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BIFURCATION FOR NON LINEAR ORDINARY DIFFERENTIAL EQUATIONS WITH SINGULAR PERTURBATION

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Abstract. We study a family of singularly perturbed ODEs with one parameter and compare their solutions to the ones of the corresponding reduced equations. The interesting characteristic here is that the reduced equations have more than one solution for a given set of initial conditions. Then we consider how those solutions are organized for different values of the parameter. The bifurcation associated to this situation is studied using a minimal set of tools from non standard analysis

1. Introduction

We study the singularly perturbed equation

$$\varepsilon\ddot{y}(t) + \dot{y}^3(t) - y^2(t) + a = 0, \quad t \in \mathbb{R}, \tag{1.1}$$

with the control parameter $a \geq 0$ finite (see [8] for the terminology in non standard analysis) and $\varepsilon > 0$ infinitely small (i.e. $\forall x \in \mathbb{R}_+^*, |\varepsilon| < x$, and we note $\varepsilon \simeq 0$). For a given a, we compare its solutions to the ones of the reduced equation

$$\dot{y}^3 - y^2 + {}^\circ a = 0 \tag{1.2}$$

(note that for any limited hyperreal x there is a unique standard real noted ${}^\circ x$ infinitely close to x, i.e. ${}^\circ x - x \simeq 0$).

As this equation has different properties for $a = 0$ and $a > 0$, we study the two cases separately then try to understand what happens when $a \to 0$.

2. Properties of (1.1)

Equation (1.1) is equivalent to the differential system

$$\begin{gathered} \dot{y} = u, \\ \varepsilon\dot{u} = y^2 - u^3 - a. \end{gathered} \tag{2.1}$$

This is a slow-fast system and its slow manifold has the equation

$$y^2 - u^3 - a = 0 \tag{2.2}$$

In the phase plane, the field is infinitely large outside of (2.2), as $1/\varepsilon$ is infinitely large (i.e. its inverse is infinitely small) and inward. On (2.2) the field is transverse and has the same sign as u. The slow manifold is attractive everywhere.

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2.1. **Case $a = 0$.** This case has been studied in [1]. Here, the system has one saddle-node equilibrium point in $(0, 0)$.

In this case, the reduced equation $\dot{y}^3 - y^2 = 0$ has an infinity of solutions, for an initial condition $y(0) \leq 0$, such that

$$y(t) = \begin{cases} \left(\frac{t}{3} + \sqrt[3]{y(0)}\right)^3, & 0 \leq t \leq -3\sqrt[3]{y(0)}, \\ 0, & -3\sqrt[3]{y(0)} \leq t \leq -3\sqrt[3]{y(0)} + \delta, \\ \left(\frac{t-\delta}{3} + \sqrt[3]{y(0)}\right)^3, & -3\sqrt[3]{y(0)} + \delta \leq t. \end{cases}$$

where $\delta \geq 0$ is the time spent on the t-axis by the solution.

In such a configuration, the main question that arises is which of these solutions of (1.1) with $\varepsilon = 0$ and $a = 0$, starting from $y(0)$ will be the closest to the unique solution of (1.1) with $a = 0$, starting from a given point infinitely close to that $y(0)$. The following theorem gives the answer to that question for the solutions of (1.1) with $a = 0$ starting from $y(0) < 0$ not infinitely small (that we call slow paths).

Theorem 2.1. *The standard part of a slow path for* (1.1) *with $a = 0$ and $\varepsilon \simeq 0$ is the solution of* (1.1) *with $\varepsilon = 0$ and $a = 0$ that starts from $^\circ y(0)$ and does not spend time on the t-axis.*

For a proof of the above theorem see [1]. This phenomenon is shown in Figure 1.

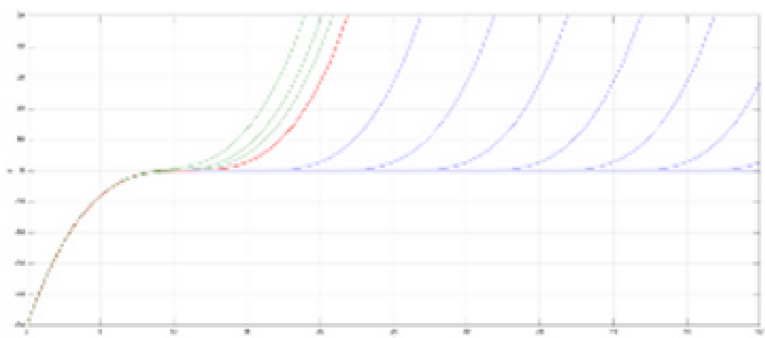

FIGURE 1. Convergence of the solution of (1.1) with $a = 0$ towards the "fastest" solution of (1.1) with $\varepsilon = 0$ and $a = 0$ (in red) for $y(0) = -50$, shown here (in green) for $\varepsilon \in \{1, 0.1, 0.01\}$.

2.2. **Case $a > 0$.** The field is shown in Figure 2.

Let us now discuss the nature of the equilibria that appear here.

Theorem 2.2. *The system has two equilibrium points:*

- $(\sqrt{a}, 0)$ *which is a saddle point;*
- $(-\sqrt{a}, 0)$ *which is a stable sink.*

Proof. The equilibria are: $(y, u) = (\pm\sqrt{a}, 0)$. The Jacobian matrix of (2.1) at these points is

$$J_{\varepsilon,a} = \begin{pmatrix} 0 & 1 \\ 2y/\varepsilon & 0 \end{pmatrix}.$$

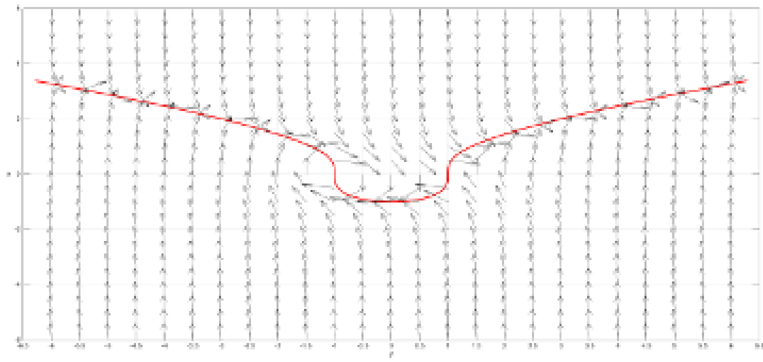

FIGURE 2. Slow manifold $(\mathcal{C}_1)$ such that: $y^2 - u^3 - 1 = 0$ and vector field, for $\varepsilon = 1$.

Nature of the equilibrium $(\sqrt{a}, 0)$**.** The eigenvalues of $J_{\varepsilon,a}$ are

$$\lambda = \pm\sqrt{2\sqrt{a}/\varepsilon}.$$

This equilibrium is a saddle point (attractive-repulsive) which is structurally stable; this means that this point will have the same dynamic in the initial non-linearised system (2.1).

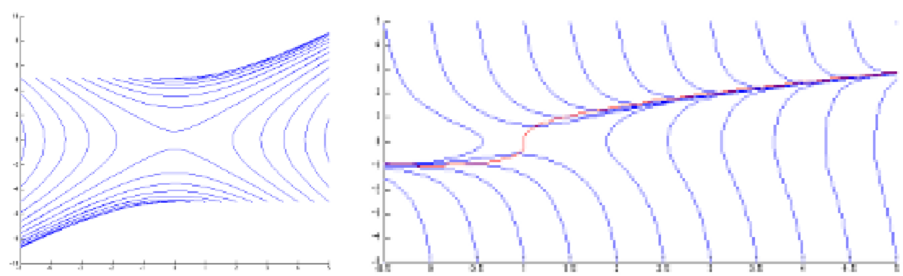

FIGURE 3. Equilibrium $(\sqrt{a}, 0)$ still acts as saddle point on the paths of (1.1).

Nature of the equilibrium $(-\sqrt{a}, 0)$**.** The eigenvalues of $J_{\varepsilon,a}$ are

$$\lambda = \pm i\sqrt{2\sqrt{a}/\varepsilon}.$$

This equilibrium is a center which is structurally unstable; this means that it will not necessarily have the same dynamic in the initial system (2.1). We thus need a further analysis to get the nature of this equilibrium.

The equilibrium point $(-\sqrt{a}, 0)$ can be of one of the following types: center, center-sink, or sink. Both a center and a center-sink have at least one periodic solution. Let us prove that this is excluded.

Lemma 2.3. *The equilibrium point* $(-\sqrt{a}, 0)$ *is asymptotically stable for a postive and finite.*

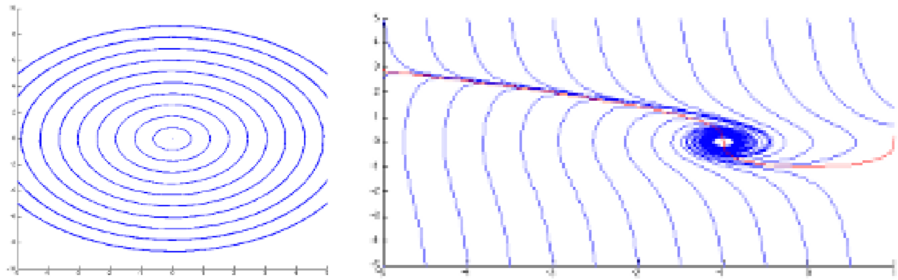

FIGURE 4. Equilibrium $(-\sqrt{a}, 0)$ no longer acts as a center on the paths of (1.1).

Proof. Let γ be a path that starts with initial conditions in the basin of attraction of $(-\sqrt{a}, 0)$. We put $y = \bar{y} - \sqrt{a}$ and use the following change of variables, with $\beta > 0$:

$$\bar{y} = \beta^3 Y, \quad u = \beta U, \quad t = \beta^2 T,$$

to write (2.1) as

$$Y' = U,$$
$$U' = \frac{\beta^4}{\varepsilon}\big(-2\sqrt{a}Y + \beta^3 Y^2 - U^3\big).$$

For $\beta = \varepsilon^{1/4}$, γ's standard part (i.e. $t \mapsto {}^\circ\gamma(t)$.) is solution of the standard system

$$Y' = U,$$
$$U' = -2\sqrt{{}^\circ a}Y - U^3,$$

as $\beta^3 = \varepsilon^{\frac{3}{4}} \ll 1$. Note that ${}^\circ a$ can be 0 if $a \simeq 0$. Multiplying the second equation by $U = Y'$ leads to

$$\frac{\mathrm{d}}{\mathrm{d}T}[K(Y,U)] = UU' + 2\sqrt{{}^\circ a}YY' = -U^4 < 0$$

for $U \neq 0$, with $K(Y,U) = \frac{U^2}{2} + \sqrt{{}^\circ a}Y^2 > 0$. Hence K is a Lyapunov function for the system so γ converges towards $(0,0)$.

The equilibrium point $(-\sqrt{a}, 0)$ is thus a stable sink. Every path in its basin of attraction converges towards $-\sqrt{a}$ as $t \to +\infty$. □

Definition 2.4. We call slow paths the ones who enter in the slow manifold $\mathcal{C}_a$ neighborhood with an abscissa lower than and non infinitely close to $-\sqrt{{}^\circ a}$.

In the phase plane, the attractive separatrix of $(\sqrt{a}, 0)$ goes, for a appreciable (i.e. a bounded non infinitely small.), from this point and is almost vertical according to the vector field description made earlier. Therefore, every slow path is in the basin of attraction of $(-\sqrt{a}, 0)$ and thus is firstly increasing than oscillating around $y = -\sqrt{a}$ to finally converge towards it. Their standard part is the only solution of the reduced equation $\dot{y}^3 - y^2 + {}^\circ a = 0$ starting from ${}^\circ y(0)$.

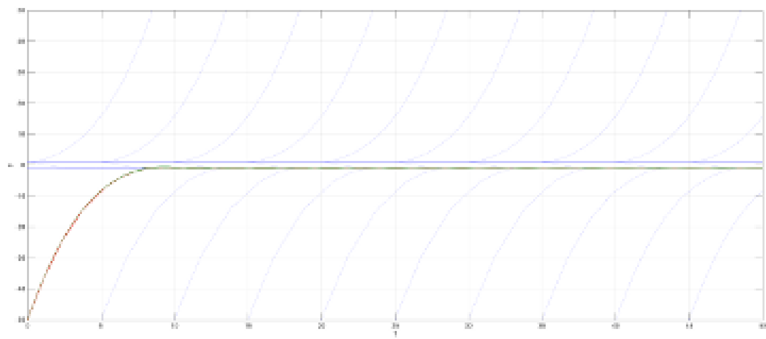

FIGURE 5. Convergence of the solution of (1.1) with $a = 1$ towards the "slow" solution of (1.1) with $\varepsilon = 0$ (in red) for $y(0) = -50$, shown here (in green) for $\varepsilon \in \{1, 0.1, 0.01\}$.

3. Case a approaches 0

Observations on the behaviors of the solutions. When $a \to 0$, the two equilibria collide into one to create a saddle-node singularity. This saddle-node bifurcation is a classical one and refers to the topological change that occurs between $a = 0$ and any $a > 0$.

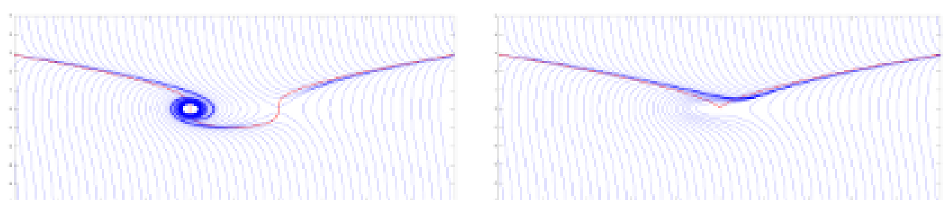

FIGURE 6. Saddle-node bifurcation for $a = 1$ and $a = 0$, and $\varepsilon = 1$.

Now let us look at what is happening with slow paths. We saw that as $\varepsilon \simeq 0$, their standard parts are the solutions of the reduced equation starting from $^\circ y(0)$ that do not stay on the time axis for $a = 0$ (cf. Theorem 2.1) and the ones that stay indefinitely on $y = -\sqrt{a}$ for $a > 0$ (cf. Theorem 2.2). Let us look at those solutions for different values of a.

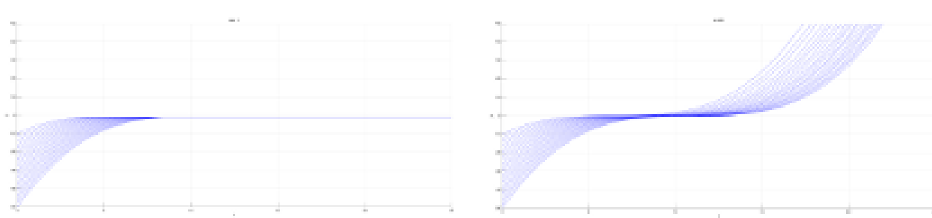

FIGURE 7. Slow paths converge towards these solutions as $\varepsilon \to 0$, shown here for $a = 1$ and $a = 0$.

The graphs above show that as a goes to 0, the shift between the "chosen" solutions of the reduced equations suddenly jumps at $a = 0$ from the ones that stay indefinitely near the time axis (as $y = -\sqrt{a}$ is getting closer to t-axis here) to the ones that do not spend time on this axis at all. From this perspective, a discontinuity appears.

Exhibiting the bifurcation. We now analyze the natural 1-dimensional foliation of $\mathbb{R}^2$ defined by the vector field of (1.1), its leaves being the integral curves. This foliation evolving continuously with the parameter $a \geq 0$, the solutions of (1.1) that converge towards the solutions of the reduced equation that do not spend time near t-axis should leave some traces around $a = 0$. Let $a > 0$ be infinitely small along with ε. For different infinitely small values of a, the global aspect of the foliation is independent on the parameter, both look identical:

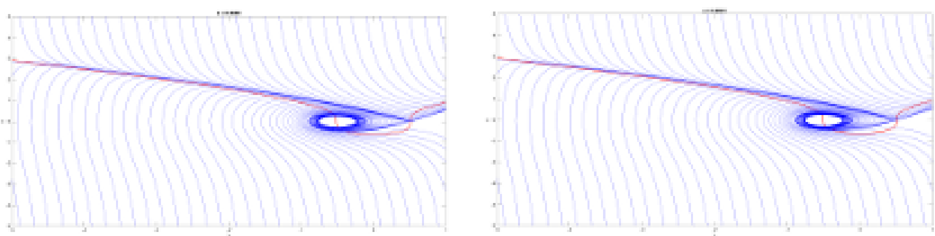

FIGURE 8. Global aspect for $a = 0.26$ (left), and $a = 0.2601$ (right), for $\varepsilon = 1$.

But if we separate the leaves based on whether they are in the basin of attractions of $(\sqrt{a}, 0)$ or of $(-\sqrt{a}, 0)$, we realise that the foliation composition is actually very dependant on the value of a.

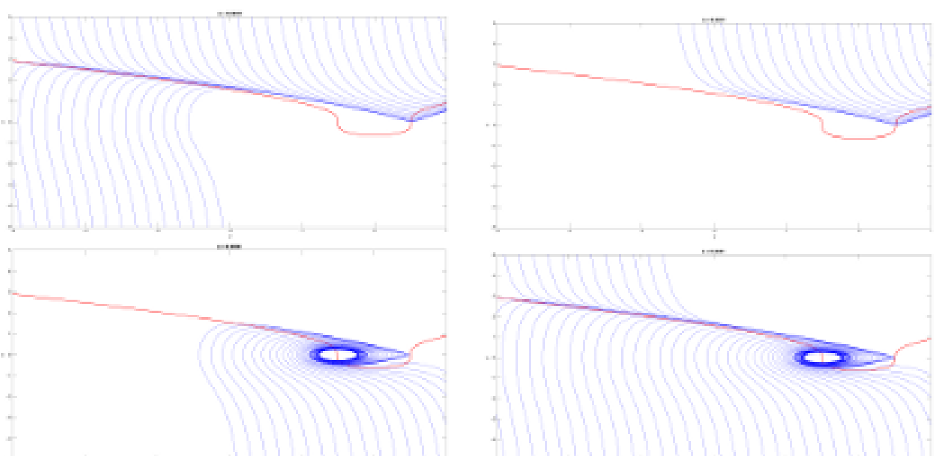

FIGURE 9. Phase plane foliation composition for those same values.

We realise here that there is an analytical bifurcation where the foliation of the part of the phase plane $y < 0$ and y not infinitely small evolves very quickly but continuously from being included in the basin of attraction of $(-\sqrt{a}, 0)$ to being included in the basin of attraction of $(\sqrt{a}, 0)$ as a decreases. The first case corresponds to when the standard part of (1.1) slow paths are the solutions of the reduced equation (1.1) with $\varepsilon = 0$ and $a = 0$ (here $^\circ a = 0$) that stay indefinitely on the time axis and the second when they are the ones of the reduced equation that do not spend time on the t-axis.

As this phenomenon takes place, we can see that as $a \to 0$, the upper attractive separatrix Σ_0 of $(\sqrt{a}, 0)$ will have to continuously go from a quasi-vertical position above the slow manifold $\mathcal{C}_a$ to crossing it just above $(-\sqrt{a}, 0)$ and going straight down. A noteworthy value associated to this bifurcation is when this Σ_0 is asymptotical to the slow manifold without crossing it, let's call it a_0. This value is necessarily infinitely small.

The following theorem is the main result of our study and gives an expression for a_0.

Theorem 3.1. *The characteristic value associated to the bifurcation described above is*

$$a_0 = s_0 \varepsilon^{6/5}$$

for $\varepsilon \simeq 0$ *and* $s_0 \in \mathbb{R}$ *standard. Simulation gives* $0.26 < s_0 < 0.2601$.

Proof. Using the change of variables

$$y = a_0^{1/2} Y, \quad u = a_0^{1/3} U, \quad t = a_0^{1/6} T,$$

System (2.1) with $a = a_0$ becomes

$$\begin{gathered} Y' = U, \\ U' = \frac{a_0^{5/6}}{\varepsilon}[Y^2 - U^3 - 1]. \end{gathered}$$

Let us discuss the values of $\beta = \frac{a_0^{5/6}}{\varepsilon}$:

β **is infinitely small** The standard part of the slow paths of (1.1) with $\alpha = a_0$ are solutions of the trivial system

$$\begin{gathered} Y' = U, \\ U' = 0. \end{gathered}$$

All its paths are horizontal in the phase plane. This forces $^\circ\Sigma_0$ to cross the slow manifold $\mathcal{C}_0 = {}^\circ\mathcal{C}_{a_0}$ at a bounded abscissa. This is excluded as Σ_0 does not cross $\mathcal{C}_{a_0}$ such that $y^2 - u^3 - a_0 = 0$, i.e. $Y^2 - U^3 - 1 = 0$ which is not horizontal in (Y, U).

β **is infinitely large** The initial system is equivalent to $(S_{\varepsilon,1})$. Σ_0 being asymptotical to $Y^2 - U^3 - 1 = 0$, i.e. $\mathcal{C}_1$ in (Y, U), a path starting on this separatrix will enter the slow manifold $\mathcal{C}_1$ neighborhood and will stay in it until it spiral-sinks into the equilibrium $(-1, 0)$. This is impossible as such a path is supposed to follow Σ_0 until it reaches $(\sqrt{a_0}, 0) \simeq (0, 0)$.

Therefore the only value possible is β appreciable (i.e. a bounded real number not infinitely small), and $a_0 = \beta^{6/5}\varepsilon^{6/5}$. □

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