MULTIPLE POSITIVE SOLUTIONS TO A FOURTH-ORDER BOUNDARY-VALUE PROBLEM

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ABSTRACT. We study the existence, localization and multiplicity of positive solutions for a nonlinear fourth-order two-point boundary value problem. The approach is based on critical point theorems in conical shells, Krasnosel'skiĭ's compression-expansion theorem, and unilateral Harnack type inequalities.

1. INTRODUCTION

The fourth-order boundary-value problems appear in the elasticity theory describing stationary states of the deflection of an elastic beam. In the last decade a lot of studies have been devoted to the existence of positive solutions for such problems, applying the Leray-Schauder continuation method, topological degree theory, fixed point theorems in cones, critical point theory and lower and upper solution methods (see, for example, [2, 3, 4, 6, 5, 10, 11, 12, 17, 18]).

In this article, we study the existence and multiplicity of positive solutions for nonlinear fourth-order two-point boundary value problem with cantilever boundary conditions. More exactly, we consider the fourth-order boundary value problem

$$\begin{aligned}
u^{(4)}(t) - f(t, u(t)) = 0, \quad 0 < t < 1, \\
u(0) = u'(0) = u''(1) = u'''(1) = 0,
\end{aligned} \tag{1.1}$$

where the function $f\colon [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous, $f(t, \mathbb{R}_+) \subset \mathbb{R}_+$ for all $t \in [0,1]$, and the solution is sought in $C^4[0,1]$.

Our approach is based on critical point theorems for functionals in conical shells (see [13, 14]) and Krasnosel'skiĭ's compression-expansion theorem. As one can see along the paper, the arguments developed here can be applied to other boundary value problems associated to fourth and sixth order differential equations. Because the estimates are connected with specific boundary conditions, we concentrate only on the model problem (1.1).

The paper is organized as follows. In Section 2 we state the critical point theorems in conical shells and Krasnosel'skiĭ's compression-expansion theorem. We also present the fixed point formulation and the variational formulation of the problem. In Section 3, the main existence and multiplicity results Theorems 3.2, 3.3, 3.5 and 3.6 are stated and proved. Their proofs are based on the mentioned above theorems

and on the inequalities proved in Lemma 3.1 and Lemma 3.4. Finally in Section 4, an example is presented.

2. Preliminaries

2.1. **Critical point theorems in conical shells.** In this subsection we introduce the results given in [13] which we are going to apply to the fourth order problem (1.1).

For any real Hilbert space H with inner product $(\cdot, \cdot)_H$ and norm $|\cdot|_H$, we let H' be its dual space. Denoting by $\langle \cdot, \cdot \rangle$ the duality between H and H', i.e. $\langle u^*, u \rangle = u^*(u)$ for $u^* \in H'$ and $u \in H$, according to the Riesz representation theorem, we can consider the canonical isomorphism $L_H : H \to H'$, given by

$$(u, v)_H = \langle L_H u, v \rangle \quad \text{for all } u, v \in H, \tag{2.1}$$

and its inverse $J_H : H' \to H$ for which

$$(J_H u, v)_H = \langle u, v \rangle \quad \text{for } u \in H', v \in H. \tag{2.2}$$

Using this isomorphism we may identify H with H', letting $L_H u \equiv u$, $J_H u \equiv u$, and so $L_H = J_H = I_H$ (identity map of H).

In what follows we consider two real Hilbert spaces, X with inner product and norm $(\cdot, \cdot)_X$, $|\cdot|_X$, and Y with inner product and norm $(\cdot, \cdot)_Y$, $|\cdot|_Y$; we assume that X is continuously embedded into Y and that Y is identified with Y'. Then, from $X \subset Y$, one has $Y' \subset X'$, and therefore

$$X \subset Y \equiv Y' \subset X'. \tag{2.3}$$

Note that for every $u \in X$, the notation $J_X u$ will be used to denote the element $J_X L_Y u$. Also, if $u, v \in Y$, then according to (2.1) and the identification $L_Y u = u$,

$$\langle u, v \rangle = (u, v)_Y.$$

This is the reason for using the symbol $\langle \cdot, \cdot \rangle$ instead of $(\cdot, \cdot)_Y$. In what follows, for simplicity, the inner product and norm will be denoted by $(\cdot, \cdot)$ and $|\cdot|$ for X, and by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ for Y. Also, we shall use the notations L and J instead of L_X and J_X.

Let K be a cone in X, i.e. a convex closed nonempty set K, $K \neq \{0\}$, with $\lambda u \in K$ for every $u \in K$ and $\lambda \geq 0$, and $K \cap (-K) = \{0\}$. For any two positive numbers R_0 and R_1, we denote by $K_{R_0 R_1}$ the *conical shell*

$$K_{R_0 R_1} := \{u \in K : \|u\| \geq R_0 \text{ and } |u| \leq R_1\}.$$

Such a set may be empty (even if $R_0 < R_1$) and may be disconnected. Let $\phi \in K \setminus \{0\}$ be a fixed element with $|\phi| = 1$. If $R_0 < \|\phi\| R_1$, then $\mu \phi \in K_{R_0 R_1}$ for every $\mu \in [R_0/\|\phi\|, R_1]$, and $\mu \phi$ is an interior point of $K_{R_0 R_1}$, in the sense that $\|\mu \phi\| > R_0$ and $|\mu \phi| < R_1$, for $\mu \in (R_0/\|\phi\|, R_1)$. In particular, any two elements of $K_{R_0 R_1}$ of the form $\mu \phi$, with $\mu \in [R_0/\|\phi\|, R_1]$, belong to the same connected component of $K_{R_0 R_1}$.

In particular if $Y = X$, the spaces X and X' are identified, so $L = J = I_X$. Also, in this case, the conical shell $K_{R_0 R_1}$ is nonempty and simply connected for every R_0, R_1 with $0 < R_0 < R_1$.

Let E be a C^1 functional defined on X. We say that E satisfies the *modified Palais-Smale-Schechter condition* (MPSS) in $K_{R_0 R_1}$, if any sequence (u_k) of elements of $K_{R_0 R_1}$ for which the sequence $(E(u_k))$ converges and one of the following conditions holds:

(i) $E'(u_k) \to 0$;

(ii) $\|u_k\| = R_0$, $(JE'(u_k), Ju_k) \geq 0$ and $JE'(u_k) - \frac{(JE'(u_k), Ju_k)}{|Ju_k|^2} Ju_k \to 0$;

(iii) $|u_k| = R_1$, $(JE'(u_k), u_k) \leq 0$ and $JE'(u_k) - \frac{(JE'(u_k), u_k)}{R_1^2} u_k \to 0$,

has a convergent subsequence.

We say that E satisfies the *compression boundary condition* in $K_{R_0 R_1}$ if

$$JE'(u) - \lambda Ju \neq 0 \quad \text{for } u \in K_{R_0 R_1}, \ \|u\| = R_0, \ \lambda > 0; \tag{2.4}$$

$$JE'(u) + \lambda u \neq 0 \text{ for } u \in K_{R_0 R_1}, \ |u| = R_1, \ \lambda > 0. \tag{2.5}$$

We say that E has a *mountain pass geometry* in $K_{R_0 R_1}$ if there exist u_0 and u_1 in the same connected component of $K_{R_0 R_1}$, and $r > 0$ such that $|u_0| < r < |u_1|$ and

$$\max\{E(u_0), E(u_1)\} < \inf\{E(u) : u \in K_{R_0 R_1}, \ |u| = r\}.$$

In this case we consider the set

$$\Gamma = \{\gamma \in C([0,1]; K_{R_0 R_1}) : \gamma(0) = u_0, \ \gamma(1) = u_1\} \tag{2.6}$$

and the number

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} E(\gamma(t)). \tag{2.7}$$

Finally, we say that E is *bounded from below* in $K_{R_0 R_1}$ if

$$m := \inf_{u \in K_{R_0 R_1}} E(u) > -\infty. \tag{2.8}$$

We assume that the following conditions are satisfied:

$$(I - JE')(K) \subset K \quad (I \text{ is the identity map on } X); \tag{2.9}$$

and there exists a constant $\nu_0 > 0$ such that

$$(JE'(u), Ju) \leq \nu_0 \quad \text{for all } u \in K \text{ with } \|u\| = R_0; \tag{2.10}$$

$$(JE'(u), u) \geq -\nu_0 \quad \text{for all } u \in K \text{ with } |u| = R_1. \tag{2.11}$$

The following theorems of localization of critical points in a conical shell appear as slight particularizations of the main results from [13, 14].

Theorem 2.1. *Assume that E is bounded from below in $K_{R_0 R_1}$ and that there is a $\rho > 0$ with*

$$E(u) \geq m + \rho \tag{2.12}$$

(m given in (2.8)) for all $u \in K_{R_0 R_1}$ which simultaneously satisfy $|u| = R_1$, $\|u\| = R_0$. In addition assume that E satisfies the (MPSS) condition and the compression boundary condition in $K_{R_0 R_1}$. Then there exists $u \in K_{R_0 R_1}$ such that

$$E'(u) = 0 \quad \text{and} \quad E(u) = m.$$

Theorem 2.2. *Assume that E has the mountain pass geometry in $K_{R_0 R_1}$ and that there is a $\rho > 0$ with*

$$|E(u) - c| \geq \rho \tag{2.13}$$

(c given in (2.7)) for all $u \in K_{R_0 R_1}$ which simultaneously satisfy $|u| = R_1$, $\|u\| = R_0$. In addition assume that E satisfies the (MPSS) condition and the compression boundary condition in $K_{R_0 R_1}$. Then there exists $u \in K_{R_0 R_1}$ such that

$$E'(u) = 0 \quad \text{and} \quad E(u) = c.$$

Remark 2.3. If the assumptions of both Theorems 2.1 and 2.2 are satisfied, since $m < c$, then E has two distinct critical points in $K_{R_0 R_1}$.

2.2. **Krasnosel'skiǐ's compression-expansion theorem.** Problem (1.1) can also be investigated by means of fixed point techniques. In this article, we are mainly concerned with the variational approach based on critical point theory. However, it deserves to comment about the applicability of fixed point methods and the surplus of information given by the variational approach.

Thus we shall report on the applicability of Krasnosel'skiǐ's compression-expansion theorem (see [7, 9]), which guarantees the existence of a fixed point of a compact operator in a conical shell of a Banach space.

Theorem 2.4 (Krasnosel'skiǐ). *Let $(X, |\cdot|)$ be a Banach space and $K \subset X$ a cone. Let R_0, R_1 be two numbers with $0 < R_0 < R_1$, $K_{R_0R_1} = \{u \in K : R_0 \le |u| \le R_1\}$, and let $N : K_{R_0R_1} \to K$ be a compact operator. Let $<$ be the strict ordering induced in X by the cone K, i.e. $u < v$ if and only if $v - u \in K \setminus \{0\}$. Assume that one of the following conditions is satisfied:*

- (a) *compression:* (i) $N(u) \not< u$ *for all* $u \in K$ *with* $|u| = R_0$, *and* (ii) $N(u) \not> u$ *for all* $u \in K$ *with* $|u| = R_1$;
- (b) *expansion:* (i) $N(u) \not> u$ *for all* $u \in K$ *with* $|u| = R_0$, *and* (ii) $N(u) \not< u$ *for all* $u \in K$ *with* $|u| = R_1$.

Then N has at least one fixed point in $K_{R_0R_1}$.

2.3. **Fixed point formulation of the problem.** For each $v \in L^2(0, 1)$, the problem

$$u^{(4)}(t) = v(t), \quad 0 < t < 1,$$
$$u(0) = u'(0) = u''(1) = u'''(1) = 0 \tag{2.14}$$

has in $H^4(0, 1)$ a unique solution u denoted by Sv, namely

$$(Sv)(t) = \int_0^1 G(t, s)\, v(s)\, ds, \quad t \in [0, 1], \tag{2.15}$$

where $G(t, s)$ is the corresponding Green's function. Obviously, $Sv \in C^4[0, 1]$ if $v \in C[0, 1]$. One can easily obtain the expression of $G(t, s)$ using the Mathematica package developed in [1], namely

$$G(t, s) = \begin{cases} \frac{s^2}{6}(3t - s), & 0 \le s \le t \le 1, \\ \frac{t^2}{6}(3s - t), & 0 \le t < s \le 1. \end{cases} \tag{2.16}$$

Then problem (1.1) is equivalent to the integral equation

$$u(t) = \int_0^1 G(t, s)\, f(s, u(s))\, ds, \quad u \in C[0, 1]. \tag{2.17}$$

Obviously, (2.17) represents a fixed point equation associated to the compact operator

$$N(u)(t) = \int_0^1 G(t, s)\, f(s, u(s))\, ds, \quad t \in [0, 1]. \tag{2.18}$$

It is clear that $N(u) = Sf(\cdot, u(\cdot))$.

2.4. **Variational formulation of the problem.** Next we describe the variational structure of problem (1.1) (see [16, 17]).

Let X be the Hilbert space

$$X := \{u \in H^2(0, 1) : u(0) = u'(0) = 0\}$$

with inner product and norm

$$(u, v) := \int_0^1 u''(t) v''(t) dt,$$

$$|u| := \Big(\int_0^1 (u''(t))^2 dt \Big)^{1/2}. \tag{2.19}$$

To problem (1.1) we associate the functional $E : X \to \mathbb{R}$ defined by

$$E(u) := \frac{1}{2} |u|^2 - \int_0^1 F(t, u(t)) dt,$$

where

$$F(t, u) = \int_0^u f(t, s) ds.$$

The functional E is C^1 and for any $u, v \in X$, and

$$\langle E'(u), v \rangle = \int_0^1 (u''(t) v''(t) - f(t, u(t)) v(t)) dt$$
$$= (u, v) - (f(\cdot, u(\cdot)), v)_{L^2}.$$

Integrating by parts twice in $(N(u), v)$ we obtain that $(N(u), v) = (f(\cdot, u(\cdot)), v)_{L^2}$.
Then

$$\langle E'(u), v \rangle = (u - N(u), v).$$

Therefore, if X is identified with X', which corresponds to the choice $Y = X$ in
(2.3), then

$$E'(u) = u - N(u).$$

If we chose $Y = Y' = L^2(0, 1)$, then since $f(\cdot, u(\cdot)) \in L^2(0, 1)$, based on (2.2),
$(f(\cdot, u(\cdot)), v)_{L^2} = \langle f(\cdot, u(\cdot)), v \rangle$. Also $(u, v) = \langle Lu, v \rangle$. Hence

$$\langle E'(u), v \rangle = \langle Lu - f(\cdot, u(\cdot)), v \rangle,$$

and so

$$E'(u) = Lu - f(\cdot, u(\cdot)),$$

or equivalently

$$JE'(u) = u - Jf(\cdot, u(\cdot)). \tag{2.20}$$

Notice that for each $v \in L^2(0, 1)$, one has

$$Jv = Sv, \tag{2.21}$$

that is, $u := Jv$ is the solution in $H^4(0, 1))$ of problem (2.14). Indeed, based on
(2.2),

$$(Jv, w) = \langle v, w \rangle, \quad w \in X. \tag{2.22}$$

For $w \in C_c^\infty(0, 1)$, one has $(u, w) = \langle u^{(4)}, w \rangle$, where $u^{(4)}$ is in the distributional
sense. Hence $\langle u^{(4)}, w \rangle = \langle v, w \rangle$ for every $w \in C_c^\infty(0, 1)$, which shows that $u^{(4)} = v$.
Since $v \in L^2(0, 1)$, we have $u \in H^4(0, 1)$. It remains to check that $u''(1) = u'''(1) = 0$. For this, we come back to (2.22), which after two successive integration by parts
in the left hand side becomes

$$u''(1) w'(1) - u'''(1) w(1) + \langle u^{(4)}, w \rangle = \langle v, w \rangle.$$

Consequently, $u''(1) w'(1) - u'''(1) w(1) = 0$. Since this holds for every $w \in X$, we
must have $u''(1) = u'''(1) = 0$, as desired.

3. Main results

3.1. **Localization in a shell defined by the energetic norm.** First we shall deal with the localization of positive solutions u of problem (1.1) in a shell defined by a single norm, more exactly

$$R_0 \leq |u| \leq R_1,$$

where $|\cdot|$ is the energetic norm given by (2.19). Therefore, in this subsection we choose $Y = X$ and consequently we identify X to X'. For this situation, the following unilateral Harnack inequality is crucial.

Lemma 3.1. *If* $u \in C^4[0,1]$ *satisfies* $u(0) = u'(0) = u''(1) = u'''(1) = 0$ *and* $u^{(4)}$ *is nonnegative and nondecreasing in* $[0,1]$, *then* u *is convex and*

$$u(t) \geq M_0(t)|u| \quad \text{for all } t \in [0,1], \tag{3.1}$$

where $M_0(t) = \sqrt{2}(1-t)t^3/6$.

Proof. From $u^{(4)} \geq 0$ it follows that u'' is convex. This together with $u''(1) = (u'')'(1) = 0$ gives that u'' is nonnegative and nonincreasing. Next, from $u'' \geq 0$ one has that u is convex, and since $u(0) = u'(0) = 0$, u must be nondecreasing and nonnegative.

On the other hand, since $u^{(4)} \geq 0$ we have that u''' is nondecreasing and since $u'''(1) = 0$, $u''' \leq 0$. Then u' is concave; it is also nondecreasing due to $u'' \geq 0$, and since $u'(0) = 0$, we have $u' \geq 0$. Now from $u'' \geq 0$, $u' \geq 0$ and $u(0) = 0$, we see that u is nonnegative, nondecreasing and convex.

Finally note that from $u^{(4)}$ nondecreasing, we have that u''' is convex, and since $u'''(1) = 0$, the graph of u''' is under the line connecting the points $(0, u'''(0))$ and $(1, 0)$, i.e.

$$u'''(t) \leq (1 - t)u'''(0), \quad t \in [0,1]. \tag{3.2}$$

Because the function u'' is nonincreasing and the function u''' is nondecreasing we have:

$$\begin{aligned}
u(t) &= \int_0^t \int_0^s u''(\tau)d\tau ds \geq \int_0^t \int_0^s u''(s)d\tau ds \\
&= \int_0^t s u''(s)ds = \frac{t^2}{2}u''(t) - \int_0^t \frac{s^2}{2}u'''(s)ds \\
&\geq -\int_0^t \frac{s^2}{2}u'''(s)ds \geq -\int_0^t \frac{s^2}{2}u'''(t)ds \\
&= -\frac{t^3}{6}u'''(t).
\end{aligned}$$

This inequality combined with (3.2) gives

$$u(t) \geq -\frac{(1-t)t^3}{6}u'''(0). \tag{3.3}$$

Next we deal with the energetic norm wishing to connect it to $u'''(0)$. One has

$$\begin{aligned}
|u|^2 &= \int_0^1 u''(t)^2 dt = u''u'|_0^1 - \int_0^1 u'''(t)u'(t)dt \\
&= -\int_0^1 u'''(t)u'(t)dt \leq -u'''(0)u'(1).
\end{aligned} \tag{3.4}$$

Also

$$u'(1) = \int_0^1 u''(t)dt = -\int_0^1 \int_t^1 u'''(s)\,ds\,dt$$

$$\leq -\int_0^1 \int_t^1 u'''(t)\,ds\,dt = -\int_0^1 (1-t)u'''(t)dt \qquad (3.5)$$

$$\leq -\int_0^1 (1-t)u'''(0)dt = -\frac{1}{2}u'''(0).$$

From (3.4) and (3.5) we deduce $|u|^2 \leq u'''(0)^2/2$, or

$$-u'''(0) \geq \sqrt{2}|u|.$$

This inequality and (3.3) prove (3.1). □

Consider the cone

$$K := \{u \in X : u \text{ convex and } u(t) \geq M_0(t)|u| \text{ on } [0,1]\}.$$

Note that $K \neq \{0\}$, since according to Lemma 3.1, for any nonzero nonnegative nondecreasing function $v \in C[0,1]$, the function $u = Sv$ given by (2.15) belongs to K and is different from zero. Also, since any convex function with $u(0) = u'(0) = 0$ is nondecreasing, all the elements of K are nondecreasing functions. Consequently, if f is nondecreasing on $[0,1] \times \mathbb{R}_+$ in each of its variables, then the composite function $f(\cdot, u(\cdot))$ is nonnegative and nondecreasing in $[0,1]$ for every $u \in K$ and so Lemma 3.1 can be applied to any solution $u \in K$ of (1.1) guaranteeing the invariance condition $N(K) \subset K$, and consequently, the condition (2.9).

Denote

$$M_1(t) := \frac{2}{3}t^{3/2}, \quad t \in [0,1].$$

Our assumptions on f are as follows:

(H1) f is nondecreasing on $[0,1] \times \mathbb{R}_+$ in each of its variables;

(H2) there exist R_0, R_1 with $0 < R_0 < R_1$ such that

(a) $\int_0^1 M_0(t)f(t, M_0(t)R_0)dt \geq R_0$,

(b) $\int_0^1 M_1(t)f(t, M_1(t)R_1)dt \leq R_1$.

(h3) there exist $u_0, u_1 \in K_{R_0 R_1} = \{u \in K : R_0 \leq |u| \leq R_1\}$ and $r > 0$ such that $|u_0| < r < |u_1|$ and

$$\max\{E(u_0), E(u_1)\} < \inf\{E(u) : u \in K, |u| = r\}.$$

Theorem 3.2. *Assume that* (H1), (H2) *are satisfied. Let* Γ, m *and* c *be defined as in* (2.6), (2.7) *and* (2.8) *respectively. Then the fourth-order problem* (1.1) *has at least one positive solution* u_m *in* $K_{R_0 R_1}$ *such that*

$$E(u_m) = m.$$

If in addition (H3) *holds, then a second positive solution* u_c *exists in* $K_{R_0 R_1}$ *with*

$$E(u_c) = c.$$

Proof. We apply Theorems 2.1 and 2.2. Recall that here we identify X to X' and thus $J = I$, the identity map on X.

First note that the (MPSS) condition holds in $K_{R_0 R_1}$ due to the compactness of the operator $N = I - E'$. Also the boundedness of $(E'(u), u)$ on the boundaries of $K_{R_0 R_1}$, i.e. (2.10) and (2.11), is guaranteed since E' maps bounded sets into bounded sets.

To check (2.9), let u be any element of K. Hence u is nonnegative and non-decreasing on $[0, 1]$. Then, from (h1) we also have that $f(t, u(t))$ is nonnegative and nondecreasing in $[0, 1]$. Now, Lemma 3.1 implies that $N(u) \in K$. But $N(u) = (I - E')(u)$. Thus (2.9) holds.

Next, let us note that for any $u \in K_{R_0 R_1}$, we have

$$u(t) = \int_0^t \int_0^s u''(\tau) d\tau ds \leq \int_0^t \sqrt{s} \Big(\int_0^s u''(\tau)^2 d\tau \Big)^{1/2} ds$$
$$\leq |u| \int_0^t \sqrt{s} ds = \frac{2}{3} t^{3/2} |u| = M_1(t) |u|. \tag{3.6}$$

Then

$$E(u) = \frac{1}{2} |u|^2 - \int_0^1 F(t, u(t)) dt \geq \frac{1}{2} R_0^2 - F(1, \frac{2}{3} R_1).$$

Hence E is bounded from below on $K_{R_0 R_1}$.

Furthermore, we check the boundary conditions (2.4). Assume that $E'(u) - \lambda u = 0$ for some $u \in K$ with $|u| = R_0$ and $\lambda > 0$. Then u solves the problem

$$u^{(4)}(t) - f(t, u(t)) - \lambda u(t) = 0, \quad 0 < t < 1,$$
$$u(0) = u'(0) = u''(1) = u'''(1) = 0$$

and

$$R_0^2 = |u|^2 = \int_0^1 [f(t, u(t)) + \lambda u(t)] u(t) dt$$
$$\geq \int_0^1 [f(t, M_0(t) R_0) + \lambda M_0(t) R_0] M_0(t) R_0 dt$$
$$> R_0 \int_0^1 f(t, M_0(t) R_0) M_0(t) dt,$$

which contradicts assumption (H2) (a). Hence $E'(u) - \lambda u \neq 0$ for all $u \in K$ with $|u| = R_0$ and $\lambda > 0$.

Assume now that $E'(u) + \lambda u = 0$ for some $u \in K$ with $|u| = R_1$ and $\lambda > 0$. Then u solves the problem

$$(1 + \lambda) u^{(4)}(t) - f(t, u(t)) = 0, \quad 0 < t < 1,$$
$$u(0) = u'(0) = u''(1) = u'''(1) = 0.$$

Then

$$R_1^2 = |u|^2 = \frac{1}{1 + \lambda} \int_0^1 f(t, u(t)) u(t) dt.$$

Using (3.6) we deduce

$$R_1^2 < \int_0^1 M_1(t) R_1 f(t, M_1(t) R_1) dt,$$

which contradicts (H2) (b). Hence $E'(u) + \lambda u \neq 0$ for all $u \in K$ with $|u| = R_1$ and $\lambda > 0$.

The conclusions follow from Theorem 2.1 and Theorem 2.2.   □

Example 3.3. We give an example of a function $f(t, u) = f(u)$ which satisfies the conditions (H1) and (H2) of Theorem 3.2. Note that

$$0 \le \frac{\sqrt{2}}{6}(1-t)t^3 < 0.03 \quad \text{if } 0 \le t \le 1,$$

$$\int_0^1 \left(\frac{\sqrt{2}}{6}(1-t)t^3\right)^2 dt = \frac{1}{4536},$$

and $4600 \times 3/100 = 138$. Define

$$f(u) = \begin{cases} 0, & u \le 0, \\ 4600u, & 0 \le u \le 0.03, \\ 138, & u \ge 0.03. \end{cases}$$

Taking $R_0 = 1$ and $R_1 = 37$, by

$$\int_0^1 138 \frac{2}{3} t^{3/2} dt = \frac{184}{5} = 36.8,$$

we obtain that the conditions (H1) and (H2) are satisfied.

For the autonomous case $f(t, u) = f(u)$, where f is nonnegative and nondecreasing on $\mathbb{R}_+$, we may replace the conditions of (H2) by a couple of simpler inequalities as shows the next result.

Theorem 3.4. *Assume that $f : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous nondecreasing and that for some numbers $a \in (0, 1)$, R_0 and R_1 with $0 < R_0 < R_1$, one has*

$$\frac{f(M_0(a)R_0)}{M_0(a)R_0} \ge \frac{1}{(1-a)M_0(a)^2}, \quad \frac{f(\frac{2}{3}R_1)}{R_1} \le \frac{15}{4}. \tag{3.7}$$

Then (1.1) has at least one positive solution u_m in $K_{R_0 R_1}$ with $E(u_m) = m$. If in addition (H3) holds, then a second positive solution u_c exists in $K_{R_0 R_1}$ with $E(u_c) = c$.

Proof. Since $M_1(t) \le 2/3$ for every $t \in [0, 1]$, we have

$$\int_0^1 M_1(t)f(M_1(t)R_1)dt \le f\left(\frac{2}{3}R_1\right) \int_0^1 \frac{2}{3}t^{3/2}dt = \frac{4}{15}f\left(\frac{2}{3}R_1\right).$$

Then the inequality

$$\frac{4}{15}f\left(\frac{2}{3}R_1\right) \le R_1,$$

or equivalently the second inequality in (3.7) is a sufficient condition for (H2)(b) to hold. As concerns the first inequality in (3.7), let us remark that if $E'(u) - \lambda u = 0$ for some $u \in K$ with $|u| = R_0$ and $\lambda > 0$, then

$$R_0^2 = |u|^2 = \int_0^1 [f(u(t)) + \lambda u(t)]u(t)dt$$

$$\ge \int_a^1 [f(u(t)) + \lambda u(t)]u(t)dt. \tag{3.8}$$

The function u being nondecreasing, one has $u(t) \ge u(a)$ for all $t \in [a, 1]$. Also, since $u \in K$, $u(a) \ge M_0(a)|u|$. Then from (3.8),

$$R_0^2 \ge (1-a)[f(M_0(a)R_0) + \lambda M_0(a)R_0]M_0(a)R_0$$

$$> (1-a)f(M_0(a)R_0)M_0(a)R_0.$$

Hence
$$R_0 > (1-a)M_0(a)f(M_0(a)R_0),$$
i.e. the opposite of the first inequality in (3.7).   □

Clearly the inequalities (3.7) express the oscillation of the function $f(t)/t$ up and down the values $1/(1-a)M_0(a)^2$ and $45/8$.

Remark 3.5 (Existence asymptotic conditions). The existence of two numbers R_0, R_1 satisfying (3.7) is guaranteed by the asymptotic conditions
$$\limsup_{\tau \to 0} \frac{f(\tau)}{\tau} > \frac{3^{11}}{32} \quad \text{and} \quad \liminf_{\tau \to \infty} \frac{f(\tau)}{\tau} < \frac{45}{8}. \tag{3.9}$$

Remark 3.6 (Multiplicity). Theorems 3.2 and 3.4 can be used to obtain multiple positive solutions. Indeed, if their assumptions are fulfilled for two pairs (R_0, R_1), $(\overline{R}_0, \overline{R}_1)$, then we obtain four solutions, provided that the sets $K_{R_0 R_1}$ and $K_{\overline{R}_0 \overline{R}_1}$ are disjoint. This happens if $0 < R_0 < R_1 < \overline{R}_0 < \overline{R}_1$. We can even obtain sequences of positive solutions; for instance, in connection with Theorem 3.4, if
$$\limsup_{\tau \to 0} \frac{f(\tau)}{\tau} > \frac{3^{11}}{32} \quad \text{and} \quad \liminf_{\tau \to 0} \frac{f(\tau)}{\tau} < \frac{45}{8}, \tag{3.10}$$
then there exists a sequence (u_k) of positive solutions with $u_k \to 0$ as $k \to \infty$. Also, if
$$\limsup_{\tau \to \infty} \frac{f(\tau)}{\tau} > \frac{3^{11}}{32} \quad \text{and} \quad \liminf_{\tau \to \infty} \frac{f(\tau)}{\tau} < \frac{45}{8}, \tag{3.11}$$
then there exists a sequence (u_k) of positive solutions with $|u_k| \to \infty$ as $k \to \infty$.

Note that $3^{11}/32 \simeq 5536$ is the minimal value of $1/(1-a)M_0(a)^2$ for $a \in (0,1)$, which is reached at $a = 2/3$. A better estimation than (3.1) would allow the replacement of this value by a smaller number.

For examples of functions satisfying asymptotic conditions of type (3.9), (3.10) or (3.11), we refer to the recent paper [8].

Remark 3.7 (Fixed point approach). Under the assumptions of Theorem 3.2, the existence of a solution in $K_{R_0 R_1}$ can also be obtained via Krasnosel'skiĭ's theorem. Indeed, problem (1.1) is equivalent to the fixed point problem (2.17) in X for the compact operator $N : K_{R_0 R_1} \to K$ given by (2.18).

Let us check the condition (a)(i) from Theorem 2.4. Assume the contrary, i.e. $N(u) < u$ for some $u \in K$ with $|u| = R_0$. Then $N(u) = u - v$ for some $v \in K \setminus \{0\}$. This means that $(u-v)^{(4)} = f(t,u)$ in the sense of distributions. Now multiply by u and integrate to obtain
$$|u|^2 - \int_0^1 u''(t)v''(t)dt = \int_0^1 f(t,u(t))u(t)dt.$$
Since $v, u - v \in K$, one has $v'' \geq 0$ and $u'' - v'' \geq 0$ in $[0,1]$. Hence
$$\int_0^1 u''(t)v''(t)dt \geq \int_0^1 v''(t)^2 dt = |v|^2 > 0.$$
Then
$$R_0^2 = |u|^2 > \int_0^1 f(t,u(t))u(t)dt.$$
Next we use (3.1) to derive a contradiction to (H2) (a).

The condition (a)(ii) from Theorem 2.4 can be proved similarly.

Notice that under assumptions (H2)(a) and (b), a solution exists in $K_{R_1 R_0}$ in case that $R_1 < R_0$. However this is not guaranteed by the variational approach.

We may conclude that, compared to the fixed point approach, the variational method gives an additional information about the solution, namely of being a minimum for the energy functional. Moreover, a second solution of mountain pass type can be guaranteed by the variational approach.

The above approach was essentially based on the monotonicity assumption on f, which was required by the Harnack type inequality (3.1). Thus a natural question is if such an inequality can be established for functions u satisfying the boundary conditions and $u^{(4)} \geq 0$, without the assumption that $u^{(4)}$ is nondecreasing. In the absence of the answer to this question, an alternative approach is possible in a shell defined by two norms as shown in the next section.

3.2. **Localization in a shell defined by two norms.** In the previous section, a unilateral Harnack inequality was established for functions u satisfying the two point boundary conditions and with $u^{(4)}$ nonnegative and nondecreasing in $[0, 1]$, in terms of the energetic norm. If we renounce to the monotonicity of $u^{(4)}$, then we have the following lemma in terms of the max norm $\| \cdot \|_\infty$, and finally in the L^2-norm $\| \cdot \|$. This allows us to find positive solutions of (1.1), first in a conical shell defined by the norm $\| \cdot \|_\infty$ using Krasnosel'skiĭ's fixed point theorem, and next in a conical shell defined by two norms, the energetic norm $| \cdot |$ of X and the norm $\| \cdot \|$ of $Y = L^2(0, 1)$, using variational methods.

Lemma 3.8. *If $u \in C^4[0, 1]$ satisfies $u(0) = u'(0) = u''(1) = u'''(1) = 0$ and $u^{(4)} \geq 0$ in $[0, 1]$, then*

$$u(t) \geq M(t)\|u\|_\infty \quad \text{for all } t \in [0, 1], \tag{3.12}$$

where $M(t) := (3 - t)t^2/3$.

Proof. We use the following estimations of Green's function:

$$\frac{t^2}{6}(3 - t)s^2 \leq G(t, s) \leq \frac{s^2}{2} \quad \text{for all } (t, s) \in [0, 1] \times [0, 1]. \tag{3.13}$$

To prove them, we first assume that $0 \leq t < s \leq 1$. Then

$$\begin{aligned}
\frac{t^2}{6}(3 - t)s^2 &\leq \frac{t^2}{6}\{(3 - t)s^2 + (1 - s)[s(1 - t) + (s - t) + s]\} \\
&= \frac{t^2}{6}(3s - t) = G(t, s) \\
&\leq \frac{s^2}{6}(3 - 0) = \frac{s^2}{2}.
\end{aligned}$$

Similarly, for $0 \leq s \leq t \leq 1$, one has

$$\begin{aligned}
\frac{t^2}{6}(3 - t)\, s^2 &\leq \frac{t^2}{6}(3 - s)\, s^2 \\
&\leq \frac{s^2}{6}\{(3 - s)t^2 + (1 - t)[t(1 - s) + (t - s) + t]\} \\
&= \frac{s^2}{6}(3t - s) = G(t, s)
\end{aligned}$$

$$\leq \frac{s^2}{6}(3-0) = \frac{s^2}{2}.$$

Hence (3.13) is proved.

Let $u \in C^4[0,1]$ satisfy $u(0) = u'(0) = u''(1) = u'''(1) = 0$ and $u^{(4)} \geq 0$ in $[0,1]$. Then

$$\begin{aligned}
u(t) &= \int_0^1 G(t,s)u^{(4)}(s)\,ds \\
&\geq \int_0^1 \frac{(3-t)\,t^2\,s^2}{6}\,u^{(4)}(s)\,ds \\
&= \frac{(3-t)\,t^2}{3}\int_0^1 \frac{s^2}{2}u^{(4)}(s)\,ds \\
&\geq \frac{(3-t)\,t^2}{3}\int_0^1 \{\max_{t\in[0,1]} G(t,s)\}\,u^{(4)}(s)\,ds \\
&\geq \frac{(3-t)\,t^2}{3}\,\max_{t\in[0,1]}\{\int_0^1 G(t,s)u^{(4)}(s)\,ds\} \\
&= \frac{(3-t)\,t^2}{3}\|u\|_\infty.
\end{aligned} \tag{3.14}$$

Hence (3.12) is proved.   □

Notice that, since $\|u\| \leq \|u\|_\infty$, where $\|\cdot\|$ is the $L^2(0,1)$-norm, inequality (3.12) also gives

$$u(t) \geq M(t)\|u\| \quad \text{for all } t \in [0,1].$$

Using Lemma 3.8, the existence of a positive solution can be immediately obtained via Krasnosel'skiĭ's compression-expansion theorem.

Theorem 3.9. *Assume that there exist positive numbers α, β, $\alpha \neq \beta$ such that*

$$\alpha \leq (S\underline{f_\alpha})(1) \quad and \quad \beta \geq (S\overline{f_\beta})(1), \tag{3.15}$$

where S is the solution operator defined by (2.15) *and*

$$\begin{aligned}
\underline{f_\alpha}(t) &= \min\{f(t,u) : M(t)\alpha \leq u \leq \alpha\}, \\
\overline{f_\beta}(t) &= \max\{f(t,u) : M(t)\beta \leq u \leq \beta\}.
\end{aligned}$$

Then problem (1.1) *has at least one positive solution u such that*

$$R_0 \leq \|u\|_\infty \leq R_1,$$

where $R_0 = \min\{\alpha, \beta\}$ and $R_1 = \max\{\alpha, \beta\}$.

Proof. As shown in Section 2.3, problem (1.1) is equivalent to the fixed point problem $N(u) = u$ in $C[0,1]$.

In the space $C[0,1]$ we consider the cone

$$K = \{u \in C[0,1] : \ u(0) = 0, \ u(t) \geq M(t)\|u\|_\infty \text{ for all } t \in [0,1]\}.$$

From Lemma 3.8 and the properties of f, it follows that $N(K) \subset K$. Also N is a compact operator.

Now we show that the required boundary conditions from Krasnosel'skiĭ's theorem are satisfied. Assume by contradiction that $N(u) < u$ for some $u \in K$ with $\|u\|_\infty = \alpha$. Then $N(u) = u - v$ for some $v \in K \setminus \{0\}$. Hence

$$u(t) - v(t) = N(u)(t). \tag{3.16}$$

We have
$$M(t)\alpha \leq u(t) \leq \alpha \quad \text{for all } t \in [0,1].$$
Hence
$$f(t, u(t)) \geq \underline{f_\alpha}(t).$$
Since Green's function is positive, the solution operator S preserves the ordering, so that $N(u)(t) \geq (S\underline{f_\alpha})(t)$. Returning to (3.16), we deduce that
$$u(t) - v(t) \geq (S\underline{f_\alpha})(t). \tag{3.17}$$
Since $v(t) \geq M(t)\|v\|_\infty > 0$, for $t > 0$, (3.17) yields
$$\alpha = \|u\|_\infty \geq u(1) > u(1) - v(1) \geq (S\underline{f_\alpha})(1),$$
a contradiction to our first assumption from (3.15).

Next assume that $N(u) > u$ for some $u \in K$ with $\|u\|_\infty = \beta$. Then $N(u) = u + v$ for some $v \in K \setminus \{0\}$ and, since $G(t,s) \leq G(1,s)$ for all $t, s \in [0,1]$, we have
$$u(t) + v(t) = N(u)(t) \leq (S\overline{f_\beta})(t) \leq (S\overline{f_\beta})(1). \tag{3.18}$$
Let t_0 be such that $u(t_0) = \|u\|_\infty = \beta > 0$. Since $u(0) = 0$, one has $t_0 > 0$ and so $v(t_0) \geq M(t_0)\|v\|_\infty > 0$. Then, for $t = t_0$, (3.18) gives
$$\beta < (S\overline{f_\beta})(1),$$
which contradicts our second assumption from (3.15). Thus Theorem 2.4 applies.

We note that if $\alpha < \beta$, then (3.15) represents the *compression* condition, while if $\alpha > \beta$, then (3.15) expresses the *expansion* condition. $\qquad \square$

Next we are interested into two positive solutions for (1.1). We shall succeed this by the variational approach based on Theorems 2.1 and 2.2 applied to the Hilbert spaces $X = \{u \in H^2(0,1) : u(0) = u'(0) = 0\}$ with norm $|\cdot|$ given by (2.19), and $Y = L^2(0,1)$ with the usual norm denoted by $\|\cdot\|$.

Let us consider the cone
$$K = \{u \in X : u(t) \geq M(t)\|u\| \text{for all } t \in [0,1]\}$$
and a fixed element $\phi \in K \setminus \{0\}$ with $|\phi| = 1$. Such an element can be Jv, where v is any nonzero nonnegative continuous function on $[0,1]$, because of Lemma 3.8. In addition, consider two numbers R_0, R_1 such that $0 < R_0 < \|\phi\|R_1$, and let
$$K_{R_0 R_1} = \{u \in K : \|u\| \geq R_0, \, |u| \leq R_1\}.$$
Denote
$$\underline{g}(t) = \min\{f(t,u) : M(t)R_0 \leq u \leq c_\infty R_1\},$$
$$\overline{g}(t) = \max\{f(t,u) : M(t)R_0 \leq u \leq c_\infty R_1\},$$
where $c_\infty > 0$ is such that $\|v\|_\infty \leq c_\infty|v|$ for all $v \in K$. For example we may take $c_\infty = 2/3$, since for any $v \in K$, Hölder's inequality gives
$$v(t) = \int_0^t \int_0^s v''(\tau)d\tau ds \leq |v| \int_0^t \sqrt{s} \leq \frac{2}{3}|v|. \tag{3.19}$$
Our assumptions are as follows:

(H1) There exist R_0, R_1 with $0 < R_0 < \|\phi\|R_1$ such that
    (a) $R_0 \leq \|S\underline{g}\|$,
    (b) $R_1 \geq c_\infty\|\overline{g}\|_{L^1(0,1)}$.

(H2) The functional E has the mountain pass geometry in $K_{R_0 R_1}$ and there exists $\rho > 0$ such that

$$E(u) \geq c + \rho \tag{3.20}$$

for all $u \in K_{R_0 R_1}$ which simultaneously satisfy $\|u\| = R_0$ and $|u| = R_1$.

Theorem 3.10. *Under assumptions* (H1), (H2), *problem* (1.1) *has at least two positive solutions* $u_m, u_c \in K_{R_0 R_1}$ *with* $E(u_m) = m$ *and* $E(u_c) = c$, *with* m *and* c *defined by* (2.8) *and* (2.7) *respectively.*

Proof. For $u \in K_{R_0 R_1}$, one has

$$M(t)R_0 \leq M(t)\|u\| \leq u(t) \leq \|u\|_\infty \leq c_\infty |u| \leq c_\infty R_1.$$

It follows that

$$F(t, u(t)) \leq \omega := \max\{F(t, u) : 0 \leq t \leq 1, \ M(t)R_0 \leq u \leq c_\infty R_1\},$$

whence, for all $u \in K_{R_0 R_1}$, it is fulfilled that

$$E(u) = \frac{1}{2}|u|^2 - \int_0^1 F(t, u(t))dt \geq -\omega,$$

and so $m > -\infty$.

Next, from $c > m$ we see that (3.20) guarantees both (2.12) and (2.13). It remains to check the compression boundary condition given by (2.4), (2.5). Assume first that (2.4) does not hold. Then $JE'(u) - \lambda Ju = 0$ for some $u \in K_{R_0 R_1}$, $\|u\| = R_0$ and $\lambda > 0$. Hence, in view of (2.20),

$$u = J(f(\cdot, u(\cdot)) + \lambda u).$$

Since $u \in C[0,1]$, one has $f(\cdot, u(\cdot)) + \lambda u \in C[0,1]$, and using (2.21) we have

$$u = S(f(\cdot, u(\cdot)) + \lambda u).$$

For $t > 0$, one has $f(t, u(t)) + \lambda u(t) > f(t, u(t)) \geq \underline{g}(t)$. As a result $S(f(\cdot, u(\cdot)) + \lambda u) > S\underline{g}$ on $(0, 1]$. Taking the L^2-norm, we deduce

$$R_0 = \|u\| > \|S\underline{g}\|,$$

which contradicts (H1)(a).

Next assume that $JE'(u) + \lambda u = 0$ for some $u \in K_{R_0 R_1}$, $|u| = R_1$ and $\lambda > 0$. Then

$$(1 + \lambda)u^{(4)} = f(t, u(t)),$$

whence, arguing as in the proof of Theorem 3.2, we deduce that

$$(1 + \lambda)R_1^2 = \int_0^1 u(t)f(t, u(t))dt.$$

Consequently

$$R_1 < c_\infty \|\overline{g}\|_{L^1(0,1)},$$

which contradicts (H1)(b). $\qquad\square$

Remark 3.11. In the autonomous case, if $f = f(u)$, and f is nondecreasing on $\mathbb{R}_+$, a sufficient condition for (H1)(a) to hold is

$$R_0 \leq f(M(a)R_0)\|S\chi_{[a,1]}\|,$$

where a is some number from $(0, 1)$ and $\chi_{[a,1]}$ is the characteristic function of the interval $[a, 1]$. Also in this case, (H1)(b) reduces to

$$R_1 \geq c_\infty f(c_\infty R_1).$$

4. AN EXAMPLE

We present an example inspired by that in [13], to which we can apply either Theorem 3.4 (working in a shell defined only by the energetic norm), or Theorem 3.10 (in a shell defined by two norms). Consider problem (1.1), where the function $f(t, u) = f(u)$ is

$$f(u) = \begin{cases} pu^p, & 0 \leq u \leq 1, \\ pu^2, & 1 \leq u \leq b, \\ p((u-b)^p + b^2), & u \geq b, \end{cases} \tag{4.1}$$

with $0 < p \leq 1/2$ and a sufficiently large number $b > 2$ as chosen below.

Note that the function f is positive and nondecreasing in $\mathbb{R}_+$. Also note as a typical behavior, that f is first *sublinear* near zero (here in $[0, 1]$), next *superlinear* (on a sufficiently large finite interval $[1, b]$), and again *sublinear* towards infinity (on $[b, \infty)$).

4.1. **Application of Theorem 3.4.** For $1 \leq u \leq 2$, we have

$$F(u) = \int_0^u f(s)ds = \frac{p}{p+1} + \frac{p}{3}(u^3 - 1) \leq \frac{p}{p+1} + \frac{7p}{3} = \frac{p(10 + 7p)}{3(p+1)}.$$

Choose $r = 2$. Then, for $u \in K$ and $|u| = 2$, as for (3.19), $|u|_\infty \leq 2|u|/3 < 2$ and so, recalling $0 < p \leq 1/2$,

$$E(u) = \frac{|u|^2}{2} - \int_0^1 F(u(t)) \, dt \geq 2 - \frac{p(10 + 7p)}{3(p+1)} \geq \frac{1}{2}.$$

We take for u_0 any element of $K \setminus \{0\}$ with $|u_0| = 1$. Clearly $|u_0| < r = 2$. Then

$$E(u_0) = \frac{|u_0|^2}{2} - \int_0^1 F(u_0(t)) \, dt = \frac{1}{2} - \int_0^1 F(u_0(t)) \, dt < \frac{1}{2}.$$

Next we take $u_1 = bu_0/\|u_0\|_\infty$. Then $|u_1| = b/\|u_0\|_\infty > 2$ if we choose $b > 2\|u_0\|_\infty$. Also

$$E(u_1) \leq \frac{b^2}{2\|u_0\|_\infty^2} - \int_{(u_1 > 1)} F(u_1(t))dt.$$

Since $\|u_1\|_\infty = b > 2$ and $u_1(0) = 0$ the level set $(u_1 > 1)$ is a proper subset of $[0, 1]$. Also $u_1(t) \leq b$ for all t. Hence on the level set $(u_1 > 1)$ we have

$$F(u_1) = \frac{p}{p+1} + \frac{p}{3}(u_1^3 - 1) > \frac{p}{3}u_1^3.$$

Then

$$E(u_1) < \frac{b^2}{2\|u_0\|_\infty^2} - \frac{pb^3}{3\|u_0\|_\infty^3} \int_{(u_1 > 1)} u_0(t)^3 dt.$$

Taking into account that the level set $(u_1 > 1)$ enlarges as b increases, we can see that the right side of the last inequality tends to $-\infty$ as $b \to +\infty$ (here we underline the role of the superlinearity of f on the interval $[1, b]$ guaranteeing the term b^3 in the last inequality). Thus we may choose b large enough to have

$$E(u_1) < \frac{1}{2}.$$

Hence the assumption (h3) of Theorem 3.4 is satisfied. Also, due to the sublinearity of f towards zero and infinity, one has

$$\lim_{\tau \to 0} \frac{f(\tau)}{\tau} = +\infty, \quad \text{and} \quad \lim_{\tau \to +\infty} \frac{f(\tau)}{\tau} = 0. \tag{4.2}$$

Consequently, we may find R_0 (small enough) and R_1 (large enough), such that u_0 and u_1 belong to $K_{R_0 R_1}$ and the conditions (3.7) hold.

Therefore, according to Theorem 3.4, problem (1.1) with f given by (4.1), $0 < p \le 1/2$ and b sufficiently large has two positive solutions.

4.2. **Application of Theorem 3.10.** The mountain pass geometry of the energy functional can be shown as above. It remains to guarantee the conditions (H1) and (3.20) from (H2).

First note that since f is nondecreasing on $\mathbb{R}_+$, we can use Remark 3.11, in view of which we may find $R_0, \overline{R}_1$ such that $u_0, u_1 \in K_{R_0 R_1}$ and (H1) holds, for every $R_1 \ge \overline{R}_1$. Recall that, this time, the conical shell $K_{R_0 R_1}$ is defined by the energetic norm and the L^2-norm.

Now we fix R_0 and we look for an $R_1 \ge \overline{R}_1$ such that (3.20) holds. For the variable parameter R_1, we denote by Γ_{R_1}, c_{R_1} the corresponding elements Γ and c in Theorem 3.10. It is clear that if $R_1 \ge \overline{R}_1$, then $\Gamma_{\overline{R}_1} \subset \Gamma_{R_1}$ and consequently, $c_{\overline{R}_1} \ge c_{R_1}$. Hence to guarantee (3.20) it suffices to find $R_1 \ge \overline{R}_1$ such that for a given number $\rho > 0$, $E(u) \ge c_{\overline{R}_1} + \rho$ for all $u \in K_{R_0 R_1}$ satisfying simultaneously $\|u\| = R_0$ and $|u| = R_1$. We shall guarantee even more, namely that

$$E(u) \ge c_{\overline{R}_1} + \rho \quad \text{for all } u \in K \text{ with } |u| = R_1. \tag{4.3}$$

Let $u \in K$ be such that $|u| = R_1$. Then

$$\begin{aligned}
E(u) &= \frac{1}{2}R_1^2 - \int_0^1 F(u(t))dt \\
&= \frac{1}{2}R_1^2 - \int_{(u \le b)} F(u(t))dt - \int_{(u \ge b)} F(u(t))dt \\
&\ge \frac{1}{2}R_1^2 - p\Big(\frac{1}{p+1} + \frac{b^3}{3}\Big) \\
&\quad - \int_{(u \ge b)} \Big(\frac{p}{p+1} + \frac{p}{3}(b^3 - 1) + \frac{p}{p+1}(u-b)^{p+1} + pb^2 u - pb^3\Big)dt \\
&\ge \frac{1}{2}R_1^2 - C_1|u|_{L^{p+1}(0,1)}^{p+1} - C_2|u|_{L^1(0,1)} - C_3,
\end{aligned}$$

where $C_1, C_2, C_3 > 0$ are constants depending only on p and b. Since X is continuously embedded into $L^q(0,1)$ for every $q \ge 1$, we deduce that

$$E(u) \ge \frac{1}{2}R_1^2 - \widetilde{C}_1 R_1^{p+1} - \widetilde{C}_2 R_1 - C_3.$$

The expression in the right side of this inequality tends to ∞ as $R_1 \to \infty$. Hence we can find $R_1 \ge \overline{R}_1$ such that (4.3) holds.

Therefore, according to Theorem 3.10, problem (1.1) with f, given by (4.1), $0 < p \le 1/2$ and b sufficiently large has two positive solutions.

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