WEIGHTED PSEUDO ALMOST AUTOMORPHIC SOLUTIONS TO FUNCTIONAL DIFFERENTIAL EQUATIONS WITH INFINITE DELAY

YONG-KUI CHANG, SHAN ZHENG

Abstract. In this article, we first establish some results on composition of Stepanov-like weighted pseudo almost automorphic functions so called class \( r \) and class infinity under a uniform continuity condition with respect to \( L^p \)-norm. And then, we study the existence and uniqueness of weighted pseudo almost automorphic solutions to an abstract partial neutral functional differential equation with infinite delay with a Stepanov-like nonlinear term.

1. Introduction

The concept of almost automorphy, which was introduced by Bochner [1], as a generalization of the classical almost periodicity in the sense of Bohr; see for example [13, 23, 25]. N’Guérékata and Pankov introduced the concept of Stepanov-like almost automorphy in [24]. Diagana [11] introduced the concept of Stepanov-like pseudo almost automorphy as a natural generalization of the pseudo almost automorphy and an implement of the Stepanov-like almost automorphy due to N’Guérékata and Pankov [24]. Blot et al [5] introduced the notion of weighted pseudo almost automorphic functions with values in a Banach space. Xia and Fan presented the notion of Stepanov-like (or \( S^p \)-) weighted pseudo almost automorphic function in [30]. To study differential equations with delay, Zhang, Chang and N’Guérékata [33] further studied new types of functions so called Stepanov-like weighted pseudo almost automorphic functions of class \( r \), and Stepanov-like weighted pseudo almost automorphic functions of class infinity. The above mentioned concepts have been considerably investigated and applied to various differential equations, see [1, 2, 3, 7, 9, 10, 13, 14, 17, 19, 27, 28, 29] and the references therein.

The main purpose of this article is to study composition results for Stepanov-like weighted pseudo almost automorphic functions of class \( r \) and Stepanov-like weighted pseudo almost automorphic functions of class infinity [33]. Considering the space of Stepanov-like weighted pseudo almost automorphic functions of class \( r \) and Stepanov-like weighted pseudo almost automorphic functions of class infinity with an integral norm coming from \( L^p \)-norm, we first prove new composition theorems for Stepanov-like weighted pseudo almost automorphic functions of class \( r \)

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under a uniform continuity condition with respect to the $L^p$-norm suggested by [10]. Similarly, we can arrive at new composition theorems for Stepanov-like weighted pseudo almost automorphic functions of class infinity. And then, we apply the obtained results to prove the existence and uniqueness of weighted pseudo almost automorphic solutions for the following abstract partial neutral functional-differential equation with infinite delay under Stepanov-like nonlinear forcing term

$$\frac{d}{dt}(u(t) + f(t, u_t)) = A(t)u(t) + g(t, u_t), \quad t \in \mathbb{R},$$

where $A(t) : D(A(t)) \subset X \to X$ is a family of densely defined closed linear operators on the domain $D = D(A(t))$, which is independent of $t$, the history $u_t : [0, \infty] \to X$ defined by $u_t(\theta) = u(t + \theta)$, belongs to some abstract phase space $\mathcal{B}$ defined axiomaticaly, and $f, g : \mathbb{R} \times \mathcal{B} \to X$ are some suitable functions.

The rest of this paper is organized as follows. In Section 2, we recall some basic definitions, lemmas, and notation which will be used throughout this paper. In Section 3, we establish some new results on composition of Stepanov-like weighted pseudo almost automorphic functions of class $r$ and Stepanov-like weighted pseudo almost automorphic functions of class infinity under $L^p$-norm uniform continuity condition. In Section 4, we prove the existence and uniqueness of weighted pseudo almost automorphic solutions to the equation $\text{(1.1)}$ under Stepanov-like nonlinear forcing term. An example is also given to illustrate the main results.

2. Preliminaries

Let $(X, \| \cdot \|)$ and $(Y, \| \cdot \|_Y)$ be two Banach spaces and $\mathbb{N}, \mathbb{R}$ stand for sets of natural numbers and real numbers, respectively. To facilitate discussions later, we introduce the following notation:

- $BC(\mathbb{R}, X)$ (respectively, $BC(\mathbb{R} \times Y, X)$): The Banach spaces of bounded continuous function from $\mathbb{R}$ to $X$ (respectively, from $\mathbb{R} \times Y$ to $X$) with the sup norm.

- $L^p(\mathbb{R}, X)$: The space of all classes of equivalence (with respect to the equality almost everywhere on $\mathbb{R}$) of measurable function $f : \mathbb{R} \to X$ such that $\|f\| \in L^p(\mathbb{R}, \mathbb{R})$.

- $L^p_{\text{loc}}(\mathbb{R}, X)$: The space of all classes of equivalence of measurable function $f : \mathbb{R} \to X$ such that the restriction of $f$ to every bounded subinterval of $\mathbb{R}$ is in $L^p(\mathbb{R}, X)$.

- $\mathcal{L}(X, Y)$: Stands for the Banach space of bounded linear operators from $X$ to $Y$ equipped with its natural topology.

**Definition 2.1** ([25]). A continuous function $f : \mathbb{R} \to X$ is said to be almost automorphic if for every sequence of real numbers $\{s_n\}_{n \in \mathbb{N}}$ there exists a subsequence $\{s'_n\}_{n \in \mathbb{N}}$ such that

$$g(t) := \lim_{n \to \infty} f(t + s_n)$$

is well defined for each $t \in \mathbb{R}$, and

$$\lim_{n \to \infty} g(t - s_n) = f(t)$$

for each $t \in \mathbb{R}$. The collection of all such functions will be denoted by $AA(X)$.

**Definition 2.2** ([22] [25]). A continuous function $f : \mathbb{R} \times X \to X$ is said to be almost automorphic if $f(t, x)$ is almost automorphic for each $t \in \mathbb{R}$ uniformly for all $x \in \mathcal{B}$, where $\mathcal{B}$ is any bounded subset of $X$. The collection of all such functions will be denoted by $AA(\mathbb{R} \times X, X)$. 
**Definition 2.3** ([32]). A continuous function \( f(t, x) : \mathbb{R} \times \mathbb{R} \to X \) is called bi-
almost automorphic if for every sequence of real numbers \( \{s_n\}_{n \in \mathbb{N}} \), there exists a subsequence \( \{s'_n\}_{n \in \mathbb{N}} \) such that

\[
g(t, s) := \lim_{n \to \infty} f(t + s_n, s + s_n)
\]
is well-defined for each \( t, s \in \mathbb{R} \), and

\[
\lim_{n \to \infty} g(t - s_n, s - s_n) = f(t, s)
\]
for each \( t, s \in \mathbb{R} \). The collection of all functions will be denoted by \( bAA(\mathbb{R} \times \mathbb{R}, X) \).

Let \( U \) denote the set of all functions \( \rho : \mathbb{R} \to (0, \infty) \), which are locally integrable
over \( \mathbb{R} \) such that \( \rho > 0 \) almost everywhere. For a given \( T > 0 \) and for each \( \rho \in U \), we set
\[
m(T, \rho) := \int_{-T}^{T} \rho(t) dt.
\]
Thus the space of weights \( U_{\infty} \) is defined by

\[
U_{\infty} := \{ \rho \in U : \lim_{T \to \infty} m(T, \rho) = \infty \}.
\]
For a given \( \rho \in U_{\infty} \), we define

\[
PAA_0(X, \rho) := \{ f \in BC(\mathbb{R}, X) : \lim_{T \to \infty} \frac{1}{m(T, \rho)} \int_{-T}^{T} \| f(t) \| \rho(t) dt = 0 \};
\]

\[
PAA_0(Y, X, \rho) := \{ f \in C(\mathbb{R} \times Y, X) : f(\cdot, y) \text{ is bounded for each } y \in Y \text{ and} \}
\]

\[
\lim_{T \to \infty} \frac{1}{m(T, \rho)} \int_{-T}^{T} \| f(t, y) \| \rho(t) dt = 0 \text{ uniformly for } y \in Y \}.
\]
To study the delay case, we introduce spaces of functions defined for each \( r > 0 \) by

\[
W(T, f, r, \rho) = \frac{1}{m(T, \rho)} \int_{-T}^{T} \left( \sup_{\theta \in [t-r,t]} \| f(\theta) \| \right) \rho(t) dt,
\]

\[
PAA_0(X, r, \rho) := \{ f \in BC(\mathbb{R}, X) : \lim_{T \to \infty} \frac{1}{m(T, \rho)} \int_{-T}^{T} \left( \sup_{\theta \in [t-r,t]} \| f(\theta) \| \right) \rho(t) dt = 0 \},
\]

\[
PAA_0(Y, X, r, \rho) := \{ f \in C(\mathbb{R} \times Y, X) : f(\cdot, y) \text{ is bounded for each } y \in Y \text{ and} \}
\]

\[
\lim_{T \to \infty} \frac{1}{m(T, \rho)} \int_{-T}^{T} \left( \sup_{\theta \in [t-r,t]} \| f(\theta, y) \| \right) \rho(t) dt = 0 \text{ uniformly for } y \in Y \}.
\]

**Definition 2.4** ([33]). Let \( \rho \in U_{\infty} \). A function \( f \in BC(\mathbb{R}, X) \) (respectively, \( f \in BC(\mathbb{R} \times Y, X) \)) is called weighted pseudo almost automorphic if it can be expressed as \( f = g + \phi \), where \( g \in AA(X) \) (respectively, \( AA(\mathbb{R} \times Y, X) \)) and \( \phi \in PAA_0(X, \rho) \) (respectively, \( PAA_0(Y, X, r, \rho) \)). We denote by \( WPAA(X) \) (respectively, \( WPAA(\mathbb{R} \times Y, X, r) \)) the set of all such functions.

**Definition 2.5** ([33]). Let \( \rho \in U_{\infty} \). A function \( f \in BC(\mathbb{R}, X) \) (respectively, \( f \in BC(\mathbb{R} \times Y, X) \)) is called weighted pseudo almost automorphic of class \( r \) if it can be expressed as \( f = g + \phi \), where \( g \in AA(X) \) (respectively, \( AA(\mathbb{R} \times Y, X) \)) and \( \phi \in PAA_0(X, r, \rho) \) (respectively, \( PAA_0(Y, X, r, \rho) \)). We denote by \( WPAA(X, r) \) (respectively, \( WPAA(\mathbb{R} \times Y, X, r) \)) the set of all such functions.
Definition 2.6 ([21] [24]). The Bochner transform \( f^b(t, s), t \in \mathbb{R}, s \in [0, 1], \) of a function \( f : \mathbb{R} \to \mathbb{X} \) is defined by

\[
f^b(t, s) := f(t + s).
\]

Remark 2.7 ([21]). (i) A function \( \varphi(t, s), t \in \mathbb{R}, s \in [0, 1], \) is the Bochner transform of a certain function \( f, \varphi(t, s) = f^b(t, s), \) if and only if \( \varphi(t + \tau, s - \tau) = \varphi(s, t) \) for all \( t \in \mathbb{R}, s \in [0, 1] \) and \( \tau \in [s - 1, s]. \)

(ii) Note that if \( f = h + \varphi, \) then \( f^b = h^b + \varphi^b. \) Moreover, \( (\lambda f)^b = \lambda f^b \) for each scalar \( \lambda. \)

Definition 2.8 ([21]). The Bochner transform \( f^b(t, s, u), t \in \mathbb{R}, s \in [0, 1], u \in \mathbb{X} \) of a function \( f : \mathbb{R} \times \mathbb{X} \to \mathbb{X} \) is defined by

\[
f^b(t, s, u) := f(t + s, u) \quad \text{for each} \quad u \in \mathbb{X}.
\]

We always denote by \( \| \cdot \| \) the norm of space \( L^p(0, 1; \mathbb{X}) \) for \( p \in [1, \infty). \)

Definition 2.9 ([21] [24]). Let \( p \in [1, \infty). \) The space \( BS^p(\mathbb{X}) \) of all Stepanov bounded functions, with the exponent \( p, \) consists of all measurable functions \( f : \mathbb{R} \to \mathbb{X} \) such that \( f^b \in L^\infty(\mathbb{R}, L^p(0, 1; \mathbb{X})). \) This is a Banach space with the norm

\[
\|f\|_{S^p} = \|f^b\|_{L^\infty(\mathbb{R}, L^p)} = \sup_{t \in \mathbb{R}} \left( \int_t^{t+1} \|f(\tau)\|^p d\tau \right)^{1/p} = \sup_{t \in \mathbb{R}} \|f(t + \cdot)\|_p.
\]

Lemma 2.10 ([33]). Let \( \rho \in U_{\infty}. \) Suppose that \( PAA_0(\mathbb{X}, r, \rho) \) is translation invariant. Then the decomposition of weighted pseudo almost automorphic functions of class \( r \) is unique.

Lemma 2.11 ([33]). Let \( \rho \in U_{\infty} \) and \( PAA_0(\mathbb{X}, r, \rho) \) be translation invariant, then \( WPAA(\mathbb{X}, r) \) is a Banach space with norm \( \| \cdot \|_{\infty}. \)

Definition 2.12 ([21] [24]). The space \( AS^p(\mathbb{X}) \) of Stepanov-like almost automorphic (or \( S^p \)-almost automorphic) functions consists of all \( f \in BS^p(\mathbb{X}) \) such that \( f^b \in AA(L^p(0, 1; \mathbb{X})). \) In other words, a function \( f \in L^p_{loc}(\mathbb{R}, \mathbb{X}) \) is said to be \( S^p \)-almost automorphic if its Bochner transform \( f^b : \mathbb{R} \to L^p(0, 1; \mathbb{X}) \) is almost automorphic in the sense that for every sequence of real numbers \( \{s_n\}_{n \in \mathbb{N}}, \) there exist a subsequence \( \{s'_n\}_{n \in \mathbb{N}}, \) a function \( g \in L^p_{loc}(\mathbb{R}, \mathbb{X}) \) such that

\[
\lim_{n \to \infty} \left( \int_t^{t+1} \|f(s + s_n) - g(s)\|^p ds \right)^{1/p} = 0,
\]

\[
\lim_{n \to \infty} \left( \int_t^{t+1} \|g(s - s_n) - f(s)\|^p ds \right)^{1/p} = 0.
\]

pointwise on \( \mathbb{R}. \)

Definition 2.13 ([21] [24]). A function \( f : \mathbb{R} \times \mathbb{Y} \to \mathbb{X}, (t, u) \to f(t, u) \) with \( f(\cdot, u) \in L^p_{loc}(\mathbb{R}, \mathbb{X}) \) for each \( u \in \mathbb{Y}, \) is said to be \( S^p \)-almost automorphic in \( t \in \mathbb{R} \) uniformly in \( u \in \mathbb{Y} \) if \( t \to f(t, u) \) is \( S^p \)-almost automorphic for each \( u \in \mathbb{Y}. \) That means, for every sequence of real numbers \( \{s'_n\}_{n \in \mathbb{N}}, \) there exist a subsequence \( \{s_n\}_{n \in \mathbb{N}} \) and a function \( g(\cdot, u) \in L^p_{loc}(\mathbb{R}, \mathbb{X}) \) such that

\[
\lim_{n \to \infty} \left( \int_t^{t+1} \|f(s + s_n, u) - g(s, u)\|^p ds \right)^{1/p} = 0,
\]

\[
\lim_{n \to \infty} \left( \int_t^{t+1} \|g(s - s_n, u) - f(s, u)\|^p ds \right)^{1/p} = 0.
\]
Stepanov-like weighted pseudo almost automorphic relatively to the weight expressed as $f$ with its Bochner transform $f$ in the sense that there exist two functions $g, \phi \in \mathcal{PAA}_0(L^p(0, 1; X), \rho)$. In other words, a function $f \in L^p_{\text{loc}}(\mathbb{R}, X)$ is said to be Stepanov-like weighted pseudo almost automorphic relatively to the weight $\rho \in \mathbb{U}_\infty$, if its Bochner transform $f^b : \mathbb{R} \to L^p(0, 1; X)$ is weighted pseudo almost automorphic in the sense that there exist two functions $g, \phi : \mathbb{R} \to X$ such that $f = g + \phi$, where $g \in \mathcal{AS}^p(X)$ and $\phi^b \in \mathcal{PAA}_0(L^p(0, 1; X), \rho)$. We denote by $\mathcal{WPAAS}^p(X)$ the set of all such functions.

Definition 2.14 ([33]). Let $\rho \in \mathbb{U}_\infty$. A function $f \in \mathcal{BS}^p(X)$ is said to be Stepanov-like weighted pseudo almost automorphic (or $S^p$-weighted pseudo almost automorphic) if it can be expressed as $f = g + \phi$, where $g \in \mathcal{AS}^p(X)$ and $\phi^b \in \mathcal{PAA}_0(L^p(0, 1; X), \rho)$. In other words, a function $f \in L^p_{\text{loc}}(\mathbb{R}, X)$ is said to be Stepanov-like weighted pseudo almost automorphic relatively to the weight $\rho \in \mathbb{U}_\infty$, if its Bochner transform $f^b : \mathbb{R} \to L^p(0, 1; X)$ is weighted pseudo almost automorphic in the sense that there exist two functions $g, \phi : \mathbb{R} \to X$ such that $f = g + \phi$, where $g \in \mathcal{AS}^p(X)$ and $\phi^b \in \mathcal{PAA}_0(L^p(0, 1; X), \rho)$. We denote by $\mathcal{WPAAS}^p(X, r)$ the set of all such functions.

Definition 2.15 ([33]). Let $\rho \in \mathbb{U}_\infty$. A function $f : \mathbb{R} \times Y \to X, (t, u) \mapsto f(t, u)$ with $f(\cdot, u) \in L^p_{\text{loc}}(\mathbb{R}, X)$ for each $u \in Y$, is said to be Stepanov-like weighted pseudo almost automorphic (or $S^p$-weighted pseudo almost automorphic) of class $r$ if it can be expressed as $f = g + \phi$, where $g \in \mathcal{AS}^p(X)$ and $\phi^b \in \mathcal{PAA}_0(L^p(0, 1; X), r, \rho)$. In other words, a function $f \in L^p_{\text{loc}}(\mathbb{R}, X, r)$ is said to be Stepanov-like weighted pseudo almost automorphic of class $r$ relatively to the weight $\rho \in \mathbb{U}_\infty$, if its Bochner transform $f^b : \mathbb{R} \to L^p(0, 1; X)$ is weighted pseudo almost automorphic of class $r$ in the sense that there exist two functions $g, \phi : \mathbb{R} \to X$ such that $f = g + \phi$, where $g \in \mathcal{AS}^p(X)$ and $\phi^b \in \mathcal{PAA}_0(L^p(0, 1; X), r, \rho)$. We denote by $\mathcal{WPAAS}^p(X, r)$ the set of all such functions.

Definition 2.17 ([33]). Let $\rho \in \mathbb{U}_\infty$. A function $f : \mathbb{R} \times Y \to X, (t, u) \mapsto f(t, u)$ with $f(\cdot, u) \in L^p_{\text{loc}}(\mathbb{R}, X)$ for each $u \in Y$, is said to be Stepanov-like weighted pseudo almost automorphic of class $r$ (or $S^p$-weighted pseudo almost automorphic of class $r$) if it can be expressed as $f = g + \phi$, where $g \in \mathcal{AS}^p(X \times Y, X)$ and $\phi^b \in \mathcal{PAA}_0(Y, L^p(0, 1; X), \rho)$. We denote by $\mathcal{WPAAS}^p(X \times Y, X, r)$ the set of all such functions.

Lemma 2.18 ([33]). Let $\rho \in \mathbb{U}_\infty$. The space $\mathcal{WPAAS}^p(X, r)$ equipped with the norm $\| \cdot \|_{SP}$ is a Banach space.

Concerning infinite delays, we introduce the following spaces of functions as in [33]:

\[ \mathcal{PAA}_0(\mathbb{X}, \infty, \rho) := \cap_{r > 0} \mathcal{PAA}_0(\mathbb{X}, r, \rho), \]
\[ \mathcal{PAA}_0(\mathbb{X}, \mathbb{Y}, \infty, \rho) := \cap_{r > 0} \mathcal{PAA}_0(\mathbb{X}, \mathbb{Y}, r, \rho), \]
\[ \mathcal{PAA}_0(L^p(0, 1; \mathbb{X}), \infty, \rho) := \cap_{r > 0} \mathcal{PAA}_0(L^p(0, 1; \mathbb{X}), r, \rho), \]
\[ \mathcal{PAA}_0(\mathbb{Y}, L^p(0, 1; \mathbb{X}), \infty, \rho) := \cap_{r > 0} \mathcal{PAA}_0(\mathbb{Y}, L^p(0, 1; \mathbb{X}), r, \rho). \]

Obviously, $\mathcal{PAA}_0(\mathbb{X}, \infty, \rho)$ and $\mathcal{PAA}_0(\mathbb{X}, \mathbb{Y}, \infty, \rho)$ are, respectively, closed subspaces of $\mathcal{PAA}_0(\mathbb{X}, r, \rho)$ and $\mathcal{PAA}_0(\mathbb{X}, \mathbb{Y}, r, \rho)$, and hence both are Banach spaces. By
the same way, \( PAA_0(L^p(0,1;X),\infty,\rho) \) and \( PAA_0(Y,L^p(0,1;X),\infty,\rho) \) are, respectively, closed subspaces of \( PAA_0(\mathbb{Y},L^p(0,1;X),r,\rho) \) and \( PAA_0(Y,L^p(0,1;X),r,\rho) \), and thus both are Banach spaces.

**Definition 2.19** (\([\text{33}]\)). Let \( \rho \in \mathbb{U}_\infty \). A function \( f \in BC(\mathbb{R},\mathbb{X}) \) (respectively, \( f \in BC(\mathbb{R} \times \mathbb{Y},\mathbb{X}) \)) is called weighted pseudo almost automorphic of class infinity if it can be expressed as \( f = g + \phi \), where \( g \in AA(\mathbb{X}) \) (respectively, \( AA(\mathbb{R} \times \mathbb{Y},\mathbb{X}) \)) and \( \phi \in PAA_0(\mathbb{X},\infty,\rho) \) (respectively, \( PAA_0(\mathbb{Y},\mathbb{X},\infty,\rho) \)). We denote by \( WPAA(\mathbb{X},\infty) \) (respectively, \( WPAA(\mathbb{R} \times \mathbb{Y},\mathbb{X},\infty) \)) the set of all such functions.

**Definition 2.20** (\([\text{33}]\)). Let \( \rho \in \mathbb{U}_\infty \). A function \( f \in B\mathbb{S}^p(\mathbb{X}) \) is said to be Stepanov-like weighted pseudo almost automorphic of class infinity (or \( S^p \)-weighted pseudo almost automorphic of class infinity) if it can be expressed as \( f = g + \phi \), where \( g \in AS^p(\mathbb{X}) \) and \( \phi^b \in PAA_0(L^p(0,1;X),\infty,\rho) \). In other words, a function \( f \in L^p_{\text{loc}}(\mathbb{R},\mathbb{X}) \) is said to be Stepanov-like weighted pseudo almost automorphic of class infinity relatively to the weight \( \rho \in \mathbb{U}_\infty \), if its Bochner transform \( f^b: \mathbb{R} \rightarrow L^p(0,1;X) \) is weighted pseudo almost automorphic of class infinity in the sense that there exist two functions \( g,\phi: \mathbb{R} \rightarrow \mathbb{X} \) such that \( f = g + \phi \), where \( g \in AS^p(\mathbb{X}) \) and \( \phi^b \in PAA_0(L^p(0,1;X),\infty,\rho) \). We denote by \( WPAA^s(\mathbb{X},\infty) \) (respectively, \( WPAA^s(\mathbb{R} \times \mathbb{Y},\mathbb{X},\infty) \)) the set of all such functions.

**Definition 2.21** (\([\text{33}]\)). Let \( \rho \in \mathbb{U}_\infty \). A function \( f : \mathbb{R} \times \mathbb{Y} \rightarrow \mathbb{X},(t,u) \rightarrow f(t,u) \) with \( f(\cdot, u) \in L^p_{\text{loc}}(\mathbb{R},\mathbb{X}) \) for each \( u \in \mathbb{Y} \), is said to be Stepanov-like weighted pseudo almost automorphic of class infinity (or \( S^p \)-weighted pseudo almost automorphic of class infinity) if it can be expressed as \( f = g + \phi \), where \( g \in AS^p(\mathbb{R} \times \mathbb{Y},\mathbb{X}) \) and \( \phi^b \in PAA_0(Y,L^p(0,1;X),\infty,\rho) \). We denote by \( WPAA^s(\mathbb{R} \times \mathbb{Y},\mathbb{X},\infty) \) the set of all such functions.

Using similar ideas to those in [16] Lemma2.7, we can easily show the following results.

**Lemma 2.22.** (i) Assume \( PAA_0(L^p(0,1;X),r,\rho) \) is translation invariant, then the decomposition of an \( S^p \)-weighted pseudo almost automorphic function of class \( r \) is unique.

(ii) The space \( WPAA^s(\mathbb{X},r) \) equipped with \( \| \cdot \|_{S^p} \) is a Banach space.

(iii) \( WPAA(\mathbb{X},r) \) is continuously embedded in \( WPAA^s(\mathbb{X},r) \).

**Lemma 2.23.** (i) Assume that \( PAA_0(L^p(0,1;X),\infty,\rho) \) is translation invariant, then the decomposition of an \( S^p \)-weighted pseudo almost automorphic function of class infinity is unique.

(ii) The space \( WPAA^s(\mathbb{X},\infty) \) equipped with \( \| \cdot \|_{S^p} \) is a Banach space.

(iii) \( WPAA(\mathbb{X},\infty) \) is continuously embedded in \( WPAA^s(\mathbb{X},\infty) \).

In this work we use an axiomatic definition of the phase space \( \mathcal{B} \), which is similar to the one introduce in (\([\text{20}]\)). \( \mathcal{B} \) is a vector space of functions mapping \((-\infty,0] \) into \( \mathbb{X} \) endowed with a seminorm \( \| \cdot \|_{\mathcal{B}} \) such that the next axioms hold by (\([\text{33}]\)).

(A1) If \( \varphi : (-\infty,\sigma + a) \rightarrow \mathbb{X}, \ a > 0, \ \sigma \in \mathbb{R}, \) is continuous on \([\sigma,\sigma + a]\) and \( x_\sigma \in \mathcal{B} \), then for every \( t \in [\sigma,\sigma + a]\) the following hold:

(i) \( x_t \) is in \( \mathcal{B} \);

(ii) \( \| x(t) \| \leq H \| x_t \|_{\mathcal{B}} \);

(iii) \( \| x(t) \|_{\mathcal{B}} \leq \tilde{K}(t - \sigma) \sup \| x(s) \| : \sigma \leq s \leq t \) + \( \tilde{M}(t - \sigma)\| x_\sigma \|_{\mathcal{B}} \), where \( H > 0 \) is a constant; \( \tilde{K},\tilde{M} : (0,\infty) \rightarrow [1,\infty) \). \( \tilde{K} \) is continuous, \( \tilde{M} \) is locally bounded and \( H,\tilde{K},\tilde{M} \) are independent of \( x(\cdot) \).
(A1') For the function \( x(\cdot) \) appearing in (A1), the function \( t \to x_t \) is continuous from \( [\sigma, \sigma + a] \) into \( \mathcal{B} \).

(A2) The space \( \mathcal{B} \) is complete.

(A3) If \( (\varphi^n)_{n \in \mathbb{N}} \) is a bounded sequence in \( BC((\infty, 0], X) \) given by functions with compact support and \( \varphi^n \to \varphi \) in the compact-open topology, then \( \varphi \in \mathcal{B} \) and \( \|\varphi^n - \varphi\|_{\mathcal{B}} \to 0 \) as \( n \to \infty \).

**Definition 2.24.** [18] Let \( \mathcal{B}_0 = \{ \varphi \in \mathcal{B} : \varphi(0) = 0 \} \) and \( S(t) : \mathcal{B} \to \mathcal{B} \) be the \( C_0 \)-semigroup defined by \( S(t)\varphi(\theta) = \varphi(0) \) on \([-t, 0] \) and \( S(t)\varphi(\theta) = \varphi(t + \theta) \) on \((\infty, -t)\). The phase space \( \mathcal{B} \) is called a fading memory space if \( \|S(t)\varphi\|_{\mathcal{B}_0} \to 0 \) as \( t \to \infty \) for every \( \varphi \in \mathcal{B}_0 \). We said that \( \mathcal{B} \) is a uniform fading memory space if \( \|S(t)\varphi\|_{\mathcal{B}_0} \to 0 \) as \( t \to \infty \).

**Remark 2.25** ([18]). In this article we assume \( \varsigma > 0 \) and \( \|\varphi\|_{\mathcal{B}} \leq \varsigma \sup_{\theta \leq 0} \|\varphi(\theta)\| \) for each \( \varphi \in \mathcal{B} \cap BC((\infty, 0], X) \), see [20] for details. Moreover, if \( \mathcal{B} \) is a fading memory, we assume that \( \max\{K(t), \dot{M}(t)\} \leq \mathcal{R} \) for all \( t \geq 0 \), see [20].

**Lemma 2.26** ([20]). The phase space \( \mathcal{B} \) is a uniform fading memory space if, and only if, axiom (A3) holds, the function \( \dot{K} \) is bounded and \( \lim_{t \to \infty} \dot{M}(t) = 0 \).

### 3. Results on Composition Theorems

The aim of this section is to establish some new results on composition of Stepanov-like weighted pseudo almost automorphic functions of class infinity. We first list the following “uniform continuity condition” with respect to the \( L^p \)-norm for a function \( h : \mathbb{R} \times X \to X \) with \( h(\cdot, u) \in L^p_{loc}(\mathbb{R}, X) \) for each \( u \in X \), which was initially adopted in [16]:

(A4) For any \( \varepsilon > 0 \), there exists \( \sigma > 0 \) such that each \( t, y \in L^p(0, 1, X) \) and \( \|x - y\|_p < \sigma \) imply that

\[
\|h(t + \cdot, x(\cdot)) - h(t + \cdot, y(\cdot))\|_p < \varepsilon, \quad t \in \mathbb{R}.
\]

In the sequel, we say that a function \( \psi \) satisfies (A4) if \( \psi \) replaces \( h \) in (A4).

Let \( f \in AS^p(\mathbb{R} \times X, X) \), then for a sequence \( \{s_n\} \subset \mathbb{R} \), there exist a subsequence \( \{\tau_n\} \) and a function \( g : \mathbb{R} \times X \to X \) with \( g(\cdot, x) \in L^p_{loc}(\mathbb{R}, X) \), \( x \in X \) such that for each \( t \in \mathbb{R} \),

\[
\lim_{n \to \infty} \|f(t + \tau_n + \cdot, x) - g(t + \cdot, x)\|_p = \lim_{n \to \infty} \|g(t - \tau_n + \cdot, x) - f(t + \cdot, x)\|_p = 0.
\]

(A5) \( f \in AS^p(\mathbb{R} \times X, X) \) satisfies (A4), and for a sequence \( \{s_n\} \subset \mathbb{R} \), there exist a subsequence \( \{\tau_n\} \) and a function \( g \) given above such that \( g \) satisfies (A4).

**Lemma 3.1** ([16]). Let \( h \) be the function in (A4), and \( x : \mathbb{R} \to X \) with \( \overline{x(\mathbb{R})} \) compact. For \( \varepsilon > 0 \), there exist a finite set \( \{x_k\}_{k=1}^m \subset \overline{x(\mathbb{R})} \) such that

\[
\|h(t + \cdot, x(t + \cdot))\|_p < \varepsilon + m \sup_{1 \leq k \leq m} \|h(t + \cdot, x_k)\|_p, \quad t \in \mathbb{R}.
\]

**Lemma 3.2** ([33]). Let \( \rho \in U_{\infty} \) and \( f \in BS^p(X) \), then \( f^b \in PAA_0(L^p(0, 1, X), \tau, \rho) \) if and only if for any \( \varepsilon > 0 \),

\[
\lim_{T \to \infty} \frac{1}{m(T, \rho)} \int_{M_T \varepsilon(t)} \rho(t) dt = 0,
\]
where

\[ M_{r,\varepsilon}(f) = \{ t \in [-T, T] : \sup_{\theta \in [t-r, t]} \left( \int_{\theta}^{\theta+1} \|f(s)\|^p ds \right)^{1/p} \geq \varepsilon \}. \]

Lemma 3.3. Assume that \( f \) satisfies (A5) and \( x \in AS^p(\mathbb{X}) \) with \( \pi(\mathbb{R}) \) compact. Then \( f(\cdot, x(\cdot)) \in AS^p(\mathbb{X}) \).

Next, we give results in the compositions of \( S^p \)-weighted pseudo almost automorphic functions of class \( r \).

Theorem 3.4. Let \( \rho \in U_\infty \), \( f = g + \phi \in WPAAS^p(\mathbb{R} \times \mathbb{X}, r) \), \( u = u_1 + u_2 \in WPAAS^p(\mathbb{R} \times \mathbb{X}, r) \), \( g \in AS^p(\mathbb{R} \times \mathbb{X}) \), \( \phi^b \in PAA_0(\mathbb{X}, L^p(0, 1, \mathbb{X}, r, \rho)) \), \( u_1 \in AS^p(\mathbb{X}) \), \( u^b_1 \in PAA_0(L^p(0, 1, \mathbb{X}), r, \rho) \), \( Q = \{ u_1(t) : t \in \mathbb{R} \} \) compact and there exist a continuous function \( \mathcal{L}_f(\cdot) : \mathbb{R} \to [0, \infty) \) satisfying

\[ \left( \int_t^{t+1} \|f(s, x_1) - f(s, x_2)\|^p ds \right)^{1/p} \leq \mathcal{L}_f(t) \|x_1 - x_2\|. \quad (3.1) \]

If \( \xi^b \in PAA_0(\mathbb{R}, L^p(0, 1, \mathbb{X}, \rho)) \), then

\begin{align*}
&\lim_{T \to \infty} \sup_{t \in [-T, T]} \frac{1}{m(T, \rho)} \int_{-T}^{T} \left( \sup_{\theta \in [t-r, t]} \mathcal{L}_f(\theta + \cdot) \right) \rho(t) dt < \infty, \quad (3.2) \\
&\lim_{T \to \infty} \sup_{t \in [-T, T]} \frac{1}{m(T, \rho)} \int_{-T}^{T} \left( \sup_{\theta \in [t-r, t]} \mathcal{L}_f(\theta + \cdot) \xi(t) \right) \rho(t) dt < \infty. \quad (3.3)
\end{align*}

If

(i) \( g(t, x) \) satisfies (A5) and
(ii) \( \phi \) satisfies (A4),

Then \( f(\cdot, u(\cdot)) \in WPAAS^p(\mathbb{X}, r) \).

Proof. Let \( G(t) = g(t, u_1(t)) \), \( H(t) = f(t, u(t)) - f(t, u_1(t)) \), \( \Lambda(t) = \phi(t, u_1(t)) \), \( t \in \mathbb{R} \). Then

\[ f(t, u(t)) = g(t, u_1(t)) + f(t, u(t)) - f(t, u_1(t)) + \phi(t, u_1(t)) = G(t) + H(t) + \Lambda(t). \]

We have \( G(t) \in AS^p(\mathbb{X}) \) by Lemma 3.3 then it remains to show that \( H^b, \Lambda^b \) is in \( PAA_0(L^p(0, 1, \mathbb{X}), r, \rho) \).

Indeed, for \( T > 0 \), using (3.1), we see that

\begin{align*}
&\frac{1}{m(T, \rho)} \int_{-T}^{T} \left( \sup_{\theta \in [t-r, t]} \left( \int_{\theta}^{\theta+1} \|H(s)\|^p ds \right)^{1/p} \right) \rho(t) dt \\
&= \frac{1}{m(T, \rho)} \int_{-T}^{T} \left( \sup_{\theta \in [t-r, t]} \left( \int_{\theta}^{\theta+1} \|f(t, u(t)) - f(t, u_1(t))\|^p ds \right)^{1/p} \right) \rho(t) dt \\
&\leq \frac{1}{m(T, \rho)} \int_{-T}^{T} \left( \sup_{\theta \in [t-r, t]} \mathcal{L}_f(\theta) \|u_2(\theta)\| \right) \rho(t) dt \\
&\leq \frac{1}{m(T, \rho)} \int_{-T}^{T} \left( \sup_{\theta \in [t-r, t]} \mathcal{L}_f(\theta) \left( \sup_{\theta \in [t-r, t]} \left( \int_{\theta}^{\theta+1} \|u_2(s)\|^p ds \right)^{1/p} \right) \right) \rho(t) dt \\
&= \frac{1}{m(T, \rho)} \int_{-T}^{T} \left( \sup_{\theta \in [t-r, t]} \mathcal{L}_f(\theta) \left( \sup_{\theta \in [t-r, t]} u_2(\theta + \cdot) \right) \rho(t) dt. \right.
\]

This implies that \( H^b(t) \in PAA_0(L^p(0, 1, \mathbb{X}), r, \rho) \) by (3.3).
Next, we prove $\Lambda^b \in PAA_0(L^p(0, 1, X), r, \rho)$. For $\varepsilon > 0$, let $\sigma$ be given by (A4) with $\phi$ in the place of $h$, by Lemma 3.1, there is a finite set $\{x_k\}_{k=1}^m \subset \{u_1(t), t \in \mathbb{R}\}$ such that for $t \in \mathbb{R},$

$$||\phi(t + \cdot, u_1(t + \cdot))||_p < \varepsilon + m \sup_{1 \leq k \leq m} ||\phi(t + \cdot, x_k)||_p,$$

Since $\phi^b \in PAA_0(\mathcal{X}, L^p(0, 1, X), r, \rho)$, for each $x \in \mathcal{X}$, there is $T > T_0$, $1 \leq k \leq m,$

$$\mathcal{W}(T, \phi^b(\cdot, x_k), r, \rho) = \frac{1}{m(T, \rho)} \int_{-T}^T \left( \sup_{\theta \in [t-r,t]} ||\phi(\theta + \cdot, x_k)||_p \right) \rho(t) dt < \varepsilon.$$

Then for $T > T_0,$

$$\mathcal{W}(T, \Lambda^b(\cdot, u_1(\cdot)), r, \rho) = \frac{1}{m(T, \rho)} \int_{-T}^T \left( \sup_{\theta \in [t-r,t]} \phi(\theta + \cdot, u_1(t + \cdot)) \right) \rho(t) dt$$

$$\leq \varepsilon + m \sup_{1 \leq k \leq m} \frac{1}{m(T, \rho)} \int_{-T}^T \left( \sup_{\theta \in [t-r,t]} ||\phi(\theta + \cdot, x_k)||_p \right) \rho(t) dt$$

$$= \varepsilon + m \varepsilon = 2 \varepsilon.$$

This yields $\lim_{T \to \infty} \mathcal{W}(T, \Lambda^b(\cdot, u_1(\cdot)), r, \rho) = 0$. That is $\Lambda^b \in PAA_0(L^p(0, 1, X), r, \rho)$. The proof is complete. \hfill \Box

**Theorem 3.5.** Let $\rho \in U_\infty$, $F = F_1 + F_2 \in WPAAS^p(\mathbb{R} \times \mathcal{X}, X, r), \phi = \phi_1 + \phi_2 \in WPAAS^p(\mathcal{X}, r)$ with $Q = \{\phi_1(t) : t \in \mathbb{R}\}$ compact, $F_1 \in AS^p(\mathbb{R} \times \mathcal{X}, X), F_2 \in PAA_0(\mathcal{X}, L^p(0, 1, X), r, \rho), \phi_1 \in AS^p(\mathcal{X}), \phi^b_2 \in PAA_0(L^p(0, 1, X), r, \rho)$. Assume that $F_1$ satisfies (A5), $F_2$ satisfies (A4) and $\{F(\cdot, z) : z \in \mathcal{Z}\}$ is bounded in $WPAAS^p(\mathcal{X}, r)$ for any bounded $\mathcal{Z} \subset \mathcal{X}$, then $t \to F(t, \phi(t)) \in WPAAS^p(\mathcal{X}, r)$.

**Proof.** Let $\Upsilon(t) = F_1(t, \phi_1(t)), \Psi(t) = F(t, \phi(t)) - F(t, \phi_1(t)), \Phi(t) = F_2(t, \phi_1(t)), t \in \mathbb{R}$. Then

$$F(t, \phi(t)) = F_1(t, \phi_1(t)) + F_2(t, \phi(t)) = F_1(t, \phi_1(t)) + F_2(t, \phi_1(t)) = \Upsilon(t) + \Psi(t) + \Phi(t).$$

We have $\Upsilon(t) \in AS^p(\mathcal{X})$ by Lemma 3.3, so we need only to prove $\Psi^b, \Phi^b \in PAA_0(L^p(0, 1, X), r, \rho)$.

It is easy to see that $\Psi \in BS^p(\mathcal{X})$ since $\phi$ and $\phi_1$ are bounded and $\{F(\cdot, z) : z \in \mathcal{Z}\}$ is bounded in $WPAAS^p(\mathcal{X}, r)$ for any bounded $\mathcal{Z} \subset \mathcal{X}$. Noticing that $F$ satisfies (A4) since $F_1$ and $F_2$ satisfy (A4), for $\varepsilon > 0$, let $\sigma > 0$ be given by (A4), then

$$||\Psi(t + \cdot)||_p = ||F(t + \cdot, \phi(t + \cdot)) - F(t + \cdot, \phi_1(t + \cdot))||_p < \varepsilon,$$

for $||\phi_2(t + \cdot)||_p < \sigma$, where $\phi_2(s) = \phi(s) - \phi_1(s)$. Hence, for each $t \in \mathbb{R},$ $||\phi_2(s)|| < \sigma$, $s \in [t, t + 1]$ implies that

$$\left( \int_t^{t+1} ||\Psi(s)||^p ds \right)^{1/p} = \left( \int_t^{t+1} ||F(s, \phi(s)) - F(s, \phi_1(s))||^p ds \right)^{1/p}$$

$$= ||F(t + \cdot, \phi(t + \cdot)) - F(t + \cdot, \phi_1(t + \cdot))||_p < \varepsilon.$$

We can obtain

$$\sup_{\theta \in [t-r,t]} \left( \int_{\theta}^{\theta+1} ||\Psi(s)||^p ds \right)^{1/p}$$
This implies \( \lim \phi \) since \( \text{Lemma 3.6} \).

Let \( M_{T, \sigma(\phi_2)} = \{ t \in [-T, T] : \sup_{\theta \in [t-r,t]} \left( \int_{\theta}^{\theta+1} \| \phi_2(s) \|^p ds \right)^{1/p} \geq \sigma \} \).

So we obtain

\[ M_{T, \sigma(\phi_2)} = M_{T, \sigma(F(\cdot, \phi(\cdot)) - F(\cdot, \phi_1(\cdot)))} \subseteq M_{T, \sigma(\phi_2)}. \]

Since \( \phi_2^\rho \in \text{PAA}_0(L^p(0,1, \mathcal{X}), r, \rho) \) by \text{Lemma 3.2}, we obtain

\[ \lim_{T \to \infty} \int_{M_{T, \sigma(\phi_2)}} \rho(t) dt = 0. \]

Thus

\[ \lim_{T \to \infty} \int_{M_{T, \sigma(\psi)}} \rho(t) dt = 0. \]

This shows that \( \Psi^b \in \text{PAA}_0(L^p(0,1, \mathcal{X}), r, \rho) \).

For \( \varepsilon > 0 \), let \( \sigma \) be given by (A4) with \( F_2 \) in the place of \( h \), by \text{Lemma 3.1} there is a finite set \( \{ x_k \}_{k=1}^m \subseteq \{ \phi_1(t) : t \in \mathbb{R} \} \) such that for \( t \in \mathbb{R} \)

\[ \| F_2(t + \cdot, \phi_1(t + \cdot)) \|_p < \varepsilon + m \sup_{1 \leq k \leq m} \| F_2(t + \cdot, x_k) \|_p. \]

Since \( F_2^b \in \text{PAA}_0(\mathcal{X}, L^p(0,1, \mathcal{X}), r, \rho) \) for each \( x \in \mathcal{X} \), there is \( T > T_0, 1 \leq k \leq m, \)

\[ \mathcal{W}(T, F_2^{b}(\cdot, x_k), r, \rho) = \frac{1}{m(T, \rho)} \int_{T}^{T} \left( \sup_{\theta \in [t-r,t]} \| F_2(\theta + \cdot, x_k) \|_p \right) \rho(t) dt < \frac{\varepsilon}{m}. \]

Then for \( T > T_0, \)

\[ \mathcal{W}(T, \Phi^b, r, \rho) = \mathcal{W}(T, F_2^{b}(\cdot, \phi_1(\cdot)), r, \rho)
\]

\[ = \frac{1}{m(T, \rho)} \int_{T}^{T} \left( \sup_{\theta \in [t-r,t]} \| F_2(\theta + \cdot, \phi_1(t + \cdot)) \|_p \right) \rho(t) dt \]

\[ \leq \varepsilon + m \sup_{1 \leq k \leq m} \frac{1}{m(T, \rho)} \int_{T}^{T} \left( \sup_{\theta \in [t-r,t]} \| \phi(\theta + \cdot, x_k) \|_p \right) \rho(t) dt \]

\[ = \varepsilon + m \mathcal{W}(T, F_2^{b}(\cdot, x_k), r, \rho) \]

\[ = \varepsilon + \frac{m \varepsilon}{m} = 2 \varepsilon. \]

This implies \( \lim_{T \to \infty} \mathcal{W}(T, \Phi^b, r, \rho) = 0. \) That is \( \Phi^b \in \text{PAA}_0(L^p(0,1, \mathcal{X}), r, \rho) \). The proof is complete. \( \Box \)

**Lemma 3.6** \((\text{3.3})\). Let \( u \in \text{WPAA}(\mathcal{X}, \infty) \) where \( \rho \in \mathbb{U}_\infty \). Assume that \( \mathfrak{B} \) is a uniform fading memory space. Then the function \( t \to u_t \) belongs to \( \text{WPAA}(\mathfrak{B}, \infty) \).

**Lemma 3.7** \((\text{3.3})\). Let \( \rho \in \mathbb{U}_\infty, u \in \text{WPAA}^p(\mathcal{X}, \infty) \) and assume that \( \mathfrak{B} \) is a uniform fading memory space. Then the function \( t \to u_t \) belongs to \( \text{WPAA}^p(\mathfrak{B}, \infty) \).

One of the consequence of \text{Lemma 3.7} is the following modified version of \text{Theorems 3.4} and \text{3.5}.
Corollary 3.8. Let \( \rho \in U_\infty \), \( f \in WPAAS_p(\mathbb{R} \times X, \infty) \) and \( u \in WPAAS_p(X, \infty) \). Assume that the condition of (i) and (ii) in the Theorem 3.4 is satisfied and there exists a continuous function \( \Omega : \mathbb{R} \rightarrow [0, \infty) \), such that (3.1) holds. If conditions (3.2) and (3.3) hold for every \( r > 0 \), then the function \( f(t,u(t)) \in WPAAS_p(X, \infty) \).

Corollary 3.9. Let \( \rho \in U_\infty \), \( F = F_1 + F_2 \in WPAAS_p(\mathbb{R} \times X, \infty) \), \( \phi = \phi_1 + \phi_2 \in WPAAS_p(X, \infty) \) with \( Q = \{ \phi_1(t) : t \in \mathbb{R} \} \) compact, \( F_1 \in AS_p(\mathbb{R} \times X, X) \), \( F_2^b \in PAA_0(X, L_p(0,1,X), \infty, \rho) \), \( \phi_1 \in AS_0(X) \), \( \phi_2^b \in PAA_0(L_p(0,1,X), \infty, \rho) \). Assume that \( F_1 \) satisfies (A5), \( F_2 \) satisfies (A4) and \( \{ F(\cdot,z) : z \in J \} \) is bounded in \( WPAAS_p(X, \infty) \) for any bounded \( J \subset X \), then \( t \rightarrow F(t,\phi(\cdot)) \in WPAAS_p(X, \infty) \).

By Lemma 2.22 (iii) and Theorem 3.5 Lemma 2.23 (iii) and Corollary 3.9 we have the following corollaries:

Corollary 3.10. Let \( \rho \in U_\infty \), \( F = F_1 + F_2 \in WPAAS_p(\mathbb{R} \times X, X, r) \) and \( \psi \in WPAA(X, r) \). Assume that \( F_1 \) satisfies (A5), \( F_2 \) satisfies (A4) and \( \{ F(\cdot,z) : z \in J \} \) is bounded in \( WPAAS_p(X, r) \) for any bounded \( J \subset X \), then \( t \rightarrow F(t,\psi(\cdot)) \in WPAAS_p(X, r) \).

Corollary 3.11. Let \( \rho \in U_\infty \), \( F = F_1 + F_2 \in WPAAS_p(\mathbb{R} \times X, \infty) \) and \( \phi \in WPAA(X, \infty) \). Assume that \( F_1 \) satisfies (A5), \( F_2 \) satisfies (A4) and \( \{ F(\cdot,z) : z \in J \} \) is bounded in \( WPAAS_p(X, \infty) \) for any bounded \( J \subset X \), then \( t \rightarrow F(t,\phi(\cdot)) \in WPAAS_p(X, \infty) \).

Corollary 3.12 (33). Let \( \rho \in U_\infty \), \( f \in WPAA(\mathbb{R} \times X, X, \infty) \) and \( u \in WPAA(X, \infty) \). Assume that the following conditions are satisfied

(i) There exist a constant \( L > 0 \) such that \( \| f(t,x) - f(t,y) \| \leq L \| x - y \| \) for all \( x, y \in X \) and \( t \in \mathbb{R} \).

(ii) \( g(t,x) \) is uniformly continuous in any bounded subset \( K' \subset X \) uniformly for \( t \in \mathbb{R} \).

Then the function \( t \rightarrow f(t,u(t)) \) belongs to \( WPAA(X, \infty) \).

4. Weighted pseudo almost automorphic mild solution

In this section, we study weighted pseudo almost automorphic mild solutions to the neutral equation (1.1). We list the following basic assumptions:

(A6) The system

\[ u'(t) = A(t)u(t), \quad t \geq s, \quad u(s) = \phi \in X. \]

has an associated evolution family of operators \( \{ U(t,s) : t \geq s \) with \( t, s \in \mathbb{R} \). Further, we assume that the domains of operators \( A(t) \) are constant in \( t \), that is, \( D(A(t)) = D = Y \) for all \( t \in \mathbb{R} \) and that the evolution family \( U(t,s) \) is asymptotically stable in the sense that there exist some constants \( M, \delta > 0 \) such that

\[ \| U(t,s) \| \leq Me^{-\delta(t-s)} \]

for all \( t, s \in \mathbb{R} \) with \( t \geq s \).

(A7) The function \( s \rightarrow A(s)U(t,s) \) defined from \( (-\infty, t) \) into \( L(Y, X) \) is strongly measurable and there exist a nonincreasing function \( H : [0, \infty) \rightarrow [0, \infty) \) and \( \delta > 0 \) with \( e^{-\delta H} \in L^1([0, \infty)) \) such that

\[ \| A(s)U(t,s) \|_{L^1(Y, X)} \leq e^{-\delta H(t-s)}, \quad t > s. \]
(A8) \( g \in WPAAS^p(\mathbb{R}, X, \infty) \) and there exist a positive constant \( L_g \) such that for \( \psi_1 \in B \), \( i = 1, 2 \), \( \| g(t, \psi_1) - g(t, \psi_2) \|_p \leq L_g \| \psi_1 - \psi_2 \|_B \).

(A9) \( f \in WPAA(\mathbb{R}, X, \infty) \) and there exist a positive constant \( L_f \) such that for \( \psi_1 \in B \), \( i = 1, 2 \), \( \| f(t, \psi_1) - f(t, \psi_2) \| \leq L_f \| \psi_1 - \psi_2 \|_B \).

(A10) The series

\[
\sum_{k=1}^{\infty} \left( \int_{t-k}^{t-k+1} e^{-\delta q(t-s)} H^q(t-s) ds \right)^{1/q}
\]

converges, \( q > 1 \), and let \( K = \left( \int_{-\infty}^{t} e^{-\delta q(t-s)} H^q(t-s) ds \right)^{1/q} \).

Let \( q > 1 \) such that \( \frac{1}{p} + \frac{1}{q} = 1 \). Denote

\[
\alpha_0 = M \left( \frac{e^{q\delta - 1}}{q\delta} \right)^{1/q}, \quad \alpha = \alpha_0 \sum_{k=1}^{\infty} e^{-\delta k}.
\]

(A11) The function \( \mathbb{R} \times \mathbb{R} \rightarrow X \), \( (t, s) \rightarrow U(t, s)x \in bAA(\mathbb{R} \times \mathbb{R}, Y) \) uniformly for \( x \in X \).

(A12) The function \( \mathbb{R} \times \mathbb{R} \rightarrow X \), \( (t, s) \rightarrow A(s)U(t, s)x \in bAA(\mathbb{R} \times \mathbb{R}, X) \) uniformly for \( x \in Y \).

**Definition 4.1** ([22]). A continuous function \( u : [\sigma, \sigma + a) \rightarrow X, a > 0 \), is a mild solution of neutral system (1.1) on \([\sigma, \sigma + a)\), if the function \( s \rightarrow A(s)U(t, s)f(s, u_s) \) is integrable on \([\sigma, t)\) for every \( \sigma < t < \sigma + a \), \( u_\sigma = \varphi \), and

\[
u(t) = U(t, \sigma)(\varphi(0) + f(\sigma, \varphi)) - f(t, u_t) - \int_{\sigma}^{t} A(s)U(t, s)f(s, u_s) ds
\]

\[
+ \int_{\sigma}^{t} U(t, s)g(s, u_s) ds, \quad t \in [\sigma, \sigma + a).
\]

Under assumptions (A6) and (A7), it can be easily shown that the function

\[
u(t) = -f(t, u_t) + \int_{-\infty}^{t} U(t, s)g(s, u_s) ds - \int_{-\infty}^{t} A(s)U(t, s)f(s, u_s) ds,
\]

for each \( t \in \mathbb{R} \) is a mild solution of (1.1).

**Lemma 4.2** ([23]). Assume that conditions (A6), (A7), (A11) hold. Let \( \rho \in U_\infty \) and \( u \in WPAAS^p(X, \infty) \), and

\[
v(t) := \int_{-\infty}^{t} U(t, s)u(s) ds, \quad t \in \mathbb{R}.
\]

Then \( v \in WPAA(X, \infty) \).

**Lemma 4.3.** Assume that conditions (A6), (A7), (A12) hold. Let \( \rho \in U_\infty \) and \( u \in WPAAS^p(X, \infty) \), and

\[
v(t) := \int_{-\infty}^{t} A(s)U(t, s)u(s) ds, \quad t \in \mathbb{R}.
\]

Then \( v \in WPAA(X, \infty) \).
Proof: Since \( u \in WPAAS^p(\mathbb{X}, \infty) \), there exist functions \( \tilde{\psi} \) in \( AS^p(\mathbb{X}) \) and \( \tilde{\omega}^b \) in \( PAA_0(L^p(0, 1; \mathbb{X}), \infty, \rho) \) such that \( u = \tilde{\psi} + \tilde{\omega}^b \).

\[
v(t) := \int_{-\infty}^{t} A(s)U(t, s)\tilde{\psi}(s)ds + \int_{-\infty}^{t} A(s)U(t, s)\tilde{\omega}^b(s)ds = \xi(t) + \eta(t).
\]

To prove that \( v \) is a weighted pseudo almost automorphic function of class infinity, we only need to verify \( \xi(t) \in AA(\mathbb{X}) \) and \( \eta(t) \in PAA_0(\mathbb{X}, \infty, \rho) \).

First, let us prove that \( \xi(t) \in AA(\mathbb{X}) \). We consider for each \( n = 1, 2, \ldots, \) the integrals

\[
\xi_n(t) = \int_{t-n}^{t-n+1} A(\sigma)U(t, \sigma)\tilde{\psi}(\sigma)d\sigma.
\]

Now, let us show that each \( \xi_n(t) \in AA(\mathbb{X}) \). Using the Hölder inequality and the exponential dissipation property of the evolution family \( U(t, s) \), it follows that

\[
\|\xi_n(t)\| \leq \int_{t-n}^{t-n+1} \|A(\sigma)U(t, \sigma)\tilde{\psi}(\sigma)\|d\sigma
\]

\[
\leq \int_{t-n}^{t-n+1} \|A(\sigma)U(t, \sigma)\|\|\tilde{\psi}(\sigma)\|d\sigma
\]

\[
\leq \int_{t-n}^{t-n+1} H(t - \sigma)e^{-\delta(t-\sigma)}\|\tilde{\psi}(\sigma)\|d\sigma
\]

\[
\leq \left( \int_{t-n}^{t-n+1} e^{-\delta q(t-\sigma)}H^q(t - \sigma)d\sigma \right)^{1/q} \left( \int_{t-n}^{t-n+1} \|\tilde{\psi}(\sigma)\|^p \right)^{1/p}d\sigma
\]

\[
\leq \left( \int_{t-n}^{t-n+1} e^{-\delta q(t-\sigma)}H^q(t - \sigma)d\sigma \right)^{1/q} \|\tilde{\psi}(\sigma)\|_{S^p}.
\]

Using the fact the series

\[
\sum_{n=1}^{\infty} \left( \int_{t-n}^{t-n+1} e^{-\delta q(t-\sigma)}H^q(t - \sigma)d\sigma \right)^{1/q}
\]

converges, we deduce from the well-known Weierstrass test that the series \( \sum_{n=1}^{\infty} \xi_n(t) \) is uniformly convergent on \( \mathbb{R} \). Let

\[
\xi(t) = \sum_{n=1}^{\infty} \xi_n(t) \quad \text{for each } t \in \mathbb{R}.
\]

Observe that

\[
\xi(t) = \int_{-\infty}^{t} A(s)U(t, s)\tilde{\psi}(s)ds \quad t \in \mathbb{R}.
\]

Clearly, \( \xi(t) \in C(\mathbb{R}, \mathbb{X}) \) and

\[
\|\xi(t)\| \leq \sum_{n=1}^{\infty} \|\xi_n(t)\| \leq \sum_{n=1}^{\infty} \left( \int_{t-n}^{t-n+1} e^{-\delta q(t-\sigma)}H^q(t - \sigma)d\sigma \right)^{1/q} \|\tilde{\psi}(\sigma)\|_{S^p}.
\]

Since \( \tilde{\psi} \in AS^p(\mathbb{X}) \) and \( A(s)U(t, s)x \in bAA(\mathbb{R}, \mathbb{X}) \), then for every sequence of real numbers \( \{s_n\}_{n \in \mathbb{N}} \) there exist a subsequence \( \{s_m\}_{m \in \mathbb{N}} \) and a function \( \tilde{\psi}(\cdot) \in L^p_{\text{loc}}(\mathbb{R}, \mathbb{X}) \) such that for each \( t \in \mathbb{R} \),

\[
\lim_{m \to \infty} \left( \int_{t}^{t+1} \|\tilde{\psi}(s + s_m) - \tilde{\psi}(s)\|^pds \right)^{1/p} = 0, \quad (4.1)
\]
\[
\lim_{m \to \infty} \left( \int_t^{t+1} \| \tilde{\psi}(s - s_m) - \tilde{\psi}(s) \|^p ds \right)^{1/p} = 0, \tag{4.2}
\]
\[
\lim_{n \to \infty} A(s + s_n) U(t + s_n, s + s_n)x = U'(t, s)x, \quad t, s \in \mathbb{R}, \ x \in \mathcal{X}, \tag{4.3}
\]
\[
\lim_{n \to \infty} U'(t - s_n, s - s_n)x = A(s) U(t, s)x, \quad t, s \in \mathbb{R}, \ x \in \mathcal{X}. \tag{4.4}
\]

Let \( \tilde{x}_n = \int_{t-n}^{t-n+1} A(\sigma) U(t, \sigma) \tilde{\psi}(\sigma) d\sigma \). Then using Hölder inequality, we have
\[
\| \tilde{x}_n(t + s_m) - \tilde{x}_n(t) \|
\]
\[
= \left\| \int_{t-n}^{t-n+1} [A(\sigma + s_m) U(t + s_m, \sigma + s_m) \tilde{\psi}(\sigma + s_m) - A(\sigma) U(t, \sigma) \tilde{\psi}(\sigma)] d\sigma \right\|
\]
\[
\leq \left\| \int_{t-n}^{t-n+1} A(\sigma + s_m) U(t + s_m, \sigma + s_m) (\tilde{\psi}(\sigma + s_m) - \tilde{\psi}(\sigma)) \right\| + \left\| \int_{t-n}^{t-n+1} [A(\sigma + s_m) U(t + s_m, \sigma + s_m) - A(\sigma) U(t, \sigma) \tilde{\psi}(\sigma)] d\sigma \right\|
\]
\[
= I_n(t) + J_n(t),
\]
where
\[
I_n(t) = \left\| \int_{t-n}^{t-n+1} A(\sigma + s_m) U(t + s_m, \sigma + s_m) (\tilde{\psi}(\sigma + s_m) - \tilde{\psi}(\sigma)) \right\|
\]
\[
J_n(t) = \left\| \int_{t-n}^{t-n+1} [A(\sigma + s_m) U(t + s_m, \sigma + s_m) - A(\sigma) U(t, \sigma) \tilde{\psi}(\sigma)] d\sigma \right\|
\]

Then using the Hölder inequality, we obtain
\[
I_n(t) \leq \int_{t-n}^{t-n+1} e^{-\delta(t-\sigma)} H(t - \sigma) \| \tilde{\psi}(\sigma + s_m) - \tilde{\psi}(\sigma) \| d\sigma
\]
\[
\leq \left( \int_{t-n}^{t-n+1} e^{-\delta(t-\sigma)} H^q(t - \sigma) d\sigma \right)^{1/q} \left( \int_{t-n}^{t-n+1} \| \tilde{\psi}(\sigma + s_m) - \tilde{\psi}(\sigma) \|^p d\sigma \right)^{1/p}.
\]

Now using (4.1), it follows that \( I_n(t) \to 0 \) as \( m \to \infty \) for each \( t \in \mathbb{R} \). Similarly, using the Lebesgue Dominated convergence Theorem and (4.3) it follows that \( J_n(t) \to 0 \) as \( m \to \infty \) for each \( t \in \mathbb{R} \). Now,
\[
\| \tilde{x}_n(t + s_m) - \tilde{x}_n(t) \| \to 0, \quad \text{as} \ m \to \infty.
\]

Similarly, using (4.2) and (4.4), it can be shown that
\[
\| \tilde{x}_n(t - s_m) - \tilde{x}_n(t) \| \to 0, \quad \text{as} \ m \to \infty.
\]

Thus, we can conclude that each \( \tilde{x}_n \in AA(\mathcal{X}) \) and consequently their uniform limit \( \tilde{x}(t) \in AA(\mathcal{X}) \).

Next we verify that \( \tilde{y}(t) \in PAA_0(\mathcal{X}, \infty, \rho) \). For each \( n=1,2,\ldots \), we consider the integral
\[
\tilde{y}_n(t) = \int_{t-n}^{t-n+1} \| A(t, s) U(t, s) \| \| \tilde{\psi}(s) \| ds.
\]
For this we have the following estimates:
\[
\sup_{t \in [t-n, t]} \| \tilde{y}_n(t) \|
\]
\[
\leq \sup_{t \in [t-n, t]} \int_{t-n}^{t-n+1} \| A(\theta, s) U(\theta, s) \| \| \tilde{\psi}(s) \| ds
\]
where pseudo almost automorphic mild solution of class infinity provided that

\[ R \text{ uniformly convergent on } E \]

Then for \( r > 0 \), we see that

\[
\frac{1}{m(T, \rho)} \int_{-T}^{T} \left( \sup_{\theta \in [t-r, t]} \left\| \mathfrak{g}_{n}(\theta) \right\| \right) \rho(t) dt
\]

\[
\leq Z \frac{1}{m(T, \rho)} \int_{-T}^{T} \left( \sup_{\theta \in [t-r, t]} \left( \int_{\theta-n}^{\theta-n+1} \left\| \mathfrak{w}(s) \right\|^p ds \right) \right)^{1/p} \rho(t) dt,
\]

where \( Z = \left( \int_{t-n}^{t-n+1} e^{-\delta q(t-s)} H^q(t-s) ds \right)^{1/q} \).

Since \( \mathfrak{w}^{h} \in PAA_{0}(L^p(0, 1, \mathcal{X}), \infty, \rho) \), we have \( \mathfrak{g}_{n}(t) \in PAA_{0}(\mathcal{X}, \infty, \rho) \) from above inequality. Then we deduce from the Weierstrass test that the series \( \sum_{n=1}^{\infty} \mathfrak{g}_{n}(t) \) is uniformly convergent on \( \mathbb{R} \). Moreover,

\[
\mathfrak{y}(t) = \int_{-\infty}^{t} A(t, s) U(t, s) ds = \sum_{n=1}^{\infty} \mathfrak{g}_{n}(t).
\]

and clearly \( \mathfrak{y}(t) \in C(\mathbb{R}, \mathcal{X}) \) and

\[
\left\| \mathfrak{y}(t) \right\| \leq \sum_{n=1}^{\infty} \left\| \mathfrak{g}_{n}(t) \right\| \leq \sum_{n=1}^{\infty} Z \left\| \mathfrak{w} \right\|_{L_p}.
\]

Applying \( \mathfrak{g}_{n} \in PAA_{0}(\mathcal{X}, \infty, \rho) \) and the inequality

\[
\frac{1}{m(T, \rho)} \int_{-T}^{T} \left( \sup_{\theta \in [t-r, t]} \left\| \mathfrak{y}(\theta) \right\| \right) \rho(t) dt
\]

\[
\leq \frac{1}{m(T, \rho)} \int_{-T}^{T} \left( \sup_{\theta \in [t-r, t]} \left\| \mathfrak{y}(\theta) - \sum_{k=1}^{n} \mathfrak{g}_{k}(\theta) \right\| \right) \rho(t) dt
\]

\[
+ \sum_{k=1}^{\infty} \frac{1}{m(T, \rho)} \int_{-T}^{T} \left( \sup_{\theta \in [t-r, t]} \left\| \mathfrak{g}_{k}(\theta) \right\| \right) \rho(t) dt.
\]

We deduce that the uniformly limit \( \mathfrak{y}(\cdot) = \sum_{n=1}^{\infty} \mathfrak{g}_{n}(t) \in PAA_{0}(\mathcal{X}, \infty, \rho) \). Therefore \( \nu(t) = \mathfrak{x}(t) + \mathfrak{y}(t) \) is weighted pseudo almost automorphic of class infinity. \( \square \)

**Theorem 4.4.** If conditions (A6)–(A12) hold, then (1.1) admits a unique weighted pseudo almost automorphic mild solution of class infinity provided that

\[
\Theta = \zeta \left( L_f + \sup_{t \in \mathbb{R}} \left( \int_{-\infty}^{t} e^{-\delta q(t-s)} H^q(t-s) ds \right)^{1/q} L_f + \alpha L_2 \right) < 1,
\]

where \( \zeta \) is defined as in Remark 2.25.

**Proof.** Define \( \mathcal{F} : WPAA(\mathcal{X}, \infty) \to WPAA(\mathcal{X}, \infty) \) as

\[
(\mathcal{F}u)(t) = -f(t, u_t) + \int_{-\infty}^{t} U(t, s) g(s, u_s) ds - \int_{-\infty}^{t} A(s) U(t, s) f(s, u_s) ds.
\]

If \( u \in WPAA(\mathcal{X}, \infty) \) by Lemma 3.6 and Corollary 3.11 \( g(s, u_s) \in WPAAAS^p(\mathcal{X}, \infty) \). By Lemma 3.9 and Corollary 3.12 \( f(s, u_s) \in WPAA(\mathcal{X}, \infty) \). Owing to Lemma 4.2.
and Lemma 4.3 it is not difficult to see that \( F(WPAA(X, \infty)) \subseteq WPAA(X, \infty) \).

For any \( u, v \in WPAA(X, \infty) \), we have

\[
\| (F u)(t) - (F v)(t) \|
= \left\| -f(t, u_t) + \int_{-\infty}^{t} U(t, s) g(s, u_s) ds - \int_{-\infty}^{t} A(s) U(t, s) f(s, u_s) ds \right. \\
+ f(t, v_t) - \int_{-\infty}^{t} U(t, s) g(s, v_s) ds + \int_{-\infty}^{t} A(s) U(t, s) f(s, v_s) ds \right\| \\
\leq L_f \| v_t - u_t \|_M + \left( \int_{-\infty}^{t} e^{-\delta q(t-s)} H^q(t-s) ds \right)^{1/q} L_f \| v_t - u_t \|_M \\
+ M \sum_{k=1}^{\infty} \left( \int_{-\infty}^{t} e^{-\delta q(s)} ds \right)^{1/q} \left( \int_{-\infty}^{t} \| g(s, u_s) - g(s, v_s) \| p ds \right)^{1/p} \\
= L_f \| v_t - u_t \|_M + \left( \int_{-\infty}^{t} e^{-\delta q(t-s)} H^q(t-s) ds \right)^{1/q} L_f \| v_t - u_t \|_M \\
+ \alpha_0 \sum_{k=1}^{\infty} e^{-\delta k} \| g(t + k - 2 + \cdot, u_{t+k-2+}) - g(t + k - 2 + \cdot, v_{t+k-2+}) \|_p \\
\leq \left( L_f + \sup_{t \in \mathbb{R}} \left( \int_{-\infty}^{t} e^{-\delta q(t-s)} H^q(t-s) ds \right)^{1/q} L_f + \alpha L_g \right) \| u - v \|_\infty \\
= \Theta \| u - v \|_\infty.
\]

Consequently,

\[
\| F u - F v \|_\infty \leq \Theta \| u - v \|_\infty.
\]

Then \( F \) is a contraction since \( \Theta < 1 \). By the Banach contraction mapping principle, \( F \) has a unique fixed point in \( WPAA(X, \infty) \), which is the unique \( WPAA \) mild solution to the problem. \( \square \)

We end this paper with a simple example. Consider the first-order boundary-value problem which was used in [8],

\[
\begin{align*}
\frac{\partial}{\partial t} \left[ u(t, \xi) + \int_{-\infty}^{0} \int_{0}^{\pi} b(s, \eta, \xi) u(t + s, \eta) d\eta ds \right] \\
= \frac{\partial^2}{\partial \xi^2} u(t, \xi) + a_0(\xi) u(t, \xi) + \int_{-\infty}^{0} a(s) u(t + s, \xi) ds, \\
u(0, t) = u(\pi, t) = 0,
\end{align*}
\]

for \( t \in \mathbb{R} \) and \( \xi \in I = [0, \pi] \).
Note that equations of type (4.5) arise in control systems described by abstract retarded functional-differential equations with feedback control governed by proportional integro-differential law, see [18] for details.

To analyze (4.5), we let $X = L^2([0, \pi])$ and $\mathfrak{B} = C((-\infty, 0], X)$. In addition, we suppose that the function $a, a_0, a_1$ are continuous and satisfy the following conditions:

(i) The function $b(\cdot), \frac{\partial^i}{\partial \xi^i} b(\tau, \eta, \xi), i=1,2$, are (Lebesgue) measurable, $b(\tau, \eta, \pi) = 0, b(\tau, \eta, 0) = 0$ for every $(\tau, \eta)$ and

\[ N_1 = \max \left\{ \int_0^\pi \int_{-\infty}^{0} \int_0^\pi \left( \frac{\partial^i}{\partial \xi^i} b(\tau, \eta, \xi) \right)^2 d\eta d\tau d\xi : i = 0, 1, 2 \right\} < \infty. \]

Define $f, g : C((-\infty, 0], X)$ by

\[ f(t, \psi)(\xi) = \int_{-\infty}^{0} \int_0^\pi b(s, \eta, \xi)\psi(s, \eta) d\eta ds, \]

\[ g(t, \psi)(\xi) = a_0(\xi) u(t, \xi) + \int_{-\infty}^{0} a(s)\psi(s, \xi) ds. \]

In view of above arguments, it is clear that (4.5) can be rewritten in the abstract form of (1.1). By direct estimation from (i), we can show that $f$ takes values in $D(A)$ and that $f(t, \cdot) : C((-\infty, 0] : X) \rightarrow [D(A)]$ is a bounded linear operator with $\|Af(t, \cdot)\| \leq (N_1 p)^{\frac{1}{2}}$ for each $t \in \mathbb{R}$. Furthermore, $g$ is a bounded linear operator on $X$ with

\[ g(t, \cdot) \leq \|a_0\|_\infty + \left( p \left( \int_{-\infty}^{0} a^2(s) ds \right) \right)^{1/2}, \]

for every $t \in \mathbb{R}$.

As a consequence of Theorem 4.4, system (4.5) has a unique weighted pseudo almost automorphic mild solution of class infinity whenever

\[ 2\sqrt{N_1 p} + \|a_0\|_\infty + \left( p \left( \int_{-\infty}^{0} a^2(s) ds \right) \right)^{1/2} < 1. \]

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