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STRUCTURE AND ASYMPTOTIC EXPANSION OF EIGENVALUES OF AN INTEGRAL-TYPE NONLOCAL PROBLEM

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Abstract. We study the structure of eigenvalues of second-order differential equations with nonlocal integral boundary conditions. Moreover, we consider the asymptotic expansion of the eigenvalues and the corresponding eigenfunctions, which shows that the eigenfunctions form a Riesz basis for $L^2([0,1],\mathbb{R})$.

1. Introduction

In recent years, many researchers studied different kinds of nonlocal boundary-value problems of ordinary differential equations, and in particular focused on the existence and multiplicity of nontrivial solutions for nonlinear nonlocal problems, see for example, [5, 8, 12, 15, 16, 19] for multi-point boundary-value problems and [1, 3, 9, 10, 20] for general nonlocal boundary-value problems.

However, the study on the eigenvalue theory of the corresponding nonlocal linear problems appears to just start. Ma and O'Regan [16] constructed all real eigenvalues of the problem

$$\begin{gathered} -y''(x) = \lambda y(x), \quad x \in (0,1), \\ y(0) = 0, \quad y(1) = \sum_{k=1}^{m} \alpha_k y(\eta_k), \end{gathered} \tag{1.1}$$

where $m \in \mathbb{N}$, $\alpha = (\alpha_1, \cdots, \alpha_m) \in \mathbb{R}_+^m$ satisfying the nondegeneracy condition $\sum_{k=1}^m |\alpha_k| < 1$ and $\eta = (\eta_1, \cdots, \eta_m) \in \Delta^m := \{(\eta_1, \cdots, \eta_m) \in \mathbb{R}^m : 0 < \eta_1 < \cdots < \eta_m < 1\}$ are taken as rational. We note that the eigenvalues of (1.1) can be analyzed using elementary method because all solutions of (1.1) can be found explicitly. However, even for (1.1), as far as we know, the first complete eigenvalue theory was proved in [4]. In particular, Gao, Sun and Zhang completely characterized the structure of eigenvalues of (1.1) for all $\alpha \in \mathbb{R}_+^m$ and $\eta \in \Delta^m$. Moreover, they gave the complete structure of eigenvalues of general multi-point boundary-value problem

$$\begin{gathered} -y''(x) + q(x)y(x) = \lambda y(x), \quad x \in (0,1), \\ y(0) = 0, \quad y(1) = \sum_{k=1}^{m} \alpha_k y(\eta_k), \end{gathered} \tag{1.2}$$

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where $q \in L^1([0,1],\mathbb{R})$ and $\alpha \in \mathbb{R}_+^m, \eta \in \Delta^m$. Note that problems (1.1) and (1.2) are non-symmetry problems. It was proved in [4] that (1.2) may admit complex eigenvalues and has always a sequence of real eigenvalues tending to infinity.

We will extend the above results to the general nonlocal integral boundary-value problem

$$\begin{gathered} -y''(x)+q(x)y(x)=\lambda y(x), \quad x\in(0,1),\\ y(0)=0, \quad y(1)=\int_0^1 k(x)y(x)dx, \end{gathered} \tag{1.3}$$

where $q \in L^1([0,1],\mathbb{R})$ and $k \in C^2([0,1],\mathbb{R})$. We will show that the eigenvalues of the problem (1.3) have the similar structure to those of (1.2). In fact, problem (1.3) can be considered as a version of (1.2) with continuous boundary condition of (1.2).

The set of all eigenvalues of (1.3) is denoted by $\Sigma_k^q \in \mathcal{C}$, which is called the spectrum of operator $\mathcal{A}$, where the linear operator $\mathcal{A} : D(\mathcal{A})(\subset L^2([0,1],\mathbb{R})) \to L^2([0,1],\mathbb{R})$ is defined by

$$\mathcal{A}(y)=-y''(x)+q(x)y(x)$$

with

$$D(\mathcal{A})=\big\{y\in H^2(0,1): y(0)=0,\ y(1)=\int_0^1 k(x)y(x)dx\big\}.$$

When $q \equiv 0$, Equation (1.3) becomes

$$\begin{gathered} -y''(x)=\lambda y(x), \quad x\in(0,1),\\ y(0)=0, \quad y(1)=\int_0^1 k(x)y(x)dx. \end{gathered} \tag{1.4}$$

We can define a linear operator $\mathcal{A}_0 : D(\mathcal{A}_0)(\subset L^2([0,1],\mathbb{R})) \to L^2([0,1],\mathbb{R})$ by

$$\mathcal{A}_0(y)=-y''(x), \tag{1.5}$$

with

$$D(\mathcal{A}_0)=\big\{y\in H^2([0,1],\mathbb{R}): y(0)=0,\ y(1)=\int_0^1 k(x)y(x)dx\big\}$$

and a bounded perturbation linear operator

$$(\mathcal{B}_0 y)(x)=q(x)y(x), \tag{1.6}$$

on $L^1([0,1],\mathbb{R})$. The eigenvalues of (1.4) are exact the eigenvalues of the operator $\mathcal{A}_0$, which can be analyzed using elementary method. However, as far as we know, even for this simple case, the spectrum theory is incomplete in the literature.

For some special functions q and k, we can adopt the backstepping method (which comes from Krstic) to obtain the existence and explicit expression of eigenvalues via transferring it into well-known eigenvalue problem, see [13] and related references. Such similar method can be used to prove Theorem 3.6 for some special functions q and k. However, for general function pair q and k, this method does not work. One motivation of this paper is to develop a method for general functions q and k. The results of (1.3) can be used to study the existence of nonlinear differential equations with integral boundary condition. Besides, the stabilization controller design of heat equation by backstepping method strongly depends on the complete spectrum analysis for the problem (1.3), which is another important motivation of this paper.

Basically, eigenvalues of (1.3) are zeros of some entire functions. To study the distributions of eigenvalues, we will consider (1.3) as a perturbation of (1.4). To obtain the existence of infinitely many real eigenvalues as in Theorem 3.7, some properties of almost periodic functions [16, 17] will be used. To pass the results of (1.4) to those with general potentials q, many techniques will be exploited. Moreover, some basic estimates for fundamental solutions of (1.4) play an important role.

This article is organized as follows. In Section 2, we will give some detailed analysis on problem (1.4). In Section 3, after developing some basic estimates, we will prove Theorems 3.6 and 3.7. In Section 4, we will give the asymptotic expansion for the eigenvalues and eigenfunctions of (1.4). In Section 5, we will prove the existence of eigenvalues for (1.3) and corresponding eigenfunctions forming Riesz basis for $L^2([0,1],\mathbb{R})$.

2. Structure of eigenvalues of the zero potential

In this section, we first consider the spectrum for (1.4), which has the zero potential. Let us use Σ_k^0 to denote the set of all eigenvalues of (1.4).

Let $\lambda\in\mathbb{C}$, the complex solutions of (1.4) satisfying $y(0)=0$ are $y(x)=cS_\lambda(x)$, where $c\in\mathbb{C}$ and

$$S_\lambda(x):=\frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}}=\sum_{k=0}^{+\infty}\frac{(-1)^k}{(2k+1)!}\lambda^k x^{2k+1},\ x\in[0,1].$$

Notice that $S_\lambda(x)$ is an entire function of $\lambda\in\mathbb{C}$. Define $C_\lambda(x):=\cos(\sqrt{\lambda}x)$ and

$$M_0(\lambda):=S_\lambda(1)-\int_0^1 k(x)S_\lambda(x)dx=\frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}}-\int_0^1 k(x)\frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}}dx. \tag{2.1}$$

Then $\lambda\in\Sigma_k^0$ if and only if λ satisfies

$$M_0(\lambda)=0.$$

We recall some properties of almost periodic functions which will be used later. We refer the readers to [2] for more information on almost periodic functions. Suppose that $f:\mathbb{R}\to\mathbb{R}$ is a bounded continuous function. We say that f is almost periodic if for any $\varepsilon>0$, there exists $l_\varepsilon>0$ such that for any $a\in\mathbb{R}$, there exists $b\in[a,a+l_\varepsilon]$ such that $\|f(\cdot+b)-f(\cdot)\|_{L^\infty}<\varepsilon$. If $f:\mathbb{R}\to\mathbb{R}$ is an almost periodic function, then for any $A\in\mathbb{R}$, we have

$$\inf_{u\in[A,\infty)}f(u)=\inf_{u\in\mathbb{R}}f(u),\qquad \sup_{u\in[A,\infty)}f(u)=\sup_{u\in\mathbb{R}}f(u).$$

Moreover, if f is non-zero and $\bar{f}=\lim_{T\to+\infty}\frac{1}{T}\int_0^T f(u)du=0$, then f is oscillatory as $u\to+\infty$. In particular, $f(u)$ has a sequence of positive zeros tending to $+\infty$.

Lemma 2.1. *If $k\in C^2([0,1],\mathbb{R})$, then $\Sigma_k^0\cap\mathbb{R}=\{\overline{\lambda}_n\}$ which satisfies*

$$\overline{\lambda}_1\le\overline{\lambda}_2\le\cdots\overline{\lambda}_n\le\cdots,\quad \lim_{n\to+\infty}\overline{\lambda}_n=+\infty.$$

Proof. Let us first consider possible positive eigenvalues $\lambda=\alpha^2$ of (1.4), where $\alpha>0$. By equation (1.4), we have

$$F(\alpha):=\sin\alpha-\int_0^1 k(x)\sin(\alpha x)dx=0$$

It is easy to check the function $F(\alpha)$ is a non-zero, almost periodic function and has mean value zero. Therefore, $F(\alpha)$ has many positive zeros tending to $+\infty$, and hence Σ_k^0 contains a sequence of positive eigenvalues tending to $+\infty$.

Next, we consider possible negative eigenvalues $\lambda = -\alpha^2$ of (1.4), where $\alpha > 0$. By the first equality of (2.3) and (1.4), we have

$$\bar{F}(\alpha) := \sinh\alpha - \int_0^1 k(x)\sinh(\alpha x)dx = 0. \tag{2.2}$$

One has

$$\lim_{\alpha\to+\infty} \frac{\bar{F}(\alpha)}{\sinh\alpha} = 1,$$

notice that $\bar{F}(\alpha)$ is analytic in α, thus $\bar{F}(\alpha) = 0$ has at most finitely many positive solutions. Hence Σ_k^0 contains at most finitely many negative eigenvalues.

Because both $F(\alpha) = 0$ and $\bar{F}(\alpha) = 0$ have only isolated solutions, the above two cases show that the result holds. □

Next we show that Σ_k^0 contains only real eigenvalues if $\int_0^1 k^2(x)dx \le 1$.

Lemma 2.2. *Assume $k \in C^2([0,1],\mathbb{R})$ satisfying $\int_0^1 k^2(x)dx \le 1$. Then Σ_k^0 contains only real eigenvalues. Moreover, $\Sigma_k^0 \subset (\frac{\pi^2}{4}, +\infty)$.*

Proof. Suppose that $\lambda = w^2 \in \Sigma_k^0$, where $w = u + iv$, $u, v \in \mathbb{R}$. We would assert that $v = 0$ under the assumption. Otherwise, assume that $v \neq 0$. We have

$$\sin w - \int_0^1 k(x)\sin(wx)dx = 0.$$

Note that the following elementary equalities hold for any $u, v \in \mathbb{R}$,

$$\sin(u+iv) = \sin u\cosh v + i\cos u\sinh v, \quad |\sin(u+iv)|^2 = \sin^2 u + \sinh^2 v. \tag{2.3}$$

Then

$$\begin{gathered} \sin u\cosh v = \int_0^1 k(x)\sin(ux)\cosh(vx)dx, \\ \cos u\sinh v = \int_0^1 k(x)\cos(ux)\sinh(vx)dx. \end{gathered}$$

It follows from Hölder inequality that

$$\begin{aligned} 1 &= \sin^2 u + \cos^2 u \qquad (2.4)\\ &= \Big(\int_0^1 k(x)\sin(ux)\frac{\cosh(vx)}{\cosh v}dx\Big)^2 + \Big(\int_0^1 k(x)\cos(ux)\frac{\sinh(vx)}{\sinh v}dx\Big)^2 \qquad (2.5)\\ &< \int_0^1 k^2(x)dx\int_0^1 \cos^2(ux)dx + \int_0^1 k^2(x)dx\int_0^1 \sin^2(ux)dx \qquad (2.6)\\ &= \int_0^1 k^2(x)dx, \qquad (2.7) \end{aligned}$$

which is a contradiction. Thus $v = 0$. On the other hand,

$$M_0(0) = 1 - \int_0^1 xk(x)dx \ge 1 - \Big(\int_0^1 k^2(x)dx\Big)^{1/2} > 0, \tag{2.8}$$

hence we have $\Sigma_k^0 \in (0, +\infty)$. Finally, for any $u \in (0, \frac{\pi}{2}]$, by the Hölder inequality, we know function $F(u)$ satisfies

$$\begin{aligned} F(u) &= \sin u - \int_0^1 k(x)\sin(ux)dx \\ &\geq \sin u - \int_0^1 |k(x)|\sin(ux)dx \\ &> \sin u - \int_0^1 |k(x)|dx \sin u \geq 0. \end{aligned}$$

Therefore, we obtain that $\Sigma_k^0 \in (\frac{\pi^2}{4}, +\infty)$. □

3. Structure of eigenvalues of non-zero potentials

Given $q \in L^1((0,1), \mathbb{R})$ and complex parameter $\lambda \in \mathbb{C}$, the fundamental solutions of (1.3) are denoted by $y_m(x, \lambda, q), m = 1, 2$, which are solutions satisfying the initial values

$$y_1(0,\lambda,q) = y_2'(0,\lambda,q) = 1, \quad y_1'(0,\lambda,q) = y_2(0,\lambda,q) = 0. \tag{3.1}$$

Notice that $y_m(x, \lambda, q)$ are entire functions of $\lambda \in \mathbb{C}$, To study (1.3), we introduce

$$M_q(\lambda) := y_2(1,\lambda,q) - \int_0^1 k(x) y_2(x,\lambda,q)dx, \quad \lambda \in \mathbb{C}. \tag{3.2}$$

We use Σ_k^q to denote the set of all eigenvalues of (1.3). Then $\lambda \in \Sigma_k^q$ if and only if $M_q(\lambda) = 0$.

We will need the following basic estimates, whose proofs are much similar to those of [4, Lemma 3.1, Lemma 3.2, Lemma 3.3, Lemma 3.4]. Here we only state them without their proofs.

Lemma 3.1. *If $\beta \in (0,1)$, one has*

$$\begin{aligned} \lim_{v\in\mathbb{R}, |v|\to+\infty} \frac{|\sin(u+iv)|}{\exp|v|} &= \frac{1}{2}, \\ \lim_{v\in\mathbb{R}, |v|\to+\infty} \frac{|\sin\beta(u+iv)|}{\exp|v|} &= 0 \end{aligned} \tag{3.3}$$

uniformly in $u \in \mathbb{R}$.

Lemma 3.2. *There exists a constant $c(k) > 0$ and a sequence a_n of increasing positive numbers such that $a_n \to +\infty$ and $(-1)^n F(a_n) > c(k)$, where $F(u) := \sin u - \int_0^1 k(x)\sin(ux)dx$.*

Lemma 3.3. *Given $q \in L^1((0,1), \mathbb{R})$ and complex parameter $\lambda \in \mathbb{C}$. Then the following inequalities hold for all $x \in [0,1]$,*

$$\begin{aligned} |y_1(x,\lambda,q) - C_\lambda(x)| &\leq \frac{1}{|\sqrt{\lambda}|}\exp(|\operatorname{Im}\sqrt{\lambda}|x + \|q\|_{L^1[0,x]}). \\ |y_2(x,\lambda,q) - S_\lambda(x)| &\leq \frac{1}{|\lambda|}\exp(|\operatorname{Im}\sqrt{\lambda}|x + \|q\|_{L^1[0,x]}). \\ |y_1'(x,\lambda,q) - C_\lambda'(x)| &\leq \|q\|\exp(|\operatorname{Im}\sqrt{\lambda}|x + \|q\|_{L^1[0,x]}). \\ |y_2'(x,\lambda,q) - S_\lambda'(x)| &\leq \frac{\|q\|}{|\sqrt{\lambda}|}\exp(|\operatorname{Im}\sqrt{\lambda}|x + \|q\|_{L^1[0,x]}). \end{aligned}$$

Lemma 3.4. *The following estimate holds for* $M_q(\lambda)$,

$$|M_q(\lambda) - M_0(\lambda)| \le \frac{B}{|w|^2}\exp(|\operatorname{Im} w|), \quad w := \sqrt{\lambda} \in \mathbb{C}. \tag{3.4}$$

where

$$B = \exp(\|q\|_{L^1[0,1]}) + \exp(\|q\|_{L^1[0,1]})\int_0^1 |k(x)|dx.$$

Lemma 3.5. *One has* $M_q(\lambda) \neq 0$ *on* $\mathbb{R}$. *Consequently, there exists* $\lambda_0 \in \mathbb{R}$ *such that* λ_0 *does not belong to* Σ_k^q.

Proof. Otherwise, we have $M_q(\lambda) \equiv 0$, Notice that

$$M_0(u^2) \equiv \frac{F(u)}{u}, \quad u > 0. \tag{3.5}$$

Let $\lambda = a_n^2$ in Lemma 3.2, we have

$$\Big|\frac{F(a_n)}{a_n}\Big| = |M_0(a_n^2)| \le \frac{B}{a_n^2}.$$

Hence, $\lim_{n\to+\infty} |F(a_n)| \le \lim_{n\to+\infty} \frac{B}{a_n} = 0$, which contradicts Lemma 3.2. $\square$

Theorem 3.6. *If* $q \in L^1([0,1],\mathbb{R})$ *and* $k \in C^2([0,1],\mathbb{R})$, *then* Σ_k^q *is composed of a sequence* $\lambda_n = \{\lambda_n(q)\} \in \mathbb{C}$ *which satisfies*

$$\operatorname{Re}\lambda_1 \le \operatorname{Re}\lambda_2 \le \cdots \operatorname{Re}\lambda_n \le \cdots, \quad \lim_{n\to+\infty} \operatorname{Re}\lambda_n = +\infty.$$

Proof. By Lemma 3.5, there exists $\lambda_0 \in \mathbb{R}$ such that $\lambda_0 \notin \Sigma_k^q$, which implies that the problem

$$\begin{gathered} -y''(x) + q(x)y(x) - \lambda_0 y(x) = 0, \quad x \in (0,1), \\ y(0) = 0, \quad y(1) = \int_0^1 k(x)y(x)dx \end{gathered} \tag{3.6}$$

has only the trivial solution $y = 0$.

Let $G_0(x,u)$ be the Green function of (3.6). Then $\lambda \in \Sigma_k^q$ if and only if $\lambda \neq \lambda_0$ and

$$\begin{gathered} -y''(x) + (q(x) - \lambda_0)y(x) = (\lambda - \lambda_0)y(x), \\ y(0) = 0, \quad y(1) = \int_0^1 k(x)y(x)\,dx, \end{gathered} \tag{3.7}$$

has a nontrivial solution y. In other words, $\lambda \in \Sigma_k^q$ if and only if the equation

$$y = (\lambda - \lambda_0)L_q y$$

has a non-trivial solution y, where

$$L_q y(x) := \int_0^1 G_0(x,z)(q(z) - \lambda_0)y(z)dz.$$

Since L_q is a compact linear operator, one sees that this happens when and only when

$$\frac{1}{\lambda - \lambda_0} \in \sigma(L_q) \subset \mathbb{C},$$

where $\sigma(L_q)$ is the spectrum of L_q. Hence Σ_k^q consists of a sequence of eigenvalues which can accumulate only at infinity.

For $\lambda \in \mathbb{C}$, denote

$$\lambda = w^2, \quad w = \sqrt{\lambda} = u + iv, \quad u, v \in \mathbb{R}.$$

Suppose that $\lambda \in \Sigma_k^q$ and $\lambda \neq 0$. Then $M_q(\lambda) = 0$ and Lemma 3.4 imply that

$$\begin{aligned} \frac{B}{|w|^2} \exp(|v|) &\geq |M_q(\lambda) - M_0(\lambda)| = |M_0(\lambda)| \\ &= \Big| \frac{\sin w - \int_0^1 k(x) \sin(wx) dx}{w} \Big| \\ &\geq \frac{|\sin(u+iv)| - \int_0^1 |k(x) \sin(u+iv)x| dx}{|w|}. \end{aligned}$$

We conclude that all non-zero eigenvalues $\lambda \in \Sigma_k^q$ satisfy

$$|w| \frac{|\sin(u+iv)| - \int_0^1 |k(x) \sin(u+iv)x| dx}{\exp|v|} \leq B. \tag{3.8}$$

Let us derive some sequences from estimate (3.8) for $\lambda \in \Sigma_k^q$.

Case 1: Since $|w| \geq |v|$, it follows from the uniform limits in (3.3) that

$$\lim_{|v| = |\operatorname{Im} w| \to +\infty} |w| \frac{|\sin(u+iv)| - \int_0^1 |k(x) \sin(u+iv)x| dx}{\exp|v|} = +\infty. \tag{3.9}$$

Thus, there exists a constant $h > 0$ such that

$$\lambda \in \Sigma_k^q \Longrightarrow w = \sqrt{\lambda} \in H_h := \{w \in \mathbb{C} : |\operatorname{Im} w| < h\}. \tag{3.10}$$

The horizontal strip H_h of it in the w-plane is transformed to the half-plane P_r, in the λ-plane:

$$\Sigma_k^q \subset P_r := \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda > r\},$$

where $r := -h^2$.

Case 2: Let $\bar{r} > -h^2$, next we assert that

$$\Sigma_k^q \cap \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda \leq \bar{r}\} = \Sigma_k^q \cap \{\lambda \in \mathbb{C} : -h^2 < Re\lambda \leq \bar{r}\}$$

contains at most finitely many eigenvalues. Otherwise, suppose that

$$\Sigma_k^q \cap \{\lambda \in \mathbb{C} : -h^2 < \operatorname{Re} \lambda \leq \bar{r}\} \tag{3.11}$$

contains infinitely many λ_n, $n \in \mathbb{N}$. Since $M_q(\lambda) = 0$ has only isolated solutions, we have necessarily $|\operatorname{Im} \lambda_n| \to +\infty$. by denoting $\sqrt{\lambda_n} = u_n + iv_n$, one has

$$-h^2 < u_n^2 - v_n^2 \leq \bar{r}, \; 2|u_n||v_n| \to +\infty.$$

In particular, $|v_n| \to +\infty$, Now estimate (3.8) reads

$$\frac{|\sin(u_n + iv_n)|}{\exp|v_n|} \leq \frac{\int_0^1 |k(x) \sin(u_n + iv_n)x| dx}{\exp|v_n|} + o(1), \quad \text{as } n \to \infty.$$

This is impossible because the estimate in (3.3). Combining Cases 1 and 2, we know that Σ_k^q can be listed as in Theorem 3.6. $\square$

Theorem 3.7. *If $q \in L^1([0,1], \mathbb{R})$ and $k \in C^2([0,1], \mathbb{R})$, then $\Sigma_k^q \cap \mathbb{R} = \{\overline{\lambda}_n = \overline{\lambda}_n(q)\}$ which satisfies*

$$\overline{\lambda}_1 \leq \overline{\lambda}_2 \leq \cdots \overline{\lambda}_n \leq \cdots, \quad \lim_{n \to +\infty} \overline{\lambda}_n = +\infty.$$

Proof. We need to only consider the positive eigenvalues of (1.3). Let $\lambda = a_n^2$ in Lemma 3.4, according to (3.5), we have

$$|M_q(a_n^2) - M_0(a_n^2)| = \Big|M_q(a_n^2) - \frac{F(a_n)}{a_n}\Big| \le \frac{B}{a_n^2}, \ \forall n \in \mathbb{N}. \tag{3.12}$$

Since $a_n \to +\infty$, w.l.o,g, we can assume that $a_n \ge \frac{2B}{c(k)}$ for all $n \in \mathbb{N}$; therefore,

$$|a_n M_q(a_n^2) - F(a_n)| \le \frac{B}{a_n} \le \frac{c(k)}{2}, \quad \forall n \in \mathbb{N}.$$

by using Lemma 3.2, we conclude that $(-1)^n M_q(a_n^2) > 0, \forall n \in \mathbb{N}$. Hence $M_q(\lambda) = 0$ has at least one positive solution $\bar{\lambda}_n$ in each interval $(a_n^2, a_{n+1}^2), n \in \mathbb{N}$. Combining with Theorem 3.6, we have $\Sigma_k^q \cap \mathbb{R}$ consists of a sequence of real eigenvalues tending to $+\infty$, hence $\Sigma_k^q \cap \mathbb{R}$ can be listed as in Theorem 3.7. □

4. Asymptotic expansion and Riesz basis

Above we have discussed the structure of eigenvalues of (1.3). In this section, we give the quantity asymptotic estimate for eigenvalues and eigenfunctions of (1.4) and (1.3). Moreover, we will show the eigenfunctions forms Resis basis of $L^2([0,1],\mathbb{R})$. We first make some preparations for the main theory.

Definition 4.1. A sequence $\{e_n\}_1^\infty \subset L^2([0,1],\mathbb{R})$ is called a basis in $L^2([0,1],\mathbb{R})$ if for any $g \in L^2([0,1],\mathbb{R})$ there exists a unique sequence $\{a_n\}_1^\infty$ of real numbers such that $g = \sum_{n=1}^\infty a_n e_n$ in $L^2([0,1],\mathbb{R})$. A basis $\{e_n\}_1^\infty$ in $L^2([0,1],\mathbb{R})$ is called a Riesz basis when the series $\sum_{n=1}^\infty a_n e_n$, with real coefficients a_n, converges in $L^2([0,1],\mathbb{R})$ if and only if $\sum_{n=1}^\infty a_n^2 < \infty$.

The following Theorem is very useful in checking the Riesz basis for the generalized eigenfunctions of $\mathcal{A}_0$.

Theorem 4.2 ([6, 7]). *Let $\mathcal{T}$ be a densely defined discrete operator, that is $(\lambda I - \mathcal{T})^{-1}$ is compact for some λ in a Hilbert space H with $\{z_n\}_1^{+\infty}$ being a Riesz basis for H. If there are an $N > 0$ and a sequence of generalized eigenvector $\{x_n\}_{N+1}^{+\infty}$ of $\mathcal{T}$ such that*

$$\sum_{n=N+1}^{\infty} \|x_n - z_n\|^2 < \infty,$$

then

(i) *There are an $M > N$ and generalized eigenvectors $\{x_{n0}\}_1^M$ of $\mathcal{T}$ such that $\{x_{n0}\}_1^M \cup \{x_n\}_{N+1}^{+\infty}$ forms a Riesz basis for H.*
(ii) *Let $\{x_{n0}\}_1^M \cup \{x_n\}_{N+1}^{+\infty}$ be eigenvalues $\{\sigma_n\}_1^{+\infty}$ of $\mathcal{T}$. Then $\sigma(\mathcal{T}) = \{\sigma_n\}_1^{+\infty}$, in which σ_n is counted according to its algebraic multiplicity.*
(iii) *If there is an $C_0 > 0$ such that $\sigma_n \ne \sigma_m$ for all $n, m > C_0$, then there is an $N_0 > C_0$ such that all $\sigma_n, n > N_0$ are algebraically simple.*

Lemma 4.3. *The eigenvalues of* (1.4) *have the asymptotic expansion*

$$\lambda_n = n^2\pi^2 + 2(k(0) - k(1)) + O(\frac{1}{n}).$$

Proof. According to Theorems 3.6 and 3.7, we know that (1.4) has a sequence of eigenvalues. In fact, the eigenvalues λ_n of (1.4) satisfy

$$\begin{aligned}\sin\sqrt{\lambda_n} &= \int_0^1 k(x)\sin\sqrt{\lambda_n}x\,dx \qquad (4.1)\\ &= -\frac{1}{\sqrt{\lambda_n}}\Big(k(1)\cos\sqrt{\lambda_n} - k(0) - \int_0^1 k'(x)\cos\sqrt{\lambda_n}x dx\Big). \qquad (4.2)\end{aligned}$$

Therefore,

$$\sqrt{\lambda_n} = n\pi + O(\frac{1}{\sqrt{\lambda_n}}). \tag{4.3}$$

At the same time, we know that

$$\sin\sqrt{\lambda_n} = O(\frac{1}{\sqrt{\lambda_n}}),$$
$$\cos\sqrt{\lambda_n} = 1 - O(\frac{1}{\sqrt{\lambda_n}}).$$

Taking them into (4.1), we have

$$O(\frac{1}{\sqrt{\lambda_n}}) = \frac{k(0)}{\sqrt{\lambda_n}} - \frac{k(1)}{\sqrt{\lambda_n}} + O(\frac{1}{\lambda_n}). \tag{4.4}$$

Hence, by (4.3), we have

$$\sqrt{\lambda_n} - n\pi = \frac{k(0)-k(1)}{\sqrt{\lambda_n}} + O(\frac{1}{\lambda_n}), \tag{4.5}$$

we can obtain

$$\lambda_n = n^2\pi^2 + 2(k(0)-k(1)) + O(\frac{1}{n}), \tag{4.6}$$

which completes the proof. □

Lemma 4.4. *Let $\{\lambda_n\}_1^\infty$ be the eigenvalues of operator $\mathcal{A}_0$. Then the corresponding eigenfunctions $\{y_n\}_1^\infty$ have the asymptotic expressions*

$$y_n(x) = \sin n\pi x + O(\frac{1}{n}).$$

Moreover, the generalized eigenfunctions of $\mathcal{A}_0$ forms a Riesz basis of $L^2([0,1],\mathbb{R})$.

Proof. According to (1.4) and Lemma 4.3 for λ_n, its corresponding eigenfunction has the asymptotic form

$$y_n(x) = \sin\sqrt{\lambda_n}x = \sin n\pi x + O(\frac{1}{n}). \tag{4.7}$$

Next, we show that $\sum_{n=1}^\infty \int_0^1 |\sin(\sqrt{\lambda_n}x) - \sin(n\pi x)|^2 dx < +\infty$. In fact,

$$\int_0^1 |\sin(\sqrt{\lambda_n}x) - \sin(n\pi x)|^2 dx \le C\cdot O(\frac{1}{n^2})$$

by the eigenvalue expansion, where C is a constant number large enough. Therefore,

$$\sum_{n=1}^{+\infty}\int_0^1 |\sin(\sqrt{\lambda_n}x) - \sin(n\pi x)|^2 dx < +\infty.$$

By Theorem 4.2, we know that the generalized eigenfunctions of $\mathcal{A}_0$ forms a Riesz basis of $L^2([0,1],\mathbb{R})$, which completes the proof. □

For obtaining the asymptotic expansion for the eigenvalue of Σ_k^q, we show the relationship between Σ_k^q and Σ_k^0. Intuitively, for the problem (1.4) and (1.3), if q is a constant, we know Σ_k^q is a constant translation of Σ_k^0, In fact, if q is not a constant, we also know the asymptotic expansion for Σ_k^q in terms of Σ_k^0, which is borrowed from the paper [7].

Definition 4.5. A linear operator $\mathcal{A}_0$ in a Hilbert space H is called discrete-type(or $[D]$-class for short), if there are Riesz basis $\{\phi_n\}_1^\infty$ of H, complex series $\{\lambda_n\}_1^\infty$ and an integer $N > 0$ such that

(i) $\lim_{n\to+\infty} |\lambda_n| = \infty$, $\lambda_n \neq \lambda_m$ as $n, m > N$.
(ii) $\mathcal{A}_0\phi_n = \lambda_n\phi_n$, $n > N$.
(iii) $\mathcal{A}_0[\phi_1, \phi_2, \cdots, \phi_N] \subset [\phi_1, \phi_2, \cdots, \phi_N]$ and $\mathcal{A}_0$ has spectrum $\{\lambda_n\}_1^N$ in $[\phi_1, \phi_2, \cdots, \phi_N]$, where $[\phi_1, \phi_2, \cdots, \phi_N]$ is the linear subspace spanned by $\{\phi_n\}_1^N$.

Lemma 4.4 show that $\mathcal{A}_0$ defined in (1.5) is a $[D]$-class. The following result can be concluded from the proof of a more general result in [14] (see also [11] and [18]).

Theorem 4.6 ([14]). *Suppose that $\mathcal{A}_0$ is of $[D]$-class satisfying conditions of definition 4.5 in a Hilbert space H. Let $d_n := \min_{n\neq m} |\lambda_n - \lambda_m|$ and assume that $\sum_{n>N}^\infty d_n^{-2} < \infty$. Then for any linear bounded perturbation operator $\mathcal{B}_0$ on H, there are constants $C, L > 0$, an integer $M > 0$, and eigenpairs $\{\mu_n, \psi_n\}_M^\infty$ of $\mathcal{A}_0 + \mathcal{B}_0$ such that*

(i) $|\mu_n - \lambda_n| \le C, \forall n \ge M$.
(ii) $\|\psi_n - \phi_n\| \le L d_n^{-1}$, *$n > M$, and hence* $\sum_M^\infty \|\psi_n - \phi_n\|^2 < \infty$.

We use Theorem 4.6 for $\mathcal{A}_0, \mathcal{B}_0$, where $\mathcal{A}_0$ is defined by (1.5), and operator $\mathcal{B}_0$ is a perturbation of $\mathcal{A}_0$, such that $\mathcal{A} = \mathcal{A}_0 + \mathcal{B}_0$, we can obtain the following result for $\mathcal{A}$.

Theorem 4.7. *Suppose that $k \in C^2([0,1], \mathbb{R})$, $q \in L^1([0,1], \mathbb{R})$, $\{\mu_n, \psi_n\}_1^\infty$ are eigenpairs of operator $\mathcal{A}$, $\{\lambda_n, y_n\}_1^\infty$ are eigenpairs of operator $\mathcal{A}_0$. Then the following results hold.*

(i) *$\mathcal{A} = \mathcal{A}_0 + \mathcal{B}_0$ is $[D]$-class.*
(ii) *The eigenvalue of $\mathcal{A}_0 + \mathcal{B}_0$ have asymptotic expansion*
$$\mu_n = \lambda_n + O(1), \quad n \to +\infty.$$
(iii) *The corresponding eigenfunctions $\{\psi_n(x)\}$ of $\mathcal{A}$ have the asymptotic expansion*
$$\psi_n(x) = y_n(x) + \varepsilon_n(x), \quad n \to +\infty, \tag{4.8}$$
where $\|\varepsilon_n\|_{L^2([0,1],\mathbb{R})} = O(\frac{1}{n})$. Moreover,
$$\sum_{n=M}^\infty \|\psi_n - y_n\|_{L^2(0,1)}^2 < \infty. \tag{4.9}$$
where $y_n(\cdot)$ is the eigenfunctions of (1.4). Moreover, the generalized eigenfunctions of $\mathcal{A}$ forms a Riesz basis of $L^2([0,1], \mathbb{R})$.

Proof. Obviously, (ii), (4.8) and (4.9) can be obtained according to Theorem 4.6. Next, we prove that the generalized eigenfunctions of $\mathcal{A}$ form a Riesz basis of $L^2([0,1], \mathbb{R})$. Combined (4.9) with Theorem 4.2, we know that the generalized eigenfunctions of $\mathcal{A}$ forms a Riesz basis of $L^2([0,1], \mathbb{R})$. Meanwhile, in terms of the definition 4.5 and Theorem 3.6, we know (i) also holds, which completes the proof. □

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