EXISTENCE OF POSITIVE SOLUTIONS FOR SINGULAR P-LAPLACIAN STURM-LIOUVILLE BOUNDARY VALUE PROBLEMS

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ABSTRACT. We prove the existence of positive solutions of the Sturm-Liouville boundary value problem

\[-(r(t)\phi'(u'))' = \lambda g(t)f(t,u), \quad t \in (0,1),\]
\[au(0) - b\phi^{-1}(r(0))u'(0) = 0, \quad cu(1) + d\phi^{-1}(r(1))u'(1) = 0,\]

where \(\phi(u') = |u'|^{p-2}u', \quad p > 1, \quad f : (0,1) \times (0,\infty) \to \mathbb{R}\) satisfies a \(p\)-sublinear condition and is allowed to be singular at \(u = 0\) with semipositone structure. Our results extend previously known results in the literature.

1. Introduction

We consider the boundary-value problem

\[-(r(t)\phi'(u'))' = \lambda g(t)f(t,u), \quad t \in (0,1),\]
\[au(0) - b\phi^{-1}(r(0))u'(0) = 0, \quad cu(1) + d\phi^{-1}(r(1))u'(1) = 0,\]

where \(\phi(u') = |u'|^{p-2}u', \quad p > 1, \quad a, b, c, d\) are nonnegative constants with \(ac + ad + bc > 0\), \(f : (0,1) \times (0,\infty) \to \mathbb{R}\) is allowed to be singular at \(u = 0\), and \(\lambda\) is a positive parameter.

When \(p = 2\) and \(f : [0,1] \times [0,\infty) \to \mathbb{R}\) is continuous, Yang and Zhou [13] prove the existence of a positive solution to (1.1) under the assumption

\[\lim_{u \to \infty} \sup_{t \in [0,1]} \frac{f(t,u)}{u} < \frac{\lambda_1}{\lambda} < \lim_{u \to 0^+} \inf_{t \in [0,1]} \frac{f(t,u)}{u},\]

where \(\lambda_1 > 0\) denotes the first eigenvalue of \(-\phi'(u')\)' = \(\lambda g(t)u\) in \((0,1)\) with Sturm-Liouville boundary conditions. Their result allows \(\lim_{u \to \infty} \sup_{t \in [0,1]} \frac{f(t,u)}{u} = -\infty\), which complements previous existence results in [1, 4, 7, 8, 9, 10, 12, 14].

In this article, we shall extend the result in [13] to the general case \(p > 1\) and also allow \(f\) to be singular at \(u = 0\). We also establish the existence of a positive solution to (1.1) for \(\lambda\) large allowing \(\lim_{u \to 0^+} \inf_{t \in (0,1)} f(t,u)/u^{p-1} = -\infty\) and \(\lim_{u \to \infty} \inf_{t \in (0,1)} f(t,u) = 0\), which does not seem to have been considered in the literature even when \(p = 2\). Note that the approach in [13] depends on the Green function and can not apply to the nonlinear case \(p > 1\) or the case when \(f\) is...
singular at \( u = 0 \). Our approach depends on a new sub- and super solutions type argument and comparison principle.

Let \( g \) satisfy condition (A2) below. Then the eigenvalue problem

\[
-(r(t)\phi(u'))' = \lambda g(t)\phi(u) \quad \text{in} \ (0, 1)
\]

has a positive first eigenvalue \( \lambda_1 \) with corresponding positive eigenfunctions (see e.g. [3, 11]).

We shall make the following assumptions:

(A1) \( r : [0, 1] \to (0, \infty) \) and \( f : (0, 1) \times (0, \infty) \to \mathbb{R} \) are continuous.

(A2) \( g \in L^1(0, 1) \) with \( g \geq 0, g \not\equiv 0 \) and there exists a constant \( \gamma \geq 0 \) such that

\[
\int_0^1 \frac{g(t)}{q(t)} \, dt < \infty,
\]

where \( q(t) = \min(b + at, d + c(1 - t)) \).

(A3) For each \( r > 0 \), there exists a constant \( K_r > 0 \) such that

\[
|f(t, u)| \leq K_r \frac{u}{u^\gamma}
\]

for \( t \in (0, 1), u \in (0, r] \), where \( \gamma \) is defined in (A2).

(A4) \( \lim_{u \to \infty} \sup_{\phi(u)} \frac{f(t, u)}{\phi(u)} < \frac{\lambda_1}{\lambda} < \lim_{u \to 0^+} \inf_{\phi(u)} \frac{f(t, u)}{\phi(u)} \), where the limits are uniform in \( t \in (0, 1) \).

(A5) \( \lim_{u \to \infty} \sup_{\phi(u)} \frac{f(t, u)}{\phi(u)} \leq \frac{\lambda_1}{\lambda} \) uniformly in \( t \in (0, 1) \).

(A6) There exist positive constants \( A, L \) such that

\[
f(t, u) \geq L \frac{u}{u^\gamma}
\]

for \( t \in (0, 1) \) and \( u \geq A \).

By a solution of (1.1), we mean a function \( u \in C^1[0, 1] \) with \( r(t)\phi(u') \) absolutely continuous on \( [0, 1] \) and satisfying (1.1).

Our main results read as follows:

**Theorem 1.1.** Let (A1)–(A4) hold. Then (1.1) has a positive solution \( u \) with \( \inf_{(0,1)}(u/q) > 0 \).

**Theorem 1.2.** Let (A1)–(A3), (A5), (A6) hold. Then there exists a constant \( \lambda_0 > 0 \) such that for \( \lambda > \lambda_0 \), Equation (1.1) has a positive solution \( u_\lambda \) with

\[
\inf_{(0,1)}(u_\lambda/q) \to \infty \quad \text{as} \quad \lambda \to \infty.
\]

Let \( \bar{\lambda} < \lambda_1 \) and consider the problem

\[
-(r(t)\phi(u'))' - \bar{\lambda}g(t)\phi(u) = \lambda g(t)f(t, u), \quad t \in (0, 1),
\]

\[
a u(0) - b \phi^{-1}(r(0))u'(0) = 0, \quad c u(1) + d \phi^{-1}(r(1))u'(1) = 0.
\]

Then, as an immediate consequence of Theorem 1.1 we obtain the following corollary.

**Corollary 1.3.** Let (A1)–(A3) hold and suppose that

\[
\lim_{u \to \infty} \sup_{\phi(u)} \frac{f(t, u)}{\phi(u)} < \frac{\lambda - \bar{\lambda}}{\lambda} < \lim_{u \to 0^+} \inf_{\phi(u)} \frac{f(t, u)}{\phi(u)}.
\]

Then (1.2) has a positive solution.

**Remark 1.4.** When \( p = 2 \) and \( f : [0, 1] \times [0, \infty) \to \mathbb{R} \) is continuous, Theorem 3.1 follows from Theorem 1.1 with \( \gamma = 0 \).
Example 1.5. Let \( g(t) \equiv 1 \equiv r(t) \) and consider the BVP
\[
-(\phi(u'))' = \lambda f(t, u), \quad t \in (0, 1),
\]
\[u(0) = u(1) = 0.\] (1.3)

Note that \( \lambda_1 = \pi_p^2 \), where
\[\pi_p = 2(p-1)^{1/p} \int_0^1 \frac{ds}{(1-s^p)^{1/p}}\]
is the first eigenvalue of \( -(\phi(u'))' \) with zero boundary conditions (see [5, 6]).

(i) Let \( f(t, u) = u^{p-1} \left( \frac{u'}{u} - u^\beta \right), \) where \( \gamma \in [0, 1), \) and \( \beta > 0. \) Suppose \( \lambda > \lambda_1 \) if \( \gamma = 0, \) and \( \lambda \) is any positive constant if \( \gamma > 0. \) Then (A1)–(A4) hold and therefore Theorem 1.1 gives the existence of a positive solution to (1.3).

(ii) Let \( f(t, u) = -\frac{1}{u^q} + \frac{1}{u^p}, \) where \( 0 < \beta < \gamma < 1. \) Then it is easy to see that the assumptions of Theorem 1.2 are satisfied and therefore (1.3) has a positive solution for \( \lambda \) large. Note that since \( \lim_{u \to 0^+} \inf_{u \in (0, 1)} f(t, u) = -\infty \) and \( \lim_{u \to \infty} \inf_{u \in (0, 1)} f(t, u) = 0, \) the results in [1, 4, 7, 8, 9, 10, 12, 13, 14] do not apply here.

(iii) Let \( f(t, u) = (1 - u^{p-1}) \cos t. \) Then
\[\lim_{u \to \infty} \sup \frac{f(t, u)}{\phi(u)} < 0 \quad \text{and} \quad \lim_{u \to 0^+} \inf \frac{f(t, u)}{\phi(u)} = \infty\]
uniformly in \( t \in (0, 1) \) and so (1.2) has a positive solution for all \( \lambda > 0, \) by Corollary 1.3

2. Preliminaries

We shall denote the norms in \( C^1[0, 1] \) and \( L^q(0, 1) \) by \( | \cdot |_1 \) and \( \| \cdot \|_q \) respectively. Here \( |u|_1 = \max(\|u\|_{\infty}, \|u'\|_{\infty}) \). We first recall the following results in [5, 6].

Lemma 2.1. Let \( h \in L^1(0, 1). \) Then the problem
\[-(r(t)\phi(u'))' = h, \quad t \in (0, 1),\]
\[au(0) - b\phi^{-1}(r(0))u'(0) = 0, \quad cu(1) + d\phi^{-1}(r(1))u'(1) = 0\]
has a unique solution \( u = Sh \in C^1[0, 1]. \) Furthermore, \( S \) is completely continuous and there exists a constant \( m > 0 \) such that
\[|u|_1 \leq m\phi^{-1}(\|h\|_1)\].

Lemma 2.2. Suppose \( u \in C^1[0, 1] \) satisfies
\[-(r(t)\phi(u'))' \geq 0, \quad t \in (0, 1),\]
\[au(0) - b\phi^{-1}(r(0))u'(0) \geq 0, \quad cu(1) + d\phi^{-1}(r(1))u'(1) \geq 0.\]
Then there exists a constant \( m_0 > 0 \) independent of \( u \) such that
\[u(t) \geq m_0\|u\|_{\infty}q(t)\]
for \( t \in [0, 1], \) where \( q \) is defined by (A2).

Remark 2.3. Lemma 2.2 is a special case of [5, Lemma 3.4] when \( h = 0. \) Note that the proof of [5, Lemma 3.4] is incorrect for \( 1 < p < 2 \) when \( h \neq 0 \) since it uses the inequality
\[|\phi^{-1}(x) - \phi^{-1}(y)| \leq 2\phi^{-1}(|x - y|) \quad \text{for all} \ x, y \in \mathbb{R},\]
which is not true when $1 < p < 2$. However, when $h = 0$, this inequality is not needed in [8] Proof of Lemma 3.4], which guarantees the validity of Lemma 2.2.

**Lemma 2.4.** There exists a constant $k > 0$ such that $|u| \le k|u|q$ in $[0, 1]$ for all $u \in C^1[0, 1]$ satisfying the Sturm-Liouville boundary conditions in (1.1).

**Proof.** Let $u \in C^1[0, 1]$. Then, if $b > 0$,

$$u(t) = u(0) + \int_0^t u' \le 2|u|_1 \le \frac{2}{b}|u|_1(b + at)$$

for $t \in [0, 1]$, while if $b = 0$ then $a > 0$, this implies $u(0) = 0$ and $u(t) \le |u|_1t$ for $t \in [0, 1]$. Hence

$$u(t) \le k_0|u|_1(b + at), \quad (2.1)$$

for $t \in [0, 1]$, where $k_0 = 2/b$ if $b > 0$, and $1/a$ if $b = 0$. Similarly, using

$$u(t) = u(1) - \int_t^1 u',$$

we obtain

$$u(t) \le k_1|u|_1(d + c(1 - t)) \quad (2.2)$$

for $t \in [0, 1]$, where $k_1 = 2/d$ if $d > 0$, and $1/c$ if $d = 0$.

Combining (2.1) and (2.2), we see that $u \le k|u|q$ in $[0, 1)$, where $k = \max(k_0, k_1)$.

By replacing $u$ by $-u$, we see that Lemma 2.4 holds. \hfill \Box

**Lemma 2.5.** Let $h_0, h_1 \in L^1(0, 1)$. Suppose $u_0, u_1 \in C^1[0, 1]$ satisfy

$$-(r(t)\phi(u_i'))' = h_i, \quad t \in (0, 1),$$

$$au_i(0) - b\phi^{-1}(r(0))u_i'(0) = 0, \quad cu_i(1) + d\phi^{-1}(r(1))u_i'(1) = 0,$$

for $i = 0, 1$. Then there exists a constant $M_0 > 0$ depending on $p, a, b, c, d, \text{ and } C$ such that

$$|u_1 - u_0|_1 \le M_0 \max\{\|h_1 - h_0\|_1, \|h_1 - h_0\|_1^{1/p}\}, \quad (2.3)$$

where $C > 0$ is such that $\|h_i\|_1 < C$ for $i = 0, 1$.

**Proof.** By integrating, we obtain

$$u_i(t) = C_i + \int_0^t \phi^{-1}\left(\frac{D_i - \int_0^s h_i}{r(s)}\right)ds \quad (2.4)$$

for $i = 0, 1$, where $C_i, D_i$ are constants satisfying

$$aC_i - b\phi^{-1}(D_i) = 0,$$

$$c\left(C_i + \int_0^1 \phi^{-1}\left(\frac{D_i - \int_0^s h_i}{r(s)}\right)ds\right) + d\phi^{-1}(D_i - \int_0^1 h_i) = 0.$$

Suppose first that $a = 0$. Then $b, c > 0, D_i = 0$, and

$$C_i = \frac{d}{c}\phi^{-1}\left(\int_0^1 h_i\right) + \int_0^1 \phi^{-1}\left(\frac{\int_0^s h_i}{r(s)}\right)ds,$$

and so

$$u_i(t) = \frac{d}{c}\phi^{-1}\left(\int_0 h_i\right) + \int_t^1 \phi^{-1}\left(\frac{\int_0^s h_i}{r(s)}\right)ds.$$

For $p \ge 2$, using the inequality

$$|\phi^{-1}(x) - \phi^{-1}(y)| \le 2\phi^{-1}(|x - y|) \quad \text{for } x, y \in \mathbb{R},$$

we obtain

$$|u_i(t)| \le C_i + \int_0^t |\phi^{-1}(\frac{D_i - \int_0^s h_i}{r(s)})|ds \le C_i + \int_0^t \frac{|D_i|}{r(s)}ds.$$
we obtain
\[
\max\{|u(t) - u_0(t)|, |u'(t) - u'_0(t)|\} \leq M_1\|h_1 - h_0\|^{1/p},
\] for \( t \in [0, 1] \), where \( r_0 = \min_{t \in [0, 1]} r(t) > 0 \), \( M_1 = 2(d/c + \phi^{-1}(1/r_0)) \).

For \( 1 < p < 2 \), using the Mean Value Theorem, we obtain
\[
|\phi^{-1}(x) - \phi^{-1}(y)| \leq (p - 1)^{-1}|x - y|(|x|, |y|)^{2 - p}
\]
for \( x, y \in \mathbb{R} \), which implies
\[
\max\{|u(t) - u_0(t)|, |u'(t) - u'_0(t)|\} \leq M_2\|h_1 - h_0\|_1,
\] for \( t \in [0, 1] \), where \( M_2 = (p - 1)^{-1}(dc^{-1} + r_0^{-1/(p - 1)})C^{2 - p} \).

Suppose next that \( a > 0 \). Then \( C_i = (b/a)\phi^{-1}(D_i) \), and \( D_i \) satisfies
\[
c_{i}(b/a)\phi^{-1}(D_i) + \int_0^1 \phi^{-1}\left(D_i - \frac{\int_0^s h_i}{r(s)}\right)ds + d\phi^{-1}\left(D_i - \int_0^1 h_i\right) = 0
\]
for \( i = 0, 1 \). Since \( \phi^{-1} \) is increasing and \( \phi^{-1}(0) = 0 \), it follows from (2.7) that
\[
|D_i| \leq \|h_i\|_1, \text{ and } |D_1 - D_0| \leq \|h_1 - h_0\|_1
\]
which, together with (2.4), imply
\[
\max\{|u(t) - u_0(t)|, |u'(t) - u'_0(t)|\} \leq M_3\max\{|h_1 - h_0|, \|h_1 - h_0\|^{1/p}\}
\] for \( t \in [0, 1] \), where \( M_3 = 2(b/a + (2/r_0)^{1/(p - 1)})C^{2 - p} \) if \( p \geq 2 \), and \( M_3 = (p - 1)^{-1}(b/a + (2/r_0)^{1/(p - 1)})C^{2 - p} \) if \( 1 < p < 2 \). Combining (2.5), (2.6), and (2.8), we obtain (2.3) with \( M_0 = \max_{1 \leq i \leq 3} M_i \), which completes the proof. \( \square \)

3. Proofs of main results

Let \( z_1 \in C^1[0, 1] \) be the normalized positive eigenfunction of \(-(r(t)\phi(u'))' = \lambda g(t)\phi(u)\) in \((0, 1)\) with Sturm-Liouville boundary conditions corresponding to \( \lambda_1 \) i.e. \( z_1 > 0 \) on \((0, 1)\) and \( \|z_1\|_\infty = 1 \). By Lemma 2.2, there exists a constant \( m_0 > 0 \) such that \( z_1 \geq m_0q \) in \((0, 1)\).

**Proof of Theorem 1.1** Since \( \lim_{z \to 0^+} \inf \frac{f(t, z)}{\phi(z)} > \frac{\lambda_1}{\lambda} \) uniformly in \( t \in (0, 1) \), there exists a constant \( c > 0 \) such that
\[
\frac{f(t, z)}{\phi(z)} > \frac{\lambda_1}{\lambda}
\]
for \( z \in (0, c) \) and \( t \in (0, 1) \). Let \( Z = cz_1 \) and \( Z_1 = Mz_1 \), where \( M > c \) is a large constant to be determined later. In view of (3.1), \( Z \) satisfies
\[
-(r(t)\phi(Z'))' = \lambda g(t)\phi(Z) \leq \lambda g(t)f(t, Z)
\]
for \( t \in (0, 1) \). For \( v \in C[0, 1] \), let \( \tilde{v} = \min\{v, Z_1\} \). Then \( Z \leq \tilde{v} \leq Z_1 \leq M \) in \((0, 1)\) and (A3) gives
\[
|g(t)f(t, \tilde{v})| \leq \frac{K_Mg(t)}{(cz_1)^{\gamma}} \leq \frac{K_Mg(t)}{(cm_0)^{\gamma}q^{\gamma}(t)}
\]
for \( t \in (0, 1) \). Hence \( g(t)f(t, \tilde{v}) \in L^1(0, 1) \) by (A2). Define \( Tv = u \), where \( u \) is the solution of
\[
-(r(t)\phi(u'))' = \lambda g(t)f(t, \tilde{v}), \quad t \in (0, 1),
\]
\[
au(0) - b\phi^{-1}(r(0))u'(0) = 0, \quad cu(1) + d\phi^{-1}(r(1))u'(1) = 0.
\]
in Lemma 2.1, it follows that $T$ is bounded. Hence, by the Schauder Fixed Point Theorem, $u(t_0) = T(u(t_0))$ has a fixed point $u$. To complete the proof, we will first show that $u$ is a fixed point of $T$.

Using (3.3) and the Lebesgue Dominated Convergence Theorem, we see that $\sup_{t \in (0, 1)} |u(t)| < \infty$. Since $u$ is bounded, it is compact. Hence, by the Schauder Fixed Point Theorem, $T$ has a fixed point $u$. To complete the proof, we will first show that $u$ is a fixed point of $T$.

Indeed, if $t_0 > 0$ then $u(t_0) = Z(t_0)$ and $u(t_0) \leq Z(t_0)$, while if $t_0 = 0$ then we have equality in (3.5). Similarly,

$$
au(t_0) - b\phi^{-1}(r(t_0))u'(t_0) = aZ(t_0) - b\phi^{-1}(r(t_0))Z'(t_0).
$$

$(3.5)$

Since $\lambda > 0$ and $\phi^{-1}$ is increasing,

$$
-(r(t)\phi(u'))' = \lambda g(t)f(t, Z(t)) \leq g(t)\left(\lambda \phi(u) + \frac{K_\lambda}{z^\gamma}\right)
$$

for $z > 0$ and $t \in (0, 1)$. Hence

$$
-(r(t)\phi(u'))' = \lambda g(t)f(t, Z(t)) \leq g(t)\left(\lambda \phi(u) + \frac{K_\lambda}{z^\gamma}\right)
$$

for $t \in (0, 1)$. Let $u_M = u/M$. Then $u_M$ satisfies

$$
-(r(t)\phi(u'_M))' = \lambda g(t)z_1^{p-1} + \frac{K_\lambda g(t)}{(cm_0)^\gamma M^{p-1} q^\gamma(t)}
$$

for $t \in (0, 1)$. Let $\bar{u}_M$ and $\tilde{u}$ satisfy

$$
-(r(t)\phi(\bar{u}_M'))' = \lambda g(t)z_1^{p-1} + \frac{K_\lambda g(t)}{(cm_0)^\gamma M^{p-1} q^\gamma(t)}
$$

and

$$
-(r(t)\phi(\tilde{u}'))' = \lambda g(t)z_1^{p-1} = h, \quad t \in (0, 1),
$$

with Sturm-Liouville boundary conditions in (1.1). Note that $\bar{u} = (\lambda/\lambda_1)^{1/(p-1)} z_1$. By the comparison principle, $u_M \leq \bar{u}_M$ in $(0, 1)$. Let $\varepsilon > 0$ be such that $(\lambda/\lambda_1)^{1/(p-1)} + \varepsilon < 1$. Since

$$
\|h_M - h\|_1 = \frac{K_\lambda}{(cm_0)^\gamma M^{p-1}} \left(\int_0^1 \frac{g(t)}{q^\gamma(t)} \, dt\right) \to 0 \quad \text{as } M \to \infty,
$$
it follows from Lemmas 2.4 and 2.5 that
\[ \bar{u}_M - \bar{u} \leq k|\bar{u}_M - \bar{u}|_q \leq km_0^{-1}|\bar{u}_M - \bar{u}|_1 \leq km_0^{-1} M_0 \max\{\|h_M - h\|_1, \|h_M - h\|_{1/2}^{1/2}\} z_1 < \varepsilon z_1, \]
provided that \( M \) is large enough. Consequently,
\[ u_M \leq \bar{u}_M \leq \bar{u} + \varepsilon z_1 = \left(\lambda/\lambda_1\right)^{1/(p-1)} + \varepsilon\right) z_1 \leq z_1 \text{ in } (0, 1), \]
i.e. \( u \leq M z_1 = Z_1 \) in \( (0, 1) \). Hence \( Z \leq u \leq Z_1 \) in \( (0, 1) \) i.e. \( u \) is a positive solution of (1.1), which completes the proof. \( \square \)

**Proof of Theorem 1.2.** By Theorem 1.1, there exists a positive solution \( w \) of the problem
\[ -(r(t)\phi(w'))' = \frac{g(t)}{w^\gamma}, \quad t \in (0, 1), \]
\[ aw(0) - b\phi^{-1}(r(0))w(0) = 0, \quad cw(1) + d\phi^{-1}(r(1))w'(1) = 0 \]
with \( w \geq \alpha q \) in \( (0, 1) \) for some \( \alpha > 0 \). Let \( w_0 \) satisfy
\[ -(r(t)\phi(w_0'))' = \begin{cases} \frac{L_1 g(t)}{w^\gamma} & \text{if } w > \frac{2AL_1^{-1/(p-1)}}{\lambda^{1/(p-1)}}, \\ -\frac{K_1 g(t)}{w^\gamma} & \text{if } w \leq \frac{2AL_1^{-1/(p-1)}}{\lambda^{1/(p-1)}} \end{cases} \equiv h_\lambda \text{ in } (0, 1), \]
with Sturm-Liouville boundary conditions, where \( \delta = (\gamma + p - 1)^{-1}, L_1 = L^{1/(p-1)}, \) and \( K_1 = 2\gamma L_1^{-\gamma/(p-1)}K_{2A} \), and \( K_{2A} \) is defined in (A3). Let \( w_1 \) satisfy
\[ -(r(t)\phi(w_1'))' = \frac{L_1 g(t)}{w^\gamma} \equiv h \text{ in } (0, 1) \]
with Sturm-Liouville boundary conditions. Then \( w_1 = L_1^{1/(p-1)} w \) and \( w_0 \leq w_1 \) in \( (0, 1) \) by the comparison principle. Since
\[ \|h_\lambda - h\|_1 = (L_1 + K_1) \int_{w \leq \frac{2AL_1^{-1/(p-1)}}{\lambda^{1/(p-1)}}} \frac{g(t)}{w^{\gamma}(t)} dt \to 0 \text{ as } \lambda \to \infty, \]
it follows from Lemma 2.5 that
\[ |w_0 - w_1|_1 \leq M_0 \max\{\|h_\lambda - h\|_1, \|h_\lambda - h\|_{1/2}^{1/2}\} \to 0 \text{ as } \lambda \to \infty. \]
Hence by Lemma 2.4 there exists a constant \( \lambda_0 > 0 \) such that
\[ w_0 \geq w_1 - k|w_0 - w_1|_q \geq \frac{L_1^{1/(p-1)} w}{2} \text{ in } (0, 1) \quad (3.7) \]
for \( \lambda > \lambda_0 \). Let \( Z = \lambda^q w_0 \) and \( Z_1 = M z_1 \) where \( M > \lambda^q km_0^{-1} \|w_1\|_1 \) (so that \( Z_1 > Z \) in \( (0, 1) \)). We shall verify that \( Z \) satisfies
\[ -(r(t)\phi(Z'))' \leq \lambda g(t)f(t, Z) \text{ in } (0, 1). \quad (3.8) \]
Indeed,
\[ -(r(t)\phi(Z'))' = \begin{cases} \frac{\lambda^q(p-1)L_1 g(t)}{w^{\gamma}} & \text{if } w > \frac{2AL_1^{-1/(p-1)}}{\lambda^{1/(p-1)}}, \\ -\frac{\lambda^q(p-1)K_1 g(t)}{w^{\gamma}} & \text{if } w \leq \frac{2AL_1^{-1/(p-1)}}{\lambda^{1/(p-1)}}. \end{cases} \]
If \( w > 2AL_{1}^{-1/(p-1)}/\lambda^{\delta} \) then by (3.7),
\[
Z \geq \frac{\lambda^{\delta}L_{1}^{1/(p-1)}w}{2} \geq A,
\]
from which (A6) gives
\[
\lambda g(t)f(t, Z) \geq \frac{\lambda Lg(t)}{Z^{\gamma}} = \frac{\lambda^{1-\gamma\delta}Lg(t)}{w^{\gamma}} \geq \frac{\lambda^{1-\gamma\delta}Lg(t)}{w_{1}^{\gamma}} = \frac{\lambda^{1-\gamma\delta}L_{1}g(t)}{w^{\gamma}} \geq \frac{\lambda^{1-\gamma\delta}L_{1}g(t)}{w^{\gamma}}.
\]
On the other hand, if \( w \leq 2AL_{1}^{-1/(p-1)}/\lambda^{\delta} \), then
\[
Z \leq \lambda^{\delta}w_{1} = L_{1}^{1/(p-1)}\lambda^{\delta}w \leq 2A,
\]
from which (A3) and (3.7) give
\[
\lambda g(t)f(t, Z) \geq -\frac{\lambda K_{2}Ag(t)}{Z^{\gamma}} = -\frac{\lambda^{1-\gamma\delta}K_{2}Ag(t)}{w_{1}^{\gamma}} \geq -\frac{\lambda^{1-\gamma\delta}K_{2}Ag(t)}{(L_{1}^{1/(p-1)/2})^{\gamma}w^{\gamma}} = -\frac{\lambda^{1-\gamma\delta}K_{2}Ag(t)}{(L_{1}^{1/(p-1)/2})^{\gamma}w^{\gamma}}.
\]
Combining (3.9) and (3.10), we see that (3.8) holds. Let \( T \) be the operator defined in the proof of Theorem 1.1 i.e. for \( v \in C[0, 1], \ u = Tv \) satisfies (3.4): i.e.,
\[
-(r(t)\phi(u'))' = \lambda g(t)f(t, \tilde{v}), \ t \in (0, 1),
\]
\[
uu(0) - b\phi^{-1}(r(0))u'(0) = 0, \ \ \ \ \ \ cu(1) + d\phi^{-1}(r(1))u'(1) = 0,
\]
where \( \tilde{v} = \min\{\max\{v, Z\}, Z_{1}\} \). Then \( T \) has a fixed point \( u_{\lambda} \) in \( C[0, 1] \). Using the same arguments as in the proof of Theorem 1.1 we see that \( u_{\lambda} \geq Z \) and, for \( M \) large enough \( u_{\lambda} \geq Z_{1} \) in \( (0, 1) \); i.e., \( u_{\lambda} \) is a positive solution of (1.1) for \( \lambda > \lambda_{0} \) with \( u_{\lambda} \geq \lambda^{\delta}(L_{1}^{1/(p-1)/2})w \) in \( (0, 1) \), which completes the proof. \( \square \)

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