alpha-2}|u|^2 dx - \lambda \int_\Omega f|u|^q dx = 0.$$

Let

$$\psi(u) = a\|u\|^2 + b\|u\|^4 - \int_\Omega |u|^6 dx - \mu \int_\Omega |x|^{\alpha-2}|u|^2 dx - \lambda \int_\Omega f|u|^q dx,$$

and then we obtain

$$\langle \psi'(u), u\rangle = 2a\|u\|^2 + 4b\|u\|^4 - 6\int_\Omega |u|^6 dx - 2\mu \int_\Omega |x|^{\alpha-2}|u|^2 dx - q\lambda \int_\Omega f|u|^q dx.$$

We split $\mathcal{N}_\lambda$ into three parts:

$$\begin{aligned}
\mathcal{N}_\lambda^+ &= \{u \in \mathcal{N}_\lambda : \langle \psi'(u), u\rangle > 0\},\\
\mathcal{N}_\lambda^0 &= \{u \in \mathcal{N}_\lambda : \langle \psi'(u), u\rangle = 0\},\\
\mathcal{N}_\lambda^- &= \{u \in \mathcal{N}_\lambda : \langle \psi'(u), u\rangle < 0\}.
\end{aligned}$$

Lemma 2.1. *Suppose $\lambda \in (0, T_1)$ with*

$$T_1 = \Big\{\frac{4-q}{6-q} b\Big[\frac{2b}{6-q}\Big]^{\frac{2}{4-q}} |\Omega|^{-\frac{6-q}{3(4-q)}} (|f|_\infty)^{-\frac{2}{4-q}} S^{\frac{2(6-q)}{4-q}}\Big\}^{\frac{4-q}{2}}.$$

Then (i) $\mathcal{N}_\lambda^\pm \neq \emptyset$, and (ii) $\mathcal{N}_\lambda^0 = \emptyset$.

Proof. (i) For a given $u \in H_0^1(\Omega)\backslash\{0\}, u \neq 0$, as $0 < \mu < a\mu_1$, one has

$$\begin{aligned}
\eta(t) :=& t^{-4} a\|u\|^2 + bt^{-2}\|u\|^4 - \mu t^{-4} \int_\Omega \frac{|u|^2}{|x|^{2-\alpha}} dx - \lambda t^{q-6} \int_\Omega f|u|^q dx\\
\geq& t^{-4}\big(a - \frac{\mu}{\mu_1}\big)\|u\|^2 + bt^{-2}\|u\|^4 - \lambda t^{q-6} \int_\Omega f|u|^q dx\\
\geq& t^{-4}\big(a - \frac{\mu}{\mu_1}\big)\|u\|^2 + bt^{-2}\|u\|^4 - \lambda t^{q-6} |f|_\infty \int_\Omega |u|^q dx.
\end{aligned}$$

We define two functions $\Phi, \Phi_1 \in C(\mathbb{R}^+, \mathbb{R})$ by

$$\begin{gathered}
\Phi(t) = t^{-4}\big(a - \frac{\mu}{\mu_1}\big)\|u\|^2 + bt^{-2}\|u\|^4 - \lambda t^{q-6}|f|_\infty \int_\Omega |u|^q dx,\\
\Phi_1(t) = bt^{-2}\|u\|^4 - \lambda t^{q-6}|f|_\infty \int_\Omega |u|^q dx.
\end{gathered}$$

Thus

$$\Phi_1'(t) = -2bt^{-3}\|u\|^4 - \lambda(q-6)t^{q-7}|f|_\infty \int_\Omega |u|^q dx.$$

Let $\Phi_1'(t) = 0$, it is simple to show that

$$t_{\max} = \Big[\frac{\lambda(6-q)|f|_\infty \int_\Omega |u|^q dx}{2b\|u\|^4}\Big]^{\frac{1}{4-q}}.$$

Easy computations show that $\Phi_1'(t) > 0$ for all $0 < t < t_{\max}$ and $\Phi_1'(t) < 0$ for all $t > t_{\max}$. Therefore, $\Phi_1(t)$ achieves its maximum at $t_{\max}$; that is,

$$\Phi_1(t_{\max}) = \frac{4-q}{6-q} b\Big[\frac{2b}{6-q}\Big]^{\frac{2}{4-q}} \frac{\|u\|^{\frac{4(6-q)}{4-q}}}{\big(\lambda|f|_\infty \int_\Omega |u|^q dx\big)^{\frac{2}{4-q}}}.$$

Then it follows from (1.4) that

$$\begin{aligned}
&\eta(t_{\max}) - \int_\Omega |u|^6 dx \\
&\geq \Phi(t_{\max}) - \int_\Omega |u|^6 dx \\
&\geq \Phi_1(t_{\max}) - \int_\Omega |u|^6 dx \\
&> \frac{4-q}{6-q} b\Big[\frac{2b}{6-q}\Big]^{\frac{2}{4-q}} \frac{\|u\|^{\frac{4(6-q)}{4-q}}}{\big(\lambda|f|_\infty \int_\Omega |u|^q dx\big)^{\frac{2}{4-q}}} - \int_\Omega |u|^6 dx \\
&> \frac{4-q}{6-q} b\Big[\frac{2b}{6-q}\Big]^{\frac{2}{4-q}} |\Omega|^{-\frac{6-q}{3(4-q)}} (\lambda|f|_\infty)^{-\frac{2}{4-q}} \frac{\|u\|^{\frac{4(6-q)}{4-q}}}{|u|_6^{\frac{2q}{4-q}}} - \int_\Omega |u|^6 dx \\
&= \Big\{\frac{4-q}{6-q} b\Big[\frac{2b}{6-q}\Big]^{\frac{2}{4-q}} |\Omega|^{-\frac{6-q}{3(4-q)}} (\lambda|f|_\infty)^{-\frac{2}{4-q}} \Big(\frac{\|u\|^2}{|u|_6^2}\Big)^{\frac{2(6-q)}{4-q}} - 1\Big\} |u|_6^6 \\
&\geq \Big\{\frac{4-q}{6-q} b\Big[\frac{2b}{6-q}\Big]^{\frac{2}{4-q}} |\Omega|^{-\frac{6-q}{3(4-q)}} (\lambda|f|_\infty)^{-\frac{2}{4-q}} S^{\frac{2(6-q)}{4-q}} - 1\Big\} |u|_6^6 > 0
\end{aligned}$$

when $0 < \lambda < T_1$, where we can choose

$$T_1 = \Big\{\frac{4-q}{6-q} b\Big[\frac{2b}{6-q}\Big]^{\frac{2}{4-q}} |\Omega|^{-\frac{6-q}{3(4-q)}} (|f|_\infty)^{-\frac{2}{4-q}} S^{\frac{2(6-q)}{4-q}}\Big\}^{\frac{4-q}{2}}.$$

Consequently, there exist constants $t^\pm$ such that $0 < t^+ = t^+(u) < t_{\max} < t^- = t^-(u)$, $t^+u \in \mathcal{N}_\lambda^+$ and $t^-u \in \mathcal{N}_\lambda^-$.

(ii) Now we show that $\mathcal{N}_\lambda^0 = \emptyset$ for all $\lambda \in (0, T_1)$. By contradiction, assume there exists $u_0 \neq 0$ such that $u_0 \in \mathcal{N}_\lambda^0$, one obtains

$$a\|u_0\|^2 + b\|u_0\|^4 = \mu \int_\Omega \frac{|u_0|^2}{|x|^{2-\alpha}} dx + \int_\Omega |u_0|^6 dx + \lambda \int_\Omega f|u_0|^q dx, \tag{2.1}$$

$$4a\|u_0\|^2 + 2b\|u_0\|^4 = 4\mu \int_\Omega \frac{|u_0|^2}{|x|^{2-\alpha}} dx + \lambda(6-q) \int_\Omega f|u_0|^q dx. \tag{2.2}$$

It follows from (2.1) and (2.2) that

$$\begin{aligned}
\lambda \int_\Omega f|u_0|^q dx &= \frac{4}{6-q}\Big(a\|u_0\|^2 - \mu \int_\Omega \frac{|u_0|^2}{|x|^{2-\alpha}} dx\Big) + \frac{2}{6-q} b\|u_0\|^4 \\
&\geq \frac{4}{6-q}\big(a - \frac{\mu}{\mu_1}\big)\|u_0\|^2 + \frac{2}{6-q} b\|u_0\|^4 \\
&> \frac{2}{6-q} b\|u_0\|^4.
\end{aligned} \tag{2.3}$$

On the one hand, since the strict inequality $\|u_0\|^2 > S|u_0|_6^2$ holds for $u_0 \in \mathcal{N}_\lambda^0\backslash\{0\}$, we use a parameter Θ by

$$\begin{aligned}\Theta &= |\Omega|^{\frac{6-q}{3(4-q)}} S^{-\frac{2(6-q)}{4-q}} |f|_\infty^{\frac{2}{4-q}} \frac{\|u_0\|^{\frac{4(6-q)}{4-q}}}{\left(\int_\Omega f(u_0^+)^q dx\right)^{\frac{2}{4-q}}} - \int_\Omega |u_0|^6 dx \\ &> |\Omega|^{\frac{6-q}{3(4-q)}} S^{-\frac{2(6-q)}{4-q}} |f|_\infty^{\frac{2}{4-q}} \frac{(S|u_0|_6^2)^{\frac{2(6-q)}{4-q}}}{|f|_\infty^{\frac{2}{4-q}} |\Omega|^{\frac{6-q}{3(4-q)}} |u_0|_6^{\frac{2q}{4-q}}} - \int_\Omega |u_0|^6 dx \\ &= \int_\Omega |u_0|^6 dx - \int_\Omega |u_0|^6 dx = 0.\end{aligned} \tag{2.4}$$

On the other hand, by (2.3), one deduces that

$$\begin{aligned}\Theta &= |\Omega|^{\frac{6-q}{3(4-q)}} S^{-\frac{2(6-q)}{4-q}} \lambda^{\frac{2}{4-q}} |f|_\infty^{\frac{2}{4-q}} \frac{\|u_0\|^{\frac{4(6-q)}{4-q}}}{\left(\lambda \int_\Omega f(u_0^+)^q dx\right)^{\frac{2}{4-q}}} - \int_\Omega |u_0|^6 dx \\ &\le |\Omega|^{\frac{26-q}{3(4-q)}} S^{-\frac{2(6-q)}{4-q}} \lambda^{\frac{2}{4-q}} |f|_\infty^{\frac{2}{4-q}} \frac{\|u_0\|^{\frac{4(6-q)}{4-q}}}{\left(\frac{2}{6-q} b\|u_0\|^4\right)^{\frac{2}{4-q}}} \\ &\quad - \frac{2-q}{6-q}\Big(a\|u_0\|^2 - \mu \int_\Omega \frac{|u_0|^2}{|x|^{2-\alpha}} dx\Big) - \frac{b(4-q)}{6-q}\|u_0\|^4 \\ &= |\Omega|^{\frac{6-q}{3(4-q)}} S^{-\frac{2(6-q)}{4-q}} \lambda^{\frac{2}{4-q}} |f|_\infty^{\frac{2}{4-q}} \big(\frac{6-q}{2b}\big)^{\frac{2}{4-q}} \|u_0\|^4 \\ &\quad - \frac{2-q}{6-q}\big(a\|u_0\|^2 - \mu \int_\Omega \frac{|u_0|^2}{|x|^{2-\alpha}} dx\big) - \frac{b(4-q)}{6-q}\|u_0\|^4 \\ &\le |\Omega|^{\frac{6-q}{3(4-q)}} S^{-\frac{2(6-q)}{4-q}} \lambda^{\frac{2}{4-q}} |f|_\infty^{\frac{2}{4-q}} \big(\frac{6-q}{2b}\big)^{\frac{2}{4-q}} \|u_0\|^4 \\ &\quad - \frac{2-q}{6-q}\big(a - \frac{\mu}{\mu_1}\big)\|u_0\|^2 - \frac{b(4-q)}{6-q}\|u_0\|^4 \\ &< \|u_0\|^4 \Big[|\Omega|^{\frac{6-q}{3(4-q)}} S^{-\frac{2(6-q)}{4-q}} \lambda^{\frac{2}{4-q}} |f|_\infty^{\frac{2}{4-q}} \big(\frac{6-q}{2b}\big)^{\frac{2}{4-q}} - \frac{b(4-q)}{6-q}\Big] < 0,\end{aligned}$$

which contradicts (2.4), where the least inequality holds when $\lambda < T_1$. The proof is complete. $\square$

Lemma 2.2. *I_λ is coercive and bounded below on $\mathcal{N}_\lambda$.*

Proof. Assume $u \in \mathcal{N}_\lambda$, then by (1.4) we obtain

$$\begin{aligned}I_\lambda(u) &= \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{\mu}{2}\int_\Omega \frac{|u|^2}{|x|^{2-\alpha}} dx - \frac{1}{6}\int_\Omega |u|^6 dx - \frac{\lambda}{q}\int_\Omega f|u|^q dx \\ &\ge \frac{a}{3}\|u\|^2 + \frac{b}{12}\|u\|^4 - \frac{\mu}{3}\int_\Omega \frac{|u|^2}{|x|^{2-\alpha}} dx - \lambda\big(\frac{1}{q} - \frac{1}{6}\big)\Big|\int_\Omega f|u|^q dx\Big| \\ &\ge \frac{a\mu_1 - \mu}{3\mu_1}\|u\|^2 + \frac{b}{12}\|u\|^4 - \lambda\big(\frac{1}{q} - \frac{1}{6}\big)|f|_\infty |\Omega|^{\frac{6-q}{6}} S^{-q/2}\|u\|^q.\end{aligned}$$

Since $1 < q < 2, 0 < \mu < a\mu_1$, it follows that I_λ is coercive and bounded below on $\mathcal{N}_\lambda$. The proof is complete. $\square$

According to Lemma 2.1, we have $\mathcal{N}_\lambda = \mathcal{N}_\lambda^+ \cup \mathcal{N}_\lambda^-$ for all $\lambda \in (0, T_1)$. Moreover, we know that $\mathcal{N}_\lambda^+$ and $\mathcal{N}_\lambda^-$ are nonempty, and by Lemma 2.2 we may define

$$\alpha_\lambda = \inf_{u\in\mathcal{N}_\lambda} I_\lambda(u), \quad \alpha_\lambda^+ = \inf_{u\in\mathcal{N}_\lambda^+} I_\lambda(u), \quad \alpha_\lambda^- = \inf_{u\in\mathcal{N}_\lambda^-} I_\lambda(u).$$

Lemma 2.3. $\alpha_\lambda \le \alpha_\lambda^+ < 0$.

Proof. Assume $u \in \mathcal{N}_\lambda^+$, then we have

$$\int_\Omega |u|^6 dx < \frac{2-q}{6-q} a\|u\|^2 + \frac{4-q}{6-q} b\|u\|^4. \tag{2.5}$$

It follows from (2.5) that

$$\begin{aligned}
I_\lambda(u) &= \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{\mu}{2}\int_\Omega \frac{|u|^2}{|x|^{2-\alpha}}dx - \frac{1}{6}\int_\Omega |u|^6 dx - \frac{\lambda}{q}\int_\Omega f|u|^q dx \\
&= \big(\frac{1}{2} - \frac{1}{q}\big)\big(a - \frac{\mu}{\mu_1}\big)\|u\|^2 + \big(\frac{1}{4} - \frac{1}{q}\big)b\|u\|^4 + \big(\frac{1}{q} - \frac{1}{6}\big)\int_\Omega |u|^6 dx \\
&< \big(\frac{a}{2} - \frac{1}{q}\big)\big(a - \frac{\mu}{\mu_1}\big)\|u\|^2 + \big(\frac{1}{4} - \frac{1}{q}\big)b\|u\|^4 \\
&\quad + \big(\frac{1}{q} - \frac{1}{6}\big)\Big(\frac{2-q}{6-q}a\|u\|^2 + \frac{4-q}{6-q}b\|u\|^4\Big) \\
&= \frac{q-2}{6q}\big(4a - \frac{3\mu}{\mu_1}\big)a\|u\|^2 + \frac{1}{3}\big(\frac{1}{4} - \frac{1}{q}\big)b\|u\|^4 < 0.
\end{aligned}$$

By the definitions of α_λ and α_λ^+, we obtain that $\alpha_\lambda \le \alpha_\lambda^+ < 0$. This completes the proof. $\square$

Lemma 2.4. *For each $u \in \mathcal{N}_\lambda$, there exist $\varepsilon > 0$ and a continuously differentiable function $\hat{f} = \hat{f}(w) > 0, w \in H_0^1(\Omega)$, $\|w\| < \varepsilon$ satisfying*

$$\hat{f}(0) = 1, \quad \hat{f}(w)(u + w) \in \mathcal{N}_\lambda, \quad \forall w \in H_0^1(\Omega),\ \|w\| < \varepsilon.$$

Proof. For $u \in \mathcal{N}_\lambda$, define $\hat{F} : \mathbb{R} \times H_0^1(\Omega) \to \mathbb{R}$ by

$$\begin{aligned}
\hat{F}(t, w) &= t^{2-q}a\int_\Omega |\nabla(u + w)|^2 dx + t^{4-q}b\Big(\int_\Omega |\nabla(u + w)|^2 dx\Big)^2 \\
&\quad - t^{2-q}\mu\int_\Omega \frac{|u + w|^2}{|x|^{2-\alpha}}dx - t^{6-q}\int_\Omega |u + w|^6 dx - \lambda\int_\Omega f|u + w|^q dx.
\end{aligned}$$

As $u \in \mathcal{N}_\lambda$, it is easy to get that $\hat{F}(1, 0) = 0$ and

$$\hat{F}_t(1, 0) = (2-q)a\|u\|^2 + (4-q)b\|u\|^4 - (2-q)\mu\int_\Omega \frac{|u|^2}{|x|^{2-\alpha}}dx - (6-q)\int_\Omega |u|^6 dx.$$

Since $u \ne 0$, by Lemma 2.1, we deduce that $\hat{F}_t(1, 0) \ne 0$. Then, applying the implicit function theorem at the point $(0, 1)$, we obtain that $\varepsilon > 0$ and a continuously differentiable function $\hat{f} : B(0, \varepsilon) \subset H_0^1(\Omega) \to \mathbb{R}^+$ satisfying that

$$\hat{f}(0) = 1, \quad \hat{f}(w) > 0, \quad f(w)(u + w) \in \mathcal{N}_\lambda, \quad \forall w \in H_0^1(\Omega) \text{ with } \|w\| < \varepsilon.$$

The proof is complete. $\square$

Lemma 2.5. *For each $u \in \mathcal{N}_\lambda^-$, there exist $\varepsilon > 0$ and a continuously differentiable function $\tilde{f} = \tilde{f}(v) > 0, v \in H_0^1(\Omega)$, $\|v\| < \varepsilon$ satisfying that*

$$\tilde{f}(0) = 1, \quad \tilde{f}(v)(u + v) \in \mathcal{N}_\lambda^-, \quad \forall v \in H_0^1(\Omega),\ \|v\| < \varepsilon.$$

Proof. Similar to the process in Lemma 2.4, for $u \in \mathcal{N}_\lambda^-$, define $\tilde{F} : \mathbb{R} \times H_0^1(\Omega) \to \mathbb{R}$ by

$$\begin{aligned}\tilde{F}(t,v) &= t^{2-q} a \int_\Omega |\nabla(u+v)|^2 dx + t^{4-q} b\Big(\int_\Omega |\nabla(u+v)|^2 dx\Big)^2 \\ &\quad - t^{2-q}\mu \int_\Omega \frac{|u+v|^2}{|x|^{2-\alpha}} dx - t^{6-q}\int_\Omega |u+v|^6 dx - \lambda \int_\Omega f|u+v|^q dx.\end{aligned}$$

As $u \in \mathcal{N}_\lambda^-$, we obtain $\tilde{F}(1,0) = 0$ and $\tilde{F}_t(1,0) < 0$. Thus, we can apply the implicit function theorem at the point $(0,1)$ to get the result. This completes the proof. □

Lemma 2.6. *If $\{u_n\} \subset \mathcal{N}_\lambda$ is a minimizing sequence of I_λ, for any $\varphi \in H_0^1(\Omega)$, then*

$$-\frac{|f_n'(0)|\|u_n\| + \|\varphi\|}{n} \le \langle I_\lambda'(u_n), \varphi\rangle \le \frac{|f_n'(0)|\|u_n\| + \|\varphi\|}{n}. \tag{2.6}$$

Proof. By Lemma 2.2, we obtain I_λ is coercive on $\mathcal{N}_\lambda$. Then, applying the Ekeland variational principle [9], there exists a minimizing sequence $\{u_n\} \subset \mathcal{N}_\lambda$ of I_λ such that

$$I_\lambda(u_n) < \alpha_\lambda + \frac{1}{n}, \quad I_\lambda(v) - I_\lambda(u_n) \ge -\frac{1}{n}\|v - u_n\|, \quad \forall v \in \mathcal{N}_\lambda. \tag{2.7}$$

Note that $I_\lambda(|u_n|) = I_\lambda(u_n)$, then we obtain that $u_n \ge 0$. Lemma 2.2 suggests that $\{u_n\}$ is bounded in $H_0^1(\Omega)$. Thus there exist a subsequence (still denoted by $\{u_n\}$) and u_* in $H_0^1(\Omega)$ such that

$$\begin{gathered} u_n \rightharpoonup u_* \quad \text{weakly in } H_0^1(\Omega), \\ u_n \to u_* \quad \text{strongly in } L^p(\Omega)\ (1 \le p < 6), \\ u_n(x) \to u_*(x) \quad \text{a.e. in } \Omega. \end{gathered}$$

Pick $t > 0$ sufficiently small, $\varphi \in H_0^1(\Omega)$, and let $u = u_n$, $w = t\varphi \in H_0^1(\Omega)$ in Lemma 2.4, then one obtains $f_n(t) = f_n(t\varphi)$ and $f_n(0) = 1$, $f_n(t)(u_n + t\varphi) \in \mathcal{N}_\lambda$. Note that

$$a\|u_n\|^2 + b\|u_n\|^4 - \int_\Omega u_n^6 dx - \mu \int_\Omega |x|^{\alpha-2} u_n^2 dx - \lambda \int_\Omega f u_n^q dx = 0. \tag{2.8}$$

From (2.7), one has

$$\begin{aligned}\frac{1}{n}[|f_n(t) - 1| \cdot \|u_n\| + t f_n(t)\|\varphi\|] &\ge \frac{1}{n}\|f_n(t)(u_n + t\varphi) - u_n\| \\ &\ge I_\lambda(u_n) - I_\lambda[f_n(t)(u_n + t\varphi)],\end{aligned} \tag{2.9}$$

and

$$\begin{aligned} & I_\lambda(u_n) - I_\lambda[f_n(t)(u_n + t\varphi)] \\ &= \frac{1 - f_n^2(t)}{2} a\|u_n\|^2 + \frac{1 - f_n^4(t)}{4} b\|u_n\|^4 + \mu\frac{f_n^2(t) - 1}{2}\int_\Omega (u_n + t\varphi)^2 |x|^{\alpha-2} dx \\ &\quad + \frac{f_n^6(t) - 1}{6}\int_\Omega (u_n + t\varphi)^6 dx + \lambda\frac{f_n^q(t) - 1}{q}\int_\Omega f(u_n + t\varphi)^q dx \\ &\quad + \frac{f_n^2(t)}{2}\Big(a + \frac{f_n^2(t)}{2} b(\|u_n\|^2 + \|u_n + t\varphi\|^2)\Big)\big(\|u_n\|^2 - \|u_n + t\varphi\|^2\big) \\ &\quad + \frac{1}{6}\Big(\int_\Omega (u_n + t\varphi)^6 dx - \int_\Omega u_n^6 dx\Big) + \frac{\lambda}{q}\int_\Omega f((u_n + t\varphi)^q - u_n^q) dx \end{aligned}$$

$$+\frac{\mu}{2}\int_\Omega |x|^{\alpha-2}((u_n+t\varphi)^2-u_n^2)dx,$$

then, by (2.8) and (2.9), dividing by t and letting $t\to 0$, we obtain

$$\begin{aligned}
&\frac{|f_n'(0)|\|u_n\|+\|\varphi\|}{n}\\
&\geq -f_n'(0)a\|u_n\|^2+f_n'(0)b\|u_n\|^4+f_n'(0)\int_\Omega u_n^6dx+\lambda f_n'(0)\int_\Omega fu_n^qdx\\
&\quad +\mu f_n'(0)\int_\Omega |x|^{\alpha-2}u_n^2dx-(a+b\|u_n\|^2)\int_\Omega(\nabla u_n\cdot\nabla\varphi)dx\\
&\quad +\int_\Omega u_n^5\varphi dx+\lambda\int_\Omega fu_n^{q-1}\varphi dx+\mu\int_\Omega |x|^{\alpha-2}u_n\varphi dx\\
&=-f_n'(0)(a\|u_n\|^2+b\|u_n\|^4-\int_\Omega u_n^6dx-\lambda\int_\Omega fu_n^qdx-\mu\int_\Omega |x|^{\alpha-2}u_n^2dx)\\
&\quad -(a+b\|u_n\|^2)\int_\Omega(\nabla u_n\cdot\nabla\varphi)dx+\int_\Omega u_n^5\varphi dx\\
&\quad +\lambda\int_\Omega fu_n^{q-1}\varphi dx+\mu\int_\Omega |x|^{\alpha-2}u_n\varphi dx\\
&=-(a+b\|u_n\|^2)\int_\Omega(\nabla u_n\cdot\nabla\varphi)dx+\int_\Omega u_n^5\varphi dx+\lambda\int_\Omega fu_n^{q-1}\varphi dx\\
&\quad +\mu\int_\Omega |x|^{\alpha-2}u_n\varphi dx.
\end{aligned}$$

Thus, it follows that

$$\begin{aligned}
-\frac{|f_n'(0)|\|u_n\|+\|\varphi\|}{n}&\leq (a+b\|u_n\|^2)\int_\Omega(\nabla u_n\cdot\nabla\varphi)dx-\int_\Omega u_n^5\varphi dx\\
&\quad -\lambda\int_\Omega fu_n^{q-1}\varphi dx-\mu\int_\Omega |x|^{\alpha-2}u_n\varphi dx\\
&=\langle I_\lambda'(u_n),\varphi\rangle
\end{aligned}\tag{2.10}$$

for any $\varphi\in H_0^1(\Omega)$. As (2.10) also holds for $-\varphi$, one sees that (2.6) holds. Moreover, Lemma 2.4 implies that there exists a constant $C>0$, such that $|f_n'(0)|\leq C$ for all $n\in N$. So, passing to the limit as $n\to\infty$ in (2.6), we have

$$\begin{aligned}
&(a+b\lim_{n\to\infty}\|u_n\|^2)\int_\Omega(\nabla u_*\cdot\nabla\varphi)dx\\
&-\int_\Omega u_*^5\varphi dx-\lambda\int_\Omega fu_*^{q-1}\varphi dx-\mu\int_\Omega |x|^{\alpha-2}u_*\varphi dx=0
\end{aligned}\tag{2.11}$$

for all $\varphi\in H_0^1(\Omega)$. The proof is complete. □

We define

$$\Lambda=\frac{abS^3}{4}+\frac{b^3S^6}{24}+\frac{(b^2S^4+4aS)^{3/2}}{24}.$$

Lemma 2.7. *Suppose $1<q<2$, $0\leq\beta<2$, and let $\{u_n\}\subset\mathcal{N}_\lambda^-$ be a minimizing sequence of I_λ with $\alpha_\lambda^-<\Lambda-D\lambda^{\frac{2}{2-q}}$ where*

$$D=\Big(\frac{(4-q)}{4q}|\Omega|^{\frac{6-q}{6}}|f|_\infty S^{-q/2}\Big)^{\frac{2}{2-q}}\Big(\frac{2q}{a}\Big)^{\frac{q}{2-q}},$$

then there exists $u \in H_0^1(\Omega)$ such that $u_n \to u$ in $L^6(\Omega)$.

Proof. Let $\{u_n\} \subset H_0^1(\Omega)$ be a $(PS)_c$ sequence for I_λ, namely

$$I_\lambda(u_n) \to c, \quad I'_\lambda(u_n) \to 0, \quad \text{as } n \to +\infty. \tag{2.12}$$

We see that $\{u_n\}$ is bounded in $H_0^1(\Omega)$. Indeed, by (2.12) and (1.4), one has

$$\begin{aligned} 1 + c + o(\|u_n\|) &\geq I_\lambda(u_n) - \frac{1}{6}\langle I'_\lambda(u_n), u_n\rangle \\ &\geq \frac{1}{3}\big(a - \frac{\mu}{\mu_1}\big)\|u_n\|^2 + \frac{b}{12}\|u_n\|^4 - \lambda\big(\frac{1}{q} - \frac{1}{6}\big)|f|_\infty \int_\Omega |u_n|^q dx \\ &\geq \frac{1}{3}\big(a - \frac{\mu}{\mu_1}\big)\|u_n\|^2 - \lambda\big(\frac{1}{q} - \frac{1}{6}\big)|f|_\infty S^{-q/2}|\Omega|^{\frac{6-q}{6}}\|u_n\|^q. \end{aligned}$$

Since $0 < \mu < a\mu_1, 1 < q < 2$, it implies that $\{u_n\}$ is bounded in $H_0^1(\Omega)$. So there exist a subsequence (still denoted by $\{u_n\}$) and $u \in H_0^1(\Omega)$ such that

$$\begin{gathered} u_n \rightharpoonup u, \quad \text{weakly in } H_0^1(\Omega), \\ u_n \to u, \quad \text{strongly in } L^p(\Omega)\ (1 \leq p < 6), \\ u_n(x) \to u(x), \quad \text{a.e. in } \Omega. \end{gathered}$$

Note that $I_\lambda(|u_n|) = I_\lambda(u_n)$, then we obtain that $u_n \geq 0$. According to the concentration compactness principle (see [20]), there exists a subsequence, say $\{u_n\}$, such that

$$\begin{gathered} |\nabla u_n|_2^2 \rightharpoonup d\eta \geq |\nabla u|_2^2 + \sum_{j\in J} \eta_j \delta_{x_j}, \\ |u_n|_6^6 \to d\nu = |u|_6^6 + \sum_{j\in J} \nu_j \delta_{x_j}, \end{gathered}$$

where J is an at most countable index set, δ_{x_j} is the Dirac mass at x_j, and let $x_j \in \Omega$ in the support of η, ν. we have

$$\eta_j, \nu_j \geq 0, \quad \eta_j \geq S\nu_j^{1/3}. \tag{2.13}$$

For any $\varepsilon > 0$ sufficiently small, let $\psi_{\varepsilon,j}(x)$ be a smooth cut-off function centered at x_j such that $0 \leq \psi_{\varepsilon,j}(x) \leq 1$,

$$\begin{gathered} \psi_{\varepsilon,j}(x) = 1 \quad \text{in } B(x_j, \frac{\varepsilon}{2}), \quad \psi_{\varepsilon,j}(x) = 0 \quad \text{in } B(x_j, \varepsilon), \\ |\nabla \psi_{\varepsilon,j}(x)| \leq \frac{4}{\varepsilon}. \end{gathered}$$

By (1.4), we obtain

$$\begin{aligned} \big|\int_\Omega f|u_n|^q \psi_{\varepsilon,j} dx\big| &\leq |f|_\infty \int_{B(x_j,\varepsilon)} |u_n|^q dx \\ &\leq |f|_\infty \Big(\int_{B(x_j,\varepsilon)} |u_n|^{q\cdot\frac{6}{q}} dx\Big)^{q/6} \Big(\int_{B(x_j,\varepsilon)} 1 dx\Big)^{\frac{6-q}{6}} \\ &\leq |f|_\infty S^{-q/2}\|u_n\|^q \varepsilon^{\frac{6-q}{2}}. \end{aligned}$$

Notice that $\{u_n\}$ is bounded in $H_0^1(\Omega)$, and $u_n \rightharpoonup u$ weakly in $L^6(\Omega)$, it implies that

$$\lim_{\varepsilon\to 0} \lim_{n\to\infty} \int_\Omega f|u_n|^q \psi_{\varepsilon,j} dx = 0.$$

Similarly, we obtain

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\mu\int_\Omega |x|^{\alpha-2}|u_n|^2\psi_{\varepsilon,j}dx=0.$$

Since $\{\psi_{\varepsilon,j}u_n\}$ is bounded in $H_0^1(\Omega)$, taking the test function $\psi_{\varepsilon,j}u_n$ in $I'_\lambda(u_n)\to 0$, one deduces that

$$\begin{aligned}
0&=\lim_{\varepsilon\to 0}\lim_{n\to\infty}\langle I'_\lambda(u_n),\psi_{\varepsilon,j}u_n\rangle\\
&=\lim_{\varepsilon\to 0}\lim_{n\to\infty}\Big\{(a+b\|u_n\|^2)\int_\Omega(\nabla u_n\cdot\nabla(\psi_{\varepsilon,j}u_n))dx\\
&\quad-\mu\int_\Omega|x|^{\alpha-2}u_n^2\psi_{\varepsilon,j}dx\\
&\quad-\int_\Omega u_n^6\psi_{\varepsilon,j}dx-\lambda\int_\Omega fu_n^q\psi_{\varepsilon,j}dx\Big\}\\
&\ge\lim_{\varepsilon\to 0}\lim_{n\to\infty}\Big\{(a+b\|u_n\|^2)\int_\Omega(|\nabla u_n|^2\psi_{\varepsilon,j}+u_n\nabla u_n\nabla\psi_{\varepsilon,j})dx\\
&\quad-\mu\int_\Omega|x|^{\alpha-2}u_n^2\psi_{\varepsilon,j}dx-\int_\Omega u_n^6\psi_{\varepsilon,j}dx-\lambda\int_\Omega fu_n^q\psi_{\varepsilon,j}dx\Big\}\\
&\ge\lim_{\varepsilon\to 0}\lim_{n\to\infty}\Big\{(a+b\|u_n\|^2)\int_\Omega(|\nabla u_n|^2\psi_{\varepsilon,j}+u_n\nabla u_n\nabla\psi_{\varepsilon,j})dx\\
&\quad-\int_\Omega u_n^6\psi_{\varepsilon,j}dx\Big\}\\
&\ge(a+b\eta_j)\eta_j-\nu_j.
\end{aligned}$$

Thus $\nu_j\ge(a+b\eta_j)\eta_j$. By (2.13) we obtain

$$\nu_j^{2/3}\ge aS+bS^2\nu_j^{1/3},\quad\text{or}\quad \eta_j=\nu_j=0. \tag{2.14}$$

Set $X=\nu_j^{1/3}$, it follows from (2.14) that

$$X^2\ge aS+bS^2X,$$

then

$$X\ge\frac{bS^2+\sqrt{b^2S^4+4aS}}{2};$$

therefore

$$\eta_j\ge SX\ge\frac{bS^3+\sqrt{b^2S^6+4aS^3}}{2}=:K.$$

Next we show that $\eta_j\ge\sqrt{aS^3}$ is impossible. So the set J is empty. Assume the contrary, there exists some $j_0\in J$ such that $\eta_{j_0}\ge\sqrt{aS^3}$. By (2.12), (1.4) and Young inequality, we obtain

$$\begin{aligned}
c&=\lim_{n\to\infty}I_\lambda(u_n)\\
&=\lim_{n\to\infty}\Big\{\frac{a}{2}\|u_n\|^2+\frac{b}{4}\|u_n\|^4-\frac{\mu}{2}\int_\Omega|x|^{\alpha-2}|u_n|^2dx-\frac16\int_\Omega|u_n|^6dx\\
&\quad-\frac{\lambda}{q}\int_\Omega f|u_n|^qdx-\frac14\Big(a\|u_n\|^2+b\|u_n\|^4-\mu\int_\Omega|x|^{\alpha-2}|u_n|^2dx\\
&\quad-\int_\Omega|u_n|^6dx-\lambda\int_\Omega f|u_n|^qdx\Big)\Big\}
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{n\to\infty} \Big\{ \big(\frac{1}{2}-\frac{1}{4}\big)a\|u_n\|^2 + b\big(\frac{1}{4}-\frac{1}{4}\big)\|u_n\|^4 \\
&\quad + \big(\frac{1}{4}-\frac{1}{6}\big)\int_\Omega |u_n|^6 dx - \lambda\big(\frac{1}{q}-\frac{1}{4}\big)\int_\Omega f|u_n|^q dx \Big\} \\
&\ge \Big\{ \big(\frac{1}{2}-\frac{1}{4}\big)a\Big(\|u\|^2 + \sum_{j\in J}\mu_j\Big) + \big(\frac{1}{4}-\frac{1}{6}\big)\Big(\int_\Omega |u|^6 dx + \sum_{j\in J}\nu_j\Big) \\
&\quad + \big(\frac{1}{4}-\frac{1}{4}\big)b\Big(\|u\|^2 + \sum_{j\in J}\mu_j\Big)^2 - \lambda\big(\frac{1}{q}-\frac{1}{4}\big)\int_\Omega |f||u|^q dx \Big\} \\
&\ge \big(\frac{1}{2}-\frac{1}{4}\big)a\eta_{j_0} + \big(\frac{1}{4}-\frac{1}{4}\big)b\eta_{j_0}^2 + \big(\frac{1}{4}-\frac{1}{6}\big)\nu_{j_0} + \frac{1}{4}a\|u\|^2 \\
&\quad - \lambda\big(\frac{1}{q}-\frac{1}{4}\big)|f|_\infty|\Omega|^{\frac{6-q}{6}} S^{-q/2}\|u\|^q \\
&\ge \frac{a}{2}K + \frac{b}{4}K^2 - \frac{K^3}{6S^3} - \frac{1}{4}\Big(aK + bK^2 - \frac{K^3}{S^3}\Big) - D\lambda^{\frac{2}{2-q}},
\end{aligned}
$$

where

$$
D = \Big(\frac{(4-q)}{4q}|\Omega|^{\frac{6-q}{6}}|f|_\infty S^{-q/2}\Big)^{\frac{2}{2-q}}\Big(\frac{2q}{a}\Big)^{\frac{q}{2-q}},
$$

$$
\frac{aK}{2} + \frac{b}{4}K^2 - \frac{K^3}{6S^3} = \Lambda, \quad K\Big(a + bK - \frac{K^2}{S^3}\Big) = 0.
$$

Indeed,

$$
\begin{aligned}
&\frac{aK}{2} + \frac{b}{4}K^2 - \frac{K^3}{6S^3} \\
&= K\big(\frac{a}{2} + \frac{bK}{4} - \frac{K^2}{6S^3}\big) \\
&= K\big[\frac{a}{2} + \frac{b}{4}\cdot\frac{bS^3 + \sqrt{b^2S^6 + 4aS^3}}{2} - \frac{2b^2S^6 + 4aS^3 + 2bS^3\sqrt{b^2S^6 + 4aS^3}}{24S^3}\big] \\
&= K\big[\frac{a}{2} + \frac{b^2S^3 + b\sqrt{b^2S^6 + 4aS^3}}{8} - \frac{b^2S^3 + 2 + b\sqrt{b^2S^6 + 4aS^3}}{12}\big] \\
&= K\big[\frac{8a + b^2S^3 + b\sqrt{b^2S^6 + 4aS^3}}{24}\big] \\
&= \frac{12abS^3 + 2b^3S^6 + (2b^2S^3 + 8a)\sqrt{b^2S^6 + 4aS^3}}{48} \\
&= \frac{abS^3}{4} + \frac{b^3S^6}{24} + \frac{(b^2S^3 + 4a)\sqrt{b^2S^6 + 4aS^3}}{24} = \Lambda.
\end{aligned}
$$

and

$$
\begin{aligned}
&a + bK - \frac{K^2}{S^3} \\
&= a + b\frac{bS^3 + \sqrt{b^2S^6 + 4aS^3}}{2} - \frac{1}{S^3}\frac{2b^2S^6 + 4aS^3 + 2bS^3\sqrt{b^2S^6 + 4aS^3}}{4} \\
&= a + \frac{b^2S^3 + b\sqrt{b^2S^6 + 4aS^3}}{2} - \frac{2S^3(b^2S^3 + 2a + b\sqrt{b^2S^6 + 4aS^3})}{4S^3} \\
&= \frac{2a + b^2S^3 + b\sqrt{b^2S^6 + 4aS^3} - b^2S^3 - 2a - b\sqrt{b^2S^6 + 4aS^3}}{2} = 0.
\end{aligned}
$$

Therefore, we obtain $\Lambda - D\lambda^{\frac{2}{2-q}} \le c < \Lambda - D\lambda^{\frac{2}{2-q}}$, which is a contradiction. It suggests that J is empty, which means that $\int_\Omega |u_n|^6 dx \to \int_\Omega |u|^6 dx$ as $n \to \infty$. This completes the proof. □

It is known that the function

$$U_\varepsilon(x) = \frac{(3\varepsilon)^{1/4}}{(\varepsilon + |x|^2)^{1/2}}, \quad x \in \mathbb{R}^3, \ \varepsilon > 0$$

satisfies

$$-\Delta U_\varepsilon = U_\varepsilon^5 \text{ in } \mathbb{R}^3.$$

Set

$$C_\varepsilon = (3\varepsilon)^{1/4}, \quad y_\varepsilon(x) = \frac{U_\varepsilon(x)}{C_\varepsilon}.$$

We select a cut-off function $\varphi \in C_0^\infty(\Omega)$ such that $\varphi(x) = 1$ for $|x| < R_0$, and $\varphi(x) = 0$ for $|x| > 2R_0$, $0 \le \varphi(x) \le 1$. Let $u_\varepsilon(x) = \varphi(x) y_\varepsilon(x)$, $v_\varepsilon(x) = \frac{u_\varepsilon(x)}{|u_\varepsilon|_6}$, then $|v_\varepsilon|_6^6 = 1$. Therefore we obtain the following results (see details in [29])

$$\begin{gathered}
\|v_\varepsilon\|^2 \le S + C\varepsilon^{1/2}, \\
\|v_\varepsilon\|^6 \le S^3 + C\varepsilon^{1/2}, \\
\|v_\varepsilon\|^{12} \le S^6 + C\varepsilon^{1/2}, \\
\|v_\varepsilon\|^{18} \le S^9 + C\varepsilon^{1/2}, \\
\|v_\varepsilon\|^{24} \le S^{12} + C\varepsilon^{1/2}.
\end{gathered} \tag{2.15}$$

and

$$\begin{gathered}
O(\varepsilon^{q/4}) \le \int_\Omega |u_\varepsilon|^q dx \le O(\varepsilon^{q/4}), \\
\int_\Omega |x|^{\alpha-2} |u_\varepsilon|^q dx = O(\varepsilon^{\alpha/2}).
\end{gathered} \tag{2.16}$$

Lemma 2.8. *Assume* $1 < q < 2$, *and* $0 < \alpha < 1$. *Then*

$$\sup_{t \ge 0} I_\lambda(t u_\varepsilon) < \Lambda - D\lambda^{\frac{2}{2-q}}.$$

Proof. We claim that there exist $t_\varepsilon > 0$ and positive constants t_0, t_1 which are independent of ε, λ, such that $\sup_{t\ge0} I_\lambda(tu_\varepsilon) = I_\lambda(t_\varepsilon u_\varepsilon)$ and

$$0 < t_0 \le t_\varepsilon \le t_1 < \infty. \tag{2.17}$$

As $\lim_{t\to+\infty} I_\lambda(tu_\varepsilon) = -\infty$, there exists $t_\varepsilon > 0$, such that

$$I_\lambda(t_\varepsilon u_\varepsilon) = \sup_{t\ge0} I_\lambda(t_\varepsilon u_\varepsilon), \quad \text{and} \quad \frac{dI_\lambda(t_\varepsilon u_\varepsilon)}{dt}\Big|_{t=t_\varepsilon} = 0.$$

Then

$$t_\varepsilon a\|u_\varepsilon\|^2 + t_\varepsilon^3 b\|u_\varepsilon\|^4 - \mu t_\varepsilon \int_\Omega |x|^{\alpha-2} u_\varepsilon^2 dx - t_\varepsilon^5 \int_\Omega u_\varepsilon^6 dx - \lambda t_\varepsilon^{q-1} \int_\Omega f u_\varepsilon^q dx = 0, \tag{2.18}$$

and

$$a\|u_\varepsilon\|^2 + 3t_\varepsilon^2 b\|u_\varepsilon\|^4 - \mu \int_\Omega |x|^{\alpha-2} u_\varepsilon^2 dx - 5t_\varepsilon^4 \int_\Omega u_\varepsilon^6 dx - \lambda(q-1) t_\varepsilon^{q-2} \int_\Omega f u_\varepsilon^q dx < 0.$$

Therefore,

$$\begin{aligned}&(2-q)t_\varepsilon a\|u_\varepsilon\|^2+(4-q)t_\varepsilon^3 b\|u_\varepsilon\|^4-(2-q)\mu t_\varepsilon\int_\Omega\frac{u_\varepsilon^2}{|x|^{2-\alpha}}dx\\&<(6-q)t_\varepsilon^5\int_\Omega u_\varepsilon^6dx.\end{aligned}\tag{2.19}$$

On the one hand, we can get easily from (2.19) that t_ε is bounded below, so, there exists a positive constant $t_0>0$ (independent of ε,λ), such that $0<t_0\le t_\varepsilon$. On the other hand, it follows from (2.18) that

$$\frac{a\|u_\varepsilon\|^2}{t_\varepsilon^2}+b\|u_\varepsilon\|^2=t_\varepsilon^2\int_\Omega u_\varepsilon^6dx+\frac{\lambda}{t_\varepsilon^{4-q}}\int_\Omega fu_\varepsilon^qdx+\frac{\mu}{t_\varepsilon^2}\int_\Omega|x|^{\alpha-2}u_\varepsilon^2dx,$$

thus, t_ε is bounded above for all $\varepsilon>0$ sufficiently small. Then (2.17) holds. Set

$$h(t_\varepsilon u_\varepsilon)=\frac{a}{2}t_\varepsilon^2\|u_\varepsilon\|^2+\frac{b}{4}t_\varepsilon^4\|u_\varepsilon\|^4-\frac{t_\varepsilon^6}{6}\int_\Omega u_\varepsilon^6dx.$$

We claim that there exists a positive constant c_7 (independent of ε,λ), such that

$$h(t_\varepsilon u_\varepsilon)\le\Lambda+c_7\varepsilon^{1/2}.\tag{2.20}$$

Indeed, set

$$g(t)=\frac{a}{2}t^2\|u_\varepsilon\|^2+\frac{b}{4}t^4\|u_\varepsilon\|^4-\frac{t^6}{6}\int_\Omega u_\varepsilon^6dx.$$

Since $\lim_{t\to\infty}g(t)=-\infty, g(0)=0$, and $\lim_{t\to 0^+}g(t)>0$, it follows that $\sup_{t\ge0}g(t)$ attained at $T_\varepsilon>0$, namely,

$$g'(t)|_{T_\varepsilon}=aT_\varepsilon\|u_\varepsilon\|^2+bT_\varepsilon^3\|u_\varepsilon\|^4-T_\varepsilon^5\int_\Omega u_\varepsilon^6dx=0.$$

Then

$$T_\varepsilon^4\int_\Omega u_\varepsilon^6dx-a\|u_\varepsilon\|^2-bT_\varepsilon^2\|u_\varepsilon\|^4=0;$$

therefore

$$T_\varepsilon=\Big(\frac{b\|u_\varepsilon\|^4+\sqrt{b^2\|u_\varepsilon\|^8+4a\|u_\varepsilon\|^2\int_\Omega u_\varepsilon^6dx}}{2\int_\Omega u_\varepsilon^6dx}\Big)^{1/2}.$$

Notice that $g(t)$ is increasing in $[0,T_\varepsilon]$, then by (2.15), one has

$$\begin{aligned}h(t_\varepsilon u_\varepsilon)&\le g(T_\varepsilon)\\&=\frac{a}{2}T_\varepsilon^2\|u_\varepsilon\|^2+\frac{b}{4}T_\varepsilon^4\|u_\varepsilon\|^4-\frac{T_\varepsilon^6}{6}\int_\Omega u_\varepsilon^6dx\\&=T_\varepsilon^2\big(\frac{a}{3}\|u_\varepsilon\|^2+\frac{b}{12}T_\varepsilon^2\|u_\varepsilon\|^4\big)\\&=T_\varepsilon^2\Big(\frac{a}{3}\|u_\varepsilon\|^2+\frac{b^2\|u_\varepsilon\|^8+b\|u_\varepsilon\|^4\sqrt{b^2\|u_\varepsilon\|^8+4a\|u_\varepsilon\|^2\int_\Omega u_\varepsilon^6dx}}{24\int_\Omega u_\varepsilon^6dx}\Big)\\&=\frac{ab\|u_\varepsilon\|^6}{6\int_\Omega u_\varepsilon^6dx}+\frac{b\|u_\varepsilon\|^6}{12\int_\Omega u_\varepsilon^6dx}+\frac{\|u_\varepsilon\|^2\sqrt{b^2\|u_\varepsilon\|^8+4a\|u_\varepsilon\|^2\int_\Omega u_\varepsilon^6dx}}{6\int_\Omega u_\varepsilon^6dx}\\&\quad+\frac{b^3\|u_\varepsilon\|^{12}}{24(\int_\Omega u_\varepsilon^6dx)^2}+\frac{b^2\|u_\varepsilon\|^8\sqrt{b^2\|u_\varepsilon\|^8+4a\|u_\varepsilon\|^2\int_\Omega u_\varepsilon^6dx}}{24(\int_\Omega u_\varepsilon^6dx)^2}\end{aligned}$$

$$
\begin{aligned}
&= \frac{ab\|u_\varepsilon\|^6}{4\int_\Omega u_\varepsilon^6 dx} + \frac{b^3\|u_\varepsilon\|^{12}}{24(\int_\Omega u_\varepsilon^6 dx)^2} + \frac{(b^2\|u_\varepsilon\|^8 + 4a\|u_\varepsilon\|^2\int_\Omega u_\varepsilon^6 dx)^{3/2}}{24(\int_\Omega u_\varepsilon^6 dx)^2} \\
&\le \frac{ab(S^{\frac{9}{2}} + c_4\varepsilon^{1/2})}{4(S^{3/2} + c_2\varepsilon^{3/2})} + \frac{b^3(S^9 + c_6\varepsilon^{1/2})}{24(S^{3/2} + c_2\varepsilon^{3/2})^2} \\
&\quad + \frac{[b^2(S^6 + c_5\varepsilon^{1/2}) + 4a(S^{3/2} + c_1\varepsilon^{1/2})(S^{3/2} + c_2\varepsilon^{3/2})]^{3/2}}{24(S^{3/2} + c_2\varepsilon^{3/2})^2} \\
&\le \frac{abS^3}{4} + \frac{b^3S^6}{24} + \frac{(b^2S^6 + 4aS^3)^{3/2}}{24S^3} + c_7\varepsilon^{1/2} \\
&= \Lambda + c_7\varepsilon^{1/2}.
\end{aligned}
$$

Consequently, there exists $c_7 > 0$ (independent of ε, λ) such that (2.20) holds.

Since $0 < \alpha < 1$, from [5], there exists a positive constant c_8 (independent of ε, λ) such that

$$
\int_\Omega |x|^{\alpha-2} u_\varepsilon^2 dx = c_8\varepsilon^{\alpha/2}. \tag{2.21}
$$

Therefore, from (2.16), (2.20) and (2.21), it holds

$$
\begin{aligned}
I_\lambda(t_\varepsilon u_\varepsilon) &= h(t_\varepsilon u_\varepsilon) - \frac{\mu t_\varepsilon^2}{2}\int_\Omega |x|^{\alpha-2}u_\varepsilon^2 dx - \lambda\frac{t_\varepsilon^q}{q}\int_\Omega f u_\varepsilon^q dx \\
&\le \Lambda + c_7\varepsilon^{1/2} - \frac{\mu}{2}t_0^2 c_8\varepsilon^{\alpha/2} + \lambda\frac{T_1^q|f|_\infty}{q}\int_\Omega u_\varepsilon^q dx \\
&= \Lambda + c_7\varepsilon^{1/2} - c_9\varepsilon^{\alpha/2} + \lambda c_{10}\varepsilon^{q/4}
\end{aligned} \tag{2.22}
$$

where $c_9 = \frac{\mu}{2}t_0^2 c_8$, $c_{10} = \frac{T_1^q|f|_\infty}{q}$. Notice that $0 < \alpha < 1$. Let

$$
\varepsilon = \lambda^{\frac{4}{2-q}}, \quad \lambda < \lambda_0 = \Big(\frac{c_9}{c_7 + c_{10} + D}\Big)^{\frac{2-q}{2(1-\alpha)}}.
$$

Then

$$
\begin{aligned}
c_7\varepsilon^{1/2} - c_9\varepsilon^{\alpha/2} + c_{10}\lambda\varepsilon^{q/4} &= c_7\lambda^{\frac{2}{2-q}} + c_{10}\lambda\lambda^{\frac{q}{2-q}} - c_9\lambda^{\frac{2\alpha}{2-q}} \\
&= \lambda^{\frac{2}{2-q}}\Big(c_7 + c_{10} - c_9\lambda^{-\frac{2(1-\alpha)}{2-q}}\Big) \\
&< -D\lambda^{\frac{2}{2-q}}.
\end{aligned}
$$

From (2.22) it follows that

$$
\begin{aligned}
I_\lambda(t_\varepsilon u_\varepsilon) &= h(t_\varepsilon u_\varepsilon) - \frac{\mu t_\varepsilon^2}{2}\int_\Omega |x|^{\alpha-2}u_\varepsilon^2 dx - \lambda\frac{t_\varepsilon^q}{q}\int_\Omega f u_\varepsilon^q dx \\
&\le \Lambda - D\lambda^{\frac{2}{2-q}}.
\end{aligned}
$$

This completes the proof. □

3. Proof of main results

There exists a constant $\delta > 0$ such that $\Lambda - D\lambda^{\frac{2}{2-q}} > 0$ for $\lambda < \delta$. we set $\lambda_* = \min\{T_1, \delta\}$, thus Lemmas 2.1–2.4, 2.6, 2.7 hold for all $0 < \lambda < \lambda_*$. We shall prove Theorem 1.1 in two steps.

Step1 By Lemma 2.6, there exists a bounded minimizing sequence $\{u_n\} \subset \mathcal{N}_\lambda$ of I_λ. Perhaps for a subsequence, still denoted by $\{u_n\}$, there exists $u_\lambda \in H_0^1(\Omega)$ such that

$$\begin{gathered} u_n \rightharpoonup u_\lambda, \quad \text{weakly in } H_0^1(\Omega), \\ u_n \to u_\lambda, \quad \text{strongly in } L^s(\Omega),\ 1 \le s < 6, \\ u_n(x) \to u_\lambda(x), \quad \text{a.e. in } \Omega, \end{gathered}$$

as $n \to \infty$. Now we shall prove that u_λ is a nonzero non-negative ground state solution of problem (1.1).

At first, we prove that u_λ is a non-negative solution of (1.1). Indeed, by (2.11) in Lemma 2.6, for all $\varphi \in H_0^1(\Omega)$, we obtain

$$\begin{aligned} &(a + b \lim_{n\to\infty} \|u_n\|^2) \int_\Omega (\nabla u_\lambda \cdot \nabla \varphi) dx \\ &- \int_\Omega u_\lambda^5 \varphi dx - \lambda \int_\Omega f u_\lambda^{q-1} \varphi dx - \mu \int_\Omega |x|^{2-\alpha} u_\lambda \varphi dx = 0. \end{aligned}$$

Setting $\lim_{n\to\infty} \|u_n\| = l$, one has

$$\begin{aligned} &(a + bl^2) \int_\Omega (\nabla u_\lambda \cdot \nabla \varphi) dx - \int_\Omega u_\lambda^5 \varphi dx \\ &- \lambda \int_\Omega f u_\lambda^{q-1} \varphi dx - \mu \int_\Omega |x|^{2-\alpha} u_\lambda \varphi dx = 0. \end{aligned} \tag{3.1}$$

Taking the test function $\varphi = u_\lambda$ in (3.1), we have

$$(a + bl^2)\|u_\lambda\|^2 - \int_\Omega u_\lambda^6 dx - \lambda \int_\Omega f u_\lambda^q dx - \mu \int_\Omega |x|^{2-\alpha} u_\lambda^2 dx = 0. \tag{3.2}$$

The fact $u_n \in \mathcal{N}_\lambda$ implies

$$(a + b\|u_n\|^2)\|u_n\|^2 - \int_\Omega u_n^6 dx - \lambda \int_\Omega f u_n^q dx - \mu \int_\Omega |x|^{2-\alpha} u_n^2 dx = 0.$$

Since $\alpha_\lambda < 0 < \Lambda - D\lambda^{\frac{2}{2-q}}$, by Lemma 2.7 we obtain

$$(a + bl^2)l^2 - \int_\Omega u_\lambda^6 dx - \lambda \int_\Omega f u_\lambda^q dx - \mu \int_\Omega |x|^{2-\alpha} u_\lambda^2 dx = 0. \tag{3.3}$$

It follows from (3.2) and (3.3) that $\|u_\lambda\| = l$, which suggests that $u_n \to u_\lambda$ in $H_0^1(\Omega)$ and u_λ is a solution of (1.1), namely,

$$\begin{aligned} &(a + b\|u_\lambda\|^2) \int_\Omega (\nabla u_\lambda \cdot \nabla \varphi) dx - \int_\Omega u_\lambda^5 \varphi dx \\ &- \lambda \int_\Omega f u_\lambda^{q-1} \varphi dx - \mu \int_\Omega |x|^{2-\alpha} u_\lambda \varphi dx = 0. \end{aligned} \tag{3.4}$$

for all $\varphi \in H_0^1(\Omega)$. Recall that $u_\lambda \ge 0$. In addition, note that $u_\lambda \in \mathcal{N}_\lambda$ (u_λ is a nontrivial solution of (1.1)) and $\alpha_\lambda < 0$ (by Lemma 2.3), then one obtains

$$\begin{aligned} &\lambda\big(\frac{1}{q} - \frac{1}{6}\big) \int_\Omega f|u_\lambda|^q dx \\ &= \frac{a}{3}\|u_\lambda\|^2 - \frac{\mu}{3} \int_\Omega |x|^{\alpha-2}|u_\lambda|^2 dx + \frac{b}{12}\|u_\lambda\|^4 - I_\lambda(u_\lambda) \\ &\ge \frac{1}{3}(a - \frac{\mu}{\mu_1})\|u_\lambda\|^2 + \frac{b}{12}\|u_\lambda\|^4 - \alpha_\lambda > 0, \end{aligned}$$

which implies that $u_\lambda \not\equiv 0$. By Lemma 2.7 we obtain

$$\alpha_\lambda = \lim_{n\to\infty} I_\lambda(u_n) = I_\lambda(u_\lambda). \tag{3.5}$$

Next, we shall show that $u_\lambda \in \mathcal{N}_\lambda^+$ and $I_\lambda(u_\lambda) = \alpha_\lambda^+$. We claim that $u_\lambda \in \mathcal{N}_\lambda^+$. On the contrary, suppose that $u_\lambda \in \mathcal{N}_\lambda^-$, by Lemma 2.1, there exist positive numbers $t^+ < t_{\max} < t^- = 1$ such that $t^+u \in \mathcal{N}_\lambda^+$, $t^-u \in \mathcal{N}_\lambda^-$ and

$$\alpha_\lambda < I_\lambda(t^+u_\lambda) < I_\lambda(t^-u_\lambda) = I_\lambda(u_\lambda) = \alpha_\lambda,$$

which is a contradiction. Thus, $u_\lambda \in \mathcal{N}_\lambda^+$. From the definition of α_λ^+, we obtain $\alpha_\lambda^+ \le I_\lambda(u_\lambda)$. It follows from Lemma 2.3 and (3.5) that

$$I_\lambda(u_\lambda) = \alpha_\lambda^+ = \alpha_\lambda < 0.$$

From the above discussion, we obtain that u_λ is a nonzero non-negative ground state solution of problem (1.1).

Step 2. We shall verify that problem (1.1) has a second solution v_λ with $v_\lambda \in \mathcal{N}_\lambda^-$.

As I_λ is also coercive on $\mathcal{N}_\lambda^-$, then we apply the Ekeland's variational principle to the minimization problem $\alpha_\lambda^- = \inf_{v\in\mathcal{N}_\lambda^-} I_\lambda(v)$ to obtain a minimizing sequence $\{v_n\} \subset \mathcal{N}_\lambda^-$ of I_λ with the following properties:

(i) $I_\lambda(v_n) < \alpha_\lambda^- + \frac{1}{n}$;

(ii) $I_\lambda(u) \ge I_\lambda(v_n) - \frac{1}{n}\|u - v_n\|$ for all $u \in \mathcal{N}_\lambda^-$.

Since $\{v_n\}$ is bounded in $H_0^1(\Omega)$, passing to a subsequence if necessary, there exists $v_\lambda \in H_0^1(\Omega)$ such that

$$\begin{gathered} v_n \rightharpoonup v_\lambda, \quad \text{weakly in } H_0^1(\Omega), \\ v_n \to v_\lambda, \quad \text{strongly in } L^s(\Omega),\ 1 \le s \le 6, \\ v_n(x) \to v_\lambda(x), \quad \text{a.e. in } \Omega, \end{gathered}$$

as $n \to \infty$. Now we shall prove that v_λ is a non-negative solution of (1.1). Similar to the proof of Theorem 1.1, we obtain $v_n \to v_\lambda$ in $H_0^1(\Omega)$ and v_λ is a non-negative solution of (1.1).

Now, we prove that $v_\lambda \not\equiv 0$ in Ω. From $v_n \in \mathcal{N}_\lambda^-$, we have

$$\begin{aligned} a(2-q)\|v_n\|^2 &\le (6-q)\int_\Omega v_n^6 dx + (2-q)\mu\int_\Omega |x|^{\alpha-2}v_n^2 dx - b(4-q)\|v_n\|^4 \\ &\le (6-q)\int_\Omega v_n^6 dx + (2-q)\mu\int_\Omega |x|^{\alpha-2}v_n^2 dx \\ &< (6-q)S^{-3}\|v_n\|^6 + (2-q)\frac{\mu}{\mu_1}\|v_n\|^2, \end{aligned}$$

so that

$$\|v_n\| > \Big(\frac{(a-\frac{\mu}{\mu_1})(2-q)S^3}{(6-q)}\Big)^{1/4}, \quad \forall v_n \in \mathcal{N}_\lambda^-. \tag{3.6}$$

Note that $v_n \to v_\lambda$ in $H_0^1(\Omega)$, (3.6) implies that $v_\lambda \not\equiv 0$.

Next, we prove that $v_\lambda \in \mathcal{N}_\lambda^-$. It suffices to prove that $\mathcal{N}_\lambda^-$ is closed. Indeed, by Lemmas 2.7 and 2.8, for $\{v_n\} \subset \mathcal{N}_\lambda^-$, we obtain

$$\lim_{n\to\infty}\int_\Omega v_n^6 dx = \int_\Omega v_\lambda^6 dx.$$

From the definition of $\mathcal{N}_\lambda^-$, it holds

$$(2-q)a\|v_n\|^2 + (4-q)b\|v_n\|^4 - (6-q)\int_\Omega v_n^6 dx - (2-q)\mu\int_\Omega |x|^{\alpha-2}v_n^2 dx < 0.$$

Then

$$(2-q)a\|v_\lambda\|^2 + (4-q)b\|v_\lambda\|^4 - (6-q)\int_\Omega v_\lambda^6 dx - (2-q)\mu\int_\Omega |x|^{\alpha-2}v_\lambda^2 dx \le 0,$$

which implies that $v_\lambda \in \mathcal{N}_\lambda^0 \cup \mathcal{N}_\lambda^-$. If $\mathcal{N}_\lambda^-$ is not closed, then one obtains $v_\lambda \in \mathcal{N}_\lambda^0$. By Lemma 2.1, it follows that $v_\lambda = 0$, which contradicts $v_\lambda \not\equiv 0$. Therefore, $v_\lambda \in \mathcal{N}_\lambda^-$. Since $\mathcal{N}_\lambda^+ \cap \mathcal{N}_\lambda^- = \emptyset$, u_λ and v_λ are different.

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