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HARNACK TYPE INEQUALITY FOR NON-NEGATIVE SOLUTIONS OF SECOND-ORDER DEGENERATE PARABOLIC EQUATIONS IN DIVERGENT FORM

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Abstract. We study a class of second-order degenerate parabolic equations in divergent form. We prove two analogues of the Harnack inequality, one for non-negative weak solutions, an another for non-negative solutions.

1. Introduction

Let $\mathbb{R}^n$ be a Euclidean space of the points $x = (x_1, x_2, \dots, x_n)$ and D be a bounded domain in $\mathbb{R}^{n+1}$ with the parabolic boundary $\Gamma(D)$, $(0,0) \in D$.

Consider the parabolic equation

$$Lu = \frac{\partial u}{\partial t} - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\Big(a_{ij}(x,t)\frac{\partial u}{\partial x_j}\Big) = 0, \quad (x,t) \in D, \tag{1.1}$$

and assume that $\{a_{ij}(x,t)\}$ is a real symmetric matrix with measurable elements and for all $(x,t) \in D$ and $\xi \in \mathbb{R}^n$ the following condition is fulfilled:

$$\gamma \sum_{i=1}^{n} \lambda_i(x,t)\xi_i^2 \le \sum_{i,j=1}^{n} a_{ij}(x,t)\xi_i\xi_j \le \gamma^{-1}\sum_{i=1}^{n} \lambda_i(x,t)\xi_i^2, \tag{1.2}$$

where $\gamma \in (0,1]$ is a constant,

$$\lambda_i(x,t) = g_i\big(\rho(x) + \sqrt{|t|}\big),$$

$$\rho(x) = \sum_{i=1}^{n} w_i(|x_i|), \quad g_i(z) = \frac{(w_i^{-1}(z))^2}{z^2}, \quad i = 1, 2, \dots, n.$$

We assume that the functions $w_i(t)$ increase strictly monotonically, $w_i(0) = 0$, $w_i^{-1}(t)$ is the function inverse to $w_i(t)$ and for $i = 1, 2, \dots, n$,

$$w_i(2t) \le 2w_i(t), \tag{1.3}$$

$$\Big(\frac{w_i(t)}{t}\Big)^{q-1} \int_0^{w_i^{-1}(t)} (\frac{w_i(z)}{z})^q dz \le c_1 t \tag{1.4}$$

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with some constant $q > n$ and positive constant c_1 independent of t. A simple example of function w_i is $w_i(t) = t^{\alpha_i}$ where

$$\alpha_i \geq \frac{-1 + \sqrt{1 + 4q(q-1)}}{2(q-1)}.$$

The principal result of this article is the Harnack type inequality for non-negative weak solutions of equation (1.1).

For uniformly second-order parabolic equations of divergent structure, with discontinuous coefficients the Harnack inequality was obtained in the well known paper by Nash [5]. Moser [4] obtained another proof of this fact. For parabolic equations of divergent structure with uniform degeneration we refer to [1, 2]. When $w_i(t)$ are power functions, the Harnack type inequality was obtained in [3].

Now we introduce some notation: let D be a cylindrical domain $\Omega \times [T_0, T]$, where Ω is a bounded domain in $\mathbb{R}^n$ and $-\infty < T_0 < T < \infty$.

By $W^{1,0}_{2,\Lambda}(D)$ and $W^{1,1}_{2,\Lambda}(D)$ we denote Banach spaces of functions $u(x,t)$ with finite norms in D,

$$\|u\|_{W^{1,0}_{2,\Lambda}(D)} = \Big(\sup_{t \in [T_0,T]} \int_\Omega u^2\,dx + \sum_{i=1}^n \int_\Omega \lambda_i(x,t) (\frac{\partial u}{\partial x_i})^2\,dx\,dt \Big)^{1/2},$$

$$\|u\|_{W^{1,1}_{2,\Lambda}(D)} = \Big(\int_D \Big(u^2 + \sum_{i=1}^n \lambda_i(x,t) (\frac{\partial u}{\partial x_i})^2 + (\frac{\partial u}{\partial t})^2 \Big)\,dx\,dt \Big)^{1/2}.$$

Let $A(D)$ be the set of all infinitely differentiable functions $u(x,t)$ on $\overline{D}$, such that $\operatorname{supp} u \subset (\overline{\Omega}_u \times [T_0,T]), \overline{\Omega}_u$ is a bounded subdomain of Ω, $u\big|_{t=T_0} = 0$. By $\mathring{W}^{1,1}_{2,\Lambda}(D)$ we denote the closure of $A(D)$ in $W^{1,1}_{2,\Lambda}(D)$. We set $u_t = \frac{\partial u}{\partial t}$, $u_{x_i} = \frac{\partial u}{\partial x_i}$, $i = 1, 2, \dots, n$.

A function $u(x,t) \in W^{1,0}_{2,\Lambda}(D)$ is called the weak solution of (1.1) in D if for any test function $\psi(x,t) \in \mathring{W}^{1,1}_{2,\Lambda}(D)$ and $t_1 \in (T_0, T]$ we have

$$\int_\Omega u(x,t_1)\psi(x,t_1)\,dx - \int_{D_{t_1}} u\psi_t\,dx\,dt + \int_{D_{t_1}} \sum_{i,j=1}^n a_{ij}(x,t) u_{x_i} \psi_{x_j}\,dx\,dt = 0, \quad (1.5)$$

where $D_{t_1} = \Omega \times (T_0, t_1)$.

2. Norm estimates of weak non-negative solutions

Here $|E|$ stands for n-dimensional (or $(n+1)$-dimensional) Lebesque measure of the measurable set $E \subset \mathbb{R}^n$ (or $E \subset \mathbb{R}^{n+1}$). We use the notation:

$$\Pi_R = \{x : |x_i| < \omega_i^{-1}(R),\ i = 1, 2, \dots, n\},$$

$$S(\rho) = \{x : |x_i| < \rho\omega_i^{-1}(R),\ i = 1, 2, \dots, n\} \times (-(1/3+\rho)R^2, -(3/4-\rho)R^2),$$

where $\rho \in (1/3, 1/2]$. We assume that $S(\rho) \subset D$. Denote

$$r_\nu = \sigma^{-\nu}(1+\sigma)^{-1}, \quad \nu = 0, 1, 2, \dots,$$

where $\sigma > 1$ will be defined later (it is the exponent of the imbedding theorem corresponding to the weights λ_i).

Now we state a Sobolev-type embedding theorem with weights, whose proof can be found in [3]. We set

$$\fint_{\Pi_R} f\,dx = \frac{1}{|\Pi_R|}\int_{\Pi_R} f\,dx, \quad \fint_{S_\rho} g\,dx\,dt = \frac{1}{|S(\rho)|}\int_{S(\rho)} g\,dx\,dt.$$

Theorem 2.1 (Sobolev theorem with weights)**.** *For any function $\varphi \in W^{1,0}_{2,\Lambda}(S(\rho))$ with zero trace on the lateral boundary of $S(\rho)$, and any $R \le R_0$ it holds*

$$\begin{aligned}\Big(\fint_{S_\rho} |\varphi|^{2\sigma}\,dx\,dt\Big)^{1/\sigma} &\le c\Big(\sup_{t\in(-(\frac13+\rho)R^2,-(\frac34-\rho)R^2)}\fint_{\Pi(R)}\varphi^2\,dx \\ &\quad + R^2\fint_{S_\rho}\sum_{i=1}^n \lambda_i(x,t)\varphi_{x_i}^2\,dx\,dt\Big),\end{aligned} \tag{2.1}$$

for $\sigma > 1$, where the constant c does not depend on φ, R and ρ.

Theorem 2.2. *Let $u(x,t)$ be a non-negative weak solution of* (1.1) *with coefficients that satisfy* (1.2)–(1.4)*. For any $r > 0$ and $1/3 \le \rho' < \rho \le 1/2$ it holds*

$$\sup_{S(\rho')} u \le c(\rho-\rho')^{-\xi}\Big(\fint_{S(\rho)} u^r\,dx\,dt\Big)^{-1/r}, \tag{2.2}$$

where positive constants c and ξ depend only on q, c_1, n, r and γ.

Proof. First, we prove the statement of this theorem for $r = 2$. The case $r > 2$ follows then by the Hölder inequality. To treat the case $r \in (0,2)$, we use an additional iteration.

Take a function η such that $\eta(x,t) = 1$ in $S(\rho')$, $\eta(x,t) = 0$ outside of $S(\rho)$, $0 \le \eta(x,t) \le 1$, and there exists a constant $c(n)$ such that

$$|\eta_{x_i}| \le \frac{c}{(\rho-\rho')w_i^{-1}(R)}, \quad i = 1,2,\dots,n; \quad |\eta_t| \le \frac{c}{(\rho-\rho')R^2}. \tag{2.3}$$

In (1.5) choose a test function $\psi = u^\beta\eta^2$, where $\beta > 0$. We obtain

$$\begin{aligned}&\sup_{t\in(-(\frac13+\rho)R^2,-(\frac34-\rho)R^2)}\frac{1}{\beta+1}\int_{\Pi(\rho)} u^{\beta+1}\eta^2\,dx + \beta\int_{S(\rho)} u^{\beta-1}\eta^2 a_{ij}u_{x_i}u_{x_j}\,dx\,dt\\ &= \frac{2}{\beta+1}\int_{S(\rho)} u^{\beta+1}\eta\eta_t\,dx\,dt - 2\int_{S(\rho)} u^\beta\eta a_{ij}u_{x_i}\eta_{x_i}\,dx\,dt.\end{aligned}$$

Let $v = u^{(\beta+1)/2}$. Using (1.2) and the Young inequality, we arrive at

$$\begin{aligned}&\sup_{t\in(-(\frac13+\rho)R^2,-(\frac34-\rho)R^2)}\int_{\Pi(\rho)} v^2\eta^2\,dx + \frac{4\beta}{\beta+1}\int_{S(\rho)}\eta^2 v_{x_i}^2\lambda_i(x,t)\,dx\,dt\\ &\le C\int_{S(\rho)} v^2\eta|\eta_t|\,dx\,dt + C(\beta+1)\beta^{-1}\int_{S(\rho)} v^2\eta_{x_i}^2\lambda_i(x,t)\,dx\,dt\end{aligned}$$

The above integral is taken over the set $S(\rho)\setminus S(\rho')$, since $\eta_{x_i} = 0$ in $S(\rho')$. But in this set

$$\rho(x) \le cR, \quad \sqrt{|t|} \le R, \quad \lambda_i(x,t) \le c\frac{(w_i^{-1}(R))^2}{R^2};$$

therefore,

$$\int_{S(\rho)}\eta^2\sum_{i=1}^n\lambda_i(x,t)v_{x_i}^2\,dx\,dt \le \frac{c(\beta+1)^2}{\beta^2(\rho-\rho')^2R^2}\int_{S(\rho)} v^2\,dx\,dt.$$

On the other hand, we have

$$\sup_{t\in(-(\frac{1}{3}+\rho)R^2,-(\frac{3}{4}-\rho)R^2)}\int_{\Pi_{\rho R}} v^2\eta^2\,dx \le C\frac{\beta+1}{\beta}(\rho-\rho')^{-2}R^{-2}\int_{S(\rho)} v^2\,dx\,dt.$$

For $\beta \ge 1$ these estimates take the form

$$\sup_{t\in(-(\frac{1}{3}+\rho)R^2,-(\frac{3}{4}-\rho)R^2)}⨍_{\Pi_{\rho R}} \eta^2 v^2\,dx \le c(\rho-\rho')^{-2}⨍_{S(\rho)} v^2\,dx\,dt, \tag{2.4}$$

$$⨍_{S(\rho)}\sum_{i=1}^n \lambda_i(x,t)v_{x_i}^2\eta^2\,dx\,dt \le c(\rho-\rho')^{-2}R^{-2}⨍_{S(\rho)} v^2\,dx\,dt. \tag{2.5}$$

Further more we assume that $\beta \ge 1$.

Applying (2.4),(2.5) and the embedding theorem (2.1) we obtain

$$\begin{aligned} &\Big(⨍_{S(\rho')} v^{2\sigma}\,dx\,dt\Big)^{1/\sigma}\\ &\le c\Big(⨍_{S(\rho)} v^{2\sigma}\eta^{2\sigma}\,dx\,dt\Big)^{1/\sigma}\\ &\le c\Big(\sup_{t\in(-(\frac{1}{3}+\rho)R^2,-(\frac{3}{4}-\rho)R^2)}⨍_{\Pi_{\rho R}}\eta^2v^2\,dx + R^2⨍_{S(\rho)}\sum_{i=1}^n\lambda_i(x,t)(v\eta)_{x_i}^2\,dx\,dt\Big)\\ &\le c(\rho-\rho')^2⨍_{S(\rho)} v^2\,dx\,dt. \end{aligned} \tag{2.6}$$

We define the sequences

$$\rho'_m = \rho' + \frac{\rho-\rho'}{2^{m+1}}, \quad \rho_m = \rho' + \frac{\rho-\rho'}{2^m},$$
$$\beta_m = 2\sigma^m - 1, \quad v_m = u^{\frac{\beta_m+1}{2}}.$$

Then from (2.6) we deduce

$$\begin{aligned} \phi_{m+1} :=&\Big(⨍_{S(\rho_{m+1})} u^{2\sigma^{m+1}}\,dx\,dt\Big)^{\frac{1}{2\sigma^{m+1}}}\\ =&\Big(⨍_{S(\rho_{m+1})} v_{m+1}^2\,dx\,dt\Big)^{\frac{1}{2\sigma^{m+1}}}\\ =&\Big(⨍_{S(\rho'_m)} v_m^{2\sigma}\,dx\,dt\Big)^{\frac{1}{2\sigma^{m+1}}}\\ \le&\Big(c(\rho_m-\rho'_m)^{-2}⨍_{S(\rho_m)} v_m^2\,dx\,dt\Big)^{\frac{1}{2\sigma^m}}\\ \le&(c2^m(\rho-\rho')^{-2})^{\frac{1}{2\sigma^m}}\phi_m. \end{aligned}$$

It easily follows that

$$\phi_{m+1} \le C(\rho-\rho')^{\sigma/(1-\sigma)}\phi_0, \quad m \ge 0.$$

Thus,

$$\limsup_{m\to\infty}\Big(⨍_{S(\rho_m)} u^{2\sigma^m}\,dx\,dt\Big)^{\frac{1}{2\sigma^m}} \le C(\rho-\rho')^{\sigma/(1-\sigma)}\Big(⨍_{S(\rho_0)} u^2\,dx\,dt\Big)^{1/2}$$

The statement of the theorem for $r = 2$ easily follows, in view of the well-known property

$$\operatorname{ess\,sup}\{u; A\} = \limsup_{q\to\infty}\Big(\int_A u^q \,dx\,dt\Big)^{1/q}.$$

The statement of the theorem for $r > 2$ follows by a direct application of the Hölder inequality

$$\Big(\fint_{S(\rho)} u^2 \,dx\,dt\Big)^{\frac{1}{2}} \le \Big(\fint_{S(\rho)} u^r \,dx\,dt\Big)^{-1/r}, \quad r > 2.$$

Now, we treat the case $r \in (0, 2)$. Here we need an additional iteration. In the integral identity (1.5) we choose the test function $\psi = u^\beta\eta^2$, where $\beta = -1+r$, and the cut-off function η has the same meaning as in (2.3). We arrive at the estimate (2.6) with the constant c, which depends on r. Iterating this relation as above, by a finite number of steps we obtain

$$\fint_{S(\rho')} u^2 \,dx\,dt \le c(\rho-\rho')^{-\xi_0}\Big(\fint_{S(\rho)} u^r \,dx\,dt\Big)^{-1/r},$$

where positive constants c and ξ_0 depend only on q, c_1, n, r and γ. Combining this inequality with the estimate (2.2) obtained earlier for $r \ge 2$, and using that ρ' can be taken arbitrarily, we arrive at the desired statement. □

Now let

$$Q(\rho) = \Pi_{\rho R} \times (-\rho^2 R^2, 0); \quad \rho \in (0, 1).$$

The following statement is proved as in the previous theorem. The only difference is that the value of β in the proof is taken to be less than -1.

Lemma 2.3. *Let $r > 0$ and $u(x, t)$ be a weak non-negative solution of* (1.1)*. Then the following estimate holds*

$$\inf_{Q(\rho')} u \ge c(\rho-\rho')^{-\xi}\Big(\fint_{Q(\rho)} u^{-r} \,dx\,dt\Big)^{-1/r}$$

where $1/3 \le \rho' < \rho \le 1/2$.

The next Lemma is a variant of Theorem 2.2 with a slightly different choice of the outer and inner cylinders.

Lemma 2.4. *Let the conditions of the previous lemma be fulfilled. Then the following estimate is valid*

$$\sup_{Q(1/3)} u \le c\Big(\fint_{Q(1/2)} u^2 \,dx\,dt\Big)^{1/2}.$$

3. Harnack type inequality

The technique of this section is based on ideas from [4].

Theorem 3.1. *Let $u(x, t)$ be a non-negative weak solution of equation* (1.1)*. Then there exist the constants $a_1(\Lambda, n)$ and $a_2(\Lambda, n)$ such that for any $s > 0$,*

$$|\{(x, t) \in D_1,\, \ln u(x, t) > s + a_1\}| \le c\frac{R^2|\Pi_R|}{s},$$

$$|\{(x, t) \in D_2,\, \ln u(x, t) < -s + a_1\}| \le c\frac{R^2|\Pi_R|}{s},$$

where
$$D_1 = \Pi_{R/2} \times (-R^2, -\frac{R^2}{2}), \quad D_2 = \Pi_{R/2} \times (-\frac{R^2}{2}, 0).$$

Proof. Assume $v(x,t) = -\ln u(x,t)$ and let $\eta(x,t) = \xi(x)w(t)$, where $w(t) = 1$ for $t \le -\tau_1 R^2$, $w(t) = 0$ for $t \ge -\frac{\tau_1}{2}R^2$, $0 \le w(t) \le 1$, $|w_t| \le \frac{c}{\tau_1 R^2}$; and $\xi(x) = 1$ for $x \in \Pi_{R/2}$, $\xi(x) = 0$ for $x \notin \Pi_{\frac{5R}{2}}$, $0 \le \xi(x) \le 1$, $|\xi_{x_i}| \le \frac{c}{w_i^{-1}(R)}$; $i = 1, 2, \dots, n$ moreover for $0 < \tau_1 < 1$, and the function $\xi(x)$ such that for an arbitrary C the set $\{x : \xi(x) \ge C\}$ is convex. From Theorem 2.2 we have (if only $\tau_1 < \tau_2 \le 1$)
$$\int_{\Pi_{\frac{5R}{6}}} v\xi^2\,dx\Big|_{-\tau_1 R^2}^{-\tau_2 R^2} + \frac{\gamma}{2}\int_{-\tau_2 R^2}^{-\tau_1 R^2} dt \int_{\Pi_{\frac{5R}{6}}} \xi^2 \sum_{i=1}^n \lambda_i(x,t) v_{x_i}^2\,dx \le c(\tau_2 - \tau_1)|\Pi_R|. \quad (3.1)$$
Indeed, since $\eta_t = 0$ for $t \in (-\tau_2 R^2, -\tau_1 R^2)$, according to Theorem 2.2, the left-hand side of (3.1) is estimated by
$$J = c(\gamma)\int_{-\tau_2 R^2}^{-\tau_1 R^2} dt \int_{\Pi_R \setminus \Pi_{R/2}} \sum_{i=1}^n \lambda_i(x,t)\xi_{x_i}^2\,dx.$$
(since $\xi_{x_i} \equiv 0$ in $\Pi_{R/2}$). Note that, for $x \in \Pi_R \setminus \Pi_{R/2}$, $w_i(|x_i|) \le cR$. Thus, $\rho(x) + \sqrt{|t|} \le cR$, i.e. $\lambda_i(x,t) \le c\frac{(w_i^{-1}(R))^2}{R^2}$. Hence we deduce that
$$\sum_{i=1}^n \lambda_i(x,t)\xi_{x_i}^2 \le c\frac{(w_i^{-1}(R))^2}{R^2}\frac{1}{(w_i^{-1}(R))^2} = \frac{c}{R^2}.$$
So,
$$J \le \frac{c}{R^2}(\tau_2 - \tau_1)R^2 = c(\tau_2 - \tau_1)$$
and (3.1) is proved. □

Now consider the functions
$$V(t) = \frac{\int_{\Pi_R} v(x,t)\xi^2(x)\,dx}{\int_{\Pi_R} \xi^2(x)\,dx}, \quad D(t) = \frac{\int_{\Pi_R} \big(v(x,t) - V(t)\big)^2 \xi^2(x)\,dx}{\int_{\Pi_R} \xi^2(x)\,dx}.$$
By the Poincare inequality [3], we have
$$D(t)\Big(\int_{\Pi_R} \xi^2(x)\,dx\Big)^2 \le cR^2|\Pi_R| \int_{\Pi_R} \xi^2(x) \sum_{i=1}^n \lambda_i(x,t) v_{x_i}^2\,dx,$$
that together with (3.1) gives
$$V(-\tau_1 \cdot R^2) - V(-\tau_2 \cdot R^2) + \frac{c}{R^2|\Pi_R|}\int_{-\tau_2 R^2}^{-\tau_1 R^2} dt \int_{\Pi_{R/2}} (v - V)^2\,dx \le c(\tau_2 - \tau_1).$$
When let τ_2 to τ_1 and assume $t = -\tau_1 R^2$. Then it follows from the above inequality that
$$R^2\frac{dV}{dt} + \frac{c}{|\Pi_R|}\int_{\Pi_{R/2}} (v - V)^2\,dx \le c. \quad (3.2)$$
Now consider the functions
$$\omega(x,t) = v(x,t) + \frac{c}{R^2}(-\frac{R^2}{2} - t),$$
$$W(t) = V(t) + \frac{c}{R^2}(-\frac{R^2}{2} - t).$$

Then from (3.2) we deduce

$$R^2\frac{dW}{dt}+\frac{c}{|\Pi_R|}\int_{\Pi_{R/2}}(\omega-W)^2\,dx\le 0\,. \tag{3.3}$$

From (3.3) it follows that the function $W(t)$ does not increase with respect to t, therefore for all $t\in(-R^2,-\frac{R^2}{2})$, we have

$$W(t)\ge W(-\frac{R^2}{2})=V(-\frac{R^2}{2}).$$

By the same reason, for $t\in(-\frac{R^2}{2},R^2)$, we have

$$W(t)\le W(-\frac{R^2}{2})=V(-\frac{R^2}{2}).$$

Assume that $s_1<V(-\frac{R^2}{2})$, and let

$$E_1(t)=\{x:x\in\Pi_{\frac{R}{2}},\,\omega(x,t)<s_1\}.$$

Then for $t\in(-R^2,-\frac{R^2}{2})$, we have

$$\begin{aligned}
0&\ge R^2\frac{dW}{dt}+\frac{c}{|\Pi_R|}\int_{E_1(t)}(w-W)^2\,dx\\
&\ge R^2\frac{dW}{dt}+\frac{c}{|\Pi_R|}\int_{E_1(t)}(W-s_1)^2\,dx\\
&=R^2\frac{dW}{dt}+c(W(t)-s_1)^2\frac{|E_1(t)|}{|\Pi_R|}.
\end{aligned}$$

Hence we deduce that

$$\begin{aligned}
R^2\int_{-R^2}^{-\frac{R^2}{2}}\frac{dW}{(W-s_1)^2}&\le-\frac{c}{|\Pi_R|}\int_{-R^2}^{-\frac{R^2}{2}}|E_1(t)|\,dt\\
&=-\frac{c}{|\Pi_R|}|\{(x,t)\in D_1;\omega(x,t)<s_1\}|\\
&=-\frac{c}{|\Pi_R|}m_1(s_1).
\end{aligned}$$

Thus,

$$-\frac{R^2}{W(t)-s_1}\Big|_{-R^2}^{-\frac{R^2}{2}}\le-\frac{c}{|\Pi_R|}m_1(s_1).$$

From this inequality we get that for all $s>0$,

$$\operatorname{meas}\big\{(x,t)\in D_1:\omega(x,t)<-s+V(-\frac{R^2}{2})\big\}\le c\frac{R^2|\Pi_R|}{s},$$

and

$$\operatorname{meas}\big\{(x,t)\in D_1:\ln u(x,t)>s-V(-\frac{R^2}{2})+\frac{c}{R^2}(-\frac{R^2}{2}-t)\big\}\le c\frac{R^2|\Pi_R|}{s}. \tag{3.4}$$

Now it suffices to take into account that $t\in(-R^2,-\frac{R^2}{2})$, and from (3.4) it follows that for $a_1=-V(-\frac{R^2}{2})+\frac{c}{2}$,

$$\begin{aligned}
&\operatorname{meas}\big\{(x,t)\in D_1:\ln u(x,t)>s+a_1\big\}\\
&\le\operatorname{meas}\big\{(x,t)\in D_1:\ln u(x,t)>s-V(-\frac{R^2}{2})+c(-\frac{R^2}{2}-t)\big\}
\end{aligned}$$

$$\leq c\frac{R^2|\Pi_R|}{s}$$

and the right side of the statement of the lemma is proved. Its second part is proved in the same way. Indeed, it suffices to obtain $s_2 > V(-\frac{R^2}{2})$ and

$$m_2(s_2) = |\{(x,t) \in D_2 : \ \omega(x,t) > s_2\}|.$$

Then

$$m_2(s_2) \leq c\frac{R^2|\Pi_R|}{(s_2 - V(-\frac{R^2}{2}))},$$

i.e. for any $s > 0$ and

$$a_2 = -V(-\frac{R^2}{2}) - \frac{c}{2}$$

we have

$$|\{(x,t) \in D_2 : \ln u(x,t) < -s + a_2\}| \leq c\frac{R^2|\Pi_R|}{s}\,.$$

The proof is complete.

It is easy to see that

$$a_1 - a_2 = c.$$

Now consider the functions $\omega_1(x,t) = u(x,t)e^{-a_1}$ and $\omega_2(x,t) = (u(x,t))^{-1}e^{a_2}$, where $u(x,t)$ is a non-negative weak solution of equation (1.1). Let $\frac{1}{3} \leq \rho' < \rho \leq \frac{1}{2}$, $r_\nu = \sigma^{-\nu}(1+\sigma)^{-1}$, $\nu = 0,1,2,\dots$; $s_1(\rho) = S(\rho)$, $s_2(\rho) = Q(\rho)$. In fact, from Theorem 3.1 it follows that

$$\sup_{s_j(\rho')} \omega_j^{r_\nu} \leq c(\rho-\rho')^{-(n+1)}\Big(\fint_{s_j(\rho)} \omega_j^2\,dx\Big)^{1/2},$$

$$\big|\big\{(x,t) \in s_j(\frac{1}{2}), \ln\omega_j > s\big\}\big| \leq c\frac{R^2|\Pi_R|}{s},$$

where $j = 1,2$.

Lemma 3.2. *If the conditions of the previous theorem are fulfilled, then the following estimates hold:*

$$\sup_{s_j(\frac{1}{3})} \omega_j \leq c, \quad j = 1,2.$$

Proof. It is obvious that it suffices to prove the lemma for $j = 1$. Consider the function $\varphi(\rho) = \sup_{s(\rho)} \ln\omega_1(x,t)$, and let $\kappa = \max\{c,1\}$. Then $\varphi(\rho)$ does not decrease with respect to ρ. If $\varphi(1/3) \leq 3\kappa$, then the lemma is proved with $c = e^{3\kappa}$.

Now let $\varphi(1/3) > 3\kappa$. Then for $\rho \in [1/3, 1/2]$,

$$\varphi(\rho) > 3\kappa$$

We show that for ρ' and ρ satisfying

$$\frac{1}{3} \leq \rho' < \rho \leq \frac{1}{2},$$

the it holds

$$\varphi(\rho') < \frac{3}{4}\varphi(\rho) + c(\rho-\rho')^{-8(n+1)}. \tag{3.5}$$

Let $s(\rho) = s^1(\rho) + s^2(\rho)$, where

$$s^1(\rho) = \{(x,t) \in s(\rho) : \frac{1}{2}\varphi(\rho) < \ln\omega_1(x,t) \leq \varphi(\rho)\},$$

$$s^2(\rho)=\{(x,t)\in s(\rho):\frac{1}{2}\varphi(\rho)\geq \ln\omega_1(x,t)\}.$$

We have

$$\begin{aligned}
\fint_{s(\rho)}\omega_1^{2r_\nu}\,dx\,dt&=\frac{1}{R^2|\Pi_\rho|}\Big(\fint_{s^1(\rho)}\omega_1^{2r_\nu}\,dx\,dt+\fint_{s^2(\rho)}\omega_1^{2r_\nu}\,dx\,dt\Big)\\
&\leq \frac{1}{R^2|\Pi_\rho|}\Big(c\frac{R^2|\Pi_R|}{\frac{1}{2}\varphi(\rho)}e^{2r_\nu\varphi(\rho)}+R^2|\Pi_\rho|e^{r_\nu\varphi(\rho)}\Big)\\
&\leq \frac{\kappa}{\varphi(\rho)}e^{2r_\nu\varphi(\rho)}+e^{r_\nu\varphi(\rho)}.
\end{aligned}$$

Since $\frac{\kappa}{\varphi(\rho)}<1/3$, then for any $\rho\in[\frac{1}{3},\frac{1}{2}]$ there exists r_ν such that

$$\frac{\kappa}{\varphi(\rho)}e^{2r_\nu\varphi(\rho)}\leq e^{r_\nu\varphi(\rho)}$$

and we can choose the non-negative integer ν so large that

$$r_\nu=\sigma^{-\nu}(1+\sigma)^{-1}\leq\frac{1}{\varphi(\rho)}\ln\frac{\varphi(\rho)}{\kappa},$$

and furthermore for any $\rho\in[\frac{1}{3},\frac{1}{2}]$

$$r_\nu\sigma=\frac{\sigma}{\sigma^\nu(1+\sigma)}>\frac{1}{\varphi(\rho)}\ln\frac{\varphi(\rho)}{\kappa},$$

since $\sigma>1$ and $\kappa\geq1$, $\frac{\varphi(\rho)}{\kappa}>3$; therefore,

$$\frac{1}{\varphi(\rho)}\ln\frac{\varphi(\rho)}{\kappa}=\frac{1}{\kappa}\cdot\frac{\ln\frac{\varphi(\rho)}{\kappa}}{\frac{\varphi(\rho)}{\kappa}}\leq\frac{\ln 3}{3}<\frac{1}{2}.$$

We have taken into account that for $x\geq3$ the function $\frac{\ln\chi}{\chi}$ decreases. Thus, we obtain

$$\begin{aligned}
\varphi(\rho')=\sup_{s(\rho')}\ln\omega_1(x,t)&=\frac{1}{2r_\nu\varphi(\rho)}\ln\sup_{s(\rho')}\omega_1^{2r_\nu}\\
&\leq\frac{1}{2r_\nu}\ln\big(c(\rho-\rho')^{-2(n+1)}\big)+\frac{\varphi(\rho)}{2}.
\end{aligned}$$

Then we have

$$\varphi(\rho')\leq\frac{1}{2}\varphi(\rho)\big(\frac{\sigma}{\ln\frac{\varphi(\rho)}{\kappa}}\ln\big(c(\rho-\rho')^{-2(n+1)}\big)+1\big).$$

From the above estimate it follows (3.5). Indeed, if the first term of the right-hand side is no greater than 1/2, then $\varphi(\rho')\leq\frac{3}{4}\varphi(\rho)$. But if

$$\frac{\sigma}{\ln\frac{\varphi(\rho)}{\kappa}}\ln\Big(c(\rho-\rho')^{-2(n+1)}\Big)>\frac{1}{2},$$

then

$$\ln\frac{\varphi(\rho)}{\kappa}<2\sigma\ln\big(c(\rho-\rho')^{-2(n+1)}\big)\leq4\ln\big(c(\rho-\rho')^{-2(n+1)}\big).$$

Hence it follows that

$$\varphi(\rho')\leq\varphi(\rho)\leq c(\rho-\rho')^{-8(n+1)},$$

and (3.5) is proved.

Now consider the sequence

$$\rho_j = \frac{1}{2} - \frac{1}{\sigma(1+j)}, \quad j = 0, 1, 2, \dots .$$

and using (3.5) we obtain

$$\begin{aligned}
\varphi(\frac{1}{3}) = \varphi(\rho_0) &< \frac{3}{4}\varphi(\rho_1) + \frac{c}{(\rho_1 - \rho_0)^{8(n+1)}} \\
&< (\frac{3}{4})^2\varphi(\rho_2) + c\Big((\rho_1 - \rho_0)^{-8(n+1)} + \frac{3}{4}(\rho_2 - \rho_1)^{-8(n+1)}\Big) \\
&< \dots < (\frac{3}{4})^m\varphi(\rho_m) + c\sum_{j=0}^{m-1}(\frac{3}{4})^j(\rho_{j+1} - \rho_j)^{-8(n+1)} \\
&= (\frac{3}{4})^m\varphi(\rho_m) + c\sum_{j=0}^{m-1}(\frac{3}{4})^j\Big(\sigma(j+1)(2+j)\Big).
\end{aligned}$$

From the continuity of the function ω_1 it follows $\varphi(\frac{1}{2}) < \infty$, thus

$$\varphi(\frac{1}{3}) \le 1 + c\sum_{j=0}^{\infty}(\frac{3}{4})^j\,(\sigma(j+1)(2+j)) \le c < \infty,$$

and the proof is complete. □

Theorem 3.3. *Let $u(x,t)$ be a non-negative weak solution of* (1.1) *whose coefficients satisfy conditions* (1.2)-(1.4). *Then there exists a constant $c = c(\gamma, n, q, c_1)$ such that*

$$\sup_{S(1/3)} u \le c \inf_{Q(1/3)} u.$$

Proof. From Lemma 3.2, we have

$$\sup_{S(1/3)} \omega_1(x,t) \sup_{Q(1/3)} \omega_2(x,t) = e^{-a_1+a_2} \sup_{S(1/3)} u(x,t) \sup_{Q(1/3)} (u(x,t))^{-1} \le c.$$

Thus,

$$\sup_{S(1/3)} u(x,t) \le c \inf_{Q(1/3)} u(x,t),$$

and the proof is complete. □

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