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NODAL SOLUTIONS FOR SCHRÖDINGER-POISSON TYPE EQUATIONS IN $\mathbb{R}^3$

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Abstract. In this article, we consider the existence of nodal solutions for the Schrodinger-Poisson type problem

$$-\Big(a+b\int_{\mathbb{R}^3}|\nabla u|^2\,dx\Big)\Delta u+V(|x|)u+\varphi u=|u|^{p-2}u,\quad \text{in } \mathbb{R}^3,$$
$$-\Delta\varphi=u^2,\quad \lim_{|x|\to\infty}\varphi(x)=0,$$

where a,b are positive constants, $p\in(4,6)$ and $V(x)$ is a radial smooth function. For each $k\in\mathbb{N}_+$, we show the existence of to nodal solution changing sign exactly k times.

1. Introduction

In this article, we consider the existence of nodal solutions for the Schrödinger-Poisson type problem

$$\begin{gathered}-\Big(a+b\int_{\mathbb{R}^3}|\nabla u|^2\,dx\Big)\Delta u+V(|x|)u+\varphi u=|u|^{p-2}u,\quad \text{in } \mathbb{R}^3,\\ -\Delta\varphi=u^2,\quad \lim_{|x|\to\infty}\varphi(x)=0,\end{gathered}\tag{1.1}$$

where $a,b>0$ are a positive constants, $p\in(4,6)$ and $V\in C(\mathbb{R}^3,\mathbb{R})$ is a radial function. The nonlocal operator $\big(a+b\int_{\mathbb{R}^3}|\nabla u|^2\,dx\big)\Delta$ appears in the Kirchhoff Dirichlet problem

$$\begin{gathered}-\Big(a+b\int_{\Omega}|\nabla u|^2\,dx\Big)\Delta u=|u|^{p-2}u,\quad \text{in } \Omega,\\ u=0,\quad \text{on } \partial\Omega\end{gathered}\tag{1.2}$$

for a domain $\Omega\subset\mathbb{R}^3$. In one dimensional case, such a problem arises in the investigation of the existence of classical D'Alembert's wave equations for free vibration of elastic strings, see [14] for details. After the work of Lions [16], higher dimensional problem (1.2) attracts attention of researchers. Various results have been appeared for Kirchhoff type problems in [1, 4, 5, 8, 12, 17, 19, 24] and references therein.

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Taking $a = 1$, $b = 0$ in (1.1), we obtain the Schröinger-Poisson equation

$$\begin{gathered} -\Delta u + V(|x|)u + \varphi u = |u|^{p-2}u, \quad \text{in } \mathbb{R}^3, \\ -\Delta\varphi = u^2, \quad \lim_{|x|\to\infty} \varphi(x) = 0. \end{gathered} \tag{1.3}$$

where

$$\varphi(x) = \int_{\mathbb{R}^3} \frac{u^2(y)}{4\pi|x-y|}\,dy$$

for all $x \in \mathbb{R}^3$. Therefore, (1.3) also involves in a nonlocal term. This problem with $p = 5/3$ stems from the Slater approximation of the exchange term in the Hartree-Fock model, see [22]. For the general exponent p, there is an interesting competition between local and nonlocal nonlinearities. This interaction yields new phenomena, and then problem (1.3) has been extensively studied in the literature, see [2, 13, 20, 21, 25, 26] and references therein.

In this paper, we intend to show the existence of radial solutions for (1.1) with prescribed k nodal for every fixed integer k.

In the case $k = 2$, solutions with two nodal domains were studied in [3, 6, 11, 25] etc. The argument generally used is to modify the method developed in [18], first one seeks a minimizer u of the minimizing problem $\inf_{\mathcal{M}} J(u)$, where

$$\mathcal{M} = \{u \in \mathcal{N} : u^{\pm} \not\equiv 0, \langle J'(u), u^+\rangle = \langle J'(u), u^-\rangle = 0\}$$

is a subset of the Nehari manifold $\mathcal{N}$. Then, one needs to show u is a critical point of J. For problem (1.3), it was proved in [3, 25] by such an argument that the problem has a sign-changing solution. Because problem (1.3) contains a nonlocal term, the corresponding functional J does not have the decomposition

$$J(u) = J(u^+) + J(u^-),$$

it brings difficulties to construct a nodal solution. On the other hand, for the Kirchhoff Dirichlet problem (1.2), besides other things, a sign-changing solution was found in [6, 11], some technique was developed in treating the nonlocal operator in the problem.

For every integer $k \geq 0$, it was proved in [7] and [9] independently that, there is a pair of solutions $u_k^{\pm}$ of

$$\begin{gathered} -\Delta u + V(|x|)u = f(|x|, u), \quad \text{in } \mathbb{R}^3, \\ u \in H^1(\mathbb{R}^N). \end{gathered} \tag{1.4}$$

Such solutions of (1.4) are obtained by gluing solutions of the equation in each annulus, including every ball and the complement of it. However, this approach cannot be applied directly to problems with nonlocal terms, such as problems (1.1)-(1.3), because nonlocal terms need the global information of u. This difficulty was overcome by regarding the problem as a system of $k+1$ equations with $k+1$ unknown functions u_i, each u_i is supported on only one annulus and vanishes at the complement of it. In this way, Kim and Seok [15] found infinitely many nodal solutions for Schröinger-Poisson system (1.3), and then Deng et at [10] treated Kirchhoff problems in $\mathbb{R}^3$ in a similar way. Inspired by [10, 15], we establish the existence of infinitely many nodal solutions for (1.1). Since problem (1.1) contains both nonlocal operator and nonlocal nonlinear term, the construction of nodal solutions become technically complicated.

We assume in this paper that, the potential function V satisfies the condition

(A1) $V(r) \in C([0,+\infty),\mathbb{R})$ is bounded from below by a positive constant V_0.

Our main result is as follows.

Theorem 1.1. *Suppose condition* (A1) *holds and* $4 < p < 6$. *For every* $k \in \mathbb{N}$, *there exists a radial solution* u_k *of* (1.1), *which changes sign exactly* k *times.*

This theorem is proved by variational approach. Denote by $H_r^1(\mathbb{R}^3)$ the set of radially symmetric functions in the Sobolev space $H^1(\mathbb{R}^3)$. We define

$$H = \Big\{u \in H_r^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} \big(a|\nabla u|^2 + V(|x|)u^2\big)\,dx < +\infty\Big\}$$

with the norm

$$\|u\|^2 = \int_{\mathbb{R}^3} \big(a|\nabla u|^2 + V(|x|)u^2\big)\,dx.$$

By assumption (A1) and the fact $a > 0$, the inclusion $H \hookrightarrow H_r^1(\mathbb{R}^3)$ is continuous, and $H \hookrightarrow L^q(\mathbb{R}^3)$ is compact for $2 < q < 6$, by the well known result of Strauss[23]. Weak solutions of (1.1) will be found as critical points of the functional

$$\begin{aligned} E(u) &= \frac{1}{2}\int_{\mathbb{R}^3} \big(a|\nabla u|^2 + V(|x|)u^2\big)\,dx + \frac{b}{4}\Big(\int_{\mathbb{R}^3} |\nabla u|^2\,dx\Big)^2 \\ &\quad + \frac{1}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{u^2(x)u^2(y)}{4\pi|x-y|}\,dxdx - \frac{1}{p}\int_{\mathbb{R}^3} |u|^p\,dx \end{aligned}$$

defined on H. The functional E belongs to $C^2(H,\mathbb{R})$, its Fréchet derivative is given by

$$\begin{aligned} \langle E'(u),\psi\rangle &= \int_{\mathbb{R}^3} \big(a\nabla u\nabla\psi + V(|x|)u\psi\big)\,dx + b\int_{\mathbb{R}^3} |\nabla u|^2\,dx \int_{\mathbb{R}^3} \nabla u\nabla\psi\,dx \\ &\quad + \int_{\mathbb{R}^3} \varphi u\psi\,dx - \int_{\mathbb{R}^3} |u|^{p-2}u\psi\,dx \end{aligned} \tag{1.5}$$

for any $u,\psi \in H$. Hence, critical points of $E(u)$ are weak solutions of (1.1), and necessarily contained in the Nehari manifold

$$\mathcal{N} = \{u \in H \ \setminus \{0\} : \langle E'(u),\varphi\rangle = 0\}.$$

In constructing nodal solutions of problem (1.1), we will modify the variational framework in accordance to fixed nodes of expected solutions. Dividing $\mathbb{R}^3$ into $k+1$ parts, we reformulate functionals and Nehari sets correspondingly. The proof of Theorem 1.1 consists of verifying the Nehari set is a manifold and finding a minimizer of the related functional on the manifold.

This paper is organized as follows. In Section 2, we present a suitable variational framework for our problem, then we prove Theorem 1.1 in Section 3.

2. Preliminaries

In this section, we develop the variational construction. Decomposing $\mathbb{R}^3$ into $k+1$ parts, we consider a system defined on $k+1$ parts. Precisely, for $k \in \mathbb{N}_+$, we define

$$\begin{gathered} \Gamma_k = \{\mathbf{r}_k = (r_1,\dots,r_k) \in \mathbb{R}^k : 0 = r_0 < r_1 < \dots < r_k < r_{k+1} = \infty\}, \\ B_i = B_i^{\mathbf{r}_k} = \{x \in \mathbb{R}^3 : r_{i-1} < |x| < r_i\} \end{gathered}$$

for each $i = 1, \dots, k+1$. By the definition, B_1 is a ball, $B_2, \dots, B_k$ are annuli and B_{k+1} is the complement of a ball. Fix $\mathbf{r}_k = (r_1, \dots, r_k) \in \Gamma_k$, there is family of $\{B_i\}_i^{k+1}$. We denote

$$H_i = \{u \in H_0^1(B_i) : u(x) = u(|x|), u(x) = 0 \text{ if } x \notin B_i\}$$

for each $i = 1, \dots, k$. H_i is Hilbert space with the norm

$$\|u\|_i^2 = \int_{B_i} \Big(a|\nabla u|^2 + V(|x|)u^2 \Big)\, dx.$$

Furthermore, we set $\mathcal{H}_k = H_1 \times H_2 \times \dots \times H_{k+1}$ and define the functional $I : \mathcal{H}_k \to \mathbb{R}$ by

$$\begin{aligned}
&I(u_1, \dots, u_{k+1})\\
&= \frac{1}{2}\sum_{i=1}^{k+1} \|u_i\|_i^2 + \frac{b}{4}\sum_{i=1}^{k+1} \Big(\int_{B_i} |\nabla u_i|^2\, dx \Big)^2 + \frac{b}{4}\sum_{i=1}^{k+1} \int_{B_i} |\nabla u_i|^2\, dx \int_{B_j} |\nabla u_j|^2\, dx\\
&\quad + \frac{1}{4}\sum_{i=1}^{k+1} \int_{B_i}\int_{B_i} \frac{u_i^2(x)u_i^2(y)}{4\pi|x-y|}\, dx\, dy + \frac{1}{4}\sum_{i\neq j}^{k+1} \int_{B_i}\int_{B_j} \frac{u_i^2(x)u_j^2(y)}{4\pi|x-y|}\, dx\, dy\\
&\quad - \frac{1}{p}\int_{B_i} |u_i|^p\, dx,
\end{aligned} \tag{2.1}$$

where $u_i \in H_i$, $i = 1, \dots, k+1$. It is readily to verify that

$$I(u_1, \dots, u_{k+1}) = E\Big(\sum_{i=1}^{k+1} u_i \Big).$$

If $(u_1, \dots, u_{k+1})$ is a critical point of I, then it satisfies

$$\begin{aligned}
&\langle I'(u_1, \dots, u_{k+1}), \psi\rangle\\
&= \int_{\mathbb{R}^3} \Big(a\sum_{i=1}^{k+1} \nabla u_i \Big) \nabla\psi\, dx + V(|x|)\sum_{i=1}^{k+1} u_i\psi\\
&\quad + b\int_{\mathbb{R}^3} \Big| \sum_{i=1}^{k+1} \nabla u_i \Big|^2 \int_{\mathbb{R}^3} \Big(\sum_{i=1}^{k+1} \nabla u_j \Big) \nabla\psi\, dx + \int_{B_i} \sum_{i=1}^{k+1} u_i\varphi\psi\, dx\\
&\quad - \int_{\mathbb{R}^3} \sum_{i=1}^{k+1} |u_i|^{p-2}u_i\psi = 0.
\end{aligned}$$

for $\psi \in H_0^1(B_i)$, where $-\Delta\varphi = \Big(\sum_{i=1}^{k+1} u_i \Big)^2$. That is, $(u_1, \dots, u_{k+1})$ is a solution of the system

$$\begin{gathered}
-\Big(a + b\sum_{j=1}^{k+1} \int_{B_j} |\nabla u_j|^2\, dx \Big)\Delta u_i + V(|x|)u_i + \varphi u_i = |u_i|^{p-2}u_i, \text{ in } B_i,\\
-\Delta\varphi = \Big(\sum_{i=1}^{k+1} u_i \Big)^2, \quad \lim_{|x|\to\infty} \varphi(x) = 0.
\end{gathered} \tag{2.2}$$

Now, we define the Nehari set

$$\mathcal{N}_k = \{(u_1, \dots, u_{k+1}) \in \mathcal{H}_k : u_i \neq 0, \partial_{u_i} I(u_1, \dots, u_{k+1})u_i = 0 \text{ for } i = 1, \dots, k+1\}$$

where

$$
\begin{aligned}
&\partial_{u_i} I(u_1,\dots,u_{k+1})u_i \\
&= \|u_i\|_i^2 + b\Big(\int_{B_i} |\nabla u_i|^2\,dx\Big)^2 + b\int_{B_i} |\nabla u_i|^2\,dx \sum_{i\neq j}^{k+1} \int_{B_i} |\nabla u_j|^2\,dx \\
&\quad + \int_{B_i}\int_{B_i} \frac{u_i^2(x)u_i^2(y)}{4\pi|x-y|}\,dx\,dy + \sum_{i\neq j}^{k+1} \int_{B_i}\int_{B_j} \frac{u_i^2(x)u_j^2(y)}{4\pi|x-y|}\,dx\,dy - \int_{B_i} |u_i|^p\,dx.
\end{aligned}
$$

Next, we show that the $\mathcal{N}_k$ is nonempty manifold in $\mathcal{H}_k$, then we seek a minimizer of the functional I constraint on $\mathcal{N}_k$. Apparently, the minimizer is a weak solution of (2.2). Finally, one needs to prove the minimizer has nonzero component. We commence with a proof of manifold for $\mathcal{N}_k$.

Lemma 2.1. *Suppose that* (A1) *holds and* $p \in (4,6)$*. For* $(u_1,\dots,u_{k+1}) \in \mathcal{H}_k$ *with* $u_i \neq 0$ *for* $i = 1,\dots,k+1$*, there exists a unique* $(k+1)$*-tuple* $(a_1,\dots,a_{k+1})$ *with positive components such that* $(a_1u_1,\dots,a_{k+1}u_{k+1}) \in \mathcal{N}_k$*.*

Proof. For a fixed $(u_1,\dots,u_{k+1}) \in \mathcal{H}_k$ with $u_i \neq 0, (a_1u_1,\dots,a_{k+1}u_{k+1})$ is contained in $\mathcal{N}_k$ if and only if

$$
\begin{aligned}
&\|u_i\|_i^2 + ba_i^2\Big(\int_{B_i} |\nabla u_i|^2\,dx\Big)^2 + b\int_{B_i} |\nabla u_i|^2\,dx \sum_{i\neq j}^{k+1} a_j^2 \int_{B_i} \nabla u_j|^2\,dx \\
&+ a_i^2 \int_{B_i}\int_{B_i} \frac{u_i^2(x)u_i^2(y)}{4\pi|x-y|}\,dx\,dy + \sum_{i\neq j}^{k+1} a_j^2 \int_{B_i}\int_{B_j} \frac{u_i^2(x)u_j^2(y)}{4\pi|x-y|}\,dx\,dy \\
&- a_i^{p-2} \int_{B_i} |u_i|^p\,dx = 0
\end{aligned}
\tag{2.3}
$$

for $i = 1,\dots,k+1$. Hence, the problem is reduced to verify that there is only one solution $(a_1,\dots,a_{k+1})$ of (2.3) such that $a_i > 0$, $i = 1,\dots,k+1$.

Fix a parameter $0 \le \alpha \le 1$, we consider the solvability of the system of $(k+1)$ equations

$$
\begin{aligned}
&\|u_i\|^2 + ba_i^2\Big(\int_{B_i} |\nabla u_i|^2\,dx\Big)^2 + \alpha b\int_{B_i} |\nabla u_i|^2\,dx \sum_{i\neq j}^{k+1} a_j^2 \int_{B_i} \nabla u_j|^2\,dx \\
&+ a_i^2 \int_{B_i}\int_{B_i} \frac{u_i^2(x)u_i^2(y)}{4\pi|x-y|}\,dx\,dy + \alpha\sum_{i\neq j}^{k+1} a_j^2 \int_{B_i}\int_{B_j} \frac{u_i^2(x)u_j^2(y)}{4\pi|x-y|}\,dx\,dy \\
&- a_i^{p-2} \int_{B_i} |u_i|^p\,dx = 0,
\end{aligned}
\tag{2.4}
$$

$i = 1,\dots,k+1$. Define

$$
D = \Big\{\alpha : 0 \le \alpha \le 1 \text{ and (2.4) is uniquely solvable in } (\mathbb{R}_{>0})^{k+1}\Big\}. \tag{2.5}
$$

We claim that $D = [0,\alpha]$. This will be done by showing that D is not empty, and D is both open and closed in $[0,1]$.

Firstly, we show that D contains 0. Let

$$
f_i(t) = \|u_i\|_i^2 + bt^2\Big(\int_{B_i} |\nabla u_i|^2\,dx\Big)^2 + t^2\int_{B_i}\int_{B_i} \frac{u_i^2(x)u_i^2(y)}{4\pi|x-y|}\,dx\,dy - t^{p-2}\int_{B_i} |u_i|^p\,dx,
$$

for $i = 1, \dots, k+1$. Without loss of generality, we need only to prove that there is a unique $t_0 > 0$ such that $f_1(t_0) = 0$.

Since $u_1 \neq 0$, we have $f_1(t) > 0$ if $t > 0$ small, and $f_1(t) < 0$ if $t > 0$ large. Therefore, there exists $t_0 > 0$ such that $f_1(t_0) = 0$.

Now we show t_0 is unique. Indeed, were it not the case, we would have $0 < t_0 < \bar{t}$ such that $f_1(t_0) = f_1(\bar{t}) = 0$. That is, the equality

$$\frac{1}{t}\|u_1\|_1^2 + b\Big(\int_{B_1} |\nabla u_1|^2\,dx\Big)^2 + \int_{B_1}\int_{B_1} \frac{u_1^2(x)u_1^2(y)}{4\pi|x-y|}\,dx\,dy - t^{p-4}\int_{B_1} |u_1|^p\,dx = 0,$$

holds for t_0 and $\bar{t}$. It yields

$$0 < (\frac{1}{t_0} - \frac{1}{\bar{t}})\|u_1\|_1^2 = (t_0^{p-4} - \bar{t}^{p-4})\int_{B_1} |u_1|^p\,dx < 0,$$

which is contradiction. Hence, $0 \in D$.

Next, we prove that D is open in $[0,1]$. Suppose that $\alpha_0 \in D$ and $(\bar{a}_1, \dots, \bar{a}_{k+1}) \in (\mathbb{R} > 0)^{k+1}$ is the unique solution of (2.4) with $\alpha = \alpha_0$. Therefore,

$$\begin{aligned}
&\|u_i\|_i^2 + bc_i\Big(\int_{B_i} |\nabla u_i|^2\,dx\Big)^2 + \alpha_0 b\int_{B_i} |\nabla u_i|^2\,dx \sum_{i\neq j}^{k+1} c_j \int_{B_i} \nabla u_j|^2\,dx \\
&+ c_i \int_{B_i}\int_{B_i} \frac{u_i^2(x)u_i^2(y)}{4\pi|x-y|}\,dx\,dy + \alpha_0 \sum_{i\neq j}^{k+1} c_j \int_{B_i}\int_{B_j} \frac{u_i^2(x)u_j^2(y)}{4\pi|x-y|}\,dx\,dy \\
&- c_i^q \int_{B_i} |u_i|^p\,dx = 0
\end{aligned} \tag{2.6}$$

for each $i = 1, \dots, k+1$, where $c_i = \bar{a}_i^2, q = \frac{p-2}{2} > 1$. To apply the implicit function theorem at α_0, we calculate the matrix

$$M = (M_{ij}) = (\partial c_j G_i)_{i,j=1,\dots,k+1},$$

where $G_i = G_i(c_1, \dots, c_{k+1}, \alpha_0)$ denotes the left-hand side of (2.6). Hence, each component of the matrix M can be represented by

$$\begin{gathered}
M_{ii} = b\Big(\int_{B_i} |\nabla u_i|^2\,dx\Big)^2 + \int_{B_i}\int_{B_i} \frac{u_i^2(x)u_i^2(y)}{4\pi|x-y|}\,dx\,dy - qc_i^{q-1}\int_{B_i} |u_i|^p\,dx, \\
M_{ij} = \alpha_0 b\int_{B_i} |\nabla u_i|^2\,dx \int_{B_i} |\nabla u_j|^2\,dx + \alpha_0 \int_{B_i}\int_{B_j} \frac{u_i^2(x)u_j^2(y)}{4\pi|x-y|}\,dx\,dy.
\end{gathered}$$

By (2.6),

$$\begin{aligned}
\hat{M}_{ii} &:= -c_i M_{ii} \\
&= q\|u_i\|_i^2 + (q-1)c_i\Big[b\Big(\int_{B_i} |\nabla u_i|^2\,dx\Big)^2 + \int_{B_i}\int_{B_i} \frac{u_i^2(x)u_i^2(y)}{4\pi|x-y|}\,dx\,dy\Big] \\
&\quad + q\alpha_0 b\int_{B_i} |\nabla u_i|^2 dx \sum_{i\neq j}^{k+1} \int_{B_j} |\nabla u_j|^2 dx \\
&\quad + q\alpha_0 \sum_{i\neq j}^{k+1} c_j \int_{B_i}\int_{B_j} \frac{u_i^2(x)u_j^2(y)}{4\pi|x-y|} dx\,dy
\end{aligned}$$

and

$$\begin{aligned}\hat{M}_{ij} := -c_j M_{ij} = -\alpha_0 b c_j \int_{B_i} |\nabla u_i|^2 dx \int_{B_j} |\nabla u_j|^2 dx \\ - \alpha_0 c_j \int_{B_i}\int_{B_j} \frac{u_i^2(x)u_j^2(y)}{4\pi|x-y|}\,dx\,dy.\end{aligned}$$

It readily verifies that

$$\hat{M}_{ij} \le 0, \quad \det(\hat{M}_{ij}) = \frac{(-1)^{k+1}}{c_1 \dots c_{k+1}} \det M$$

and

$$\begin{aligned}\hat{M}_{ii} + \sum_{i\neq j}^{k+1} \hat{M}_{ij} &= q\|u_i\|_i^2 + (q-1)bc_i\Big(\int_{B_i} |\nabla u_i|^2\,dx\Big)^2 \\ &\quad + (q-1)c_i \int_{B_i}\int_{B_i} \frac{u_i^2(x)u_i^2(y)}{4\pi|x-y|}\,dx\,dy \\ &\quad + (q-1)\alpha_0 b \int_{B_i} |\nabla u_i|^2\,dx \sum_{i\neq j}^{k+1} \int_{B_i} \nabla u_j|^2\,dx \\ &\quad + (q-1)\alpha_0 \sum_{i\neq j}^{k+1} c_j \int_{B_i}\int_{B_j} \frac{u_i^2(x)u_j^2(y)}{4\pi|x-y|}\,dx\,dy.\end{aligned}$$

Since $q > 1$, we obtain

$$\hat{M}_{ii} + \sum_{i\neq j}^{k+1} \hat{M}_{ij} > 0 \quad \text{and} \quad \det \hat{M}_{ii} \neq 0$$

which implies

$$0 \neq \det(\hat{M}_{ij}) = \frac{(-1)^{k+1}}{c_1 \dots c_{k+1}} \det M.$$

The implicit function theorem yields that there exist a neighborhood U_0 of α_0 and a neighborhood $B_0 \subset (\mathbb{R} > 0)^{k+1}$ of $(\bar{a}_1, \dots, \bar{a}_{k+1})$ such that the equation of (2.4) is uniquely solvable in $U_0 \times B_0$.

Now we show that (2.4) is uniquely solvable in $U_0 \times (\mathbb{R}_{>0})^{k+1}$, this means $U_0 \subset D$. Suppose, on the contrary, that there is $\alpha_0 \in U_0$ such that there exists the second solution $(\hat{a}_1, \dots, \hat{a}_{k+1}) \in (\mathbb{R}_{>0})^{k+1} \setminus B_0$ of (2.4). By the implicit function theorem, we can find a solution curve $(\alpha, (\hat{a}_1, \dots, \hat{a}_{k+1}))$ in $(\alpha_0 - \varepsilon, \alpha + \varepsilon) \times (\mathbb{R}_{>0})^{k+1} \setminus B_0$. Assume $\alpha_0 < \alpha_1$ for a while and extend this curve as much as possible. Since it cannot be defined at α_0 and enter into $U_0 \times B_0$, there should have a point $\alpha_2 \in [\alpha_0, \alpha_1)$ such that $(a_1(\alpha), \dots, a_{k+1}(\alpha))$ exists in $\alpha \in (\alpha_2, \alpha_1]$ and blows up as $\alpha \to \alpha_2^+$. However, this is impossible, since if $(a_1, \dots, a_{k+1})$ has sufficiently large norm, the left-hand side of (2.4) is strictly negative for at least one i. This gives a contradiction. Thus, $U_0 \subset D$. The case $\alpha_0 > \alpha_1$ can be proved in the same way.

Next, we prove that D is closed in $[0, 1]$. Let $\{\alpha_n\}$ be a sequence in D converging to $\alpha_0 \in [0, 1]$ and $(a_1^n, \dots, a_{k+1}^n) \in (\mathbb{R}_{>0})^{k+1}$ be the solution of (2.4) corresponding to α_n. By the preceding argument, the sequence $(a_1^n, \dots, a_{k+1}^n) \in (\mathbb{R}_{>0})^{k+1}$ is bounded above. Thus we may assume that $(a_1^n,\dots,a_{k+1}^n)$ converges to a solution $(a_1^0,\dots,a_{k+1}^0)\in(\mathbb{R}>0)^{k+1}$ of (2.4) for α_0. Since $H_i\hookrightarrow L^p$, we obtain

$$(a_i^n)^2\|u_i\|_i^2\le\int_{B_i}|a_1^nu_i|^pdx\le C(a_i^n)^p\|u_i\|_i^p,$$

which implies that $0<C_i\le a_i^n$ uniformly in n. Thus $a_i^0\ge C_i>0$ for $i\in\mathbb{N}$. So $(a_1^0,\dots,a_{k+1}^0)\in(\mathbb{R}_{>0})^{k+1}$. By the implicit function theorem again, $(a_1^0,\dots,a_{k+1}^0)$ is the unique solution in $(\mathbb{R}_{>0})^{k+1}$. Hence, D is closed. The conclusion of Lemma 2.1 then follows. □

Now, we show that $\mathcal{N}_k$ is a differentiable manifold.

Lemma 2.2. *$\mathcal{N}_k$ is a differentiable manifold in $\mathcal{H}_k$. Moreover, all critical points of the restriction $I|_{\mathcal{N}_k}$ of I to $\mathcal{N}_k$ are critical point of I with no zero component.*

Proof. We observe that

$$\mathcal{N}_k=\{(u_1,\dots,u_{k+1})\in\mathcal{H}_k:u_i\ne0,\ \mathbf{F}(u_1,\dots,u_{k+1})=\mathbf{0}\},$$

where $\mathbf{F}=(F_1,\dots,F_{k+1}):\mathcal{H}_k\to\mathbb{R}^{k+1}$ is given by

$$\begin{aligned}&F_i(u_1,\dots,u_{k+1})\\&=\|u_i\|_i^2+b\Big(\int_{B_i}|\nabla u_i|^2\,dx\Big)^2+b\int_{B_i}|\nabla u_i|^2\,dx\sum_{i\ne j}^{k+1}\int_{B_i}|\nabla u_j|^2\,dx\\&+\int_{B_i}\int_{B_i}\frac{u_i^2(x)u_i^2(y)}{4\pi|x-y|}\,dx\,dy+\sum_{i\ne j}^{k+1}\int_{B_i}\int_{B_j}\frac{u_i^2(x)u_j^2(y)}{4\pi|x-y|}\,dx\,dy-\int_{B_i}|u_i|^p\,dx,\end{aligned}\tag{2.7}$$

$i=1,\dots,k+1$. To prove that $\mathcal{N}_k$ is a differentiable manifold in $\mathcal{H}_k$, it suffices to verify that the matrix

$$N=(N_{ij})=(\partial_{u_i}F_j(u_1,\dots,u_{k+1}),u_i)_{i,j=1,\dots,k+1},$$

is nonsingular at each point $(u_1,\dots,u_{k+1})\in\mathcal{N}_k$, since it implies that 0 is a regular value of $\mathbf{F}$. Straightforwardly, we have

$$\begin{aligned}N_{ii}&=2\|u_i\|_i^2+4b\Big(\int_{B_i}|\nabla u_i|^2\,dx\Big)^2+2b\int_{B_i}|\nabla u_i|^2\,dx\sum_{i\ne j}^{k+1}\int_{B_i}|\nabla u_j|^2\,dx\\&\quad+4\int_{B_i}\int_{B_i}\frac{u_i^2(x)u_i^2(y)}{4\pi|x-y|}\,dx\,dy+2\sum_{i\ne j}^{k+1}\int_{B_i}\int_{B_j}\frac{u_i^2(x)u_j^2(y)}{4\pi|x-y|}\,dx\,dy\\&\quad-p\int_{B_i}|u_i|^p\,dx\end{aligned}$$

and

$$N_{ij}=2b\int_{B_i}|\nabla u_i|^2dx\int_{B_j}|\nabla u_j|^2\,dx+2\int_{B_i}\int_{B_j}\frac{u_i^2(x)u_j^2(y)}{4\pi|x-y|}\,dx\,dy.$$

By (2.4), we have

$$\begin{aligned}N_{ii}&=(2-p)\|u_i\|_i^2+(4b-bp)\Big(\int_{B_i}|\nabla u_i|^2\,dx\Big)^2\\&\quad+(2b-bp)\int_{B_i}|\nabla u_i|^2\,dx\sum_{i\ne j}^{k+1}\int_{B_i}|\nabla u_j|^2\,dx\end{aligned}$$

$$+ (4-p)\int_{B_i}\int_{B_i}\frac{u_i^2(x)u_i^2(y)}{4\pi|x-y|}\,dx\,dy + (2-p)\sum_{i\neq j}^{k+1}\int_{B_i}\int_{B_j}\frac{u_i^2(x)u_j^2(y)}{4\pi|x-y|}\,dx\,dy.$$

Let $\hat{N}_{ii} = -N_{ii}$ and $\hat{N}_{ij} = -N_{ij}$. Then

$$\begin{aligned}\hat{N}_{ii} + \sum_{i\neq j}^{k+1}\hat{N}_{ij} &= (p-2)\|u_i\|_i^2 + b(p-4)\Big(\int_{B_i}|\nabla u_i|^2\,dx\Big)^2\\ &\quad + b(p-4)\int_{B_i}|\nabla u_i|^2\,dx\sum_{i\neq j}^{k+1}\int_{B_i}|\nabla u_j|^2\,dx\\ &\quad + (p-4)\int_{B_i}\int_{B_i}\frac{u_i^2(x)u_i^2(y)}{4\pi|x-y|}\,dx\,dy\\ &\quad + (p-4)\sum_{i\neq j}^{k+1}\int_{B_i}\int_{B_j}\frac{u_i^2(x)u_j^2(y)}{4\pi|x-y|}\,dx\,dy > 0\end{aligned}$$

provided $4 < p < 6$. Hence,

$$\det(\hat{N}) \neq 0, \quad \det N = (-1)^{k+1}\det(\hat{N}) \neq 0,$$

and the matrix N is invertible at each $(u_1,\dots,u_{k+1}) \in \mathcal{N}_k$. So $\mathcal{H}_k$ is a differential manifold.

If $(u_1,\dots,u_{k+1})$ is a critical point of $I|_{\mathcal{N}_k}$, then there are Lagrange multipliers $\mu_1,\dots,\mu_{k+1}$ satisfying

$$\mu_1 F_1'(u_1,\dots,u_{k+1}) + \dots + \mu_{k+1}F_{k+1}'(u_1,\dots,u_{k+1}) = I'(u_1,\dots,u_{k+1}). \tag{2.8}$$

Inserting $(u_1,0,\dots,0)$, $(0,u_2,\dots,0)$, $\dots$, $(0,0,\dots,0,u_{k+1})$ into (2.8), we obtain

$$N\begin{pmatrix}\mu_1\\ \vdots\\ \mu_{k+1}\end{pmatrix} = \begin{pmatrix}0\\ \vdots\\ 0\end{pmatrix}.$$

Because N is nonsingular, we find that $\mu_1 = \mu_2 = \dots = \mu_{k+1} = 0$ and $(u_1,\dots,u_{k+1})$ is a critical point of I.

Next, by the Sobolev embedding $H_i \hookrightarrow L^p$, the inequality

$$\|u_i\|_i^2 \le \int_{B_i}|u_i|^2 \le C\|u_i\|_i^p \tag{2.9}$$

implies that $u_i \neq 0$ for all i. Thus, all components of critical points of I in $\mathcal{N}_k$ are nontrivial. This completes the proof. □

Lemma 2.3. *For fixed $(u_1,\dots,u_{k+1}) \in \mathcal{H}_k$ with nonzero component, the vector $(a_1,\dots,a_{k+1})$ which is obtained in Lemma 2.1 is the unique maximum point of the function $\eta : (\mathbb{R}_{>0})^{k+1} \to \mathbb{R}$ defined as $\eta(d_1,\dots,d_{k+1}) = I((d_1u_1,\dots,d_{k+1}u_{k+1}))$.*

Proof. By the proof of Lemma 2.1, $(a_1,\dots,a_{k+1})$ is the unique critical point of ψ in $(\mathbb{R}_{>0})^{k+1}$. If $|(d_1,\dots,d_{k+1})| \to \infty$, it is readily to verify that $\psi(d_1,\dots,d_{k+1}) \to -\infty$, so it is sufficient to show that $(a_1,\dots,a_{k+1})$ is not on the boundary of $(\mathbb{R}_{>0})^{k+1}$.

Choose $(d_1^0,\dots,d_{k+1}^0) \in \partial(\mathbb{R}_{>0})^{k+1}$, without loss of generality, we may assume that $d_1^0 = 0$. Since

$$\eta(t, d_2^0\dots,d_{k+1}^0)$$

$$
\begin{aligned}
&= I((tu_1, d_2^0 u_2 \dots, d_{k+1}^0 u_{k+1})) \\
&= \frac{t^2}{2}\|u_1\|_1^2 + \frac{bt^4}{4}\Big(\int_{B_1} |\nabla u_1|^2\,dx\Big)^2 + \frac{bt^2}{2}\sum_{j=2}^{k+1} {d_j^0}^2 \int_{B_1} |\nabla u_i|^2\,dx \int_{B_j} |\nabla u_j|^2\,dx \\
&\quad + \frac{t^4}{4}\int_{B_1}\int_{B_1} \frac{u_1^2(x)u_1^2(y)}{4\pi|x-y|}\,dx\,dy + \frac{t^2}{2}\sum_{j=2}^{k+1} {d_j^0}^2 \int_{B_1}\int_{B_j} \frac{u_1^2(x)u_j^2(y)}{4\pi|x-y|}\,dx\,dy \\
&\quad - \frac{t^p}{p}\int_{B_1} |u_1|^p\,dx + \sum_{i=2}^{k+1} {d_i^0}^2 \|u_i\|^2 + \frac{b}{4}\sum_{i,j=2}^{k+1} (d_i^0 d_j^0)^2 \int_{B_1} |\nabla u_i|^2\,dx \int_{B_j} |\nabla u_j|^2\,dx \\
&\quad + \frac{1}{4}\sum_{i=1}^{k+1} (d_i^0 d_j^0)^2 \int_{B_i}\int_{B_j} \frac{u_i^2(x)u_j^2(y)}{4\pi|x-y|}\,dx\,dy - \frac{1}{p}\sum_{i=2}^{k+1} {d_i^0}^p \int_{B_i} |u_i|^p\,dx
\end{aligned}
$$

is increasing if $t > 0$ is small enough, $(0, \dots, d_{k+1}^0)$ is not a maximum point of η in ${\mathbb{R}_{>0}}^{k+1}$. The proof is complete. $\square$

Next, we have existence results.

Lemma 2.4. *For fixed $\mathbf{r}_k = (r_1, \dots, r_{k+1}) \in \mathbf{\Gamma}_k$, there is a minimizer $(v_1, \dots, v_{k+1})$ of its corresponding energy $I|_{\mathcal{N}_k}$ on $\mathcal{N}_k$ such that $(-1)^{i+1} v_i$ is positive on B_i for $i = 1, \dots, k+1$. Moreover, $(v_1, \dots, v_{k+1})$ satisfies (2.2).*

Proof. For $(u_1, \dots, u_{k+1}) \in \mathcal{N}_k$, we have

$$
\begin{aligned}
I(u_1, \dots, u_{k+1}) &= \Big(\frac{1}{2} - \frac{1}{p}\Big)\sum_{i=1}^{k+1} \|u_i\|_i^2 + (\frac{b}{4} - \frac{b}{p})\sum_{i=1}^{k+1}\Big(\int_{B_i} |\nabla u_i|^2\,dx\Big)^2 \\
&\quad + \Big(\frac{b}{4} - \frac{b}{p}\Big)\sum_{i=1}^{k+1}\int_{B_i} |\nabla u_i|^2\,dx \int_{B_j} |\nabla u_j|^2\,dx \\
&\quad + \Big(\frac{1}{4} - \frac{1}{p}\Big)\sum_{i=1}^{k+1}\int_{B_i}\int_{B_i} \frac{u_i^2(x)u_j^2(y)}{4\pi|x-y|}\,dx\,dy \qquad (2.10) \\
&\quad + \Big(\frac{1}{4} - \frac{1}{p}\Big)\sum_{i\neq j}^{k+1}\int_{B_i}\int_{B_j} \frac{u_i^2(x)u_j^2(y)}{4\pi|x-y|}\,dx\,dy \\
&\geq \frac{1}{4}\sum_{i=1}^{k+1} \|u_i\|_i^2.
\end{aligned}
$$

Let $\{(u_1^n, \dots, u_{k+1}^n)\} \subset \mathcal{N}_k$ be a minimizing sequence of $I|_{\mathcal{N}_k}$; that is,

$$
\lim_{n\to\infty} I(u_1^n, \dots, u_{k+1}^n) = \inf_{(u_1,\dots,u_{k+1})\in\mathcal{N}_k} I(u_1, \dots, u_{k+1}).
$$

By (2.10), $\{u_i^n\}$ is bounded. Hence, we may assume that

$$
(u_1^n, \dots, u_{k+1}^n) \rightharpoonup (u_1^0, \dots, u_{k+1}^0) \quad \text{weakly in } \mathcal{H}_k.
$$

We claim that $u_i^0 \neq 0$ in $\mathcal{H}_i$ for $i = 1, \dots, k+1$. Indeed, if $u_i^n \to u_i^0$ in $\mathcal{H}_i$, we may show as (2.9) that

$$
\|u_i^n\|_i^2 \leq \int_{B_i} |u_i^n|^p\,dx \leq C\|u_i^n\|_i^p,
$$

which implies $\|u_i^n\|_i \geq C_i > 0$, and the strongly convergence yields $u_i^0 \neq 0$.

If $u_i^n \rightharpoonup u_i^0$ in $\mathcal{H}_i$, but $u_i^n \not\to u_i^0$ in $\mathcal{H}_i$, we have

$$\|u_i^0\|_i < \lim_{n\to\infty} \inf \|u_i^n\|_i. \tag{2.11}$$

Therefore,

$$\begin{aligned}
\|u_i^0\|_i^2 &\le \|u_i^0\|_i^2 + b\Big(\int_{B_i} |\nabla u_i^0|^2\,dx\Big)^2 + b\int_{B_i} |\nabla u_i^0|^2\,dx \sum_{i\neq j}^{k+1}\int_{B_j} |\nabla u_j^0|^2\,dx \\
&\quad + \int_{B_i}\int_{B_i} \frac{(u_i^0)^2(x)(u_i^0)^2(y)}{4\pi|x-y|}\,dx\,dy + \sum_{i\neq j}^{k+1}\int_{B_i}\int_{B_j} \frac{(u_i^0)^2(x)(u_j^0)^2(y)}{4\pi|x-y|}\,dx\,dy \\
&< \int_{B_i} |u_i^0|^p\,dx.
\end{aligned}$$

This and the Sobolev embedding $\mathcal{H}_r^1(\mathbb{R}^3) \hookrightarrow L^p(\mathbb{R}^3)$ yield

$$\|u_i^0\|_i^2 < \int_{B_i} |u_i^0|^p\,dx \le C\|u_i^0\|_i^p.$$

Thus, $u_i^0 \neq 0$ for each i.

Next we prove that $(u_1^n,\dots,u_{k+1}^n) \to (u_1^0,\dots,u_{k+1}^0)$ in $\mathcal{H}_k$ as $n\to\infty$. Suppose on the contrary that $(u_1^n,\dots,u_{k+1}^n)$ converges weakly to $(u_1^0,\dots,u_{k+1}^0)$ but does not converge strongly in $\mathcal{H}_k$ as $n\to\infty$. Thus, there exists at least one i such that

$$\|u_i^0\|_i < \lim_{n\to\infty} \inf \|u_i^n\|_i$$

and $\|u_i^0\| \neq 0$. By Lemma 2.1, there exists a unique $(a_1^0,\dots,a_{k+1}^0) \neq (1,\dots,1) \in (\mathbb{R}_{>0})^{k+1}$ such that $(a_1^0u_1^0,\dots,a_{k+1}^0u_{k+1}^0) \in \mathcal{N}_k$.

If $(a_1,\dots,a_{k+1}) = (1,\dots,1)$, we have

$$\begin{aligned}
\inf I(u_1,\dots,u_{k+1}) &= \lim_{n\to\infty} \inf I(u_1^n,\dots,u_{k+1}^n) \\
&\ge I(u_1^0,\dots,u_{k+1}^0) \\
&\ge \inf I(u_1,\dots,u_{k+1}),
\end{aligned}$$

which implies

$$I(u_1^0,\dots,u_{k+1}^0) = \inf I(u_1,\dots,u_{k+1}).$$

On the other hand, Lemma 2.3 leads to

$$\begin{aligned}
&\inf_{(u_1,\dots,u_{k+1})\in\mathcal{N}_k} I(u_1,\dots,u_{k+1}) \\
&\le I(a_1^0u_1^0,\dots,a_{k+1}^0u_{k+1}^0) \\
&< \frac12\sum_{i=1}^{k+1}(a_i^0)^2 \liminf_{n\to\infty}\|u_i^n\|_i^2 + \frac{b}{4}\sum_{i=1}^{k+1}(a_i^0)^4 \liminf_{n\to\infty}\Big(\int_{B_i}|\nabla u_i^n|^2\,dx\Big)^2 \\
&\quad + \frac{b}{4}\sum_{i\neq j}^{k+1}(a_i^0)^2(a_j^0)^2 \liminf_{n\to\infty}\int_{B_i}|\nabla u_i^n|^2dx\int_{B_j}|\nabla u_j^n|^2\,dx \\
&\quad + \frac14\sum_{i=1}^{k+1}(a_i^0)^4 \liminf_{n\to\infty}\int_{B_i}\int_{B_i}\frac{(u_i^n)^2(x)(u_i^n)^2(y)}{4\pi|x-y|}\,dx\,dy
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{4}\sum_{i\neq j}^{k+1}(a_i^0)^2(a_j^0)^2\liminf_{n\to\infty}\int_{B_i}\int_{B_j}\frac{(u_i^n)^2(x)(u_j^n)^2(y)}{4\pi|x-y|}\,dx\,dy\\
&-\frac{1}{p}\sum_{i=1}^{k+1}(a_i^0)^p\liminf_{n\to\infty}\int_{B_i}|u_i^n|^p\,dx\\
&<\liminf_{n\to\infty}I(u_1^n,\dots,u_{k+1}^n)=\inf_{(u_1,\dots,u_{k+1})\in\mathcal{N}_k}I(u_1,\dots,u_{k+1}).
\end{aligned}
$$

This is a contradiction. Therefore, $(u_1^n,\dots,u_{k+1}^n)\to(u_1^0,\dots,u_{k+1}^0)$ in $\mathcal{H}_k$, and $(u_1^0,\dots,u_{k+1}^0)$ is a minimizer of $I|_{\mathcal{N}_k}$.

Let $(v_1,\dots,v_{k+1})=(|u_1^0|,-|u_2^0|,\dots,(-1)^{k+1}|u_{k+1}^0|)$, we can check that

$$
(v_1,\dots,v_{k+1})\in\mathcal{N}_k \quad\text{and}\quad I(v_1,\dots,v_{k+1})=I(u_1^0,\dots,u_{k+1}^0).
$$

Hence $(v_1,\dots,v_{k+1})$ is a minimizer of $I|_{\mathcal{N}_k}$. By the Lemma 2.1, $(v_1,\dots,v_{k+1})$ is a critical point of $I|_{\mathcal{N}_k}$, and it satisfies (2.2). By the strong maximum principle, each $(-1)^{i+1}|v_i^0|$ is positive in B_i, $i=1,\dots,k+1$. Hence, $(v_1,\dots,v_{k+1})$ is the solution we want. □

3. Existence of sign-changing radial solutions

It is known that for any $\mathbf{r}_k=(r_1,\dots,r_k)\in\mathbf{\Gamma}_k$, there is a solution $v^{\mathbf{r}_k}=(v_1^{\mathbf{r}_k},\dots,v_{k+1}^{\mathbf{r}_k})$ of (2.2) which consists of sign changing components. We shall find a $\tilde{\mathbf{r}}_k=(\tilde r_1,\dots,\tilde r_k)\in\mathbf{\Gamma}_k$ such that $v^{\tilde{\mathbf{r}}_k}=(v_1^{\tilde{\mathbf{r}}_k},\dots,v_{k+1}^{\tilde{\mathbf{r}}_k})$ is a solution of (2.2) which is characterized as a least energy solution among all elements in $\mathbf{\Gamma}_k$ with nonzero components. Using this solution as a building block, we will construct a radial solution of (1.1) that changes sign exactly k times. Denote by $B_i^{\tilde{\mathbf{r}}_k}$ the nodal domain and by $I^{\tilde{\mathbf{r}}_k}$ the functional related to $\tilde{\mathbf{r}}_k$. Note that $v_i^{\tilde{\mathbf{r}}_k}$ is $\mathcal{C}^2(B_i^{\tilde{\mathbf{r}}_k})$ by standard elliptic regularity results. Hence it is enough to match the first derivative with respect to the radial variable, of adjacent components $v_i^{\tilde{\mathbf{r}}_k}$ and $v_{i+1}^{\tilde{\mathbf{r}}_k}$ at the point r_i to ensure the existence of a solution of (1.1) with k times sign changing.

To find a least energy radial solution of (2.2) among elements in $\mathbf{\Gamma}_k$ with nonzero components, we need to estimate the energy of the solution $(v_1^{\mathbf{r}_k},\dots,v_{k+1}^{\mathbf{r}_k})$ of (2.2). To this end, we first define the function $\chi:\mathbf{\Gamma}_k\to\mathbb{R}$ by

$$
\chi(\mathbf{r}_k)=\chi(r_1,\dots,r_k)=I^{\mathbf{r}_k}(v_1^{\mathbf{r}_k},\dots,v_{k+1}^{\mathbf{r}_k})=\inf_{(u_1^{\mathbf{r}_k},\dots,u_{k+1}^{\mathbf{r}_k})\in\mathcal{N}_k}I(u_1^{\mathbf{r}_k},\dots,u_{k+1}^{\mathbf{r}_k}).
\tag{3.1}
$$

Lemma 3.1. *For any positive integer k, let $\mathbf{r}_k=(r_1,\dots,r_k)\in\mathbf{\Gamma}_k$. Then*

(i) *if $r_i-r_{i-1}\to0$ for some $i\in\{1,\dots,k\}$, then $\chi(\mathbf{r}_k)\to+\infty$.*
(ii) *if $r_k\to\infty$, then $\chi(\mathbf{r}_k)\to+\infty$.*
(iii) *χ is continuous in $\mathbf{\Gamma}_k$.*

In particular, there is a $\tilde{\mathbf{r}}_k=(\tilde{\mathbf{r}}_1,\dots,\tilde{\mathbf{r}}_k)\in\mathbf{\Gamma}_k$ such that

$$
\chi(\tilde{\mathbf{r}}_k)=\inf_{\mathbf{r}_k\in\mathbf{\Gamma}_k}\chi(\mathbf{r}_k)
$$

Proof. (i) Suppose $r_{i_0}-r_{i_0-1}\to0$ for some $i_0\in\{1,\dots,k\}$, by the Hölder inequality and Sobolev inequality, we obtain

$$
\|v_{i_0}^{\mathbf{r}_k}\|_{i_0}^2\le\int_{B_{i_0}^{\mathbf{r}_k}}|v_{i_0}^{\mathbf{r}_k}|^p\,dx\le\Big(\int_{B_{i_0}^{\mathbf{r}_k}}|v_{i_0}^{\mathbf{r}_k}|^6\,dx\Big)^{p/6}|B_{i_0}^{\mathbf{r}_k}|^{1-\frac{p}{6}}\le C\|v_{i_0}^{\mathbf{r}_k}\|_{i_0}^p|B_{i_0}^{\mathbf{r}_k}|^{1-\frac{p}{6}};
$$

that is,

$$|B_{i_0}^{\mathbf{r}_k}|^{\frac{p}{6}-1} \le C\|v_{i_0}^{\mathbf{r}_k}\|_{i_0}^{p-2}.$$

Since $4 < p < 6$, $\|v_{i_0}^{\mathbf{r}_k}\|_{i_0} \to \infty$. Inequality (2.10) implies

$$I(v_1^{\mathbf{r}_k}, \dots, v_{k+1}^{\mathbf{r}_k}) \ge \big(\frac{1}{2} - \frac{1}{p}\big) \sum_{i=1}^{k+1} \|v_i^{\mathbf{r}_k}\|_i^2 \to \infty,$$

and then

$$\chi(\mathbf{r}_k) = \chi(r_1, \dots, r_k) = I_b^{\mathbf{r}_k}(v_1^{\rho_{\mathbf{k}}}, \dots, v^{\mathbf{r}_k}) \to \infty.$$

Thus the first item holds.

(ii) By the Strauss inequality [6], that is, for $u \in H_r^1(\mathbb{R}^3)$, there exists $C > 0$, such that

$$|u(x)| \le c\frac{\|u\|}{|x|}, \quad \text{a.e. in } \mathbb{R}^3,$$

we obtain

$$\|v_{k+1}^{\mathbf{r}_k}\|_{k+1}^2 \le \int_{B_{k+1}^{\mathbf{r}_k}} |v_{k+1}^{\mathbf{r}_k}|^p\,dx \le C\int_{B_{k+1}^{r_k}} \frac{\|v_{k+1}^{\mathbf{r}_k}\|_{k+1}^p}{|x|^p} \le C\|v_{k+1}^{\mathbf{r}_k}\|_{k+1}^p r_k^{3-p}.$$

Therefore,

$$r_k^{p-3} \le C\|v_{k+1}^{\mathbf{r}_k}\|_{k+1}^{p-2}$$

yielding $\chi(\mathbf{r}_k) \to \infty$. The conclusion follows.

(iii) Take a sequence $\{r_k^n\} = \{(r_1^n, \dots, r_k^n)\} \subset \mathbf{\Gamma}_k$ such that

$$r_k^n \to \bar{\mathbf{r}}_k = (\bar{r}_1, \dots, \bar{r}_k) \in \mathbf{\Gamma}_k.$$

The assertion follows by showing that

$$\chi(\bar{\mathbf{r}}_k) \ge \lim_{n\to\infty} \sup \chi(\mathbf{r}_{\mathbf{k}}^{\mathbf{n}}), \quad \chi(\bar{\mathbf{r}}_{\mathbf{k}}) \le \lim_{\mathbf{n}\to\infty} \inf \chi(\mathbf{r}_{\mathbf{k}}^{\mathbf{n}}). \tag{3.2}$$

We first prove $\chi(\bar{\mathbf{r}}_{\mathbf{k}}) \ge \lim_{n\to\infty} \sup \chi(\mathbf{r}_k^n)$. Define $\xi_i^{\mathbf{r}_k^n} : [r_{i-1}^n, r_i^n] \to \mathbb{R}$ by

$$\xi_i^{\mathbf{r}_k^n} = a_i^n v_i^{\bar{\mathbf{r}}_k}\Big(\frac{\bar{r}_i - \bar{r}_{i-1}}{r_i^n - r_{i-1}^n}(t - r_{i-1}^n) + \bar{r}_{i-1}\Big)$$

for $i = 1, \dots, k$, and

$$\xi_{k+1}^{\mathbf{r}_k^n} = a_{k+1}^n v_{k+1}^{\bar{\mathbf{r}}_{\mathbf{k}}}\Big(\frac{\bar{r}}{r_k^n} t\Big),$$

where $r_0^n = 0$, $r_{k+1}^n = \infty$ and each $(a_1^n, \dots, a_{k+1}^n)$ is the unique positive vector such that $(\xi_1^{\mathbf{r}_k^n}, \dots, \xi_k^{\mathbf{r}_{\mathbf{k}}^{\mathbf{n}}}) \in \mathcal{N}_k^{\mathbf{r}_k^n}$. By the definition of $(v_1^{\mathbf{r}_k^n}, \dots, v_{k+1}^{\mathbf{r}_{\mathbf{k}}^{\mathbf{n}}})$, we have

$$I^{\mathbf{r}_k^n}(\xi_1^{\mathbf{r}_k^n}, \dots, \xi_{k+1}^{\mathbf{r}_k^n}) \ge I^{\mathbf{r}_k^n}(v_1^{\mathbf{r}_k^n}, \dots, v_{k+1}^{\mathbf{r}_k^n}) = \chi(\mathbf{r}_k^n).$$

Therefore, for n large enough we have

$$\|\xi_i^{\mathbf{r}_k^n}\|_{B_i^{\mathbf{r}_k^n}}^2 = (a_i^n)^2\|v_i^{\bar{\mathbf{r}}_{\mathbf{k}}}\|_{B_i^{\mathbf{r}_k^n}} + o(1),$$

$$\begin{aligned}
&\int_{B_i^{\mathbf{r}_k^n}} |\nabla \xi_i^{\mathbf{r}_k^n}|^2\,dx \int_{B_j^{\mathbf{r}_k^n}} |\nabla \xi_j^{\mathbf{r}_k^n}|^2\,dx \\
&= (a_i^n)^2(a_j^n)^2 \int_{B_i^{\mathbf{r}_k^n}} |\nabla v_i^{\bar{\mathbf{r}}_{\mathbf{k}}}|^2\,dx \int_{B_j^{\bar{\mathbf{r}}_{\mathbf{k}}}} |\nabla v_j^{\mathbf{r}_k^n}|^2\,dx + o(1),
\end{aligned}$$

$$
\begin{aligned}
&\int_{B_i^{\mathbf{r}_k^n}}\int_{B_j^{\mathbf{r}_k^n}} \frac{(\xi_i^{\mathbf{r}_k^n})^2(x)(\xi_j^{\mathbf{r}_k^n})^2(y)}{4\pi|x-y|}\,dx\,dy \\
&= (a_i^n)^2(a_j^n)^2\int_{B_i^{\bar{\mathbf{r}}_k}}\int_{B_j^{\bar{\mathbf{r}}_\mathbf{k}}} \frac{(v_i^{\bar{\mathbf{r}}_\mathbf{k}})^2(x)(v_j^{\bar{\mathbf{r}}_k})^2(y)}{4\pi|x-y|}\,dx\,dy + o(1),
\end{aligned}
$$

$$
\int_{B_i^{\mathbf{r}_k^n}} |\xi_i^{\mathbf{r}_k^n}|^p\,dx = (a_i^n)^p\int_{B_i^{\bar{\mathbf{r}}_k}} |v_i^{\bar{\mathbf{r}}_k}|^p\,dx + o(1).
$$

Since $(\xi_1^{\mathbf{r}_k^n},\dots,\xi_{k+1}^{\mathbf{r}_k^n}) \in \mathcal{N}_k{}^{\mathbf{r}_k^n}$, we have

$$
\begin{aligned}
&(a_i^n)^2\|v_i^{\bar{\mathbf{r}}_k}\|^2_{B_i^{\mathbf{r}_k^n}} + b(a_i^n)^2\int_{B_i^{\bar{\mathbf{r}}_\mathbf{k}}} |\nabla v_i^{\bar{\mathbf{r}}_k}|^2\,dx\sum_{i=1}^{k+1}(a_j^n)^2\int_{B_j^{\bar{\mathbf{r}}_\mathbf{k}}} |\nabla v_j^{\bar{\mathbf{r}}_\mathbf{k}}|^2\,dx \\
&+\sum_{i\neq j}^{k+1}(a_i^n)^2(a_j^n)^2\int_{B_i^{\bar{\mathbf{r}}_\mathbf{k}}}\int_{B_j^{\bar{\mathbf{r}}_\mathbf{k}}} \frac{(v_i^{\bar{\mathbf{r}}_k})^2(x)(v_j^{\bar{\mathbf{r}}_k})^2(y)}{4\pi|x-y|}\,dx\,dy - (a_k^n)^p\int_{B_i^{\bar{\mathbf{r}}_k}} |v_i^{\bar{\mathbf{r}}_k}|^p\,dx \\
&= o(1)
\end{aligned}
\tag{3.3}
$$

and

$$
\begin{aligned}
&\|v_i^{\bar{\mathbf{r}}_k}\|^2_{B_i^{\mathbf{r}_k^n}} + b\int_{B_i^{\bar{\mathbf{r}}_\mathbf{k}}} |\nabla v_i^{\bar{\mathbf{r}}_k}|^2\,dx\sum_{j=1}^{k+1}\int_{B_j^{\bar{\mathbf{r}}_k}} |\nabla v_j^{\bar{\mathbf{r}}_k}|^2\,dx \\
&+\sum_{i\neq j}^{k+1}\int_{B_i^{\bar{\mathbf{r}}_k}}\int_{B_j^{\bar{\mathbf{r}}_k}} \frac{(v_i^{\bar{\mathbf{r}}_k})^2(x)(v_j^{\bar{\mathbf{r}}_k})^2(y)}{4\pi|x-y|}\,dx\,dy - \int_{B_i^{\bar{\rho}_\mathbf{k}}} |v_i^{\bar{\mathbf{r}}_k}|^p\,dx = o(1)
\end{aligned}
\tag{3.4}
$$

for $i = 1,\dots,k+1$. From (3.3) and (3.4), we deduce that $\lim_{n\to\infty} a_i^n = 1$. Thus,

$$
\begin{aligned}
\chi(\bar{\mathbf{r}}_k) &= I^{\bar{\mathbf{r}}_k}(v_1^{\bar{\mathbf{r}}_k},\dots,v_{k+1}^{\bar{\mathbf{r}}_k}) = \limsup_{n\to\infty} I^{\bar{\mathbf{r}}_k}(v_1^{\bar{\mathbf{r}}_k},\dots,v_{k+1}^{\bar{\mathbf{r}}_k}) \\
&\ge \limsup_{n\to\infty} I^{\bar{\mathbf{r}}_k}(\xi_1^{\bar{\mathbf{r}}_k},\dots,\xi_{k+1}^{\bar{\mathbf{r}}_k}) = \limsup_{n\to\infty}\chi(\mathbf{r}_k^n).
\end{aligned}
$$

This also implies

$$
\begin{gathered}
\limsup_{n\to\infty}\|v_i^{\mathbf{r}_k^n}\|^2_{B_i^{\mathbf{r}_k^n}} < \infty, \\
\limsup_{n\to\infty}\int_{B_i^{\mathbf{r}_k^n}} |\nabla \nu_i^{\mathbf{r}_k^n}|^2\,dx\int_{B_j^{\rho_\mathbf{k}^\mathbf{n}}} |\nabla v_j^{\mathbf{r}_k^n}|^2\,dx < \infty, \\
\limsup_{n\to\infty}\int_{B_i^{\mathbf{r}_k^n}}\int_{B_j^{\mathbf{r}_k^n}} \frac{(v_i^{\mathbf{r}_k^n})^2(x)(v_j^{\mathbf{r}_k^n})^2(y)}{4\pi|x-y|}\,dx\,dy < \infty.
\end{gathered}
\tag{3.5}
$$

Next, we prove $\chi(\bar{\mathbf{r}}_k) \le \liminf_{n\to\infty}\chi(\mathbf{r}_k^n)$. As above, we define $\xi_i^{\mathbf{r}_k^n} : [\bar{r}_{i-1},\bar{r}_i] \to \mathbb{R}$ by

$$
\xi_i^{\mathbf{r}_k^n} = a_i^n v_i^{\mathbf{r}_k^n}\Big(\frac{r_i^n - r_{i-1}^n}{\bar{r}_i - \bar{r}_{i-1}}(t - \bar{r}_{i-1}) + r_{i-1}^n\Big)
$$

for $i = 1,\dots,k+1$ and

$$
\xi_{k+1}^{\mathbf{r}_k^n} = a_{k+1}^n v_{k+1}^{\mathbf{r}_k^n}\Big(\frac{r_k^n}{\bar{r}_k}\Big)t,
$$

where $r_0^n = 0, r_{k+1}^n = \infty$ and each $(a_1^n,\dots,a_{k+1}^n)$ is the unique $(k+1)$-tuple of positive real numbers such that $(\xi_1^{\mathbf{r}_k^n},\dots,\xi_{k+1}^{\mathbf{r}_k^n}) \in \mathcal{N}_k{}^{\mathbf{r}_k^n}$. Then, from (3.5) it also follows that

$$\begin{aligned} &(a_i^n)^2\|v_i^{\mathbf{r}_k^n}\|^2_{B_i^{\mathbf{r}_k^n}} + b(a_i^n)^2\int_{B_i^{\mathbf{r}_k^n}}|\nabla v_i^{\mathbf{r}_k^n}|^2\,dx\sum_{j=1}^{k+1}(a_j^n)^2\int_{B_j^{\mathbf{r}_k^n}}|\nabla v_j^{\mathbf{r}_k^n}|^2\,dx \\ &+\sum_{i\neq j}^{k+1}(a_i^n)^2(a_j^n)^2\int_{B_i^{\mathbf{r}_k^n}}\int_{B_j^{\mathbf{r}_k^n}}\frac{(v_i^{\mathbf{r}_k^n})^2(x)(v_j^{\mathbf{r}_k^n})^2(y)}{4\pi|x-y|}\,dx\,dy-(a_i^n)^p\int_{B_i^{\mathbf{r}_k^n}}|v_i^{\mathbf{r}_k^n}|^p\,dx \\ &= o(1) \end{aligned} \tag{3.6}$$

as well as

$$\begin{aligned} &\|v_i^{\mathbf{r}_k^n}\|^2_{B_i^{\mathbf{r}_k^n}} + b\int_{B_i^{\mathbf{r}_k^n}}|\nabla v_i^{\mathbf{r}_k^n}|^2\,dx\sum_{j=1}^{k+1}\int_{B_j^{\mathbf{r}_k^n}}|\nabla v_j^{\mathbf{r}_k^n}|^2\,dx \\ &+\sum_{i\neq j}^{k+1}\int_{B_i^{\mathbf{r}_k^n}}\int_{B_j^{\mathbf{r}_k^n}}\frac{(v_i^{\mathbf{r}_k^n})^2(x)(v_j^{\mathbf{r}_k^n})^2(y)}{4\pi|x-y|}\,dx\,dy-\int_{B_i^{\mathbf{r}_k^n}}|v_i^{\mathbf{r}_k^n}|^p\,dx = o(1) \end{aligned} \tag{3.7}$$

for $i=1,\dots,k+1$.

From (3.6), (3.7) and $\liminf_{n\to\infty}\|v_i^{\mathbf{r}_k^n}\|_{B_i^{\mathbf{r}_k^n}}>0$ we deduce that $\lim_{n\to\infty}a_i^n=1$ for all i. So we have

$$\begin{aligned} \chi(\bar{\mathbf{r}}_k) &= I^{\bar{\mathbf{r}}_k}(v_1^{\bar{\mathbf{r}}_k},\dots,v_{k+1}^{\bar{\mathbf{r}}_k}) \le \liminf_{n\to\infty} I^{\mathbf{r}_k^n}(v_1^{\mathbf{r}_k^n},\dots,v_{k+1}^{\mathbf{r}_k^n}) \\ &= \liminf_{n\to\infty} I^{\mathbf{r}_k^n}(\xi_1^{\mathbf{r}_k^n},\dots,\xi_{k+1}^{\mathbf{r}_k^n}) = \liminf_{n\to\infty}\chi(\mathbf{r}_k^n). \end{aligned}$$

We completed the proof of (iii).

As a result, from (i)–(iii) we deduce that there is a minimum point $\tilde{\mathbf{r}_k}=(\tilde r_1,\dots,\tilde r_k)\in\mathbf{\Gamma}_k$ of χ. $\square$

Next, we show that $(v_1^{\tilde{\mathbf{r}}_k},\dots,v_{k+1}^{\tilde{\mathbf{r}}_k})$ found in the previous lemma, corresponding to $(\tilde r_1,\dots,\tilde r_k)\in\mathbf{\Gamma}_k$ for every $k\in\mathbb{N}$, is a k-times sign-changing and radial solution of (1.1).

Proof of Theorem 1.1. Suppose to the contrary that $\sum_{i=1}^{k+1}v_i^{\tilde{\mathbf{r}}_k}$ is not a solution of (1.1), there would exist $l\in\{1,\dots,k\}$ such that

$$v_-=\lim_{r\to\tilde r_l^-}\frac{dv_l^{\tilde{\mathbf{r}}_k}(r)}{dr}\neq\lim_{r\to\tilde r_l^+}\frac{dv_l^{\tilde{\mathbf{r}}_k}(r)}{dr}=v_+,$$

Fix a positive number δ small enough and set

$$\tilde y(r)=\begin{cases} v_l(r), & \text{for } r\in(\tilde r_{l-1},\tilde r_l-\delta),\\ v_l(\tilde r_l-\delta)+\frac{v_{l+1}(\tilde r+\delta)-v_l(\tilde r+\delta)}{2\delta}, & \text{for } r\in(\tilde r_l-\delta,\tilde r_{l+1}+\delta)\\ v_{l+1}(r), & \text{for } r\in(\tilde r_l+\delta,\tilde r_{l+1}). \end{cases}$$

There exists $\tilde s_l\in(\tilde r_{l-1},\tilde r_{l+1})$ such that

$$\tilde y(r)\big|_{r=\tilde s_l}=0$$

Define the $k+1$-tuple of functions $(\tilde z_l,\dots,\tilde z_{k+1})$,

$$\begin{gathered} \tilde z_l(r)=\tilde y(r),\quad \text{for } r\in(\tilde r_{l-1},\hat s_l),\\ \tilde z_i(r)=v_i(r),\quad \text{for } r\in(\tilde r_{i-1},\tilde r_{i+1}),\quad i\neq l,l+1\\ \tilde z_{l+1}(r)=\tilde y(r),\quad \text{for } r\in(\hat s_l,\tilde r_{l+1}). \end{gathered}$$

By Lemma 2.1, there exists $(\tilde{a}_l, \dots, \tilde{a}_{k+1}) \in (\mathbb{R}_{>0})^{k+1}$ such that

$$(z_1, \dots, z_{k+1}) = (\tilde{a}_1 \tilde{z}_1, \dots, \tilde{a}_{k+1} \tilde{z}_{k+1}) \in \mathcal{N}^{\bar{\mathbf{r}}_k}$$

with $\bar{\mathbf{r}}_k = (\tilde{r}_1, \dots, \tilde{r}_l, \tilde{s}_l, \tilde{r}_{l+1} \dots, \tilde{r}_k)$. On the other hand, we can verify that

$$(\tilde{a}_l, \dots, \tilde{a}_{k+1}) \to (1, \dots, 1) \tag{3.8}$$

as $\delta \to 0$. Let $W(r) = \sum_{i=1}^{k+1} v_i(r) \in H$ and $Z(r) = \sum_{i=1}^{k+1} z_i(r) \in H$. Then

$$I(W) = I^{\bar{\mathbf{r}}_k}(v_l^{\bar{\mathbf{r}}_k}, \dots, v_{k+1}^{\bar{\mathbf{r}}_k}) \le I^{\bar{\mathbf{r}}_k}(z_1^{\bar{\mathbf{r}}_k}, \dots, z_{k+1}^{\bar{\mathbf{r}}_k}) = I(Z). \tag{3.9}$$

On the other hand, for any $u \in H$, the solution of $-\Delta\varphi = u^2$ is radial and it can be expressed as

$$\varphi(t) = \frac{1}{t} \int_0^\infty u^2(s) s \min\{s, t\}\, ds$$

for $t > 0$. Therefore,

$$\begin{aligned} I(Z) - I(W) = &\Big(\int_0^{\tilde{r}_l - \delta} + \int_{\tilde{r}_l + \delta}^\infty \Big) \Big(\frac{a}{2} z'^2 + \frac{1}{2} V(r) z^2 - \frac{1}{p} |z|^p \Big) r^2\, dr \\ &+ \int_{\tilde{r}_l - \delta}^{\tilde{r}_l + \delta} \Big(\frac{a}{2} z'^2 + \frac{1}{2} V(r) z^2 - \frac{1}{p} |z|^p \Big) r^2\, dr \\ &- \int_0^\infty \Big(\frac{a}{2} v'^2 + \frac{1}{2} V(r) v^2 \Big) r^2\, dr + \frac{b}{4} \Big(\int_0^\infty z' r^2\, dr \Big)^2 \\ &- \frac{b}{4} \Big(\int_0^\infty v' r^2\, dr \Big)^2 + \frac{1}{4} \int_0^\infty \int_0^\infty z^2(s) z^2(t) st \min\{s, t\}\, ds\, dt \\ &- \frac{1}{4} \int_0^\infty \int_0^\infty v^2(s) v^2(t) st \min\{s, t\}\, ds\, dt. \end{aligned}$$

Since $\frac{1}{p}|u|^p$ is convex,

$$\frac{1}{p}|z|^p \ge \frac{1}{p}|v|^p + \frac{z^2 - \nu^2}{2} v u^{p-2} \tag{3.10}$$

for $zv > 0$, and we have

$$\begin{aligned} &\int_{\tilde{r}_l - \delta}^{\tilde{r}_l + \delta} (\frac{a}{2} z'^2 + \frac{1}{2} V(r) z^2 - \frac{1}{p} |z|^p) r^2\, dr \\ &\le \int_{\tilde{r}_l - \delta}^{\tilde{r}_l + \delta} \Big(\frac{a}{2} z'^2 + \frac{1}{2} V(r) z^2 + \frac{1}{2} |v|^p - \frac{1}{2} |v|^p - \frac{1}{p} |z|^p \Big) r^2\, dr. \end{aligned} \tag{3.11}$$

By the definition of W, we have

$$\begin{aligned} &\int_0^\infty (a v'^2 + V(r) v u^2) r^2\, dr + b(\int_0^\infty v' r^2\, dr)^2 \\ &+ \int_0^\infty \int_0^\infty v^2(s) v^2(t) st \min\{s, t\}\, ds\, dt \\ &= \int_0^\infty |v|^p r^2\, dr. \end{aligned} \tag{3.12}$$

Set $A=\int_0^\infty v'^2r^2\,dr$. By (3.10)–(3.11), we have

$$
\begin{aligned}
I(Z)-I(W)\le A_1+A_2+A_3+\frac14\int_0^\infty\int_0^\infty z^2(s)z^2(t)st\min\{s,t\}\,ds\,dt\\
+\frac14\int_0^\infty\int_0^\infty v^2(s)v^2(t)st\min\{s,t\}\,ds\,dt,
\end{aligned}
\tag{3.13}
$$

where

$$
\begin{gathered}
A_1=\Big(\int_0^{\tilde r_l-\delta}+\int_{\tilde r_l+\delta}^\infty\Big)\Big(\frac{a+bA}{2}z'^2+\frac12V(r)z^2-\frac12|z|^2|v|^{p-2}\Big)r^2\,dr,\\
A_2=\int_{\tilde r_l-\delta}^{\tilde r_l+\delta}\Big(\frac{a+bA}{2}z'^2+\frac12V(r)z^2-\frac1p|z|^p+\frac1p|v|^p\Big)r^2\,dr,\\
A_3=\frac b4\Big(\int_0^\infty z'r^2\,dr\Big)^2-\frac b2\int_0^\infty v'r^2\,dr\int_0^\infty z'r^2\,dr+\frac b4\Big(\int_0^\infty v'r^2\,dr\Big)^2.
\end{gathered}
$$

From

$$
v_-=\lim_{\delta\to0}\frac{v(\tilde r_l-\delta)-v(\tilde r_l)}{-\delta}
$$

we deduce that $v(\tilde r_l-\delta)=-\delta v_-+o(\delta)$, and using that v is a solution of (1.1), that is v satisfies

$$
\begin{gathered}
-\Big(a+b\int_0^\infty v'^2r^2\,dr\Big)\Big(v''+\frac2rv'\Big)+(V(r)v+\varphi v)r^2=r^2|v|^{p-2}v,\\
-\Delta\varphi=v^2,\ \lim_{|x|\to\infty}\varphi(x)=0,
\end{gathered}
\tag{3.14}
$$

we obtain

$$
(\tilde r_l-\delta)^2v'(\tilde r_l-\delta)=\tilde r^2v_-+o(\delta).
$$

Integrating by part, we find

$$
\begin{aligned}
&\int_0^{\tilde r_l-\delta}\frac{a+bA}{2}v'^2r^2\,dr\\
&=\frac{a+bA}{2}\Big(v'vr^2\Big|_0^{\tilde r_l-\delta}-\int_0^{\tilde r_l-\delta}v\,d(v'r^2)\Big)\\
&=\frac{a+bA}{2}v'(\tilde r_l-\delta)v(\tilde r_l-\delta)(\tilde r_l-\delta)^2-\frac{a+bA}{2}\int_0^{\tilde r_l-\delta}v[r^2v''+2rv']\,dr\\
&=\frac{a+bA}{2}v'(\tilde r_l-\delta)v(\tilde r_l-\delta)(\tilde r_l-\delta)^2-\frac12\int_0^{\tilde r_l-\delta}V(r)v^2r^2\,dr\\
&\quad-\frac12\int_0^{\tilde r_l-\delta}\int_0^\infty v^2(s)v^2(t)st\min\{s,t\}\,ds\,dt+\frac12\int_0^{\tilde r_l-\delta}|v|^{p-2}vr^2\,dr.
\end{aligned}
\tag{3.15}
$$

We deduce from (3.13)–(3.15) that

$$
\begin{aligned}
&\int_0^{\tilde r_l-\delta}\Big(\frac{a+bA}{2}v'^2+\frac12 V(r)v^2-\frac12 v^2|v|^{p-2}\Big)r^2\,dr\\
&=\frac{a+bA}{2}v'(\tilde r_l-\delta)v(\tilde r_l-\delta)(\tilde r_l-\delta)^2\\
&\quad-\frac12\int_0^{\tilde r_l-\delta}\int_0^\infty v^2(s)v^2(t)st\min\{s,t\}\,ds\,dt\\
&=-\frac{a+bA}{2}\tilde r_l^2v_-^2\delta+o(\delta)-\frac12\int_0^{\tilde r_l-\delta}\int_0^\infty v^2(s)v^2(t)st\min\{s,t\}\,ds\,dt.
\end{aligned}
\tag{3.16}
$$

By (3.8), we obtain

$$
\begin{aligned}
&\int_0^{\tilde r_l-\delta}\Big(\frac{a+bA}{2}z'^2+\frac12 V(r)z^2-\frac12 z^2|z|^{p-2}\Big)r^2\,dr\\
&=(1+o(1))\int_0^{\tilde r_l-\delta}\Big(\frac{a+bA}{2}v'^2+\frac12 V(r)v^2-\frac12 v^2|v|^{p-2}\Big)r^2\,dr\\
&=-\frac{a+bA}{2}\tilde r_l^2v_-^2\delta+o(\delta)\\
&\quad-\Big(\frac12+o(1)\Big)\int_0^{\tilde r_l-\delta}\int_0^\infty v^2(s)v^2(t)st\min\{s,t\}\,ds\,dt.
\end{aligned}
\tag{3.17}
$$

In the same way, we have

$$
\begin{aligned}
&\int_{\tilde r_l+\delta}^{\infty}\Big(\frac{a+bA}{2}z'^2+\frac12 V(r)z^2-\frac12 z^2|z|^{p-2}\Big)r^2\,dr\\
&=-\frac{a+bA}{2}\tilde r_l^2v_+^2\delta+o(\delta)\\
&\quad-\Big(\frac12+o(1)\Big)\int_{\tilde r_l+\delta}^\infty\int_0^\infty v^2(s)v^2(t)st\min\{s,t\}\,ds\,dt.
\end{aligned}
\tag{3.18}
$$

Equations (3.17) and (3.18) lead to

$$
\begin{aligned}
A_1&=-\frac{a+bA}{2}\tilde r_l^2\delta(v_+^2+v_-^2)+o(\delta)\\
&\quad-\Big(\frac12+o(1)\Big)\Big(\int_0^{\tilde r_l-\delta}+\int_{\tilde r_l+\delta}^\infty\Big)\int_0^\infty v^2(s)v^2(t)st\min\{s,t\}\,ds\,dt.
\end{aligned}
\tag{3.19}
$$

Next, we estimate A_2. It is readily to verify that

$$
\int_{\tilde r_l-\delta}^{\tilde r_l+\delta}(\frac12 V(r)z^2-\frac1p|z|^p+\frac1p|v|^p)r^2\,dr=o(\delta). \tag{3.20}
$$

If $r\in[\tilde r_l-\delta,\tilde r_l+\delta]$, we have

$$
\tilde y'(t)=\frac{v(\tilde r_l+\delta)-v(\tilde r_l-\delta)}{2\delta}.
$$

Therefore,

$$\begin{aligned} &\frac{a+bA}{2}\int_{\tilde r_l-\delta}^{\tilde r_l+\delta}\tilde y'(t)r^2\,dr \\ &=\frac{a+bA}{2}\int_{\tilde r_l-\delta}^{\tilde r_l+\delta}\frac{[v(\tilde r_l+\delta)-v(\tilde r_l-\delta)]^2}{4\delta^2}r^2\,dr \\ &=\frac{a+bA}{8}\delta\frac{[v(\tilde r_l+\delta)-v(\tilde r_l+\delta)]^2}{\delta^2}\frac{(\tilde r_l+\delta)^3-(\tilde r_l-\delta)^3}{3\delta}. \end{aligned} \tag{3.21}$$

Since

$$v_+=\lim_{\delta\to 0}\frac{v(\tilde r_l+\delta)-v(\tilde r_l)}{\delta}\quad\text{and}\quad v_-=\lim_{\delta\to 0}\frac{v(\tilde r_l)-v(\tilde r_l-\delta)}{\delta},$$

we have

$$(v_++v_-)^2=\lim_{\delta\to 0}\frac{[v(\tilde r_l+\delta)-v(\tilde r_l-\delta)]^2}{\delta^2}. \tag{3.22}$$

Obviously,

$$\lim_{\delta\to 0}\frac{(\tilde r_l+\delta)^3-(\tilde r_l-\delta)^3}{3\delta}=\lim_{\delta\to 0}\frac{3(\tilde r_l+\delta)^2+3(\tilde r_l-\delta)^2}{3}=2\tilde r_l^2. \tag{3.23}$$

Hence, by (3.21), (3.22) and (3.23), for $\delta\to 0$ we obtain

$$\int_{\tilde r_l-\delta}^{\tilde r_l+\delta}\frac{a+bA}{2}y'(r)r^2\,dr=\frac{a+bA}{4}\delta\tilde r_l^2(v_++v_-)^2+o(\delta).$$

By (3.8),

$$\int_{\tilde r_l-\delta}^{\tilde r_l+\delta}\frac{a+bA}{2}z'(r)r^2dr=\frac{a+bA}{4}\delta\tilde r_l^2(v_++v_-)^2+o(\delta). \tag{3.24}$$

Finally, we claim that

$$\tilde a_i=1+o(\delta^{\frac12})$$

as $\delta\to 0$ for $i=1,\dots,k+1$. Suppose by a contradiction, for some i_0, that

$$\lim_{\delta\to 0^+}|\delta^{-\frac12}(\tilde a_{i_0}-1)|=\max_{1\le i\le k+1}\lim_{\delta\to 0^+}|\delta^{-\frac12}(\tilde a_{i_0}-1)|=B\neq 0.$$

Since $(v_1,\dots,v_{k+1})\in\mathcal N_k^{\tilde{\mathbf r}_k}$, $(\tilde a_1\tilde z_1,\dots,\tilde a_{k+1}\tilde z_{k+1})\in\mathcal N_k^{\bar{\mathbf r}_k}$ and $\bar{\mathbf r}_k\to\tilde{\mathbf r}_k$ as $\delta\to 0^+$, we have

$$\begin{aligned} 0&=\lim_{\delta\to 0^+}\delta^{-\frac12}(F_{i_0}(\tilde a_1\tilde z_1,\dots,\tilde a_{k+1}\tilde z_{k+1})-F_{i_0}(\tilde z_1,\dots,\tilde z_{k+1}))\Big(\delta^{-\frac12}(\tilde a_{i_0}-1)\Big)^{-1}\\ &\le 2\|v_{i_0}\|_{i_0}^2+4b\Big(\int_{B_{i_0}^{\bar{\mathbf r}_k}}|\nabla v_{i_0}|^2\,dx\Big)^2+4b\int_{B_{i_0}^{\bar{\mathbf r}_k}}|\nabla v_{i_0}|^2\,dx\sum_{j\neq i_0}^{k+1}\int_{B_j^{\bar{\mathbf r}_k}}|\nabla v_j|^2\,dx\\ &\quad+4\int_{B_{i_0}^{\bar{\mathbf r}_k}}\int_{B_{i_0}^{\bar{\mathbf r}_k}}\frac{v_{i_0}^2(x)v_{i_0}^2(y)}{4\pi|x-y|}\,dx\,dy+4\sum_{j\neq i_0}^{k+1}\int_{B_{i_0}^{\bar{\mathbf r}_k}}\int_{B_j^{\bar{\mathbf r}_k}}\frac{v_{i_0}^2(x)v_j^2(y)}{4\pi|x-y|}\\ &\quad-p\int_{B_{i_0}^{\bar{\mathbf r}_k}}|v_{i_0}|^{(p-1)}\,dx, \end{aligned}$$

where F_{i_0} is given by (2.7). This leads to a contradiction since $(v_1, \dots, v_{k+1}) \in \mathcal{N}_k^{\hat{\rho}}$ and $p > 4$. Thus,

$$\begin{aligned}
A_3 &= \frac{b}{4}\Big(\int_0^\infty v' r^2\, dr - \int_0^\infty z' r^2\, dr\Big)^2 \\
&= \frac{b}{4}\Big[o(\delta^{\frac{1}{2}})\Big(\int_0^{\tilde{r}_l-\delta} + \int_{\tilde{r}_l+\delta}^\infty\Big) v'^2 r^2\, dr + \frac{1}{2}(v_+ + v_-)^2 \tilde{r}_l^2 \delta \\
&\quad - {v_-}^2 \tilde{r}_l^2 \delta - {v_+}^2 \tilde{r}_l^2 \delta\Big]^2 = o(\delta).
\end{aligned} \tag{3.25}$$

By the estimates on A_1, A_2 and A_3, we obtain

$$\begin{aligned}
&I(Z) - I(W) \\
&\le -\frac{a+bA}{4}(v_+ - v_-)^2 \tilde{r}_l^2 \delta + o(\delta) + \frac{1}{4}\int_0^\infty \int_0^\infty z^2(s) z^2(t) st \min\{s,t\}\, ds\, dt \\
&\quad + \frac{1}{4}\int_0^\infty \int_0^\infty v^2(s) v^2(t) st \min\{s,t\} ds\, dt \\
&\quad - \Big(\frac{1}{2} + o(1)\Big)\big(\int_0^{\tilde{r}_l-\delta} + \int_{\tilde{r}_l+\delta}^\infty) v'^2 r^2 dr\big) \int_0^\infty v^2(s) v^2(t) st \min\{s,t\}\, ds\, dt \\
&= -\frac{a+bA}{4}(v_+ - v_-)^2 \tilde{r}_l^2 \delta + o(\delta) \\
&\quad - (\frac{1}{4} + o(1)) \int_{\tilde{r}_l-\delta}^{\tilde{r}_l+\delta} \int_0^\infty z^2(s) z^2(t) st \min\{s,t\}\, ds\, dt \\
&\quad - (\frac{1}{4} + o(1)) \int_{\tilde{r}_l-\delta}^{\tilde{r}_l+\delta} \int_0^\infty v^2(s) v^2(t) st \min\{s,t\}\, ds\, dt.
\end{aligned}$$

So we obtain $I(Z) < I(W)$ if $\delta > 0$ is sufficiently small, which is a contradiction to the definition of W. Consequently, $v_- = v_+$, and then $v_k = \sum_{i=1}^{k+1} v_i^{\tilde{r}_k}$ is a solution of (1.1) changing sign exactly k times. The proof is complete. □

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