SOLVABILITY OF SINGULAR SECOND-ORDER INITIAL-VALUE PROBLEMS

PETIO KELEVEDJIEV

Abstract. This article concerns the solvability of the initial-value problem
\[ x'' = f(t, x, x'), \quad x(0) = A, \quad x'(0) = B, \]
where the scalar function \( f \) may be unbounded as \( t \to 0 \). Using barrier strip type arguments, we establish the existence of monotone and/or positive solutions in \( C^1[0, T] \cap C^2(0, T) \).

1. Introduction

In this article we study the solvability of the initial value problem (IVP)
\[ x'' = f(t, x, x'), \quad x(0) = A, \quad x'(0) = B, \]
where the scalar function \( f(t, x, p) \) is defined for \( (t, x, p) \in D_t \times D_x \times D_p \), and \( D_t, D_x, D_p \subseteq \mathbb{R} \), but there may be sets \( X \subseteq D_x \) and \( P \subseteq D_p \) such that \( f \) is unbounded as \( t \to 0 \) and \( (x, p) \in X \times P \).

The solvability of various nonsingular and singular second order IVPs has been studied by Aslanov [3], Agarwal and O’Regan [1, 2], Bobisud and O’Regan [4], Bobisud and Lee [5], Cabada et al. [6, 7, 8], Cid [9], Maagli and Masmoudi [13], Rachůnková and Tomeček [14, 15, 16], Yang [17, 18] and Zhao [19]. Yang [17, 18], for example, established the solvability in \( C^1[0, 1] \) and \( C[0, 1] \times C^2(0, 1) \) in the case \( A = B = 0 \). In these works the function \( f(t, x, p) \in C((0, 1), (0, \infty))^2 \) is allowed to be singular at \( t = 0, t = 1, x = 0 \) or \( p = 0 \) and is such that
\[ 0 < f(t, x, p) \leq k(t)F(x)G(p) \quad \text{for} \quad (t, x, p) \in (0, 1) \times (0, \infty)^2, \]
where \( k, F \) and \( G \) are suitable functions.

Here we present sufficient conditions guaranteeing monotone and/or positive solutions to (1.1) in \( C^1[0, T] \times C^2(0, T) \). They are established by adapting ideas from Kelevedjiev and Popivanov [10] and Kelevedjiev et al. [11] (see also Kelevedjiev [12]), where (1.1) may be singular at \( x = A \) and/or \( p = B \). The results in these works rely on a combination of a barrier type condition with the assumption that there is a constant \( k < 0 \) such that
\[ f(t, x, p) \leq k \]
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on a suitable bounded subset of the domain of \( f \). It turned out, however, that (1.2) is not necessary when (1.1) is singular only at \( t = 0 \), that is why we pay a special attention to this case.

In our considerations we use two results from [11] for the nonsingular problem

\[
x'' = f(t, x, x'),
\]

\[
x(a) = A, \quad x'(a) = B,
\]

where \( f : D_t \times D_x \times D_p \to R, \quad D_t, D_x, D_p \subseteq R \). They are based on the assumption

\((A1)\) There are constants \( T > a, m_1, M_1, \overline{M}_1 \) and a sufficiently small \( \tau > 0 \) such that

\[\overline{M}_1 - \tau \geq M_1 \geq B \geq m_1 \geq \overline{m}_1 + \tau,\]

\[\{a, T\} \subseteq D_t, [m_0 - \tau, M_0 + \tau] \subseteq D_x, \quad [m_1, \overline{M}_1] \subseteq D_p,\]

where \( M_0 = \max\{|m_1|, |M_1|\}(T - a) + |A| \), and \( m_0 = -M_0 \),

\[f(t, x, p) \in C([a, T] \times [m_0 - \tau, M_0 + \tau] \times [m_1 - \tau, M_1 + \tau]),\]

\[f(t, x, p) \leq 0 \quad \text{for} \quad (t, x, p) \in [a, T] \times D_x \times [M_1, \overline{M}_1],\]

\[f(t, x, p) \geq 0 \quad \text{for} \quad (t, x, p) \in [a, T] \times D_{M_0} \times [\overline{m}_1, m_1],\]

where \( D_{M_0} = D_x \cap (-\infty, M_0]\).

So, we need the following result.

**Lemma 1.1 (11).** Let (A1) hold and \( x \in C^2[a, T] \) be a solution to (1.3). Then

\[m_0 \leq x(t) \leq M_0, \quad m_1 \leq x'(t) \leq M_1, \quad m_2 \leq x''(t) \leq M_2 \quad \text{for} \quad t \in [a, T],\]

where \( m_2 = \min f(t, x, p) \) and \( M_2 = \max f(t, x, p) \) for \( (t, x, p) \in [a, T] \times [m_0, M_0] \times [m_1, M_1] \).

This lemma was used in the proof of the following theorem.

**Theorem 1.2 (11).** Let (A1) hold. Then nonsingular IVP (1.3) has at least one solution in \( C^2[a, T] \).

## 2. Existence results

Returning our attention to singular problem (1.1), we assume that

\((A2)\) There are constants \( T > 0, m_1, \overline{m}_1, M_1, \overline{M}_1 \) and a sufficiently small \( \tau > 0 \) such that

\[\overline{M}_1 - \tau \geq M_1 \geq B \geq m_1 \geq \overline{m}_1 + \tau,\]

\[(0, T] \subseteq D_t, [\overline{m}_0 - \tau, \overline{M}_0 + \tau] \subseteq D_x, \quad [m_1, \overline{M}_1] \subseteq D_p,\]

where \( \overline{M}_0 = \max\{|m_1|, |M_1|\}T + |A| \), and \( \overline{m}_0 = -\overline{M}_0 \),

\[f(t, x, p) \in C((0, T] \times [\overline{m}_0 - \tau, \overline{M}_0 + \tau] \times [m_1 - \tau, M_1 + \tau]),\]

\[f(t, x, p) \leq 0 \quad \text{for} \quad (t, x, p) \in (0, T] \times D_x \times [M_1, \overline{M}_1],\]

\[f(t, x, p) \geq 0 \quad \text{for} \quad (t, x, p) \in (0, T] \times D_{\overline{M}_0} \times [\overline{m}_1, m_1],\]

where \( D_{\overline{M}_0} = D_x \cap (-\infty, \overline{M}_0]\).

We are now in a position to state our first existence theorem.
Theorem 2.1. Let (A2) hold. Then \( (2.2) \) has at least one solution in \( C^1[0,T] \cap C^2(0,T) \) such that

\[
\begin{align*}
m_1 t + A & \leq x(t) \leq M_1 t + A \quad \text{for } t \in [0,T], \\
m_1 & \leq x'(t) \leq M_1 \quad \text{for } t \in [0,T].
\end{align*}
\]

Proof. We will do the proof in several steps considering the family of nonsingular problems

\[
x'' = f(t,x,x'), \quad x(n^{-1}) = A, \quad x'(n^{-1}) = B,
\]

where \( n \in N_T = \{n \in N : n^{-1} < T\} \).

**Step 1** Construction of a sequence \( \{x_n\} \) of \( C^2[n^{-1},T] \)-solutions to \( (2.2) \). It is not hard to check that each problem of \( (2.2) \) satisfies (A1) for \( a = n^{-1} \), \( M_0 = \max\{|m_1|,|M_1|,(T^{-n})+|A|<\hat{M}_0\}, \) and \( m_0 = -\hat{M}_0 \). Thus, according to Theorem 1.1, \( (2.2) \) has a solution

\[
x_n \in C^2[n^{-1},T] \quad \text{for each } n \in N_T.
\]

In addition, for each \( n \in N_T \) Lemma 1.1 guarantees the bounds

\[
\begin{align*}
\tilde{m}_0 < m_0 & \leq x_n(t) \leq M_0 < \hat{M}_0 \quad \text{for } t \in [n^{-1},T], \\
\tilde{m}_1 & \leq x_n'(t) \leq M_1 \quad \text{for } t \in [n^{-1},T].
\end{align*}
\]

**Step 2** Construction of a \( C^2(0,T) \)-solution to the differential equation. Now, we introduce a numerical sequence \( \{\theta_i\}, i \in N, \) having the properties

\[
\theta_i \in (0,T), \quad \theta_{i+1} < \theta_i \quad \text{for } i \in N \quad \text{and} \quad \lim_{i \to \infty} \theta_i = 0,
\]

and consider the sequence \( \{x_n\} \) of \( C^2[n^{-1},T] \)-solutions of family \( (2.2) \) only for \( n \in N_1 = \{n \in N_T : n^{-1} < \theta_1\} \). Clearly, the bounds

\[
\begin{align*}
\tilde{m}_0 < x_n(t) & \leq \hat{M}_0 \quad \text{for } t \in [\theta_1,T], \quad (2.3) \\
\tilde{m}_1 & \leq x_n'(t) \leq M_1 \quad \text{for } t \in [\theta_1,T], \quad (2.4)
\end{align*}
\]

independent of \( n \in N_1 \). In view of \( (2.1) \), \( f(t,x,p) \) is continuous on the set \( [\theta_1,T] \times [\tilde{m}_0,\hat{M}_0] \times [m_1,M_1] \) and so there is a constant \( M_{1,2} \), independent on \( n \), such that

\[
|x_n''(t)| \leq M_{1,2} \quad \text{for } t \in [\theta_1,T].
\]

The obtained bounds for \( x_n(t), x_n'(t) \) and \( x_n''(t) \) on the interval \( [\theta_1,T] \) allows us to apply the Arzela-Ascoli theorem on the sequence \( \{x_n\} \) to conclude that there are a subsequence \( \{x_{1,n_k}\}, k \in N, n_k \in N_1 \), and a function \( x_{\theta_1} \in C^2[\theta_1,T] \) such that

\[
\|x_{1,n_k} - x_{\theta_1}\|_1 \to 0 \quad \text{on } t \in [\theta_1,T];
\]

that is, the sequences \( \{x_{1,n_k}\} \) and \( \{x_{1,n_k}'\} \) converge uniformly on \([\theta_1,T]\) to \(x_{\theta_1}\) and \(x_{\theta_1}'\), respectively. Since \( (2.3) \) and \( (2.4) \) are valid in particular for the elements of \( \{x_{1,n_k}\} \) and \( \{x_{1,n_k}'\} \), letting \( k \to \infty \), we obtain

\[
\begin{align*}
\tilde{m}_0 & \leq x_{\theta_1}(t) \leq \hat{M}_0 \quad \text{for } t \in [\theta_1,T], \quad (2.5) \\
\tilde{m}_1 & \leq x_{\theta_1}'(t) \leq M_1 \quad \text{for } t \in [\theta_1,T]. \quad (2.6)
\end{align*}
\]
On the other hand, on using that the functions \(x_{1,n_k}(t), n_k \in N_1\), are solutions of the differential equation \((2.2)\), we have

\[
x'_{1,n_k}(t) = x'_{1,n_k}(\theta_1) + \int_{\theta_1}^{t} f(s, x_{1,n_k}(s), x'_{1,n_k}(s))ds, \quad t \in (\theta_1, T].
\]

Next, we need to show that the sequence \(\{f(s, x_{1,n_k}(s), x'_{1,n_k}(s))\}\), \(n_k \in N_1\), converges uniformly on the interval \([\theta_1, T]\). To this aim we observe at first that since \(f(t,x,p)\) is uniformly continuous on the compact set \([\theta_1, T] \times [\tilde{m}_0, \tilde{M}_0] \times [m_1, M_1]\), for each \(\varepsilon > 0\) there is a \(\delta > 0\) such that

\[
|f(t_0, x_0, p_0) - f(t_1, x_1, p_1)| < \varepsilon \quad \text{if} \quad (t_0, x_0, p_0), (t_1, x_1, p_1) \in [\theta_1, T] \times [\tilde{m}_0, \tilde{M}_0] \times [m_1, M_1]
\]

and

\[
\sqrt{(t_0 - t_1)^2 + (x_0 - x_1)^2 + (p_0 - p_1)^2} < \delta.
\]

Now, from the uniform convergence of \(\{x_{1,n_k}\}\) and \(\{x'_{1,n_k}\}\) on \([\theta_1, T]\) it follows that there is a \(N_{\delta(\varepsilon)}\) with the properties

\[
|x_{1,n_k} - x_{\theta_1}| < \frac{\delta}{\sqrt{2}} \quad \text{and} \quad |x'_{1,n_k} - x'_{\theta_1}| < \frac{\delta}{\sqrt{2}} \quad \text{for} \quad t \in [\theta_1, T]
\]

and each \(n_k > N_{\delta(\varepsilon)}\). As a result, for \(t \in [\theta_1, T]\) we obtain

\[
\sqrt{(t - t')^2 + (x_{1,n_k} - x_{\theta_1})^2 + (x'_{1,n_k} - x'_{\theta_1})^2} < \delta. \quad (2.8)
\]

Finally, for \(t \in [\theta_1, T]\) and \(n_k > N_{\delta(\varepsilon)}\) from \((2.3), (2.6)\) we obtain

\[
(t, x_{1,n_k}(t), x'_{1,n_k}(t)), (t, x_{\theta_1}(t), x'_{\theta_1}(t)) \in [\theta_1, T] \times [\tilde{m}_0, \tilde{M}_0] \times [m_1, M_1]. \quad (2.9)
\]

On combining \((2.8)\) and \((2.9)\) with \((2.7)\), we establish that for an arbitrary \(\varepsilon > 0\) there exists \(N_{\delta(\varepsilon)}\) such that for \(n_k > N_{\delta(\varepsilon)}\) we have

\[
|f(s, x_{1,n_k}(s), x'_{1,n_k}(s)) - f(s, x_{\theta_1}(s), x'_{\theta_1}(s))| < \varepsilon \quad \text{for} \quad t \in [\theta_1, T],
\]

i.e. the sequence \(\{f(s, x_{1,n_k}(s), x'_{1,n_k}(s))\}\), \(n_k \in N_1\), converges uniformly on the interval \([\theta_1, T]\) to \(f(s, x_{\theta_1}(s), x'_{\theta_1}(s))\). Then, returning to the integral equation and letting \(k \to \infty\) yield

\[
x'_{\theta_1}(t) = x'_{\theta_1}(t) + \int_{\theta_1}^{t} f(s, x_{\theta_1}(s), x'_{\theta_1}(s))ds, \quad t \in (\theta_1, T],
\]

from where it follows that \(x_{\theta_1}(t)\) is a \(C^2[\theta_1, T]\)-solution to the differential equation \(x'' = f(t,x,x')\) on \([\theta_1, T]\).

Further, we consider the sequence \(\{x_{1,n_k}\}\) on the new interval \([\theta_2, T]\) and for \(n_k \in N_2 = \{n_k \in N_T, k \in N : n_k^{-1} < \theta_2\}\). Obviously, for \(n_k \in N_2\) we have

\[
\tilde{m}_0 \leq x_{1,n_k}(t) \leq \tilde{M}_0 \quad \text{for} \quad t \in [\theta_2, T],
\]

\[
m_1 \leq x'_{1,n_k}(t) \leq M_1 \quad \text{for} \quad t \in [\theta_2, T].
\]

Besides, there is a constant \(M_{2,2}\), independent on \(n_k\), such that

\[
|x''_{1,n_k}(t)| \leq M_{2,2} \quad \text{for} \quad t \in [\theta_2, T].
\]

Having obtained bounds, we apply the Arzela-Ascoli theorem on the sequence \(\{x_{1,n_k}\}\) to conclude that there exist a subsequence \(\{x_{2,n_k}\}, k \in N, n_k \in N_2\), and a function \(x_{\theta_2} \in C^2[\theta_2, T]\) such that

\[
\|x_{2,n_k} - x_{\theta_2}\|_1 \to 0 \quad \text{on the new interval} \ [\theta_2, T].
\]
As above we establish also that \( x_{\theta_i}(t) \) is a \( C^2[\theta_2,T] \)-solution to the differential equation \( x'' = f(t,x,x') \) on \([\theta_2,T]\) and
\[
\hat{m}_0 \leq x_{\theta_i}(t) \leq \hat{M}_0 \quad \text{for } t \in [\theta_2,T],
\]
\[
m_1 \leq x'_{\theta_i}(t) \leq M_1 \quad \text{for } t \in [\theta_2,T].
\]
In addition, since \( \{x_{2,n_k}\} \) is a subsequence of \( \{x_{1,n_k}\} \), then \( \{x_{2,n_k}\} \) converges uniformly to \( x_{\theta_1} \) on the interval \([\theta_1,T]\) which means
\[
x_{\theta_2}(t) \equiv x_{\theta_1}(t) \quad \text{for } t \in [\theta_1,T].
\]
Applying the same procedure repeatedly for \( \theta_i \to 0 \), we establish that for each \( i \in N \) there exists a function \( x_{\theta_i}(t) \) which is a \( C^2[\theta_i,T] \)-solution to the equation \( x'' = f(t,x,x') \) on the interval \([\theta_i,T]\),
\[
\|x_{i,n_k} - x_{\theta_i}\|_1 \to 0 \quad \text{on the interval } [\theta_i,T]
\]
as \( k \to \infty \) and \( n_k \in N_i = \{n_k \in N_T, k \in N : n_k^{-1} < \theta_i\} \),
\[
\hat{m}_0 \leq x_{\theta_i}(t) \leq \hat{M}_0 \quad \text{for } t \in [\theta_i,T],
\]
\[
m_1 \leq x'_{\theta_i}(t) \leq M_1 \quad \text{for } t \in [\theta_i,T],
\]
\[
x_{\theta_i}(t) \equiv x_{\theta_i}(t) \quad \text{for } t \in [\theta_i,T].
\]
Thanks to the properties of the functions of \( \{x_{\theta_i}\} \), we conclude that there is some function \( x_0(t) \) which is a \( C^2(0,T) \)-solution to the equation \( x'' = f(t,x,x') \) on the interval \((0,T)\),
\[
\hat{m}_0 \leq x_0(t) \leq \hat{M}_0 \quad \text{for } t \in (0,T),
\]
\[
m_1 \leq x'_{0}(t) \leq M_1 \quad \text{for } t \in (0,T),
\]
\[
x_0(t) \equiv x_{\theta_i}(t) \quad \text{for } t \in [\theta_i,T].
\]

**Step 3** Construction of a \( C^1[0,T] \cap C^2(0,T) \)-solution to \([1.1]\). To define a \( C[0,T] \)-solution to \([1.1]\) we need to show that
\[
\lim_{t \to 0^+} x_0(t) = A.
\]
To this aim, we assume firstly on the contrary that for some \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that \( (0,\delta) \subset (0,T) \) and
\[
x_0(t) \notin (A - \varepsilon, A + \varepsilon) \quad \text{for } t \in (0,\delta).
\]
Returning our attention to the sequence \( \{x_n\} \), from \( x_n \in C[0,T] \) and \( x_n(n^{-1}) = A \) we deduce that there is a number \( n_\delta \) such that for each \( n \geq n_\delta, n \in N \), there exists a sufficiently small \( \delta_n > n^{-1} \) with the properties \( (n^{-1},\delta_n) \subset (0,\delta) \) and
\[
x_n(t) \in (A - \varepsilon/2, A + \varepsilon/2) \quad \text{for } t \in (n^{-1},\delta_n).
\]
On the other hand, there exists a number \( n^* \) such that for each \( n \geq n^*, n \in N \), there exists some \( \theta_i^* \in N \) for which
\[
[\theta_i^*,\theta_i^* - 1] \subset (n^{-1},\delta_n) \subset (0,\delta);
\]
the assumption that the interval \( [\theta_i^*,\theta_i^* - 1] \) does not exist contradicts to the fact that \( t = 0 \) is an accumulation point of the sequence \( \{\theta_i\} \). As a result, for each \( n \geq \max\{n_\delta, n^*\} \) there exists \( \theta_i^* \in N \) such that
\[
A - \varepsilon/2 < x_n(t) < A + \varepsilon/2 \quad \text{for } t \in [\theta_i^*,\theta_i^* - 1] \subset (0,\delta).
\]
\[
\frac{A - \varepsilon}{2} < x_n(t) < \frac{A + \varepsilon}{2} \quad \text{for } t \in [\theta_i^*,\theta_i^* - 1] \subset (0,\delta).
\]
\[
\frac{A - \varepsilon}{2} < x_n(t) < \frac{A + \varepsilon}{2} \quad \text{for } t \in [\theta_i^*,\theta_i^* - 1] \subset (0,\delta).
\]
It is easy to see, for every $i^*$ there is a number $n_{i^*}$ such that \( (2.15) \) holds for each $n_k \in N_{i^*}, k \in N,$ with $n_k \geq \max\{n_{i^*}, n_{i^*}, n^*\},$ that is,
\[
A - \varepsilon/2 < x_{i^*, n_k}(t) < A + \varepsilon/2 \quad \text{for } t \in [\theta_{i^*}, \theta_{i^*} - 1] \subset (0, \delta).
\] (2.16)

Further, from \( (2.10) \) and \( (2.12) \) for each $i \in N$ we obtain
\[
||x_{i, n_k} - x_0||_1 \to 0 \quad \text{on } [\theta_i, T] \text{ when } k \to \infty \text{ and } n_k \in N_i,
\] (2.17)
which means that for each $i \in N$ there is a number $\pi_i$ such that for each $n_k \in N_i$ with $n_k \geq \pi_i$ we have
\[
-x/2 < x_{i, n_k}(t) - x_0(t) < \varepsilon/2 \quad \text{for } t \in [\theta_i, T]
\] or
\[
x_{i, n_k}(t) - \varepsilon/2 < x_0(t) < x_{i, n_k}(t) + \varepsilon/2 \quad \text{for } t \in [\theta_i, T].
\]
In particular, for each $n_k \in N_{i^*}$ with $n_k \geq \max\{n_{i^*}, \pi_{i^*}, n_{i^*}, n^*\}, k \in N,$ we obtain
\[
x_{i^*, n_k}(t) - \varepsilon/2 < x_0(t) < x_{i^*, n_k}(t) + \varepsilon/2 \quad \text{for } t \in [\theta_{i^*}, T].
\]
This combined with \( (2.16) \) yields
\[
A - \varepsilon < x_0(t) < A + \varepsilon \quad \text{for } t \in [\theta_{i^*}, \theta_{i^*} - 1] \subset (0, \delta),
\]
which contradicts to \( (2.15) \) and so \( (2.13) \) holds.

By exactly the same reasoning applied on the sequence $\{x'_i\}$ we establish
\[
\lim_{t \to 0^+} x'_0(t) = B.
\]
Moreover, now we use that for each $i \in N$ and sufficiently large $n_k \in N_i, k \in N,$ \( (2.17) \) yields
\[
-x/2 < x'_{i, n_k}(t) - x'_0(t) < \varepsilon/2 \quad \text{for } t \in [\theta_i, T].
\]
Next, introduce the function
\[
x(t) = \begin{cases} A & \text{for } t = 0, \\ x_0(t) & \text{for } t \in (0, T]. \end{cases}
\]
Clearly, $x'(t) = x'_0(t)$ for $t \in (0, T].$ Besides,
\[
x'(0) = \lim_{t \to 0^+} \frac{x(t) - x(0)}{t - 0} = \lim_{t \to 0^+} x'(t) = \lim_{t \to 0^+} x'_0(t) = B.
\]
Thus, $x' \in C[0, T]$ and so $x(t)$ is a $C^1[0, T] \cap C^2(0, T)$-solution to \( (1.1) \).

The inequalities \( (2.11) \) give immediately
\[
m_1 \leq x'(t) \leq M_1 \quad \text{for } t \in [0, T],
\]
from where by integration from $0$ to $t \in (0, T]$ we obtain the bounds for $x(t)$. \( \square \)

As an elementary consequence of Theorem \( 2.1 \) we obtain results guaranteeing important properties of the solutions.

**Theorem 2.2.** Let $B \geq 0$ and let \( (A2) \) hold for $m_1 = 0.$ Then problem \( (1.1) \) has at least one nondecreasing solution in $C^1[0, T] \cap C^2(0, T]$.

**Theorem 2.3.** Let $B > 0$ and let \( (A2) \) hold for $m_1 > 0.$ Then problem \( (1.1) \) has at least one strictly increasing solution in $C^1[0, T] \cap C^2(0, T]$.

**Theorem 2.4.** Let $A > 0 \quad (A = 0), \quad B \geq 0$ and let \( (A2) \) hold for $m_1 = 0.$ Then problem \( (1.1) \) has at least one positive (nonnegative) nondecreasing solution in $C^1[0, T] \cap C^2(0, T]$. 
Theorem 2.5. Let $A \geq 0$, $B > 0$ and let (A2) hold for $m_1 > 0$. Then problem (1.1) has at least one strictly increasing solution in $C^1[0,T] \cap C^2(0,T)$ having positive values for $t \in (0,T]$.

3. Example

Consider the IVP

$$x'' = t^\alpha P_k(x'),$$

$$x(0) = A, \quad x'(0) = B,$$

where $A \geq 0$, $B > 0$, $m, n \in \mathbb{N}$, and the polynomial $P_k(p), k \geq 2$, has simple zeros $p_1$ and $p_2$ such that $P'_k(p_1) < 0$ and $0 < p_1 < B < p_2$.

Let $\theta > 0$ be so small that $p_1 - \theta > 0$, $p_1 + \theta < B < p_2 - \theta$ and

$$P_k(p) \neq 0 \quad \text{for} \quad p \in [p_1 - \theta, p_1) \cup (p_1, p_1 + \theta) \cup [p_2 - \theta, p_2) \cup (p_2, p_2 + \theta].$$

Then $P'_k(p_1) < 0$ implies

$$P_k(p) > 0 \quad \text{for} \quad p \in [p_1 - \theta, p_1) \quad \text{and} \quad P_k(p) < 0 \quad \text{for} \quad p \in (p_1, p_1 + \theta].$$

Besides, we see easily that if

$$P_k(p) < 0 \quad \text{for} \quad p \in [p_2 - \theta, p_2),$$

then (A2) holds for an arbitrary $T > 0$,

$$m_1 = p_1 - \theta, \quad m_1 = p_1, \quad M_1 = p_2 - \theta, \quad \bar{M}_1 = p_2, \quad \tau = \theta/2,$$

moreover $M_0 = (p_2 - \theta)T + A$, and if

$$P_k(p) < 0 \quad \text{for} \quad p \in (p_2, p_2 + \theta],$$

it is satisfied for an arbitrary $T > 0$,

$$m_1 = p_1 - \theta, \quad m_1 = p_1, \quad M_1 = p_2, \quad \bar{M}_1 = p_2 + \theta, \quad \tau = \theta/2,$$

moreover $M_0 = p_2T + A$. So, it follows from Theorem 2.5 that for each $T > 0$ the considered problem has a strictly increasing solution in $C^2[0,T] \cap C^2(0,T]$ which is positive on $(0,T]$.

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References


PETO KELEVEDJIEV
TECHNICAL UNIVERSITY OF SOFIA, BRANCH SLIVEN, BULGARIA
E-mail address: keleved@mailcity.com