RIEMANN-LIOUVILLE FRACTIONAL COSINE FUNCTIONS

ZHAN-DONG MEI, JI-GEN PENG

Abstract. In this article, we present the notion of Riemann-Liouville fractional cosine function. We prove that a Riemann-Liouville \( \alpha \)-order fractional cosine function is equivalent to the Riemann-Liouville \( \alpha \)-order fractional resolvent introduced in [15].

1. Introduction

Let \( X \) be a Banach space, and \( A : D(A) \subset X \to X, B : D(B) \subset X \to X \) be closed linear operators. It is well-known that \( C_0 \)-semigroups are important tools to study the abstract Cauchy problem of first order

\[
\frac{du(t)}{dt} = Au(t), \quad t > 0
\]

\[ u(0) = x, \]

and that the cosine function essentially characterizes the abstract Cauchy problem of second order

\[
\frac{d^2u(t)}{dt^2} = Bu(t), \quad t > 0
\]

\[ u(0) = x, u'(0) = 0. \]

Here a \( C_0 \)-semigroup is a family \( \{T(t)\}_{t \geq 0} \) of strongly continuous and bounded linear operators defined on \( X \) satisfying \( T(0) = I \) and \( T(t+s) = T(t)T(s), t, s \geq 0; \) a cosine function is a family \( \{S(t)\}_{t \geq 0} \) of strongly continuous and bounded linear operators defined on \( X \) satisfying \( S(0) = I \) and \( 2S(t)S(s) = S(t+s), t \geq s \geq 0. \)

Concretely, system (1.1) is well-posed if and only if \( A \) generates a \( C_0 \)-semigroup \( \{T(t)\}_{t \geq 0} \), namely, \( Ax = \lim_{t \to 0^+} t^{-1}(T(t)x - x) \) with domain \( D(A) = \{ x \in D(A) : \lim_{t \to 0^+} t^{-1}(T(t)x - x) \text{ exists} \}; \) system (1.2) is well-posed if and only if \( B \) generates a cosine function \( \{S(t)\}_{t \geq 0} \), namely, \( Bx = 2\lim_{t \to 0^+} t^{-2}(S(t)x - x) \) with domain \( D(B) = \{ x \in D(B) : \lim_{t \to 0^+} t^{-2}(S(t)x - x) \text{ exists} \}. \) Therefore, pure algebraic methods can be used to study abstract Cauchy problems of first and second orders. For details, we refer to [5, 7].

However, equations of integer order such as (1.1) and (1.2) cannot exactly describe the behavior of many physical systems; fractional differential equations maybe more suitable for describing anomalous diffusion on fractals (physical objects of fractional dimension, like some amorphous semiconductors or strongly porous...
materials; see [14] and the references therein), fractional random walk [6] [18], etc. Fractional derivatives appear in the theory of fractional differential equations; they describe the property of memory and heredity of materials, and it is the major advantage of fractional derivatives compared with integer order derivatives. Let \( \alpha > 0 \) and \( m = \lfloor \alpha \rfloor \), the smallest integer larger than or equal to \( \alpha \). There are mainly two types of \( \alpha \)-order fractional differential equations, which are most used in the real problems.

(1) Caputo fractional abstract Cauchy problem

\[
C D_t^\alpha u(t) = Au(t), \quad t > 0, \quad u(0) = x, u^{(k)}(0) = 0, \quad k = 1, 2, \ldots, m - 1.
\tag{1.3}
\]

where \( C D_t^\alpha \) is the Caputo fractional differential operator defined as follows:

\[
C D_t^\alpha u(t) = \frac{1}{\Gamma(m - \alpha)} \int_0^t (t - \sigma)^{-\alpha} u^{(m)}(\sigma) d\sigma;
\]

(2) Riemann-Liouville fractional abstract Cauchy problem

\[
D_t^\alpha u(t) = Au(t), \quad (g_{2-\alpha} * u)(0) = \lim_{s \to 0^+} \int_0^s \frac{(s - \sigma)^{m-1-\alpha}}{\Gamma(2-\alpha)} u(\sigma) d\sigma = x, \quad (g_{2-\alpha} * u)^{(k)}(0) = \lim_{s \to 0^+} \int_0^s \frac{d^k}{dt^k} \frac{(s - \sigma)^{m-1-\alpha}}{\Gamma(m - \alpha)} u(\sigma) d\sigma = 0, \quad k = 1, 2, \ldots, m - 1.
\tag{1.4}
\]

where the Riemann-Liouville fractional differential operator is

\[
D_t^\alpha u(t) = \frac{1}{\Gamma(m - \alpha)} \frac{d}{dt} \int_0^t (t - \sigma)^{m-1-\alpha} u(\sigma) d\sigma.
\]

Obviously, (1.1) is just the limit state of equations (1.3) and (1.4) as \( \alpha \to 1 \), and (1.2) is just the limit state of equations (1.3) and (1.4) as \( \alpha \to 2 \). Initial conditions for the Caputo fractional derivatives are expressed in terms of initials of integer order derivatives [4, 14, 17]. For some real materials, initial conditions should be expressed in terms of Riemann-Liouville fractional derivatives, and it is possible to obtain initial values for such initial conditions by appropriate measurements [8, 9].

To study Caputo fractional abstract Cauchy problem (1.3), Bajlekova [2] introduced the important notion of solution operator for equations (1.3) as follows.

**Definition 1.1.** A family \( \{T(t)\}_{t \geq 0} \) of bounded linear operators of \( X \) is called a solution operator for (1.3) if the following three conditions are satisfied:

(a) \( T(t) \) is strongly continuous for \( t \geq 0 \) and \( T(0) = I \),
(b) \( T(t) D(A) \subset D(A) \) and \( AT(t)x = T(t)Ax \) for all \( x \in D(A) \) and \( t \geq 0 \),
(c) for any \( x \in D(A) \), it holds

\[
T(t)x = x + J_t^\alpha T(t)Ax, \quad t \geq 0,
\]

where

\[
J_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - \sigma)^{\alpha-1} f(t) dt.
\]
Chen and Li [3] developed a notion of $\alpha$-resolvent operator function, which was proved to be a new characteristic of solution operator. Hence, Caputo fractional abstract Cauchy problem can be studied by pure algebraic methods. The definition of $\alpha$-resolvent operator function is as follows.

**Definition 1.2.** Let $\{S(t)\}_{t \geq 0}$ be a family of bounded linear operators on $X$. Then $\{S(t)\}_{t \geq 0}$ is called to be an $\alpha$-resolvent operator function, if the following assumptions are satisfied:

1. $S(t)$ is strongly continuous and $S(0) = I$. 
2. $S(s)S(t) = S(t)S(s)$ for all $t, s \geq 0$. 
3. $S(s)J^\alpha_t S(t) - J^\alpha_t S(s)S(t) = J^\alpha_t S(t) - J^\alpha_t S(s)$ for all $t, s \geq 0$.

Li and Peng [12] proposed the following notion of fractional resolvent to study Riemann-Liouville $\alpha$-order fractional abstract Cauchy problem (1.4) with $\alpha \in (0, 1)$.

**Definition 1.3** ([12]). Let $0 < \alpha < 1$. A family $\{T(t)\}_{t > 0}$ of bounded linear operators on Banach space $X$ is called an $\alpha$-order fractional resolvent if it satisfies the following assumptions:

1. For any $x \in X$, $T(\cdot)x \in C((0, \infty), X)$, and
   \[
   \lim_{t \to 0^+} \Gamma(\alpha) t^{1-\alpha} T(t)x = x \quad \text{for all } x \in X; \quad (1.5)
   \]
2. $T(s)T(t) = T(t)T(s)$ for all $t, s > 0$;
3. For all $t, s > 0$, it holds
   \[
   T(t)J^\alpha_t S(s) - J^\alpha_t T(t)S(s) = \frac{t^{\alpha-1}}{\Gamma(\alpha)} J^\alpha_t S(s) - \frac{s^{\alpha-1}}{\Gamma(\alpha)} J^\alpha_t T(t). \quad (1.6)
   \]

In [13], we studied the Riemann-Liouville $\alpha$-order fractional Cauchy problem (1.4) with order $\alpha \in (1, 2)$. There Riemann-Liouville $\alpha$-order fractional resolvent defined as follows.

**Definition 1.4.** A family $\{T(t)\}_{t > 0}$ of bounded linear operators is called Riemann-Liouville $\alpha$-order fractional resolvent if it satisfies the following assumptions:

(a) For any $x \in X$, $T_\alpha(\cdot)x \in C((0, \infty), X)$, and
   \[
   \lim_{t \to 0^+} \Gamma(\alpha - 1) t^{2-\alpha} T(t)x = x \quad \text{for all } x \in X; \quad (1.7)
   \]
(b) $T(s)T_\alpha(t) = T(t)T_\alpha(s)$ for all $t, s > 0$;
(c) For all $t, s > 0$, it holds
   \[
   T(s)J^\alpha_s T(t) - J^\alpha_s T(s)T(t) = \frac{s^{\alpha-2}}{\Gamma(\alpha - 1)} J^\alpha_s T(t) - \frac{t^{\alpha-2}}{\Gamma(\alpha - 1)} J^\alpha_s T(s). \quad (1.8)
   \]

The linear operator $A$ defined by

\[
Ax = \lim_{t \to 0^+} \frac{t^{1-\alpha} T(t)x - \frac{1}{\Gamma(\alpha)} x}{t^{2\alpha}},
\]

for $x \in D(A) = \{x \in X : \lim_{t \to 0^+} \frac{t^{1-\alpha} T(t)x - \frac{1}{\Gamma(\alpha)} x}{t^{2\alpha}} \text{ exists}\}$.

Operator $A$ generates a Riemann-Liouville $\alpha$-order fractional resolvent $\{T(t)\}_{t > 0}$ in Definition 1.3.

Also, we proved that $\{T(t)\}_{t > 0}$ is a Riemann-Liouville $\alpha$-order fractional resolvent if and only if it is a solution operator defined as follows.
Definition 1.5. A family \( \{T(t)\}_{t>0} \) of bounded linear operators of \( X \) is called a solution operator for (1.4) if the following three conditions are satisfied:

(a) \( T(t) \) is strongly continuous for \( t > 0 \) and \( \lim_{t \to 0^+} \Gamma(\alpha - 1)t^{2-\alpha}T(t)x = x, \quad x \in X \),
(b) \( T(t)D(A) \subset D(A) \) and \( AT(t)x = T(t)Ax \) for all \( x \in D(A) \) and \( t > 0 \),
(c) for any \( x \in D(A) \), it holds

\[
T(t)x = \frac{t^{1-\alpha}}{\Gamma(2-\alpha)}x + J_\alpha^t T(t)Ax, \quad t > 0.
\]

However, the above functional equations for fractional differential equations are not expressed in terms of the sum of time variables: \( s + t \). This is very important in concrete applications of the functional equation, just like \( C_0 \)-semigroups, cosine functions. Motivated by this, Peng and Li [17] established the characterization of \( \alpha \)-order fractional semigroup with \( \alpha \in (0, 1) \):

\[
\int_0^{t+s} \frac{T(\tau)}{(t + s - \tau)^\alpha} d\tau - \int_0^t \frac{T(\tau)}{(t + s - \tau)^\alpha} d\tau - \int_0^s \frac{T(\tau)}{(t + s - \tau)^\alpha} d\tau = \int_0^t \int_0^s \frac{T(r_1)T(r_2)}{(t + s - r_1 - r_2)^{1+\alpha}} dr_1 dr_2, \quad t, s \geq 0,
\]

where the integrals are in the sense of strong operator topology. Concretely, they proved that \( \alpha \)-order fractional semigroup is closely related to the solution operator of Caputo fractional abstract Cauchy problem (1.3).

Mei, Peng and Zhang [13] developed the notion of Riemann-Liouville fractional semigroup as follows.

Definition 1.6. A family \( \{T(t)\}_{t>0} \) of bounded linear operators is called Riemann-Liouville \( \alpha \)-order fractional semigroup on Banach space \( X \), if the following conditions are satisfied:

(i) for any \( x \in X \), \( t \mapsto T(t)x \) is continuous over \( (0, \infty) \) and

\[
\lim_{t \to 0^+} \Gamma(\alpha)t^{1-\alpha}T(t)x = x;
\]

(ii) for all \( t, s > 0 \), it holds

\[
\Gamma(1-\alpha)T(t+s) = \int_0^t \int_0^s \frac{T(r_1)T(r_2)}{(t + s - r_1 - r_2)^{1+\alpha}} dr_1 dr_2, \quad t, s \geq 0,
\]

where the integrals are in the sense of strong operator topology.

It is proved in [13] that \( A \) generates a Riemann-Liouville fractional semigroup if and only if it generates a fractional resolvent developed in [11].

To study Caputo fractional Cauchy problem of order \( \alpha \in (1, 2) \), we recently studied in [14] the notion of fractional cosine function as follows.
Definition 1.7. A family \( \{T(t)\}_{t \geq 0} \) of bounded and strongly continuous operators is called an \( \alpha \)-fractional cosine function if \( T(0) = I \) and it holds

\[
\int_0^{t+s} \int_0^\sigma \frac{T(\tau)}{(t+s-\sigma)^{\alpha-1}} \, d\sigma \,d\tau - \int_0^t \int_0^\sigma \frac{T(\tau)}{(t+s-\sigma)^{\alpha-1}} \, d\sigma \,d\tau = \int_0^t \int_0^s \frac{T(\sigma)T(\tau)}{(s-\tau)^{\alpha-1}} \, d\tau \, d\sigma - \int_0^t \int_0^s \frac{T(\sigma)T(\tau)}{(t-s-\tau)^{\alpha-1}} \, d\sigma \,d\tau, \quad t, s \geq 0,
\]

(1.11)

where the integrals are in the sense of strong operator topology.

There, we proved that \( A \) generates a fractional cosine function \( \{T(t)\}_{t \geq 0} \) if and only if it generates an \( \alpha \)-resolvent operator function; that is, the following equalities hold:

\[
T(s)J_0^\alpha T(t) - J_0^\alpha T(s)T(t) = J_0^\alpha T(t) - J_0^\alpha T(s), \quad t, s \geq 0.
\]

As stated above, functional equations involving \( t, s \) and \( t+s \) have been discussed for Caputo fractional differential equations (1.3) with \( \alpha \in (0, 1) \) and \( \alpha \in (1, 2) \), Riemann-Liouville fractional equation (1.4) with \( \alpha \in (0, 1) \). To close the gap, we will discuss the residual case, that is, functional equations involving \( t, s \) and \( t + s \) for Riemann-Liouville fractional equation (1.4) with \( \alpha \in (1, 2) \). To this end, we first consider the special case that \( T(\cdot) \) is exponentially bounded (hence it is Laplace transformatable). Take laplace transform on both sides of (1.6) with respect to \( s \) and \( t \) to obtain

\[
(\lambda^{-\alpha} - \mu^{-\alpha}) \hat{T}(\mu) \hat{T}(\lambda) = \lambda^{1-\alpha} \mu^{1-\alpha} (\lambda^{-1} \hat{T}(\lambda) - \mu^{-1} \hat{T}(\mu)).
\]

(1.12)

It follows from [14] (3.8) that the Laplace transform of the right-hand side of (1.10) satisfies

\[
\int_0^\infty e^{-\mu t} \int_0^\infty e^{-\lambda s} \left( \frac{1}{t} \int_0^t \frac{T(\sigma)T(\tau)}{(t-\sigma)^{\alpha-1}} \, d\sigma \, d\tau + \int_0^t \int_0^s \frac{T(\sigma)T(\tau)}{(s-\tau)^{\alpha-1}} \, d\tau \, d\sigma \right) \, ds \, dt = \frac{\Gamma(2-\alpha)(\lambda^{1-\alpha} - \mu^{1-\alpha})}{\lambda \mu (\lambda - \mu)} T(\mu) \hat{T}(\lambda).
\]

(1.13)

The combination of (1.12) and (1.13) implies

\[
\int_0^\infty e^{-\mu t} \int_0^\infty e^{-\lambda s} \left( \frac{1}{t} \int_0^t \frac{T(\sigma)T(\tau)}{(t-\sigma)^{\alpha-1}} \, d\sigma \, d\tau + \int_0^t \int_0^s \frac{T(\sigma)T(\tau)}{(s-\tau)^{\alpha-1}} \, d\tau \, d\sigma \right) \, ds \, dt = \frac{\Gamma(2-\alpha)(\lambda^{-1} \hat{T}(\lambda) - \mu^{-1} \hat{T}(\mu))}{\mu - \lambda}.
\]

Let \( m(t) = \int_0^t T(\sigma) \, d\sigma \), by similar proof of [10] (4.2), it holds

\[
\int_0^\infty e^{-\mu t} \int_0^\infty e^{-\lambda s} m(t+s) \, ds \, dt = \frac{\hat{m}(\mu) - \hat{m}(\lambda)}{\lambda - \mu} = \frac{\lambda^{-1} \hat{T}(\lambda) - \mu^{-1} \hat{T}(\mu)}{\mu - \lambda}.
\]
By the Laplace transform, it follows that
\[
\Gamma(2 - \alpha) \int_0^{t+s} T(\sigma) \, d\sigma \\
= \int_0^t \int_0^s \frac{T(\tau)T(\tau)}{(t-\sigma)^{\alpha-1}} \, d\tau \, d\sigma + \int_0^t \int_0^s \frac{T(\sigma)T(\tau)}{(s-\tau)^{\alpha-1}} \, d\tau \, d\sigma \\
- \int_0^t \int_0^s \frac{T(\sigma)T(\tau)}{(t+s-\sigma-\tau)^{\alpha-1}} \, d\tau \, d\sigma,
\] (1.14)

In the following two sections, we show that (1.14) also holds without the assumption that \(\{T(t)\}_{t>0}\) is exponentially bounded and it essentially describes a Riemann-Liouville fractional resolvent.

2. RIEMANN-LIOUVILLE FRACTIONAL COSINE FUNCTION

Equality (1.6) is an important functional equation for the solution of (1.4) with \(\alpha \in (1, 2)\). However, as stated in the introduction, (1.6) does not write the functional equation in terms of the sum of time variables: \(s + t\). This is very important in concrete applications of the algebraic functional equation. Therefore, it is very valuable to study functional equation (1.14), which appears in the following definitions.

**Definition 2.1.** A family \(\{T(t)\}_{t>0}\) of bounded linear operators is called Riemann-Liouville \(\alpha\)-order fractional cosine function on a Banach space \(X\), if the following conditions are satisfied:

(i) \(T(t)\) is strongly continuous, that is, for any \(x \in X\), the mapping \(t \mapsto T(t)x\) is continuous over \((0, \infty)\);

(ii) it holds
\[
\lim_{t \to 0^+} t^{2-\alpha}T(t)x = \frac{x}{\Gamma(\alpha-1)} \quad \text{for all} \quad x \in X; \quad (2.1)
\]

(iii) for all \(t, s > 0\), it holds
\[
\Gamma(2 - \alpha) \int_0^{t+s} T(\sigma) \, d\sigma \\
= \int_0^t \int_0^s \frac{T(\sigma)T(\tau)}{(t-\sigma)^{\alpha-1}} \, d\tau \, d\sigma + \int_0^t \int_0^s \frac{T(\sigma)T(\tau)}{(s-\tau)^{\alpha-1}} \, d\tau \, d\sigma \\
- \int_0^t \int_0^s \frac{T(\sigma)T(\tau)}{(t+s-\sigma-\tau)^{\alpha-1}} \, d\tau \, d\sigma, \quad (2.2)
\]

where the integrals are in the sense of strong operator topology.

**Lemma 2.2.** Let \(\{T(t)\}_{t>0}\) be a Riemann-Liouville \(\alpha\)-order fractional cosine on a Banach space \(X\). Then \(\{T(t)\}_{t>0}\) is commutative, i.e. \(T(t)T(s) = T(s)T(t)\) for all \(t, s > 0\).

**Proof.** Observe that the left-hand side of (2.2) is symmetric with respect to \(t\) and \(s\). Hence we can obtain the equality
\[
\int_0^t \int_0^s \frac{T(\sigma)T(\tau)}{(t-\sigma)^{\alpha-1}} \, d\tau \, d\sigma + \int_0^t \int_0^s \frac{T(\sigma)T(\tau)}{(s-\tau)^{\alpha-1}} \, d\tau \, d\sigma \\
- \int_0^t \int_0^s \frac{T(\sigma)T(\tau)}{(t+s-\sigma-\tau)^{\alpha-1}} \, d\tau \, d\sigma
\]
Obviously, for $0 < \alpha < 1$, we have
\[
= \int_0^t \int_0^s \frac{T(\sigma)T(\tau)}{(s - \tau)^{\alpha - 1}} \, d\tau \, d\sigma + \int_0^s \int_0^t \frac{T(\sigma)T(\tau)}{(t - \tau)^{\alpha - 1}} \, d\tau \, d\sigma \\
- \int_0^s \int_0^t \frac{T(\sigma)T(\tau)}{(t + s - \tau)^{\alpha - 1}} \, d\tau \, d\sigma, \quad t, s > 0.
\]

The commutative property is proved as in [13 Proposition 3.4]. \qed

**Definition 2.3.** Let $\{T(t)\}_{t > 0}$ be a Riemann-Liouville $\alpha$-order fractional cosine function on Banach space $X$. Denote by $D(A)$ the set of all $x \in X$ such that the limit
\[
\lim_{t \to 0^+} \Gamma(\alpha + 1)t^{-\alpha}J_i^2(\alpha) \left( T(t)x - \frac{t^{\alpha - 2}}{\Gamma(\alpha - 1)}x \right)
\]
e exists. Then the operator $A : D(A) \to X$ defined by
\[
Ax = \lim_{t \to 0^+} \Gamma(\alpha + 1)t^{-\alpha}J_i^2(\alpha) \left( T(t)x - \frac{t^{\alpha - 2}}{\Gamma(\alpha - 1)}x \right)
\]
is called the generator of $\{T(t)\}_{t > 0}$. \par

**Proposition 2.4.** Assume $\{T(t)\}_{t > 0}$ is a Riemann-Liouville $\alpha$-order fractional cosine function on Banach space $X$. Suppose that $A$ is the generator of $\{T(t)\}_{t > 0}$. Then
\begin{enumerate}
\item For any $x \in X$ and $t > 0$, it holds $J_i^\alpha T(t)x \in D(A)$ and
\[
T(t)x = \frac{t^{\alpha - 2}}{\Gamma(\alpha - 1)}x + AJ_i^\alpha T(t)x; \tag{2.3}
\]
\item $T(t)D(A) \subset D(A)$ and $T(t)Ax = AT(t)x$, for all $x \in D(A)$.
\item For all $x \in D(A)$, we have
\[
T(t)x = \frac{t^{\alpha - 2}}{\Gamma(\alpha - 1)}x + J_i^\alpha T(t)Ax;
\]
\item $A$ is equivalently defined by
\[
Ax = \Gamma(2\alpha - 1) \lim_{t \to 0^+} \frac{T(t)x - \frac{t^{\alpha - 2}}{\Gamma(\alpha - 1)}x}{t^{2\alpha - 2}} \tag{2.4}
\]
and $D(A)$ is just consists of those $x \in X$ such that the above limit exists.
\item $A$ is closed and densely defined.
\item $A$ admits at most one Riemann-Liouville $\alpha$-order fractional cosine function.
\end{enumerate}

**Proof.** (a) Let $x \in X$ and $b > 0$ be fixed. Denote by $g_b(\cdot)$ the truncation of $T(\cdot)$ at $b$; that is,
\[
g_b(\sigma) = \begin{cases} T(\sigma), & \text{if } 0 < \sigma \leq b \\
0, & \text{if } \sigma > b.
\end{cases}
\]
Define the function $H_b(r, s)$ for $r, s > 0$ by
\[
H_b(r, s) = \left( g_b(r) - \frac{r^{\alpha - 2}}{\Gamma(\alpha - 1)} \right) J_i^\alpha g_b(s)x. \tag{2.5}
\]
Obviously, for $0 < r \leq t$,
\[
H_t(r, t) = \left( T(r) - \frac{r^{\alpha - 2}}{\Gamma(\alpha - 1)} \right) J_i^\alpha T(t)x. \tag{2.6}
\]
Take Laplace transform with respect to \( r \) and \( s \) successively for both sides of (2.5) to obtain
\[
\hat{H}_b(\mu, \lambda) = \lambda^{-\alpha} \hat{g}_b(\mu) \hat{g}_b(\lambda) x - \lambda^{-\alpha} \mu^{-1} \hat{g}_b(\lambda) x. \tag{2.7}
\]
Denote by \( L(t, s) \) and \( R(t, s) \) the left and right sides of (2.2), respectively. Moreover, denote by \( R_b(t, s) \), and \( L_b(t, s) \) the quantities resulted by replacing \( T(t) \) with \( g_b(t) \) in \( R(t, s) \), \( L(t, s) \), respectively.

It follows from [14, (3.7)] that the Laplace transform of \( R_b(t, s) \) with respect to \( t \) and \( s \) is given by
\[
\hat{R}_b(\mu, \lambda) = \frac{\Gamma(2 - \alpha)(\lambda^\alpha - \mu^\alpha)}{\lambda \mu (\lambda - \mu)} \hat{g}_b(\mu) \hat{g}_b(\lambda). \tag{2.8}
\]
For all \( t > 0 \), the Laplace transform of \( \hat{L}_b(t, s) \) with respect to \( s \) and \( t \) can be obtained as
\[
\hat{L}_b(\mu, \lambda) = \frac{\lambda^{-\alpha} \hat{g}_b(\mu) \hat{g}_b(\lambda)^x - \lambda^{-\alpha} \mu^{-1} \hat{g}_b(\mu) x}{\lambda \mu - \lambda} \tag{2.9}
\]
Combine (2.7), (2.8) and (2.9) to derive
\[
\hat{H}_b(\mu, \lambda) = \mu^{-\alpha} \hat{g}_b(\mu) \hat{g}_b(\lambda) x - \lambda^{-\alpha} \lambda^{1-\alpha} \hat{g}_b(\mu) x \\
+ \frac{\lambda^{1-\alpha} \mu^{1-\alpha} (\lambda - \mu)}{\Gamma(2 - \alpha)} (\hat{L}_b(\mu, \lambda) - \hat{R}_b(\mu, \lambda)) x.
\]
Take inverse Laplace transform to obtain
\[
H_b(r, s) = \left( g_b(s) - \frac{s^{\alpha-2}}{\Gamma(\alpha - 1)} f(t) \right) J^\alpha_r g_b(r) x \\
+ \frac{[(D^2 s^{-\alpha}) J^\alpha_r - (D^2 r^{-\alpha}) J^\alpha_s - 1] \cdot [L_b(r, s) - R_b(r, s)] x}{\Gamma(2 - \alpha)}.
\]
Here the Laplace transform formula
\[
\hat{D}^\alpha_x f(\lambda) = \lambda^\beta \hat{f}(\lambda) - \lim_{t \to 0^+} J^\alpha_x^{-1} f(t), \quad 0 < \beta < 1, \ f \in C([0, \infty), X)
\]
is used.

From the definition of \( g_b \), it follows that \( L_b(r, s) = R_b(r, s) \) for \( 0 < s, r \leq b \). Then we have
\[
H_b(r, s) = \left( T(s) - \frac{s^{\alpha-2}}{\Gamma(\alpha - 1)} f(t) \right) J^\alpha_r T(r) x, \quad \forall 0 < r, s \leq b.
\]
This implies
\[
H_b(r, t) = \left( T(t) - \frac{t^{\alpha-2}}{\Gamma(\alpha - 1)} f(t) \right) J^\alpha_r T(r) x, \quad \forall 0 < r \leq t. \tag{2.10}
\]
Combining (2.6) with (2.10), we obtain
\[
\lim_{r \to 0^+} \Gamma(\alpha + 1) r^{-\alpha} J^{2-\alpha}_r \left( T(t) - \frac{t^{\alpha-2}}{\Gamma(\alpha - 1)} f(t) \right) J^\alpha_r T(r) x \\
= \lim_{r \to 0^+} \Gamma(\alpha + 1) r^{-\alpha} \left( T(t) - \frac{t^{\alpha-2}}{\Gamma(\alpha - 1)} f(t) \right) J^2_r T(r) x \\
= \Gamma(\alpha + 1) \left( T(t) - \frac{t^{\alpha-2}}{\Gamma(\alpha - 1)} f(t) \right) \lim_{r \to 0^+} r^{-\alpha} \int_0^r (r - \sigma) T(\sigma) x d\sigma
\]
exists. Let \( x \in D(A) \) and
\[
A J^\alpha_r T(t)x = T(t)x - \frac{t^{\alpha-2}}{\Gamma(\alpha-1)} x.
\]
Conditions (b) and (c) are directly obtained by Lemma 2.2 and (a).

(d) Denote by \( D \) the set of those \( x \in X \) such that the limit
\[
\lim_{t \to 0^+} \frac{T(t)x - \frac{t^{\alpha-2}}{\Gamma(\alpha-1)} x}{t^{2\alpha-2}}
\]
exists. Let \( x \in D(A) \). Then, by (b), we have
\[
\Gamma(2\alpha-1) \lim_{t \to 0^+} \frac{T(t)x - \frac{t^{\alpha-2}}{\Gamma(\alpha-1)} x}{t^{2\alpha-2}}
\]
\[
= \Gamma(2\alpha-1) \lim_{t \to 0^+} \frac{J^\alpha_r T(t)x}{t^{2\alpha-2}}
\]
\[
= \frac{\Gamma(2\alpha-1)}{\Gamma(\alpha)} \lim_{t \to 0^+} \frac{1}{t^{2\alpha-2}} \int_0^t (t-s)^{\alpha-1} T(s)Ax ds
\]
\[
= \frac{\Gamma(2\alpha-1)}{\Gamma(\alpha)} \int_0^1 (1-\sigma)^{\alpha-1} \sigma^{\alpha-2} T(\sigma) Ax d\sigma.
\]
The dominated convergence theorem and (b) of Definition 2.1 indicate that
\[
\Gamma(2\alpha-1) \lim_{t \to 0^+} \frac{T(t)x - \frac{t^{\alpha-2}}{\Gamma(\alpha-1)} x}{t^{2\alpha-2}}
\]
\[
= \frac{\Gamma(2\alpha-1)}{\Gamma(\alpha)} \int_0^1 (1-\sigma)^{\alpha-1} \sigma^{\alpha-2} T(\sigma) Ax d\sigma
\]
\[
= \frac{\Gamma(2\alpha-1) \Gamma(\alpha)}{\Gamma(\alpha-1) \Gamma(\alpha)} \int_0^1 (1-\sigma)^{\alpha-1} \sigma^{\alpha-2} Ax d\sigma
\]
\[
= \frac{\Gamma(2\alpha-1) \Gamma(\alpha)}{\Gamma(\alpha-1) \Gamma(\alpha)} Ax = Ax.
\]
This implies that $x \in D$ and then $D(A) \subset D$. Now we prove the converse inclusion. Let $x \in D$, that is, the limit
\[
\lim_{t \to 0^+} \frac{T(t)x - t^{\alpha-2}_{\Gamma(\alpha-1)} x}{t^{2\alpha-2}}.
\]
exists. By the dominated convergence theorem, it follows that
\[
\lim_{t \to 0^+} \Gamma(\alpha + 1)t^{-\alpha}J_{t}^{2-\alpha}\left(T(t)x - \frac{t^{\alpha-2}}{\Gamma(\alpha-1)} x\right)
= \lim_{t \to 0^+} \Gamma(\alpha + 1) \int_{0}^{1} (1 - \sigma)^{1-\sigma} \sigma^{2\alpha-2} \frac{T(t \sigma)x - \frac{(t \sigma)^{\alpha-2}}{\Gamma(\alpha-1)} x}{(t \sigma)^{2\alpha-2}} d\sigma
= \frac{\Gamma(\alpha + 1)}{\Gamma(2 - \alpha)} \int_{0}^{1} (1 - \sigma)^{1-\alpha} \sigma^{2\alpha-2} \lim_{t \to 0^+} \frac{T(t \sigma)x - \frac{(t \sigma)^{\alpha-2}}{\Gamma(\alpha-1)} x}{(t \sigma)^{2\alpha-2}} d\sigma
= \frac{\Gamma(\alpha + 1)}{\Gamma(2 - \alpha)} \Gamma(2\alpha - 1) \Gamma(\alpha + 1) \lim_{t \to 0^+} \frac{T(t \sigma)x - \frac{(t \sigma)^{\alpha-2}}{\Gamma(\alpha-1)} x}{t^{2\alpha-2}}.
\]
Hence, $x \in D(A)$ and
\[
Ax = \Gamma(2\alpha - 1) \lim_{t \to 0^+} \frac{T(t \sigma)x - \frac{(t \sigma)^{\alpha-2}}{\Gamma(\alpha-1)} x}{t^{2\alpha-2}}.
\]
(e) The properties that $A$ is closed and densely defined are followed directly from the combination of (d) and [12].

(f) Assume that both $\{T(t)\}_{t > 0}$ and $\{S(t)\}_{t > 0}$ are Riemann-Liouville $\alpha$-order fractional resolvent generated by $A$. Then, by (c), for all $x \in D(A)$, we have
\[
\frac{t^{\alpha-2}}{\Gamma(\alpha-1)} * T(t)x = (S(t) - J_{t}^{\alpha} AS(t)) * T(t)x
= S(t) * T(t)x - (J_{t}^{\alpha} AS(t)) * T(t)x
= S(t) * (T(t)x - J_{t}^{\alpha} AT(t)x)
= \frac{t^{\alpha-2}}{\Gamma(\alpha-1)} * S(t)x.
\]
By Titchmarsh’s Theorem, for any $t > 0$, $T(t) = S(t)$ on $D(A)$. The result is obtained by the density of $A$. \qed

**Corollary 2.5.** Assume that $A$ generates a Riemann-Liouville $\alpha$-order fractional cosine function on Banach space $X$. Then $\{T(t)\}_{t > 0}$ is a Riemann-Liouville $\alpha$-order fractional resolvent.

**Proof.** In (a) of Theorem 2.4 replacing $x$ with $J_{s}^{\alpha} T(s)x$, and using Lemma 2.2 we obtain
\[
T(t)J_{s}^{\alpha} T(s)x = \frac{t^{\alpha-2}}{\Gamma(\alpha-1)} J_{s}^{\alpha} T(s)x + AJ_{s}^{\alpha} T(t)J_{s}^{\alpha} T(s)x
= \frac{t^{\alpha-2}}{\Gamma(\alpha-1)} J_{s}^{\alpha} T(s)x + AJ_{s}^{\alpha} T(s)J_{s}^{\alpha} T(t)x
= \frac{t^{\alpha-2}}{\Gamma(\alpha-1)} J_{s}^{\alpha} T(s)x + (T(s) - \frac{t^{\alpha-2}}{\Gamma(\alpha-1)}) J_{s}^{\alpha} T(t)x,
\]
which is just (1.6). The proof is complete. \qed
3. Equivalence of Riemann-Liouville fractional resolvent

In this section, we prove that equality \[1.6\] essentially describes a Riemann-Liouville \(\alpha\)-order fractional cosine function.

**Theorem 3.1.** Suppose that \(\{T(t)\}_{t>0}\) is a Riemann-Liouville \(\alpha\)-order fractional resolvent on Banach space \(X\). Then, the family is a Riemann-Liouville \(\alpha\)-order fractional cosine function.

**Proof.** Denote by \(L(t, s)\) and \(R(t, s)\) the left and right sides of equality \[2.2\], respectively. Obviously, we need to prove that \(L(t, s) = R(t, s)\) for all \(t, s > 0\). For brevity, we introduce the following notation. Let

\[
H(t, s) = T(t)J^\alpha_t(s) - J^\alpha_s(t)T(s),
\]

\[
K(t, s) = \frac{t^{\alpha-2}}{\Gamma(\alpha - 1)} J^\alpha_s(s) - \frac{s^{\alpha-2}}{\Gamma(\alpha - 1)} J^\alpha_s(t), t, s > 0.
\]

Moreover, for sufficiently large \(b > 0\) denote by \(g_b(t)\) the truncation of \(T(t)\) at \(b\), and by \(R_b(t, s), L_b(t, s), H_b(t, s)\) and \(K_b(t, s)\) the quantities resulted by replacing \(T(t)\) with \(g_b(t)\) in \(R(t, s), L(t, s), H(t, s)\) and \(K(t, s)\), respectively.

We set

\[
P_b(t, s) = \int_0^t \int_0^s \frac{H_b(\sigma, \tau)}{(t-\sigma)^{\alpha-1}} d\tau d\sigma + \int_0^t \int_0^s \frac{H_b(\sigma, \tau)}{(s-\tau)^{\alpha-1}} d\tau d\sigma
\]

\[
- \int_0^t \int_0^s \frac{H_b(\sigma, \tau)}{(t+s-\sigma-\tau)^{\alpha-1}} d\tau d\sigma
\]

and

\[
Q_b(t, s) = \int_0^t \int_0^s \frac{K_b(\sigma, \tau)}{(t-\sigma)^{\alpha-1}} d\tau d\sigma + \int_0^t \int_0^s \frac{K_b(\sigma, \tau)}{(s-\tau)^{\alpha-1}} d\tau d\sigma
\]

\[
- \int_0^t \int_0^s \frac{K_b(\sigma, \tau)}{(t+s-\sigma-\tau)^{\alpha-1}} d\tau d\sigma.
\]

Observe that the equality \[1.6\] implies \(H(t, s) = K(t, s)\) for any \(t, s > 0\). Thus, for all \(t, s > 0\),

\[
\lim_{b \to \infty} P_b(t, s) = \lim_{b \to \infty} Q_b(t, s).
\]

By \[14\] (3.13)], it follows that

\[
P_b(t, s) = (J_t^\alpha - J_s^\alpha)R_b(t, s), \quad \forall \ t, s > 0.
\]

We now compute Laplace transform of the first term of \(Q_b(t, s)\) with respect to \(s\) and \(t\) as follows,

\[
\int_0^\infty e^{-\mu t} \int_0^\infty e^{-\lambda s} \int_0^t \int_0^s \frac{K_b(\sigma, \tau)}{(t-\sigma)^{\alpha-1}} d\tau d\sigma ds dt
\]

\[
= \int_0^\infty e^{-\mu t} \int_0^\infty e^{-\lambda s} \int_0^t \int_0^s \frac{\sigma^{\alpha-2}}{\Gamma(\alpha - 1)} J^\alpha_s(\tau) - \frac{\tau^{\alpha-2}}{\Gamma(\alpha - 1)} J^\alpha_s(\sigma) d\tau d\sigma ds dt
\]

\[
= \int_0^\infty e^{-\mu t} \int_0^t \int_0^\infty e^{-\lambda s} \int_0^s \frac{\sigma^{\alpha-2}}{\Gamma(\alpha - 1)} J^\alpha_s(\tau) - \frac{\tau^{\alpha-2}}{\Gamma(\alpha - 1)} J^\alpha_s(\sigma) d\sigma ds d\tau dt
\]

\[
= \int_0^\infty e^{-\mu t} \int_0^\infty e^{-\lambda s} \int_0^t \int_0^s \frac{\sigma^{\alpha-2}}{\Gamma(\alpha - 1)} J^\alpha_s(\tau) - \frac{\tau^{\alpha-2}}{\Gamma(\alpha - 1)} J^\alpha_s(\sigma) d\tau d\sigma ds dt
\]
We compute the Laplace transform of the third term of $Q_b(t,s)$ with respect to $s$ and $t$ as follows.

\[
\begin{align*}
&= \int_0^{\infty} e^{-\mu t} \int_0^t e^{-\lambda s} \int_0^s \frac{K_b(\sigma,\tau)}{(s-\tau)^{\alpha-1}} d\tau d\sigma d\mu \\
&= \int_0^{\infty} e^{-\mu t} \int_0^t e^{-\lambda s} \int_0^s \frac{\Gamma(\alpha-1) J^\alpha_x g_b(\tau)}{(t+s-\sigma-\tau)^{\alpha-1}} d\tau d\sigma d\mu \\
&= \int_0^{\infty} e^{-\mu t} \int_0^t e^{-\lambda s} \int_0^s \frac{J^\alpha_x g_b(\tau)}{(t+s-\sigma-\tau)^{\alpha-1}} d\tau d\sigma ddt \\
&+ \int_0^{\infty} e^{-\mu t} \int_0^t J^\alpha_x g_b(\sigma) \int_0^s e^{-\lambda s} \int_0^\infty \frac{1}{(t+s-\sigma-\tau)^{\alpha-1}} ds d\sigma dt \lambda^{-\alpha} g_b(\lambda) \\
&+ \lambda^{1-\alpha} \int_0^{\infty} e^{-\mu t} \int_0^t J^\alpha_x g_b(\sigma) \int_0^\infty e^{-\lambda s} \frac{1}{(t+s-\sigma-\tau)^{\alpha-1}} ds d\sigma dt \\
&= \int_0^{\infty} e^{-\mu t} \int_0^t e^{-\lambda s} \int_0^s \frac{K_b(\sigma,\tau)}{(s-\tau)^{\alpha-1}} d\tau d\sigma ddt \\
&+ \int_0^{\infty} e^{-\mu t} \int_0^t J^\alpha_x g_b(\sigma) \int_0^s e^{-\lambda s} \int_0^\infty \frac{1}{(t+s-\sigma-\tau)^{\alpha-1}} ds d\sigma dt \lambda^{-\alpha} g_b(\lambda) \\
&+ \lambda^{1-\alpha} \int_0^{\infty} e^{-\mu t} \int_0^t J^\alpha_x g_b(\sigma) e^{\lambda(t-\sigma)} \\
&\times \left( \int_0^t e^{-\lambda r} r^{\alpha-\alpha} dr - \int_0^s e^{-\lambda r} r^{\alpha-\alpha} dr \right) d\sigma dt \lambda^{-\alpha} g_b(\lambda) \\
&\times \left( \int_0^t e^{-\lambda r} r^{\alpha-\alpha} dr - \int_0^s e^{-\lambda r} r^{\alpha-\alpha} dr \right) d\sigma dt
\end{align*}
\]
Using (2.9), we obtain

\[ Q_b(t,s) = (J_s^\alpha - J_t^\alpha)\hat{g}_b(t,s), \quad \forall t, s > 0. \] (3.5)

Form (3.3) and (3.5), we have

\[ (J_s^\alpha - J_t^\alpha)L(t,s) = (J_s^\alpha - J_t^\alpha)R(t,s), \quad \forall t, s > 0. \]

Therefore, \( L(t,s) = R(t,s). \) This completes the proof. \( \square \)

Combining Corollary 2.5 and Theorem 3.1, we can obtain the equivalent of Riemann-Liouville \( \alpha \)-order fractional resolvents and Riemann-Liouville \( \alpha \)-order fractional cosine functions.

Acknowledgments. This work was supported by the Natural Science Foundation of China (grant nos. 11301412 and 11131006), Research Fund for the Doctoral Program of Higher Education of China (grant no. 20130201120053), Shaanxi Province Natural Science Foundation of China (grant no. 2014JQ1017), Project funded by China Postdoctoral Science Foundation (grant nos. 2014M550482 and 2015T81011), the Fundamental Research Funds for the Central Universities (grant no. 2012jdhz52).

References


Zhan-Dong Mei (corresponding author)
School of Mathematics and Statistics, Xi’an Jiaotong University, Xi’an 710049, China
E-mail address: zhdmei@mail.xjtu.edu.cn

Ji-Gen Peng
School of Mathematics and Statistics, Xi’an Jiaotong University, Xi’an 710049, China
E-mail address: jgpeng@mail.xjtu.edu.cn