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LEAST ENERGY SIGN-CHANGING SOLUTIONS FOR NONLINEAR PROBLEMS INVOLVING FRACTIONAL LAPLACIAN

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Abstract. In this article, we study the existence of least energy sign-changing solutions for nonlinear problems involving fractional Laplacian. By introducing some new ideas and combining constraint variational method with the quantitative deformation lemma, we prove that the problem possesses one least energy sign-changing solution.

1. Introduction

This article concerns the nonlinear problem involving fractional Laplacian

$$\begin{gathered} (-\Delta)^\alpha u = f(x,u), \quad x \in \Omega, \\ u = 0, \quad x \in \mathbb{R}^N \backslash \Omega, \end{gathered} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary, $0 < \alpha < 1$, $N > 2\alpha$, $(-\Delta)^\alpha$ is the fractional Laplacian of order α, $f \in C(\Omega \times \mathbb{R}, \mathbb{R})$.

To prove our results, we use the following assumptions:

- (A1) $\lim_{s\to 0} f(x,s)/s = 0$, uniformly in $x \in \Omega$;
- (A2) $\lim_{|s|\to\infty} f(x,s)/s^{2^*_\alpha - 1} = 0$, uniformly in $x \in \Omega$, where $2^*_\alpha = \frac{2N}{N-2\alpha}$;
- (A3) $\lim_{|s|\to\infty} f(x,s)/|s| = +\infty$ for a.e. $x \in \Omega$;
- (A4) $f(x,s)/|s|$ is increasing in s on $\mathbb{R}\backslash\{0\}$ for every $x \in \Omega$.

In recent years, nonlinear problems involving fractional Laplacian have been investigated extensively. Indeed, they have impressive applications in many fields, such as thin obstacle problem, optimization, finance, phase transitions, anomalous diffusion and so on. For previous related results see [1, 6, 8, 9, 11, 14, 15, 16, 17, 18, 25, 27, 29, 40, 41] and the references therein. Precisely, under the assumption that the nonlinearity satisfies the Ambrosetti-Rabinowitz condition or is indeed of perturbative type, the author proved some existence results of solutions for fractional Schrödinger equations in [25]. Using mountain pass theorem, Raffaella and Servadei studied the existence of solutions for equations driven by a non-local integrodifferential operator with homogeneous Dirichlet boundary conditions in [26]. In fact, by the extension theorem in [7] Caffarelli and Silvestrein made greatest achievement in overcoming the difficulty, which is the nonlocality of fractional Laplacian $(-\Delta)^\alpha$ in the fractional Schrödinger equation. Moreover, a great deal of progress

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has been made to the fractional Laplacian equations after the work [7]. We refer to [10, 12, 30, 31, 37, 43] for the existence results and multiplicity results of solutions, and to [4, 5] for the regularity results, maximum principle, uniqueness result and other properties.

As we know, a great attention has been devoted to the existence and multiplicity of positive and nodal solutions of elliptic problems in recent years, see for example [2, 3, 13, 22, 32, 33, 42] and the references therein. Actually, with the descended flow method and harmonic extension techniques, Chang and Wang studied the existence and multiplicity of sign-changing solutions in [12]. Via costrained minimization method, Tang [34, 35, 36, 19, 20] obtained the existence of Nehari-type ground state positive solutions. By combing minimax method with invariant sets of descending flow, some results about nodal solutions have been obtained in [21].

Motivated by papers above, and we especially borrow some ideas from [19]. What is more, we are interested in Problem (1.1) with constraint variational method and quantitative deformation lemma, and study the existence of a least energy sign-changing solution.

For any measurable function $u : \mathbb{R}^N \to \mathbb{R}$ with respect to the Gagliardo norm

$$[u]_\alpha = \Big(\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{2\alpha+N}} \mathrm{d}x\mathrm{d}y \Big)^{1/2}.$$

We introduce the fractional Sobolev space

$$H^\alpha(\mathbb{R}^N) = \{u \in L^2(\mathbb{R}^N) : [u]_\alpha < +\infty\},$$

which is a Hilbert space. A complete introduction to fractional Sobolev spaces can be found in [24]. We also define a closed subspace

$$X(\Omega) = \{u \in H^\alpha(\mathbb{R}^N) : u = 0 \text{ a.e. in } \mathbb{R}^N \backslash \Omega\}.$$

Then, by [25], $X(\Omega)$ is a Hilbert space with the inner product

$$(u, v) = \iint_{\Omega\times\Omega} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{2\alpha+N}} \mathrm{d}x\mathrm{d}y, \quad \forall u, v \in X(\Omega),$$

and the corresponding norm $\|\cdot\|_X = [\cdot]_\alpha$. For $u \in X(\Omega)$, set

$$\Phi(u) = \frac{1}{2}\|u\|_X^2 - \int_\Omega F(x, u)\mathrm{d}x, \tag{1.2}$$

where $F(x, u) = \int_0^u f(x, t)\mathrm{d}t$. Then $\Phi \in C^1(X(\Omega), \mathbb{R})$ and

$$\langle \Phi'(u), v\rangle = \iint_{\Omega\times\Omega} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{2\alpha+N}} \mathrm{d}x\mathrm{d}y - \int_\Omega f(x, u)v\mathrm{d}x, \tag{1.3}$$

for all $u, v \in X(\Omega)$. Obviously, its critical points are weak solutions of Problem (1.1). Furthermore, if $u \in X(\Omega)$ is a solution of (1.1) with $u^\pm \neq 0$, then u is a sign-changing solution, where

$$u^+(x) := \max\{u(x), 0\} \quad \text{and} \quad u^-(x) =: \min\{u(x), 0\}.$$

We set

$$\mathcal{M} := \{u \in X(\Omega) : u^\pm \neq 0,\ \langle \Phi'(u), u^+\rangle = \langle \Phi'(u), u^-\rangle = 0\},$$

and define

$$m = \inf_{u\in\mathcal{M}} \Phi(u).$$

Throughout this paper, $\|\cdot\|_p$ denotes the usual norm in $L^p(\Omega)$.

Theorem 1.1. *Assume that conditions* (A1)–(A4) *hold. Then* (1.1) *possesses one least energy sign-changing solution* $u \in \mathcal{M}$ *such that* $\inf_{u\in\mathcal{M}} \Phi(u) = m > 0$.

The rest of this article is organized as follows. In Section 2, we prove several lemmas, which are crucial to investigate our main result. The proof of Theorem 1.1 is given in Section 3.

2. Preliminary results

Lemma 2.1 ([38, Lemma 2.1])**.** *For any* $a, b \in \mathbb{R}$, *we have*

(i) $(ka)^{\pm} = ka^{\pm}$, *for all* $k \geq 0$, $|a^{\pm} - b^{\pm}| \leq |a - b|$;

(ii) $(a-b)(a^{+} - b^{+}) \geq (a^{+} - b^{+})^2$ *and* $(a-b)(a^{-} - b^{-}) \geq (a^{-} - b^{-})^2$;

(iii) $(a^{+} - b^{+})(a^{-} - b^{-}) \geq 0$.

By simple computations from the above lemma, we obtain the following lemma.

Lemma 2.2. *Under assumptions* (A1) *and* (A2), *for any* $u \in X(\Omega)$, *the following facts hold:*

(i) $\|u^{\pm}\|_X \leq \|u\|_X$;

(ii)

$$
\begin{aligned}
(u, u^{\pm}) &= (u^{\pm}, u^{\pm}) - \iint_{\Omega\times\Omega} \frac{u^{+}(x)u^{-}(y)}{|x-y|^{2\alpha+N}} \mathrm{d}x\mathrm{d}y - \iint_{\Omega\times\Omega} \frac{u^{-}(x)u^{+}(y)}{|x-y|^{2\alpha+N}} \mathrm{d}x\mathrm{d}y \\
&= (u^{\pm}, u^{\pm}) - 2\iint_{\Omega\times\Omega} \frac{u^{+}(x)u^{-}(y)}{|x-y|^{2\alpha+N}} \mathrm{d}x\mathrm{d}y;
\end{aligned}
$$

(iii)

$$
\begin{aligned}
\langle \Phi'(u), u^{\pm} \rangle &= \langle \Phi'(u^{\pm}), u^{\pm} \rangle - \iint_{\Omega\times\Omega} \frac{u^{+}(x)u^{-}(y)}{|x-y|^{2\alpha+N}} \mathrm{d}x\mathrm{d}y - \iint_{\Omega\times\Omega} \frac{u^{-}(x)u^{+}(y)}{|x-y|^{2\alpha+N}} \mathrm{d}x\mathrm{d}y \\
&= \langle \Phi'(u^{\pm}), u^{\pm} \rangle - 2\iint_{\Omega\times\Omega} \frac{u^{+}(x)u^{-}(y)}{|x-y|^{2\alpha+N}} \mathrm{d}x\mathrm{d}y.
\end{aligned}
$$

In what follows, we denote

$$
B(u) := -\iint_{\Omega\times\Omega} \frac{u^{+}(x)u^{-}(y)}{|x-y|^{2\alpha+N}} \mathrm{d}x\mathrm{d}y\,.
$$

It is obvious that $B(u) \geq 0$.

Lemma 2.3. *Assume* (A1) *and* (A2), *and let* $\{u_n\}$ *be a bounded sequence in* $X(\Omega)$. *Then up to a subsequence, still denoted by* $\{u_n\}$, *there exists* $u \in X(\Omega)$ *such that*

(i)

$$
\lim_{n\to\infty} \int_\Omega |u_n^{\pm}|^p \mathrm{d}x = \int_\Omega |u^{\pm}|^p \mathrm{d}x, \quad \forall p \in [2, 2^*_\alpha);
$$

(ii)

$$
\lim_{n\to\infty} \int_\Omega u_n f(x, u_n) \mathrm{d}x = \int_\Omega u f(x, u) \mathrm{d}x;
$$

(iii)

$$
\lim_{n\to\infty} \int_\Omega F(x, u_n) \mathrm{d}x = \int_\Omega F(x, u) \mathrm{d}x;
$$

(iv)

$$
\liminf_{n\to\infty} \langle \Phi'(u_n), u_n^{\pm} \rangle \geq \langle \Phi'(u), u^{\pm} \rangle.
$$

Proof. (i)–(iii) are easily proved; so we omit their proofs.

(iv)From (ii), Fatou's Lemma and (iii) of Lemma 2.2, it follows that

$$\begin{aligned}
&\langle \Phi'(u), u^{\pm}\rangle \\
&= \langle \Phi'(u^{\pm}), u^{\pm}\rangle - 2\iint_{\Omega\times\Omega} \frac{u^{+}(x)u^{-}(y)}{|x-y|^{2\alpha+N}}\mathrm{d}x\mathrm{d}y \\
&= \iint_{\Omega\times\Omega} \frac{[u^{\pm}(x)-u^{\pm}(y)]^2}{|x-y|^{2\alpha+N}}\mathrm{d}x\mathrm{d}y - 2\iint_{\Omega\times\Omega} \frac{u^{+}(x)u^{-}(y)}{|x-y|^{2\alpha+N}}\mathrm{d}x\mathrm{d}y - \int_\Omega u^{\pm} f(x,u^{\pm})\mathrm{d}x \\
&\le \liminf_{n\to\infty} \Big\{ \iint_{\Omega\times\Omega} \frac{[u_n^{\pm}(x)-u_n^{\pm}(y)]^2}{|x-y|^{2\alpha+N}}\mathrm{d}x\mathrm{d}y - 2\iint_{\Omega\times\Omega} \frac{u_n^{+}(x)u_n^{-}(y)}{|x-y|^{2\alpha+N}}\mathrm{d}x\mathrm{d}y \Big\} \\
&\quad - \lim_{n\to\infty} \int_\Omega u_n^{\pm} f(x,u_n^{\pm})\mathrm{d}x \\
&= \liminf_{n\to\infty} \langle \Phi'(u_n), u_n^{\pm}\rangle.
\end{aligned}$$

This shows that (iv) holds. □

Lemma 2.4. *Under assumptions* (A1) *and* (A2)*, if* $\{u_n\}$ *is a bounded sequence in* $\mathcal{M}$ *and* $q \in (2, 2^*_\alpha)$*, we have*

$$\liminf_{n\to\infty} \int_\Omega |u_n^{\pm}|^q dx > 0.$$

Proof. From (A1) and (A2), for any $\varepsilon > 0$ and fixed $\tau \in [2, 2^*_\alpha)$, there exists $C_\varepsilon > 0$ such that

$$|sf(x,s)| \le \varepsilon|s|^2 + C_\varepsilon|s|^\tau + \varepsilon|s|^{2^*_\alpha}, \quad \forall x\in\Omega, s\in\mathbb{R}. \tag{2.1}$$

For $u_n \in \mathcal{M}$, we have $\langle \Phi'(u_n), u_n^{\pm}\rangle = 0$. From (iii) of Lemma 2.2, we have

$$\langle \Phi'(u_n^{\pm}), u_n^{\pm}\rangle - 2\iint_{\Omega\times\Omega} \frac{u_n^{+}(x)u_n^{-}(y)}{|x-y|^{2\alpha+N}}\mathrm{d}x\mathrm{d}y = 0,$$

which, together with Sobolev embedding and (2.1), for $q \in (2, 2^*_\alpha)$, yields

$$\begin{aligned}
\|u_n^{\pm}\|_X^2 &\le \int_\Omega u_n^{\pm} f(x,u_n^{\pm})\mathrm{d}x \\
&\le \varepsilon \int_\Omega |u_n^{\pm}|^2\mathrm{d}x + C_\varepsilon \int_\Omega |u_n^{\pm}|^q\mathrm{d}x + \varepsilon\int_\Omega |u_n^{\pm}|^{2^*_\alpha}\mathrm{d}x \\
&\le \varepsilon\gamma_2^{-2}\|u_n^{\pm}\|_X^2 + C_\varepsilon\gamma_q^{-2}\|u_n^{\pm}\|_X^2\|u_n^{\pm}\|_q^{q-2} + \varepsilon\gamma_{2^*_\alpha}^{-2^*_\alpha}\|u_n^{\pm}\|_X^{2^*_\alpha},
\end{aligned} \tag{2.2}$$

where $\gamma_s := \inf_{\|u\|_s=1}\|u\|_X$, $2 \le s \le 2^*_\alpha$. From the boundedness of $\{u_n\}$, there is M such that

$$\|u_n^{\pm}\|_X^{2^*_\alpha-2} \le M.$$

From (2.2), taking $\varepsilon = \min\{\gamma_2^2/4, \gamma_{2^*_\alpha}^{2^*_\alpha}/4M\}$, $C_0 \ge C_\varepsilon$, it follows that

$$\frac{1}{2} \le C_0\gamma_q^{-2}\|u_n^{\pm}\|_q^{q-2}.$$

Then

$$\liminf_{n\to\infty}\int_\Omega |u_n^{\pm}|^q\mathrm{d}x \ge \Big(\frac{\gamma_q^2}{2C_0}\Big)^{\frac{1}{q-2}} > 0.$$

□

Lemma 2.5. *Under assumptions* (A1), (A2), (A4), *for any* $u \in X(\Omega)$ *with* $u^{\pm} \neq 0$, $s, t \geq 0$ *and* $(s-1)^2 + (t-1)^2 \neq 0$, *we have*

$$\Phi(u) > \Phi(su^+ + tu^-) + \frac{1-s^2}{2}\langle \Phi'(u), u^+\rangle + \frac{1-t^2}{2}\langle \Phi'(u), u^-\rangle + B(u)(s-t)^2.$$

Proof. For $\tau \neq 0$, (A4) yields

$$\begin{gathered} f(x,s) < \frac{f(x,\tau)}{|\tau|}|s|, \quad |s| < |\tau|; \\ f(x,s) > \frac{f(x,\tau)}{|\tau|}|s|, \quad |s| > |\tau|. \end{gathered}$$

It follows that

$$\frac{1-\theta^2}{2}\tau f(x,\tau) > \int_{\theta\tau}^{\tau} f(x,s)\mathrm{d}s, \quad \forall x \in \Omega,\ \tau \neq 0,\ \theta \geq 0 \ \text{ and } \theta \neq 1.$$

Thus, we deduce that

$$\begin{aligned} &\Phi(u) - \Phi(su^+ + tu^-) \\ &= \frac{1-s^2}{2}\langle \Phi'(u), u^+\rangle + \frac{1-t^2}{2}\langle \Phi'(u), u^-\rangle \\ &\quad + \int_\Omega \big[\frac{1-s^2}{2} f(x,u^+)u^+ - \big(F(x,u^+) - F(x,su^+)\big)\big]\mathrm{d}x \\ &\quad + \int_\Omega \big[\frac{1-t^2}{2} f(x,u^-)u^- - \big(F(x,u^-) - F(x,tu^-)\big)\big]\mathrm{d}x + B(u)(s-t)^2 \\ &= \frac{1-s^2}{2}\langle \Phi'(u), u^+\rangle + \frac{1-t^2}{2}\langle \Phi'(u), u^-\rangle + \int_\Omega \big[\frac{1-s^2}{2} f(x,u^+)u^+ \\ &\quad - \int_{su^+}^{u^+} f(x,\xi)\mathrm{d}\xi\big]\mathrm{d}x + \int_\Omega \Big[\frac{1-t^2}{2} f(x,u^-)u^- - \int_{tu^-}^{u^-} f(x,\xi)\mathrm{d}\xi\Big]\mathrm{d}x \\ &\quad + B(u)(s-t)^2 \\ &> \frac{1-s^2}{2}\langle \Phi'(u), u^+\rangle + \frac{1-t^2}{2}\langle \Phi'(u), u^-\rangle + B(u)(s-t)^2, \end{aligned}$$

for all $s, t \geq 0$, $(s-1)^2 + (t-1)^2 \neq 0$. □

From Lemma 2.5, we have the following two corollaries.

Corollary 2.6. *Under assumptions* (A1), (A2), (A4), *we have*

$$\Phi(u) \geq \Phi(tu) + \frac{1-t^2}{2}\langle \Phi'(u), u\rangle, \quad \forall u \in X(\Omega),\ t \geq 0. \tag{2.3}$$

Corollary 2.7. *Under assumptions* (A1),(A2), (A4), *we have*

$$\Phi(u) \geq \Phi(su^+ + tu^-), \quad \forall u \in \mathcal{M},\ s, t \geq 0. \tag{2.4}$$

Lemma 2.8. *Assume* (A1)–(A4) *hold; if* $u \in X(\Omega)$ *with* $u^{\pm} \neq 0$, *then there exists a unique pair* (s_u, t_u) *of positive numbers such that* $s_u u^+ + t_u u^- \in \mathcal{M}$.

Proof. Let

$$\begin{aligned} g_1(s,t) &= \langle \Phi'(su^+ + tu^-), su^+\rangle \\ &= s^2\|u^+\|^2 - \int_\Omega f(x,su^+)su^+\mathrm{d}x + 2B(u)st, \end{aligned} \tag{2.5}$$

$$\begin{aligned} g_2(s,t) &= \langle \Phi'(su^+ + tu^-), tu^- \rangle \\ &= t^2\|u^-\|^2 - \int_\Omega f(x,tu^-)tu^- \mathrm{d}x + 2B(u)st. \end{aligned} \tag{2.6}$$

From (A1), (A2) and (A3), a straightforward computation yields that there are $r > 0$ small enough and $R > 0$ large enough such that

$$\begin{gathered} g_1(r,r) > 0, \quad g_2(r,r) > 0, \\ g_1(R,R) < 0, \quad g_2(R,R) < 0. \end{gathered}$$

Notice that for any fixed $s > 0$, $g_1(s,t)$ is increasing in t on $[0,+\infty)$, then

$$\begin{gathered} g_1(r,t) \geq g_1(r,r) > 0, \quad \forall t \in [r,R], \\ g_1(R,t) \leq g_1(R,R) < 0, \quad \forall t \in [r,R]. \end{gathered}$$

Analogously, for $g_2(s,t)$, one has

$$\begin{gathered} g_2(s,r) \geq g_2(r,r) > 0, \quad \forall s \in [r,R], \\ g_2(s,R) \leq g_2(R,R) < 0, \quad \forall s \in [r,R]. \end{gathered}$$

The above inequalities and the Miranda theorem [23] imply that there is a pair $(s_u,t_u) \in (r,R) \times (r,R)$ such that $g_1(s_u,t_u) = g_2(s_u,t_u) = 0$, and then, $s_u u^+ + t_u u^- \in \mathcal{M}$.

Next, we prove the uniqueness. Let $(\hat{s}_1,\hat{t}_1)$ and $(\hat{s}_2,\hat{t}_2)$ such that $\hat{s}_i u^+ + \hat{t}_i u^- \in \mathcal{M}$, $i = 1,2$. We assume that $(\frac{\hat{s}_2}{\hat{s}_1} - 1)^2 + (\frac{\hat{t}_2}{\hat{t}_1} - 1)^2 \neq 0$, then Lemma 2.5 implies

$$\begin{gathered} \Phi(\hat{s}_1 u^+ + \hat{t}_1 u^-) > \Phi(\hat{s}_2 u^+ + \hat{t}_2 u^-), \\ \Phi(\hat{s}_2 u^+ + \hat{t}_2 u^-) > \Phi(\hat{s}_1 u^+ + \hat{t}_1 u^-). \end{gathered}$$

This contradiction shows $(\hat{s}_1,\hat{t}_1) = (\hat{s}_2,\hat{t}_2)$, this completes the proof. □

Corollary 2.9. *Under assumptions* (A1)–(A4),

$$m := \inf_{u\in\mathcal{M}} \Phi(u) = \inf_{u\in X(\Omega), u^\pm \neq 0} \max_{s,t\geq 0} \Phi(su^+ + tu^-).$$

Lemma 2.10. *Assume that* (A1)–(A4) *hold. If* $u_0 \in \mathcal{M}$*, and* $\Phi(u_0) = m$*, then* u_0 *is a critical point of* Φ.

Proof. Arguing by contradiction, $\Phi(u_0) = m$ and $\Phi'(u_0) \neq 0$. Therefore, there exist $\delta > 0$ and $\rho > 0$ such that

$$v \in X(\Omega),\ \|v - u_0\| \leq 3\delta \Rightarrow \|\Phi'(v)\| \geq \rho.$$

Let $D = (\frac{1}{2},\frac{3}{2}) \times (\frac{1}{2},\frac{3}{2})$. It follows from Lemma 2.5 that

$$\bar{m} := \max_{(s,t)\in\partial D} \Phi(su_0^+ + tu_0^-) < m.$$

For $\varepsilon := \min\{(m-\bar{m})/3, 1, \rho\delta/8\}$, $S := B(u_0,\delta)$, [39, Lemma 2.3] yields a deformation $\eta \in C([0,1] \times X(\Omega), X(\Omega))$ such that

(i) $\eta(1,u) = u$ if $\Phi(u) < m - 2\varepsilon$ or $\Phi(u) > m + 2\varepsilon$;
(ii) $\eta(1,\Phi^{m+\varepsilon} \cap B(u_0,\delta)) \subset \Phi^{m-\varepsilon}$;
(iii) $\Phi(\eta(1,u)) \leq \Phi(u)$, for all $u \in X(\Omega)$.

By Corollary 2.7, $\Phi(su_0^+ + tu_0^-) \le \Phi(u_0) = m$, for $s, t \ge 0$, then from (ii) it follows that

$$\Phi(\eta(1, su_0^+ + tu_0^-)) \le m - \varepsilon, \quad \forall s, t \ge 0,\ |s-1|^2 + |t-1|^2 < \frac{\delta^2}{2\|u_0\|_X^2}. \tag{2.7}$$

On the other hand, by (iii) and Lemma 2.5, one has

$$\Phi(\eta(1, su_0^+ + tu_0^-)) \le \Phi(su_0^+ + tu_0^-) < \Phi(u_0) = m, \tag{2.8}$$

for all $s, t \ge 0$, $|s-1|^2 + |t-1|^2 \ge \frac{\delta^2}{2\|u_0\|_X^2}$. Combining (2.7) and (2.8), we have

$$\max_{(s,t)\in\bar{D}} \Phi(\eta(1, su_0^+ + tu_0^-)) < m.$$

By the similar method in [28], we can prove that $\eta(1, su_0^+ + tu_0^-) \cap \mathcal{M} \ne \emptyset$ for some $(s,t) \in \bar{D}$, which contradicts the definition of m . □

3. Proof of main result

Proof of Theorem 1.1. We shall show that $m > 0$ can be achieved to get a critical point of Φ. Let u_n be a sequence in $\mathcal{M}$ such that

$$\lim_{n\to\infty} \Phi(u_n) = m.$$

First of all, we claim that $\{u_n\}$ is bounded in $X(\Omega)$. To this end, suppose by contradiction that $\|u_n\|_X \to \infty$, and set $v_n = \frac{u_n}{\|u_n\|}$. Since $\|v_n\|_X = 1$, passing to a subsequence, there exists $v \in X(\Omega)$ such that $v_n \rightharpoonup v$ in $X(\Omega)$, $v_n \to v$ in $L^p(\Omega)$, for $2 \le p < 2_\alpha^*$, and $v_n(x) \to v(x)$ a.e. on Ω. If $v = 0$, then we have $v_n \to 0$ in $L^p(\Omega)$, for $2 \le p < 2_\alpha^*$. Fix $\tau \in [2, 2_\alpha^*)$ and $R = \sqrt{2(m+1)}$. By (A1) and (A2), given $\varepsilon > 0$, there exists $C_\varepsilon > 0$, such that

$$|F(x,s)| \le \varepsilon|s|^2 + C_\varepsilon|s|^\tau + \varepsilon|s|^{2_\alpha^*}, \quad \forall x \in \Omega, s \in \mathbb{R}. \tag{3.1}$$

By (3.1), Corollary 2.6 and Lebesgue's dominated convergence theorem, it follows that

$$\begin{aligned}
m &= \Phi(u_n) + o(1) \\
&\ge \Phi(Rv_n) + \Big(\frac{1}{2} - \frac{R^2}{2\|u_n\|^2}\Big)\langle \Phi'\langle u_n), u_n\rangle + o(1) \\
&= \frac{R^2}{2} - \int_\Omega F(x, Rv_n)\mathrm{d}x + o(1) \\
&\ge \frac{R^2}{2} - \int_\Omega |F(x, Rv_n)|\mathrm{d}x + o(1) \\
&\ge \frac{R^2}{2} - \int_\Omega \big[\varepsilon|Rv_n|^2 + C_\varepsilon|Rv_n|^\tau + \varepsilon|Rv_n|^{2_\alpha^*}\big]\mathrm{d}x + o(1) \\
&= m + 1 - \big\{\varepsilon\big[R^2\|v_n\|_2^2 + R^{2_\alpha^*}\|v_n\|_{2_\alpha^*}^{2_\alpha^*}\big] + C_\varepsilon R^\tau \|v_n\|_\tau^\tau\big\} + o(1) \\
&\ge m + 1 - C_1\varepsilon + o(1),
\end{aligned}$$

the contradiction is obvious due to the arbitrariness of ε. Thus, $v \ne 0$. Denote $A = \{x \in \Omega : v(x) \ne 0\}$. Then for $x \in A$, we have $\lim_{n\to\infty} |u_n(x)| = \infty$. By (A3), (A4) and Fatou's Lemma

$$0 = \lim_{n\to\infty} \frac{\Phi(u_n)}{\|u_n\|^2}$$

$$
\begin{aligned}
&= \lim_{n\to\infty}\Big[\frac{1}{2}-\int_A \frac{F(x,u_n)}{u_n^2}v_n^2\mathrm{d}x\Big]\\
&\le \frac{1}{2}-\liminf_{n\to\infty}\int_A \frac{F(x,u_n)}{u_n^2}v_n^2\mathrm{d}x\\
&\le \frac{1}{2}-\int_A \liminf_{n\to\infty}\frac{F(x,u_n)}{u_n^2}v_n^2\mathrm{d}x=-\infty.
\end{aligned}
$$

The contradiction shows that $\{u_n\}$ is bounded in $X(\Omega)$. Passing to a subsequence, there exists $u\in X(\Omega)$ such that $u_n \rightharpoonup u$ in $X(\Omega)$, $u_n\to u$ in $L^p(\Omega)$, for $2\le p<2_\alpha^*$, and $u_n(x)\to u(x)$ a.e. on Ω.

Next, we show that $m>0$ is attained. From Lemma 2.4, it follows that $u^\pm\neq 0$. Then by Lemma 2.8, there are $s,t>0$ such that $su^+ + tu^- \in \mathcal{M}$. By Lemmas 2.3 and 2.5, we have

$$
\begin{aligned}
m &\le \Phi(su^+ + tu^-)\\
&\le \Phi(u)-\frac{1-s^2}{2}\langle \Phi'(u),u^+\rangle-\frac{1-t^2}{2}\langle \Phi'(u),u^-\rangle\\
&= \Phi(u)-\frac{1}{2}\langle \Phi'(u),u\rangle+\frac{s^2}{2}\langle \Phi'(u),u^+\rangle+\frac{t^2}{2}\langle \Phi'(u),u^-\rangle\\
&\le \lim_{n\to\infty}\int_\Omega \big[\frac{1}{2}f(x,u_n)-F(x,u_n)\big]\mathrm{d}x\\
&\quad +\liminf_{n\to\infty}\big\{\frac{s^2}{2}\langle \Phi'(u_n),u_n^+\rangle+\frac{t^2}{2}\langle \Phi'(u_n),u_n^-\rangle\big\}\\
&= \lim_{n\to\infty}\big[\Phi(u_n)-\frac{1}{2}\langle \Phi'(u_n),u_n\rangle\big]=m,
\end{aligned}
$$

which implies $\Phi(su^+ + tu^-)=m$. From Lemma 2.10, $\Phi'(su^+ + tu^-)=0$, and then $su^+ + tu^-$ is a sign-changing solution of (1.1). $\square$

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