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CARLEMAN ESTIMATES AND NULL CONTROLLABILITY OF DEGENERATE/SINGULAR PARABOLIC SYSTEMS

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Abstract. We study null controllability properties for parabolic coupled systems with degeneracy and singularity occurring in the interior of the spatial domain. This article is the first to consider a problem with singular coupling terms; previous result cannot be adapted to this situation. In particular, we focus on the well posedness of the problem and then we prove Carleman estimates for the associated adjoint problem.

1. Introduction

This article concerns the null controllability for the coupled degenerate singular parabolic system

$$u_t - (a(x)u_x)_x - \frac{\lambda_1}{b_1(x)}u - \frac{\mu}{d(x)}v = h1_\omega, \quad (t,x) \in Q, \tag{1.1}$$

$$v_t - (a(x)v_x)_x - \frac{\lambda_2}{b_2(x)}v - \frac{\mu}{d(x)}u = 0, \quad (t,x) \in Q, \tag{1.2}$$

$$u(t,0) = u(t,1) = v(t,0) = v(t,1) = 0, \quad t \in (0,T), \tag{1.3}$$

$$u(0,x) = u_0(x), \quad v(0,x) = v_0(x), \quad x \in (0,1), \tag{1.4}$$

where ω is an open subset of $(0,1)$, $T > 0$ fixed, $Q := (0,T) \times (0,1)$, $h \in L^2(Q)$ and 1_ω denotes the characteristic function of the set ω.

Moreover, we assume that the constants λ_i, μ, $i = 1,2$, satisfy suitable assumptions described below, and the functions a, b_i, d, $i = 1,2$, degenerate at the same interior point x_0 of the spatial domain $(0,1)$ that can belong to the control set ω (for the precise assumptions we refer to section 2).

Let us note that it is just for the sake of simplicity that we focus on problem (1.1)-(1.4): in fact all the stated results are still valid for the system

$$\begin{gathered} u_t - (a(x)u_x)_x - \frac{\lambda_1}{b_1(x)}u - c_{11}u - c_{12}v - \frac{\mu}{d(x)}v = h1_\omega, \quad (t,x) \in Q, \\ v_t - (a(x)v_x)_x - \frac{\lambda_2}{b_2(x)}v - c_{22}v - c_{21}u - \frac{\mu}{d(x)}u = 0, \quad (t,x) \in Q, \\ u(t,0) = u(t,1) = v(t,0) = v(t,1) = 0, \quad t \in (0,T), \end{gathered}$$

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$$u(0,x) = u_0(x), \quad v(0,x) = v_0(x), \quad x \in (0,1),$$

where $c_{ij} \in L^\infty(Q)$, $i,j = 1,2$.

In recent years an increasing interest has been devoted to the study of degenerate and/or singular parabolic equations. Indeed many problems coming from physics, biology and economics are described by degenerate/singular parabolic equations, whose linear prototype is

$$\frac{\partial u}{\partial t} - Au - \frac{\lambda}{b(x)}u = h(t,x), \quad (t,x) \in (0,T) \times (0,1) \tag{1.5}$$

with the associated boundary conditions. Here h belongs to a suitable Lebesgue space and $Au = A_1 u := (au_x)_x$ or $Au = A_2 u := au_{xx}$, where a and b can be degenerate functions.

A common strategy in showing controllability results for (1.5) is to show that certain global Carleman estimates hold for the operator which is the adjoint of the given operator.

More recently, several works were done in the controllability of purely ($\lambda = 0$) degenerate equations in divergence or in non divergence form with boundary degeneracy. The main result of these works is the development of adequate Carleman inequalities, see [3, 16, 23]. For related systems of degenerate equations we refer to [1, 2]. To our best knowledge, the first results on Carleman estimates for purely degenerate problems with an interior degenerate point are obtained in [19], for a regular degeneracy, and in [18], for a globally non smooth degeneracy.

Another interesting situation is the case of parabolic operators with singular lower order terms. For instance, in [6, 10, 21], the reader will find a motivating example of the so-called inverse-square potential that arises for example in quantum mechanics or linearized combustion problems. Furthermore, while in [5, 27] only the existence of a solution for the uniformly ($a > 0$) parabolic problem with singular potential is considered, in [13, 26] the authors analyze in detail the question of whether it is possible to control heat equations involving singular inverse-square potentials.

Moreover, a full analysis has been developed for operators that couple a degenerate diffusion coefficient with a singular potential in the case of degeneracy and singularity located on the boundary of the spatial domain (see [14, 15, 25]).

In this article, we investigate the null controllability of system (1.1)-(1.4). More precisely, we use the new Carleman estimates for the interior degenerate/singular parabolic equations in [17] to develop Carleman estimate for the adjoint degenerate/singular system associated with (1.1)-(1.4) which yields in turn an observability inequality. Also, by a standard argument, we deduce the null controllability of system (1.1)-(1.4) from any open subset ω. To our knowledge, this is the first null controllability result for such kind of systems.

In particular, the main result of this paper will be the following.

Theorem 1.1. *If hypothesis 2.10 is satisfied, then the coupled degenerate/singular parabolic system* (1.1)-(1.4) *with one control force is null controllable.*

For our further results, it is important to remind the following fundamental Hardy-Poincaré inequality.

Theorem 1.2 ([19, Proposition 2.6])**.** *Assume that p is any continuous function in $[0,1]$, with $p > 0$ on $[0,1] \setminus \{x_0\}$, $p(x_0) = 0$ and such that there exists $\vartheta > 1$ such* *that the function*

$$x \mapsto \frac{p(x)}{|x-x_0|^{\vartheta}} \tag{1.6}$$

is nonincreasing on the left of $x = x_0$, and is nondecreasing on the right of $x = x_0$. Then, there exists a constant $C_{HP} > 0$ such that for any function w locally absolutely continuous on $[0, x_0) \cup (x_0, 1]$ and satisfying

$$w(0) = w(1) = 0 \quad \textit{and} \quad \int_0^1 p(x)|w'(x)|^2\,dx < \infty,$$

the following inequality holds

$$\int_0^1 \frac{p(x)}{(x-x_0)^2} w^2(x)\,dx \le C_{HP} \int_0^1 p(x)|w'(x)|^2\,dx.$$

This article is organized as follows. In Section 2, we study the well-posedness of the system (1.1)-(1.4). In Section 3, we derive a Carleman estimate with boundary terms for the adjoint problem to (1.1)-(1.4). In Section 4, we provide an ω-Carleman estimate that will be useful to study the null controllability of (1.1)-(1.4) with one control force. Using the previous Carleman estimates we will deduce in Section 5 observability inequality and null controllability results. The last section is an appendix which is devoted to the proof of a Caccioppoli's inequality which plays a crucial role in the proof of the Carleman estimate.

2. Assumptions and well-posedness

To study the well-posedness of system (1.1)-(1.4), we distinguish four different types of degeneracy. Towards this end, as in [17], we consider the following cases

Hypothesis 2.1. Double weakly degenerate case (WWD). There exists $x_0 \in (0,1)$ such that $a(x_0) = b_i(x_0) = 0$, $a, b_i > 0$ in $[0,1] \setminus \{x_0\}$, $a, b_i \in C^1([0,1] \setminus \{x_0\})$ and there exists $K, L_i \in (0,1)$ such that $(x-x_0)a' \le Ka$ and $(x-x_0)b_i' \le L_i b_i$ a.e. in $[0,1]$.

Hypothesis 2.2. Weakly strongly degenerate case (WSD). There exists $x_0 \in (0,1)$ such that $a(x_0) = b_i(x_0) = 0$, $a > 0$ in $[0,1] \setminus \{x_0\}$, $a \in C^1([0,1] \setminus \{x_0\})$, $b_i \in C^1([0,1] \setminus \{x_0\}) \cap W^{1,\infty}(0,1)$ there exists $K \in (0,1)$, $L_i \in [1,2)$ such that $(x-x_0)a' \le Ka$ and $(x-x_0)b_i' \le L_i b_i$ a.e. in $[0,1]$.

Hypothesis 2.3. Strongly weakly degenerate case (SWD). There exists $x_0 \in (0,1)$ such that $a(x_0) = b_i(x_0) = 0$, $a > 0$ in $[0,1] \setminus \{x_0\}$, $a \in C^1([0,1] \setminus \{x_0\}) \cap W^{1,\infty}(0,1)$, $b_i \in C^1([0,1] \setminus \{x_0\})$, $\exists K \in [1,2)$, $L_i \in (0,1)$ such that $(x-x_0)a' \le Ka$ and $(x-x_0)b_i' \le L_i b_i$ a.e. in $[0,1]$ and, if $K > 4/3$, then there exists $\theta \in (0,K]$ such that $\frac{a}{|x-x_0|^{\theta}}$ is nonincreasing on the left of $x = x_0$ and nondecreasing on the right of $x = x_0$.

Hypothesis 2.4. Double strongly degenerate case (SSD). There exists $x_0 \in (0,1)$ such that $a(x_0) = b_i(x_0) = 0$, $a > 0$ in $[0,1] \setminus \{x_0\}$, $a, b_i \in C^1([0,1] \setminus \{x_0\}) \cap W^{1,\infty}(0,1)$, there exist $K, L_i \in [1,2)$ such that $(x-x_0)a' \le Ka$ and $(x-x_0)b_i' \le L_i b_i$ a.e. in $[0,1]$.

For the next results we shall make the following hypothesis on the coupling term.

Hypothesis 2.5. The function d is weakly degenerate, that is, there exists $x_0 \in (0,1)$ such that $d(x_0) = 0$, $d > 0$ on $[0,1] \setminus \{x_0\}$, $d \in C^1([0,1] \setminus \{x_0\})$ and there exists $M \in (0,1)$ such that $(x - x_0)d' \leq Md$ a.e. in $[0,1]$.

Hypothesis 2.6. *The function d is strongly degenerate, that is, there exists $x_0 \in (0,1)$ such that $d(x_0) = 0$, $d > 0$ on $[0,1] \setminus \{x_0\}$, $d \in C^1([0,1] \setminus \{x_0\}) \cap W^{1,\infty}(0,1)$ and there exists $M \in [1,2)$ such that $(x - x_0)d' \leq Md$ a.e. in $[0,1]$.*

As in [17], we start introducing the following weighted Hilbert spaces, which are suitable to study all situations, namely the (WWD), (SSD), (WSD) and (SWD) cases:

$$H^1_a(0,1) := \big\{u \in W^{1,1}_0(0,1) : \sqrt{a}u_x \in L^2(0,1)\big\},$$
$$H^1_{a,b_i}(0,1) := \big\{u \in H^1_a(0,1) : \frac{u}{\sqrt{b_i}} \in L^2(0,1)\big\}$$

endowed with the inner products

$$\langle u, v\rangle_{H^1_a} := \int_0^1 au'v'\,dx + \int_0^1 uv\,dx,$$
$$\langle u, v\rangle_{H^1_{a,b_i}} := \int_0^1 au'v'\,dx + \int_0^1 uv\,dx + \int_0^1 \frac{uv}{b_i}\,dx,$$

respectively.

In our situation, due to the presence of singular coupling terms, we shall also introduce the following Hilbert space

$$H^1_{a,b_i,d}(0,1) := \big\{u \in H^1_{a,b_i}(0,1) : \frac{u}{\sqrt{d}} \in L^2(0,1)\big\}.$$

To study well-posedness of the problem (1.1)-(1.4), we use the following crucial weighted Hardy-Poincaré inequality.

Theorem 2.7 ([17, Proposition 2.2])**.** *If one of the Hypotheses 2.1, 2.2, 2.3 holds with $K + L_i \leq 2$, then there exists a constant $C_i > 0$ such that for all $w \in H^1_{a,b_i}(0,1)$ we have*

$$\int_0^1 \frac{w^2}{b_i(x)}\,dx \leq C_i \int_0^1 a(x)|w'|^2\,dx. \tag{2.1}$$

The function d playing a crucial role, it is non surprising that the following lemma is crucial as well.

Theorem 2.8. *If Hypotheses 2.5 or 2.6 holds with $K + M \leq 2$, then there exists a constant $C^*_{HP} > 0$ such that for all $w \in H^1_{a,b_i,d}(0,1)$ we have*

$$\int_0^1 \frac{w^2}{d(x)}\,dx \leq C^*_{HP} \int_0^1 a(x)|w'|^2\,dx. \tag{2.2}$$

A key step in the proof of Carleman estimates and observability inequalities is the correct choice of the weight function space in which we will work and a key ingredient in the proof takes the form of special Hardy-Poincaré inequalities; such estimates are valid in the following suitable Hilbert space

$$\mathcal{H}_i := H^1_{a,b_i,d}(0,1).$$

Using the lemmas given above one can prove the next inequality, which turn out to be key tool to obtain well-posedness and observability properties.

Proposition 2.9 ([17, Proposition 2.2]). *Assume hypothesis 2.10. Then there exists $\Lambda_i \in (0,1]$ such that for all $w \in \mathcal{H}_i$,*

$$\int_0^1 a(x)|w'|^2\,dx - \lambda_i \int_0^1 \frac{w^2}{b_i(x)}\,dx \geq \Lambda_i \int_0^1 a(x)|w'|^2\,dx. \tag{2.3}$$

From now on, we make the following assumptions on a, b_i, d, λ_i and μ:

Hypothesis 2.10.
(1) One among the definitions 2.1, 2.2 or 2.3 holds with $K + L_i \leq 2$.
(2) We shall also admit Hypothesis 2.5 or 2.6 with $K + M \leq 2$.
(3) Setting C_i^* the best constant of (2.1) in $\mathcal{H}_i$, we assume that $\lambda_i, \mu \neq 0$ and

$$\lambda_i \in \big(0, \frac{1}{C_i^*}\big), \tag{2.4}$$

$$\mu \in \big(0, \frac{\sqrt{\Lambda_1\Lambda_2}}{C_{HP}^*}\big), \tag{2.5}$$

where Λ_i, $i = 1, 2$ is given in (2.3).

Remark 2.11.
(1) It is well known that when $K = L_i = 1$, an inequality of the form (2.1) does not hold (for other comments on this argument we refer to [17, Remark 4] and [25]).
(2) The upper bound for λ_i and μ is required for the well-posedness of the problem, as will be discussed with more details later.
(3) Under the assumptions of Theorem 2.7 and 2.8, the standard norm $\|\cdot\|_{\mathcal{H}_i}$ is equivalent to $\|w\|_o := (\int_0^1 a w_x^2 dx)^{1/2}$ for all $w \in \mathcal{H}_i$, $i = 1, 2$.

In the Hilbert space $\mathbb{H} = \mathrm{L}^2(0,1) \times \mathrm{L}^2(0,1)$, the system (1.1)-(1.4) can be transformed into the Cauchy problem

$$X'(t) - \mathbb{A}X(t) = f(t), X(0) = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}, \tag{2.6}$$

where $X = \begin{pmatrix} u(t) \\ v(t) \end{pmatrix}$,

$$\mathbb{A} = \mathcal{A} + \mathcal{B}, \tag{2.7}$$

with

$$\mathcal{D}(\mathbb{A}) := \{X^T \in \mathcal{H}_1 \times \mathcal{H}_2 : (\mathbb{A}X)^T \in \mathbb{H}\}, \tag{2.8}$$

where $\mathcal{A} = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$, $A_i w := (a w_x)_x + \frac{\lambda_i}{b_i} w$, with

$$\begin{aligned} D(A_i) :=& H^2_{a,b_i}(0,1) \\ :=& \{w \in H^1_a(0,1) : aw' \in H^1(0,1) \text{ and } A_i w \in L^2(0,1)\}, \end{aligned}$$

$$\mathcal{B} = \begin{pmatrix} 0 & \frac{\mu}{d} \\ \frac{\mu}{d} & 0 \end{pmatrix}, \quad f(t) = \begin{pmatrix} h(t,\cdot)1_\omega \\ 0 \end{pmatrix}.$$

Remark 2.12. Observe that if $X^T \in \mathcal{D}(\mathbb{A})$, then $(\frac{u}{d}, \frac{v}{d}) \in \mathbb{H}$.

As in [20, Lemma 2.1], one can prove the following formula of integration by parts which is a crucial tool for the rest of this article.

Lemma 2.13. *For all $(u,v)\in H^2_{a,b_i}(0,1)\times H^1_a(0,1)$ one has*

$$\int_0^1 (au')'v dx = -\int_0^1 au'v'dx. \tag{2.9}$$

To show that the operator $(\mathbb{A},\mathcal{D}(\mathbb{A}))$ defined by (2.7)-(2.8) generates a contraction semigroup on the Hilbert $\mathbb{H}$, we need the following technical lemma.

Lemma 2.14. *Assume that hypothesis 2.10 is satisfied. Then, the operator $\mathbb{A}:\mathcal{D}(\mathbb{A})\to\mathbb{H}$ is nonpositive and self-adjoint on $\mathbb{H}$.*

Proof. Observe that $\mathcal{D}(\mathbb{A})$ is dense in $\mathbb{H}$.
(i) $\mathbb{A}$ is symmetric. Indeed, for any $X=\binom{w_1}{w_2}$, $Y=\binom{z_1}{z_2}\in\mathcal{D}(\mathbb{A})$, one has

$$\begin{aligned}
\langle Y,\mathbb{A}X\rangle_{\mathbb{H}} &= \langle Y,\mathcal{A}X+\mathcal{B}X\rangle_{\mathbb{H}},\\
&= \Big\langle \begin{pmatrix} z_1\\ z_2\end{pmatrix}, \begin{pmatrix} A_1 & 0\\ 0 & A_2\end{pmatrix}\begin{pmatrix} w_1\\ w_2\end{pmatrix}\Big\rangle_{\mathbb{H}} + \Big\langle \begin{pmatrix} z_1\\ z_2\end{pmatrix}, \begin{pmatrix} 0 & \frac{\mu}{d}\\ \frac{\mu}{d} & 0\end{pmatrix}\begin{pmatrix} w_1\\ w_2\end{pmatrix}\Big\rangle_{\mathbb{H}},\\
&= \Big\langle \begin{pmatrix} A_1 & 0\\ 0 & A_2\end{pmatrix}\begin{pmatrix} z_1\\ z_2\end{pmatrix}, \begin{pmatrix} w_1\\ w_2\end{pmatrix}\Big\rangle_{\mathbb{H}} + \Big\langle \begin{pmatrix} 0 & \frac{\mu}{d}\\ \frac{\mu}{d} & 0\end{pmatrix}\begin{pmatrix} z_1\\ z_2\end{pmatrix}, \begin{pmatrix} w_1\\ w_2\end{pmatrix}\Big\rangle_{\mathbb{H}},\\
&= \langle \mathbb{A}Y,X\rangle_{\mathbb{H}}.
\end{aligned}$$

(ii) $\mathbb{A}$ is nonpositive. By Proposition 2.9 and Lemma (2.9), it follows that, for any $X=\binom{w_1}{w_2}\in\mathcal{D}(\mathbb{A})$ we have

$$\begin{aligned}
-\langle \mathbb{A}X,X\rangle_{\mathbb{H}} &= -\langle \mathcal{A}X+\mathcal{B}X,X\rangle_{\mathbb{H}},\\
&= -\Big\langle \begin{pmatrix} A_1 & 0\\ 0 & A_2\end{pmatrix}\begin{pmatrix} w_1\\ w_2\end{pmatrix}, \begin{pmatrix} w_1\\ w_2\end{pmatrix}\Big\rangle_{\mathbb{H}} - \Big\langle \begin{pmatrix} 0 & \frac{\mu}{d}\\ \frac{\mu}{d} & 0\end{pmatrix}\begin{pmatrix} w_1\\ w_2\end{pmatrix}, \begin{pmatrix} w_1\\ w_2\end{pmatrix}\Big\rangle_{\mathbb{H}},\\
&= \int_0^1 a(w_1')^2\,dx - \lambda_1\int_0^1 \frac{w_1^2}{b_1}\,dx + \int_0^1 a(w_2')^2\,dx - \lambda_2\int_0^1 \frac{w_2^2}{b_2}dx\\
&\quad - 2\mu\int_0^1 \frac{w_1w_2}{d}\,dx,\\
&\ge \Lambda_1\int_0^1 a(w_1')^2\,dx + \Lambda_2\int_0^1 a(w_2')^2\,dx - 2\mu\int_0^1 \frac{w_1w_2}{d}\,dx.
\end{aligned}$$

Moreover,

$$\begin{aligned}
\Big|\int_0^1 \frac{w_1w_2}{d(x)}\,dx\Big| &\le \int_0^1 \frac{|w_1|}{\sqrt{d(x)}}\frac{|w_2|}{\sqrt{d(x)}}\,dx,\\
&\le \delta\int_0^1 \frac{w_1^2}{d(x)}\,dx + \frac{1}{4\delta}\int_0^1 \frac{w_2^2}{d(x)}\,dx,\\
&\le \delta C^*_{HP}\int_0^1 a(x)|w_1'|^2\,dx + \frac{C^*_{HP}}{4\delta}\int_0^1 a(x)|w_2'|^2\,dx.
\end{aligned}$$

Hence,

$$-\langle \mathbb{A}X,X\rangle_{\mathbb{H}} \ge (\Lambda_1 - 2\mu\delta C^*_{HP})\int_0^1 a(w_1')^2\,dx + (\Lambda_2 - 2\mu\frac{C^*_{HP}}{4\delta})\int_0^1 a(w_2')^2\,dx.$$

Now, by (2.5) one can find δ such that

$$\frac{\mu C^*_{HP}}{2\Lambda_2} < \delta < \frac{\Lambda_1}{2\mu C^*_{HP}}.$$

So, there exists $\Sigma > 0$ such that

$$-\langle \mathbb{A}X, X\rangle_{\mathbb{H}} \geq \Sigma \|X\|^2_{\mathcal{H}_1\times\mathcal{H}_2}.$$

(iii) $\mathbb{A}$ is self-adjoint. Let $T : \mathbb{H} \to \mathbb{H}$ be the mapping defined in the following usual way: to each $f \in \mathbb{H}$ associate the weak solution $X = T(f) \in \mathcal{H}_1 \times \mathcal{H}_2$ of

$$-\langle \mathbb{A}X, Y\rangle_{\mathbb{H}} = \langle f, Y\rangle_{\mathbb{H}},$$

for every $Y \in \mathcal{H}_1 \times \mathcal{H}_2$. Note that T is well defined by Lax-Milgram Lemma via the part *(ii)*, which also implies that T is continuous. Now, it is easy to see that T is injective and symmetric. Thus it is self adjoint. As a consequence, $\mathbb{A} = T^{-1} : \mathcal{D}(\mathbb{A}) \to \mathbb{H}$ is self-adjoint (for example, see [24, Proposition A.8.2]).
(iv) $\mathbb{A}$ is m-dissipative. Being $\mathbb{A}$ non positive and selfadjoint, this is a straightforward consequence of [11, Corollary 2.4.8]. Then $(\mathbb{A}, \mathcal{D}(\mathbb{A}))$ generates a cosine family and an analytic contractive semigroup of angle $\frac{\pi}{2}$ on $\mathbb{H}$ (see, for instance, [4, Examples 3.14.16 and 3.7.5]). ☐

As a consequence of the previous lemmas, we have the following well-posedness and regularity results

Proposition 2.15. *(i) The operator $\mathbb{A}$ generates a contraction strongly continuous semigroup $(T(t))_{t\geq 0}$.*

(ii) For all $h \in L^2(Q)$ and $u_0, v_0 \in L^2(0,1)$, there exists a unique weak solution $(u,v) \in C([0,T];\mathbb{H}) \cap L^2(0,T;\mathcal{H}_1 \times \mathcal{H}_2)$ of (1.1)-(1.4) and

$$\sup_{t\in[0,T]} \|(u,v)(t)\|^2_{\mathbb{H}} + \int_0^T \Big(\|u\|^2_{\mathcal{H}_1} + \|v\|^2_{\mathcal{H}_2}\Big)dt \leq C_T(\|(u_0,v_0)\|^2_{\mathbb{H}} + \|h\|^2_{L^2(Q)}), \tag{2.10}$$

for a positive constant C_T.

(iii) Moreover, if $(u_0, v_0) \in \mathcal{D}(\mathbb{A})$, then

$$(u,v) \in H^1(0,T;\mathbb{H}) \cap L^2(0,T;\mathcal{D}(\mathbb{A})) \cap C([0,T];\mathcal{H}_1 \times \mathcal{H}_2), \tag{2.11}$$

and there exists a positive constant C such that

$$\begin{aligned} &\sup_{t\in[0,T]} \big(\|(u,v)(t)\|^2_{\mathcal{H}_1\times\mathcal{H}_2}\big) + \int_0^T \Big(\big\|(u_t,v_t)\big\|^2_{\mathbb{H}} + \big\|(u,v)\big\|^2_{\mathcal{D}(\mathbb{A})}\Big)dt \\ &\leq C\Big(\|(u_0,v_0)\|^2_{\mathcal{H}_1\times\mathcal{H}_2} + \|h\|^2_{L^2(Q)}\Big). \end{aligned} \tag{2.12}$$

3. Carleman estimates with boundary observation

In this section we prove one of the main result of this paper, i.e. a new Carleman estimate with boundary terms for solutions of the singular/degenerate problem

$$U_t + (a(x)U_x)_x + \frac{\lambda_1}{b_1}U + \frac{\mu}{d}V = h_1, \quad (t,x) \in Q, \tag{3.1}$$

$$V_t + (a(x)V_x)_x + \frac{\lambda_2}{b_2}V + \frac{\mu}{d}U = h_2, \quad (t,x) \in Q, \tag{3.2}$$

$$U(t,1) = U(t,0) = V(t,1) = V(t,0) = 0, \quad t \in (0,T), \tag{3.3}$$

$$U(T,x) = U_T(x), V(T,x) = V_T(x), \quad x \in (0,1), \tag{3.4}$$

which is the adjoint of problem (1.1)–(1.4).

To prove our Carleman estimates, as in [17] or in [19], let us introduce the function $\varphi(t,x) := \theta(t)\psi(x)$, where

$$\theta(t) := \frac{1}{[t(T-t)]^4} \quad \text{and} \quad \psi(x) := \mathfrak{c}\Big[\int_{x_0}^{x} \frac{y-x_0}{a(y)}dy - \mathfrak{d}\Big], \tag{3.5}$$

with $\mathfrak{d} > \mathfrak{d}^* := \sup_{[0,1]} \int_{x_0}^{x} \frac{y-x_0}{a(y)}\, dy$ and $\mathfrak{c} > 0$.

The main result of this section reads as follows.

Theorem 3.1. *Let $T > 0$ be given. Assume Hypothesis 2.10 is satisfied. Then there exist two positive constants C and s_0 such that every solution (U,V) of (3.1)-(3.4) in*

$$\mathcal{V} = \mathrm{L}^2\Big(0,T; D(\mathbb{A})\Big) \cap \mathrm{H}^1\Big(0,T; \mathcal{H}_1 \times \mathcal{H}_2\Big) \tag{3.6}$$

satisfies, for all $s \geq s_0$,

$$\begin{aligned} &\int_0^T \int_0^1 \Big[s\theta a(U_x^2 + V_x^2) + s^3\theta^3 \frac{(x-x_0)^2}{a}(U^2+V^2)\Big] e^{2s\varphi(t,x)}\, dx\, dt \\ &\leq C\Big(\int_0^T \int_0^1 \Big[h_1^2 + h_2^2\Big] e^{2s\varphi}\, dx\, dt + s\mathfrak{c}\int_0^T \big[a\theta e^{2s\varphi}(x-x_0)(U_x^2+V_x^2)\big]_{x=0}^{x=1}\, dt\Big). \end{aligned}$$

Proof. First, let us re-write the problem (3.1)-(3.4) as follows:

$$\begin{gathered} Y_t + \mathcal{A}Y + \mathcal{B}Y = H, \\ Y(t,0) = Y(t,1) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \\ Y(T,x) = Y_T(x) = \begin{pmatrix} U_T(x) \\ V_T(x) \end{pmatrix}, \end{gathered} \tag{3.7}$$

where $Y = \left(\begin{smallmatrix} U \\ V \end{smallmatrix}\right)$ and $H = \left(\begin{smallmatrix} h_1 \\ h_2 \end{smallmatrix}\right)$. Now, for $s > 0$, define the function

$$Z(t,x) = e^{s\varphi(t,x)}Y(t,x) := \begin{pmatrix} w \\ z \end{pmatrix},$$

where Y is any solution of (3.7). Then Z satisfies

$$\begin{gathered} (e^{-s\varphi}Z)_t + \mathcal{A}(e^{-s\varphi}Z) + \mathcal{B}(e^{-s\varphi}Z) = H, \quad (t,x) \in Q, \\ Z(t,0) = Z(t,1) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad t \in (0,T), \\ Z(T,x) = Z(0,x) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad x \in (0,1). \end{gathered} \tag{3.8}$$

The previous problem can be recast as follows: setting

$$\mathcal{L}W = W_t + \mathcal{A}W + \mathcal{B}W \quad \text{and} \quad \mathcal{L}_s Z = e^{s\varphi}\mathcal{L}(e^{-s\varphi}Z),$$

then (3.8) becomes

$$\begin{gathered} \mathcal{L}_s Z = e^{s\varphi}H, \quad (t,x) \in Q, \\ Z(t,0) = Z(t,1) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad t \in (0,T), \\ Z(T,x) = Z(0,x) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad x \in (0,1). \end{gathered}$$

Computing $\mathcal{L}_s Z$, one has

$$\mathcal{L}_s Z = \mathcal{L}_s^+ Z + \mathcal{L}_s^- Z,$$

where

$$\mathcal{L}_s^+ = \begin{pmatrix} L_s^+ & 0 \\ 0 & L_s^+ \end{pmatrix} + \mathcal{B} \quad \text{and} \quad \mathcal{L}_s^- = \begin{pmatrix} L_s^- & 0 \\ 0 & L_s^- \end{pmatrix},$$

with

$$\begin{gathered} L_s^+ \bar{u} := (a\bar{u}_x)_x + \lambda_i \frac{\bar{u}}{b_i} - s\varphi_t \bar{u} + s^2 a\varphi_x^2 \bar{u}, \\ L_s^- \bar{u} := \bar{u}_t - 2sa\varphi_x \bar{u}_x - s(a\varphi_x)_x \bar{u}. \end{gathered}$$

Moreover,

$$\begin{aligned} 2\langle \mathcal{L}_s^+ Z, \mathcal{L}_s^- Z\rangle_{\mathbb{H}_T} &\le 2\langle \mathcal{L}_s^+ Z, \mathcal{L}_s^- Z\rangle_{\mathbb{H}_T} + \|\mathcal{L}_s^+ Z\|_{\mathbb{H}_T}^2 + \|\mathcal{L}_s^- Z\|_{\mathbb{H}_T}^2 \\ &= \|\mathcal{L}_s Z\|_{\mathbb{H}_T}^2 = \|e^{s\varphi} H\|_{\mathbb{H}_T}^2, \end{aligned} \tag{3.9}$$

where $\langle \cdot, \cdot \rangle_{\mathbb{H}_T}$ denotes the usual scalar product in $\mathbb{H}_T := (L^2(Q))^2$. Of course,

$$\begin{aligned} \langle \mathcal{L}_s^+ Z, \mathcal{L}_s^- Z\rangle_{\mathbb{H}_T} &= \Big\langle \begin{pmatrix} L_s^+ & \frac{\mu}{d} \\ \frac{\mu}{d} & L_s^+ \end{pmatrix} \begin{pmatrix} w \\ z \end{pmatrix}, \begin{pmatrix} L_s^- & 0 \\ 0 & L_s^- \end{pmatrix} \begin{pmatrix} w \\ z \end{pmatrix} \Big\rangle_{\mathbb{H}_T}, \\ &= \Big\langle \begin{pmatrix} L_s^+ w + \frac{\mu}{d} z \\ L_s^+ z + \frac{\mu}{d} w \end{pmatrix}, \begin{pmatrix} L_s^- w \\ L_s^- z \end{pmatrix} \Big\rangle_{\mathbb{H}_T}, \\ &= \langle L_s^+ w, L_s^- w\rangle_{L^2(Q)} + \langle L_s^+ z, L_s^- z\rangle_{L^2(Q)} \\ &\quad + \mu\langle \frac{z}{d}, L_s^- w\rangle_{L^2(Q)} + \mu\langle \frac{w}{d}, L_s^- z\rangle_{L^2(Q)}. \end{aligned}$$

Observe that the operators L_s^+ and L_s^- are exactly the ones of [17]. Using [17, Lemmas 3.2 and 3.3], we deduce immediately that there exist two positive constants C and s_0, such that for all $s \ge s_0$,

$$\begin{aligned} &\langle \mathcal{L}_s^+ Z, \mathcal{L}_s^- Z\rangle_{\mathbb{H}_T} \\ &\ge C \int_0^T \int_0^1 \big[s\theta a w_x^2 + s^3\theta^3 \frac{(x-x_0)^2}{a} w^2 \big]\, dx\, dt \\ &\quad + C \int_0^T \int_0^1 \big[s\theta a z_x^2 + s^3\theta^3 \frac{(x-x_0)^2}{a} z^2 \big]\, dx\, dt \\ &\quad - s \int_0^T \big[\theta a^2 \big(w_x^2 + z_x^2\big)\psi'\big]_{x=0}^{x=1} dt + \mu\langle \frac{z}{d}, L_s^- w\rangle_{L^2(Q)} + \mu\langle \frac{w}{d}, L_s^- z\rangle_{L^2(Q)}. \end{aligned}$$

By several integrations by parts in space and in time, the scalar product

$$\langle \frac{z}{d}, L_s^- w\rangle_{L^2(Q)} + \langle \frac{w}{d}, L_s^- z\rangle_{L^2(Q)},$$

may be written as a sum of a distributed term DT and a boundary term BT where

$$\begin{gathered} \mathrm{DT} = -2s \int_0^T \int_0^1 \frac{a\varphi_x d'}{d^2} wz dx\, dt, \\ \mathrm{BT} = \int_0^1 \frac{1}{d} [wz]_{t=0}^{t=T} dx - 2s \int_0^T \big[\frac{a\varphi_x}{d} wz\big]_{x=0}^{x=1} dt. \end{gathered}$$

Proceeding as in [17], using the definition of φ and the boundary conditions on $w = e^{s\varphi} U$ and $z = e^{s\varphi} V$, one has that

$$\mathrm{BT} = 0.$$

We conclude now the proof of our Carleman inequality by producing a lower bound for the distributed term DT. It is simply a matter of computation to show that, with this choice of φ and the assumption on d, one has

$$\mathrm{DT} = -2\mathfrak{c}s\int_0^T\int_0^1 \theta\frac{(x-x_0)d'}{d^2}wz\,dx\,dt \geq -2\mathfrak{c}Ms\int_0^T\int_0^1 \theta\frac{wz}{d}dx\,dt.$$

Now, using Young inequality, we can estimate $\int_0^T\int_0^1 s\theta\frac{wz}{d}dx\,dt$ thanks to the Hardy-Poincaré inequality (2.2),

$$\begin{aligned}
\int_0^T\int_0^1 s\theta\frac{wz}{d}dx\,dt &= \int_0^T\int_0^1 \Big(\sqrt{s\theta}\frac{w}{\sqrt{d}}\Big)\Big(\sqrt{s\theta}\frac{z}{\sqrt{d}}\Big)dx\,dt,\\
&\leq \frac{1}{2}\int_0^T\int_0^1 s\theta\frac{w^2}{d}dx\,dt + \frac{1}{2}\int_0^T\int_0^1 s\theta\frac{z^2}{d}dx\,dt,\\
&\leq \frac{C^*_{HP}}{2}\int_0^T\int_0^1 s\theta a w_x^2 dx\,dt + \frac{C^*_{HP}}{2}\int_0^T\int_0^1 s\theta a z_x^2 dx\,dt.
\end{aligned}$$

Hence, putting all together and taking into account the fact that one can assume C as large as desired, provided that s_0 increases as well, it is straightforward to check that, taking s large enough, one has

$$\begin{aligned}
&\langle \mathcal{L}_s^+ Z, \mathcal{L}_s^- Z\rangle_{\mathbb{H}_T}\\
&\geq C\int_0^T\int_0^1 \Big[s\theta a w_x^2 + s^3\theta^3\frac{(x-x_0)^2}{a}w^2\Big]dx\,dt\\
&\quad + C\int_0^T\int_0^1 \Big[s\theta a z_x^2 + s^3\theta^3\frac{(x-x_0)^2}{a}z^2\Big]dx\,dt\\
&\quad - 2\mu\mathfrak{c}M\frac{C^*_{HP}}{2}\int_0^T\int_0^1 s\theta a w_x^2 dx\,dt - 2\mu\mathfrak{c}M\frac{C^*_{HP}}{2}\int_0^T\int_0^1 s\theta a z_x^2 dx\,dt\\
&\quad - s\int_0^T \big[\theta a^2\big(w_x^2+z_x^2\big)\psi'\big]_{x=0}^{x=1}dt\\
&\geq C\int_0^T\int_0^1 \Big[s\theta a w_x^2 + s^3\theta^3\frac{(x-x_0)^2}{a}w^2\Big]dx\,dt\\
&\quad + C\int_0^T\int_0^1 \Big[s\theta a z_x^2 + s^3\theta^3\frac{(x-x_0)^2}{a}z^2\Big]dx\,dt\\
&\quad - \frac{C}{2}\int_0^T\int_0^1 s\theta a w_x^2 dx\,dt - \frac{C}{2}\int_0^T\int_0^1 s\theta a z_x^2 dx\,dt\\
&\quad - s\int_0^T \big[\theta a^2(w_x^2+z_x^2)\psi'\big]_{x=0}^{x=1}dt\\
&= \frac{C}{2}\int_0^T\int_0^1 s\theta a(w_x^2+z_x^2)\,dx\,dt + C\int_0^T\int_0^1 s^3\theta^3\frac{(x-x_0)^2}{a}(w^2+z^2)\,dx\,dt\\
&\quad - s\int_0^T \big[\theta a^2(w_x^2+z_x^2)\psi'\big]_{x=0}^{x=1}dt.
\end{aligned} \tag{3.10}$$

From (3.9) and (3.10), we finally obtain

$$\begin{aligned}
&\int_0^T\int_0^1 \Big[s\theta a(w_x^2+z_x^2)+s^3\theta^3\frac{(x-x_0)^2}{a}(w^2+z^2)\Big]\,dx\,dt \\
&\le C\Big(\int_0^T\int_0^1 \big[h_1^2+h_2^2\big]e^{2s\varphi}\,dx\,dt + s\int_0^T \big[\theta a^2\Big(w_x^2+z_x^2\Big)\psi'\big]_{x=0}^{x=1}dt\Big).
\end{aligned}\tag{3.11}$$

Recalling the definition of w and z, we have

$$U=e^{-s\varphi}w,$$
$$V=e^{-s\varphi}z,$$

and

$$U_x=-s\theta\psi' e^{-s\varphi}w+e^{-s\varphi}w_x,$$
$$V_x=-s\theta\psi' e^{-s\varphi}z+e^{-s\varphi}z_x.$$

Thus, substituting in (3.11), Theorem 3.1 follows. □

4. Carleman estimates with distributed observation

As it is by now classical, for proving Theorem 1.1 we will apply the Hilbert Uniqueness Method (HUM, [22]); hence the controllability property will be equivalent to the observability of the homogeneous adjoint system associated to (1.1)-(1.4), namely

$$U_t+(a(x)U_x)_x+\frac{\lambda_1}{b_1}U+\frac{\mu}{d}V=0,\quad (t,x)\in Q,\tag{4.1}$$
$$V_t+(a(x)V_x)_x+\frac{\lambda_2}{b_2}V+\frac{\mu}{d}U=0,\quad (t,x)\in Q,\tag{4.2}$$
$$U(t,1)=U(t,0)=V(t,1)=V(t,0)=0,\quad t\in(0,T),\tag{4.3}$$
$$U(T,x)=U_T(x),\quad V(T,x)=V_T(x),\quad x\in(0,1).\tag{4.4}$$

We show now an intermediate Carleman-type estimate which could be used to show the null controllability for parabolic systems with two control forces. As a first step, consider the adjoint problem with more regular final datum

$$U_t+(a(x)U_x)_x+\frac{\lambda_1}{b_1}U+\frac{\mu}{d}V=0,\quad (t,x)\in Q,\tag{4.5}$$
$$V_t+(a(x)V_x)_x+\frac{\lambda_2}{b_2}V+\frac{\mu}{d}U=0,\quad (t,x)\in Q,\tag{4.6}$$
$$U(t,1)=U(t,0)=V(t,1)=V(t,0)=0,\quad t\in(0,T),\tag{4.7}$$
$$\big(U(T,x)=U_T(x),\quad V(T,x)=V_T(x)\big)\in\mathcal{D}(\mathbb{A}^2),\quad x\in(0,1).\tag{4.8}$$

where $\mathcal{D}(\mathbb{A}^2)=\{X^T\in\mathcal{D}(\mathbb{A}):(\mathbb{A}X)^T\in\mathcal{D}(\mathbb{A})\}$. Observe that $\mathcal{D}(\mathbb{A}^2)$ is densely defined in $\mathcal{D}(\mathbb{A})$ for the graph norm (see, e.g., [9, Lemma 7.2]) and hence in $\mathbb{H}$. As in [18] or [19], letting (U_T,V_T) vary in $\mathcal{D}(\mathbb{A}^2)$, we define the following class of functions:

$$\mathcal{W}:=\big\{(U,V)\text{ is a solution of (4.5)-(4.8)}\big\}.$$

Obviously (see, e.g., [9, Theorem 7.5]) $\mathcal{W}\subset C^1\big([0,T];D(\mathbb{A})\big)\subset\mathcal{V}\subset\mathcal{U}$, where $\mathcal{V}$ is defined in (3.6) and

$$\mathcal{U}:=C\big([0,T];\mathbb{H}\big)\cap L^2\big(0,T;\mathcal{H}_1\times\mathcal{H}_2\big).$$

Now we are ready to state Carleman estimates with distributed observation of U and V related to (4.5)-(4.8).

Theorem 4.1. *Let $T > 0$ be given. Assume Hypothesis 2.10 is satisfied. Then there exist two positive constants C and s_0 such that every solution (U, V) of (4.5)-(4.8) satisfies, for all $s \geq s_0$,*

$$\begin{aligned}&\int_0^T \int_0^1 \Big[s\theta a(U_x^2 + V_x^2) + s^3\theta^3 \frac{(x-x_0)^2}{a}(U^2+V^2)\Big] e^{2s\varphi(t,x)}\,dx\,dt\\&\leq C\int_0^T\int_\omega s^3\theta^3[U^2+V^2]e^{-2s\Phi(t,x)}\,dx\,dt.\end{aligned}$$

For the proof of the previous Theorem, we shall use the following non degenerate non singular classical Carleman estimate in suitable interval (A, B) (see [19]).

Proposition 4.2. *Let z be the solution of*

$$\begin{gathered}z_t + (az_x)_x + \frac{\lambda}{b(x)}z = h \in L^2((0,T)\times(A,B)), \quad x \in (A,B),\ t\in(0,T),\\ z(t,A) = z(t,B) = 0, \quad t \in (0,T),\end{gathered}$$

where $a \in C^1([A,B])$ is a strictly positive function and $b \in C([A,B])$ is such that $b \geq b_0 > 0$ in $[A,B]$. Then there exist two positive constants r and s_0 such that for any $s > s_0$

$$\begin{aligned}&\int_0^T\int_A^B s\theta e^{r\zeta} z_x^2 e^{-2s\Phi}\,dx\,dt + \int_0^T\int_A^B s^3\theta^3 e^{3r\zeta} z^2 e^{-2s\Phi}\,dx\,dt\\&\leq c\Big(\int_0^T\int_A^B h^2 e^{-2s\Phi}\,dx\,dt - \int_0^T \Big[\sigma(t,\cdot)z_x^2(t,\cdot)e^{-2s\Phi(t,\cdot)}\Big]_{x=A}^{x=B} dt\Big),\end{aligned} \tag{4.9}$$

for some positive constant C. Here the functions θ, Φ and ζ are defined as follows: For $x \in [A,B]$:

$$\begin{gathered}\Phi(t,x) = \theta(t)\Psi(x), \quad \Psi(x) = e^{2\rho} - e^{r\zeta(x)},\\ \text{where} \quad \zeta(x) = \int_x^B \frac{dy}{\sqrt{a(y)}}, \quad \rho = r\zeta(A),\end{gathered} \tag{4.10}$$

and $\sigma(t,x) := rs\theta(t)e^{r\zeta(x)}$, for $r, s > 0$.

Proof of Theorem 4.1. First of all, to simplify the presentation, we assume that $\omega = (\alpha, \beta) \subset (0,1)$ is lying on one side of the degeneracy point x_0, that can always be done, taking if necessary a smaller set. Let us suppose that $0 < x_0 < \alpha < \beta < 1$ (the proof is analogous if we assume that $0 < \alpha < \beta < x_0 < 1$ with obvious adaptation); moreover, set $\tilde\alpha := \frac{2\alpha+\beta}{3}$ and $\tilde\beta := \frac{\alpha+2\beta}{3}$, so that $\alpha < \tilde\alpha < \tilde\beta < \beta$.

Now, we consider a smooth function $\eta : [0,1] \to [0,1]$ such that

$$\begin{gathered}\eta(x) = 1, \quad x \in [\tilde\beta, 1]\\ \eta(x) = 0, \quad x \in [0, \tilde\alpha].\end{gathered}$$

Then, define $\hat p = \eta U$ and $\hat q = \eta V$, where (U, V) is the solution of (4.5)-(4.8).

Hence, $\hat p$ and $\hat q$ satisfy the system

$$\hat p_t + (a\hat p_x)_x + \frac{\lambda_1}{b_1}\hat p = -\frac{\mu}{d}\hat q + (a\eta_x U)_x + a\eta_x U_x, \quad (t,x) \in Q,$$

$$
\begin{gathered}
\hat{q}_t + (a\hat{q}_x)_x + \frac{\lambda_2}{b_2}\hat{q} = -\frac{\mu}{d}\hat{p} + (a\eta_x V)_x + a\eta_x V_x, \quad (t,x) \in Q,\\
\hat{p}(t,\alpha) = \hat{p}(t,1) = \hat{q}(t,\alpha) = \hat{q}(t,1) = 0, \quad t \in (0,T).
\end{gathered}
$$

Observe that the system above is a nondegenerate nonsingular problem, hence, we can apply the classical Carleman estimate stated in Proposition 4.2, with $A = \alpha$ and $B = 1$, obtaining

$$
\begin{aligned}
&\int_0^T \int_\alpha^1 \Big[s\theta e^{r\zeta}\hat{p}_x^2 + s^3\theta^3 e^{3r\zeta}\hat{p}^2\Big] e^{-2s\Phi}\,dx\,dt\\
&\le \tilde{c}\int_0^T \int_\alpha^1 \hat{q}^2 e^{-2s\Phi}\,dx\,dt + C\int_0^T \int_{\hat{\omega}} [U^2 + U_x^2] e^{-2s\Phi}\,dx\,dt,
\end{aligned}
$$

for all $s \ge s_0$ with $\hat{\omega} = [\tilde{\alpha}, \tilde{\beta}]$. Let us remark that the boundary term in $x = 1$ is nonpositive, while the one in $x = \alpha$ is 0, so that they can be neglected in the classical Carleman estimate.

Analogously, one can prove that $\hat{q}$ satisfies

$$
\begin{aligned}
&\int_0^T \int_\alpha^1 \big[s\theta e^{r\zeta}\hat{q}_x^2 + s^3\theta^3 e^{3r\zeta}\hat{q}^2\big] e^{-2s\Phi}\,dx\,dt\\
&\le \tilde{c}\int_0^T \int_\alpha^1 \hat{p}^2 e^{-2s\Phi}\,dx\,dt + C\int_0^T \int_{\hat{\omega}} [V^2 + V_x^2] e^{-2s\Phi}\,dx\,dt.
\end{aligned}
$$

Thus combining the last two inequalities, it follows

$$
\begin{aligned}
&\int_0^T \int_\alpha^1 \Big[s\theta e^{r\zeta}(\hat{p}_x^2 + \hat{q}_x^2) + s^3\theta^3 e^{3r\zeta}(\hat{p}^2 + \hat{q}^2)\Big] e^{-2s\Phi}\,dx\,dt\\
&\le \tilde{C}\int_0^T \int_\alpha^1 [\hat{p}^2 + \hat{q}^2] e^{-2s\Phi}\,dx\,dt + C\int_0^T \int_{\hat{\omega}} [U^2 + V^2 + U_x^2 + V_x^2] e^{-2s\Phi}\,dx\,dt.
\end{aligned}
$$

Taking s such that $\tilde{C} \le \frac{1}{2}s^3\theta^3 e^{3r\zeta}$ and using Caccioppoli inequality (6.1), we obtain

$$
\begin{aligned}
&\int_0^T \int_\alpha^1 \Big[s\theta e^{r\zeta}(\hat{p}_x^2 + \hat{q}_x^2) + s^3\theta^3 e^{3r\zeta}(\hat{p}^2 + \hat{q}^2)\Big] e^{-2s\Phi}\,dx\,dt\\
&\le C\int_0^T \int_\omega s^2\theta^2 [U^2 + V^2] e^{-2s\Phi}\,dx\,dt.
\end{aligned}
$$

Now, choose the constant $\mathfrak{c}$ in (3.5) so that

$$
\mathfrak{c} \ge \max\Big\{\frac{e^{2r\zeta(\alpha)} - 1}{\mathfrak{d} - \frac{(1-x_0)^2}{a(1)(2-k_1)}}, \frac{e^{2r\zeta(\alpha)} - 1}{\mathfrak{d} - \frac{x_0^2}{a(0)(2-k_1)}}\Big\} \tag{4.11}
$$

Then, by definition of φ and the choice of $\mathfrak{c}$, one can prove that there exists a positive constant C such that for every $(t,x) \in [0,T] \times [\alpha,1]$

$$
a(x)e^{2s\varphi(t,x)} \le Ce^{r\zeta}e^{-2s\Phi}, \quad \frac{(x-x_0)^2}{a(x)}e^{2s\varphi(t,x)} \le Ce^{3r\zeta}e^{-2s\Phi}. \tag{4.12}
$$

Consequently,

$$
\begin{aligned}
&\int_0^T \int_\alpha^1 \Big[s\theta a(\hat{p}_x^2 + \hat{q}_x^2) + s^3\theta^3\frac{(x-x_0)^2}{a}(\hat{p}^2 + \hat{q}^2)\Big] e^{2s\varphi}\,dx\,dt\\
&\le C\int_0^T \int_\omega s^2\theta^2 [U^2 + V^2] e^{-2s\Phi}\,dx\,dt.
\end{aligned}
$$

By the definition of $\hat{p}$ and $\hat{q}$, we obtain

$$\begin{aligned}&\int_0^T\int_{\tilde{\beta}}^1 \Big[s\theta a(U_x^2+V_x^2)+s^3\theta^3\frac{(x-x_0)^2}{a}(U^2+V^2)\Big]e^{2s\varphi}\,dx\,dt\\&\le C\int_0^T\int_\omega s^2\theta^2[U^2+V^2]e^{-2s\Phi}\,dx\,dt,\end{aligned}\tag{4.13}$$

for a positive constant C and for s large enough.

On the other hand, by the properties of the weight functions, calculations show that

$$s^3\theta^3\frac{(x-x_0)^2}{a}e^{2s\varphi}\le Cs^2\theta^2e^{-2s\Phi},\quad \forall(t,x)\in(0,T)\times(\tilde{\alpha},\tilde{\beta})\tag{4.14}$$

for a positive constant C. In addition, arguing as in the proof of Caccioppoli inequality 6.1, one can easily show that

$$\int_0^T\int_{\tilde{\alpha}}^{\tilde{\beta}} s\theta[U_x^2+V_x^2]e^{2s\varphi}dxdt\le C\int_0^T\int_\omega s^3\theta^3[U^2+V^2e^{-2s\Phi}dxdt,\tag{4.15}$$

for some constant $C>0$.

By (4.14) and (4.15) we can find a positive constant C such that

$$\begin{aligned}&\int_0^T\int_{\tilde{\alpha}}^{\tilde{\beta}} \Big[s\theta a(U_x^2+V_x^2)+s^3\theta^3\frac{(x-x_0)^2}{a}(U^2+V^2)\Big]e^{2s\varphi}\,dx\,dt\\&\le C\int_0^T\int_\omega s^3\theta^3[U^2+V^2]e^{-2s\Phi}\,dx\,dt.\end{aligned}\tag{4.16}$$

Thus (4.13) and (4.16) imply

$$\begin{aligned}&\int_0^T\int_{\tilde{\alpha}}^1 \Big[s\theta a(U_x^2+V_x^2)+s^3\theta^3\frac{(x-x_0)^2}{a}(U^2+V^2)\Big]e^{2s\varphi}\,dx\,dt\\&\le C\int_0^T\int_\omega s^3\theta^3[U^2+V^2]e^{-2s\Phi}\,dx\,dt,\end{aligned}\tag{4.17}$$

for a positive universal constant C and for s large enough.

To complete the proof, it is sufficient to prove a similar inequality on the interval $[0,\tilde{\alpha}]$. To this aim, we follow a reflection procedure. Consider the functions

$$W(t,x):=\begin{cases}U(t,x), & x\in[0,1],\\ U(t,-x), & x\in[-1,0],\end{cases}\qquad Z(t,x):=\begin{cases}V(t,x), & x\in[0,1],\\ V(t,-x), & x\in[-1,0],\end{cases}$$

where (U,V) solves (4.5)-(4.8) and

$$\tilde{\psi}(x):=\begin{cases}\psi(x), & x\in[0,1],\\ \mathfrak{c}[\int_{-x_0}^x\frac{y+x_0}{\tilde{a}(y)}\,dy-\mathfrak{d}], & x\in[-1,0],\end{cases}\qquad \tilde{a}(x)=\begin{cases}a(x), & x\in[0,1],\\ a(-x), & x\in[-1,0],\end{cases}$$

$$\tilde{b}_i(x):=\begin{cases}b_i(x), & x\in[0,1],\\ b_i(-x), & x\in[-1,0],\end{cases}\qquad \tilde{d}(x)=\begin{cases}d(x), & x\in[0,1],\\ d(-x), & x\in[-1,0].\end{cases}$$

Therefore, (W,Z) solves the system

$$
\begin{gathered}
W_t + (\tilde{a}W_x)_x + \frac{\lambda_1}{\tilde{b}_1}W + \frac{\mu}{\tilde{d}}Z = 0, \quad x \in (-1,1),\ t \in (0,T), \\
Z_t + (\tilde{a}Z_x)_x + \frac{\lambda_2}{\tilde{b}_2}Z + \frac{\mu}{\tilde{d}}W = 0, \quad x \in (-1,1),\ t \in (0,T), \\
W(t,-1) = W(t,1) = Z(t,-1) = Z(t,1) = 0, \quad t \in (0,T).
\end{gathered} \tag{4.18}
$$

Now, consider a smooth function $\tau : [-1,1] \to [0,1]$ such that

$$
\tau(x) = \begin{cases} 1, & x \in [-x_0/3, \tilde{\alpha}], \\ 0, & x \in [-1, -x_0/2] \cup [\tilde{\beta}, 1], \end{cases}
$$

and define the functions $\bar{p} = \tau W$ and $\bar{q} = \tau Z$, where (W,Z) is the solution of (4.18). Then $(\bar{p},\bar{q})$ satisfies

$$
\begin{gathered}
\bar{p}_t + (\tilde{a}\bar{p}_x)_x + \frac{\lambda_1}{\tilde{b}_1}\bar{p} + \frac{\mu}{\tilde{d}}\bar{q} = (\tilde{a}\tau_x W)_x + \tilde{a}\tau_x W_x, \quad x \in (-1,1),\ t \in (0,T), \\
\bar{q}_t + (\tilde{a}\bar{q}_x)_x + \frac{\lambda_2}{\tilde{b}_2}\bar{q} + \frac{\mu}{\tilde{d}}\bar{p} = (\tilde{a}\tau_x Z)_x + \tilde{a}\tau_x Z_x, \quad x \in (-1,1),\ t \in (0,T), \\
\bar{p}(t, -\frac{2x_0}{3}) = \bar{p}(t,1) = \bar{q}(t, -\frac{2x_0}{3}) = \bar{q}(t,1) = 0, \quad t \in (0,T).
\end{gathered}
$$

Now, define $\tilde{\varphi} := \theta(t)\tilde{\psi}(x)$, where $\tilde{\psi}$ is defined as above. Using the analogue of Theorem 3.1 on $(-\frac{2x_0}{3}, 1)$ in place of $(0,1)$ and with φ replaced by $\tilde{\varphi}$, by the equalities $\bar{p}_x(t, -\frac{2x_0}{3}) = \bar{p}_x(t,1) = \bar{q}_x(t, -\frac{2x_0}{3}) = \bar{q}_x(t,1) = 0$ and the definition of (W,Z), we obtain

$$
\begin{aligned}
&\int_0^T \int_{-2x_0/3}^1 \Big[s\theta\tilde{a}\big(\bar{p}_x^2 + \bar{q}_x^2\big) + s^3\theta^3 \frac{(x-x_0)^2}{\tilde{a}} \Big(\bar{p}^2 + \bar{q}^2\Big)\Big] e^{2s\tilde{\varphi}}\,dx\,dt \\
&\le C \int_0^T \int_{-\frac{x_0}{2}}^{-\frac{x_0}{3}} [W^2 + W_x^2 + Z^2 + Z_x^2] e^{2s\tilde{\varphi}}\,dx\,dt \\
&\quad + C \int_0^T \int_{\tilde{\alpha}}^{\tilde{\beta}} [W^2 + W_x^2 + Z^2 + Z_x^2] e^{2s\varphi}\,dx\,dt \\
&\le \underbrace{C \int_0^T \int_{\frac{x_0}{3}}^{\frac{x_0}{2}} [U^2 + U_x^2 + V^2 + V_x^2] e^{2s\varphi}\,dx\,dt}_{J} \\
&\quad + C \int_0^T \int_{\tilde{\alpha}}^{\tilde{\beta}} [U^2 + U_x^2 + V^2 + V_x^2] e^{2s\varphi}\,dx\,dt.
\end{aligned}
$$

To absorb J, let $\epsilon > 0$ be small enough. Since

$$
\inf_{t \in [0,T]} \theta(t) > 0, \quad \inf_{x \in [\frac{x_0}{3}, \frac{x_0}{2}]} a(x) > 0, \quad \inf_{x \in [\frac{x_0}{3}, \frac{x_0}{2}]} \frac{(x-x_0)^2}{a(x)} > 0,
$$

taking s large enough, it follows that

$$
\int_0^T \int_{\frac{x_0}{3}}^{\frac{x_0}{2}} [U^2 + U_x^2 + V^2 + V_x^2] e^{2s\varphi}\,dx\,dt
$$

$$
\begin{aligned}
&\le \epsilon \int_0^T \int_{\frac{x_0}{3}}^{\frac{x_0}{2}} \Big[s\theta a\big(U_x^2+V_x^2\big)+s^3\theta^3\frac{(x-x_0)^2}{a}\Big(U^2+V^2\Big)\Big]e^{2s\varphi}\,dx\,dt\\
&\le \epsilon \int_0^T \int_0^{\tilde\alpha} \Big[s\theta a\big(U_x^2+V_x^2\big)+s^3\theta^3\frac{(x-x_0)^2}{a}\Big(U^2+V^2\Big)\Big]e^{2s\varphi}\,dx\,dt.
\end{aligned}
$$

Therefore, by Caccioppoli inequality (6.1), we obtain

$$
\begin{aligned}
&\int_0^T\int_{-2x_0/3}^1 \Big[s\theta\tilde a\big(\bar p_x^2+\bar q_x^2\big)e^{2s\tilde\varphi}+s^3\theta^3\frac{(x-x_0)^2}{\tilde a}\big(\bar p^2+\bar q^2\big)\Big]e^{2s\tilde\varphi}\,dx\,dt\\
&\le C\int_0^T\int_\omega s^2\theta^2[U^2+V^2]e^{-2s\Phi}\,dx\,dt \qquad (4.19)\\
&\quad+\epsilon\int_0^T\int_0^{\tilde\alpha}\Big[s\theta a\Big(U_x^2+V_x^2\Big)+s^3\theta^3\frac{(x-x_0)^2}{a}\Big(U^2+V^2\Big)\Big]e^{2s\varphi}\,dx\,dt,
\end{aligned}
$$

for a universal positive constant C. Hence, by (4.19), the definition of W, Z, $\bar p$ and $\bar q$, we obtain

$$
\begin{aligned}
&\int_0^T\int_0^{\tilde\alpha}\Big[s\theta a(U_x^2+V_x^2)+s^3\theta^3\frac{(x-x_0)^2}{a}(U^2+V^2)\Big]e^{2s\varphi}\,dx\,dt\\
&=\int_0^T\int_0^{\tilde\alpha}\Big[s\theta a(W_x^2+Z_x^2)+s^3\theta^3\frac{(x-x_0)^2}{a}(W^2+Z^2)\Big]e^{2s\varphi}\,dx\,dt\\
&\le\int_0^T\int_{-x_0/3}^{\tilde\alpha}\Big[s\theta\tilde a(W_x^2+Z_x^2)+s^3\theta^3\frac{(x-x_0)^2}{\tilde a}(W^2+Z^2)\Big]e^{2s\tilde\varphi}\,dx\,dt\\
&=\int_0^T\int_{-x_0/3}^{\tilde\alpha}\Big[s\theta\tilde a(\bar p_x^2+\bar q_x^2)+s^3\theta^3\frac{(x-x_0)^2}{\tilde a}(\bar p^2+\bar q^2)\Big]e^{2s\tilde\varphi}\,dx\,dt \qquad (4.20)\\
&\le\int_0^T\int_{-2x_0/3}^{1}\Big[s\theta\tilde a(\bar p_x^2+\bar q_x^2)+s^3\theta^3\frac{(x-x_0)^2}{\tilde a}(\bar p^2+\bar q^2)\Big]e^{2s\tilde\varphi}\,dx\,dt\\
&\le C\int_0^T\int_\omega s^2\theta^2(U^2+V^2)e^{-2s\Phi}\,dx\,dt\\
&\quad+\epsilon\int_0^T\int_0^{\tilde\alpha}\Big[s\theta a(U_x^2+V_x^2)+s^3\theta^3\frac{(x-x_0)^2}{a}(U^2+V^2)\Big]e^{2s\varphi}\,dx\,dt.
\end{aligned}
$$

Finally adding up (4.17) and (4.20), the proof is complete. □

To study the null controllability of the parabolic system (4.5)-(4.8) with one control force, we need to show the following Carleman estimate.

Theorem 4.3. *Let $T>0$. Then there exist two positive constants C and s_0 such that, for all $s\ge s_0$, the solution $(U,V)\in\mathcal W$ of (4.5)-(4.8) satisfies*

$$
\begin{aligned}
&\int_0^T\int_0^1\Big[s\theta a(U_x^2+V_x^2)+s^3\theta^3\frac{(x-x_0)^2}{a}(U^2+V^2)\Big]e^{2s\varphi(t,x)}\,dx\,dt\\
&\le C\int_0^T\int_\omega U^2\,dx\,dt. \qquad (4.21)
\end{aligned}
$$

The above theorem is a consequence of Theorem 4.1 applied to some open subset $\omega_1\subset\subset\omega$ and of the following Lemma.

Lemma 4.4. *For each $\varepsilon > 0$ there is $C_\varepsilon > 0$ such that*
$$\int_0^T \int_{\omega_1} s^3\theta^3 V^2 e^{-2s\Phi(t,x)}\,dx\,dt \le \varepsilon J(v) + C_\varepsilon \int_0^T \int_\omega U^2\,dx\,dt,$$
where $\varepsilon > 0$ is small enough, s is large enough and
$$J(V) = \int_0^T \int_0^1 (s\theta a V_x^2 + s^3\theta^3 \frac{(x-x_0)^2}{a} V^2) e^{2s\varphi}\,dx\,dt.$$

As usual, in order to prove such a Lemma, the parameters $\mathfrak{d}$, ρ, and $\mathfrak{c}$ will be chosen such that
$$\mathfrak{d} > 16\mathfrak{d}^\star, \quad \rho > 2\ln(2), \tag{4.22}$$
$$\frac{e^{2\rho}-1}{\mathfrak{d}-\mathfrak{d}^\star} \le \mathfrak{c} < \frac{4}{3\mathfrak{d}}(e^{2\rho}-e^\rho). \tag{4.23}$$

Remark 4.5. The interval
$$\Big[\frac{e^{2\rho}-1}{\mathfrak{d}-\mathfrak{d}^\star}, \frac{4(e^{2\rho}-e^\rho)}{3\mathfrak{d}}\Big)$$
is not empty. In fact, from $\rho > 2\ln 2$, and $\mathfrak{d} > 16\mathfrak{d}^\star$, we have
$$\begin{aligned}
\frac{\mathfrak{d}^\star}{\mathfrak{d}} < \frac{1}{16} &\Leftrightarrow \frac14 < \frac13 - \frac{4\mathfrak{d}^\star}{3\mathfrak{d}} \\
&\Leftrightarrow e^{-\rho} < \frac13 - \frac{4\mathfrak{d}^\star}{3\mathfrak{d}} \\
&\Leftrightarrow \frac{e^{2\rho}-1}{e^{2\rho}-e^\rho} < \frac{4(\mathfrak{d}-\mathfrak{d}^\star)}{3\mathfrak{d}} \\
&\Leftrightarrow \frac{e^{2\rho}-1}{\mathfrak{d}-\mathfrak{d}^\star} < \frac{4}{3\mathfrak{d}}(e^{2\rho}-e^\rho).
\end{aligned}$$

Lemma 4.6. *By* (4.22)-(4.23)*, for $(t,x) \in [0,T]\times[0,1]$, we have*
$$\begin{gathered}
\varphi(t,x) \le -\Phi(t,x) \text{ and} \\
4\Phi(t,x) + 3\varphi(t,x) > 0.
\end{gathered} \tag{4.24}$$

Proof. (1) $\varphi \le -\Phi$: Since $\mathfrak{c} \ge \frac{e^{2\rho}-1}{\mathfrak{d}-\mathfrak{d}^\star}$, we have $\max\{\psi(0),\psi(1)\} \le -\Psi(1)$ and the conclusion follows immediately.

(2) $4\Phi(t,x)+3\varphi(t,x) > 0$: This follows easily from the assumption $\mathfrak{c}\mathfrak{d} < 4\Psi(0)/3$. $\square$

Proof of Lemma 4.4. The choice of the weight functions satisfying (4.24) will play a crucial role. Let $\chi \in C^\infty(0,1)$, such that $\operatorname{supp}(\chi) \subset \omega$ and $\chi \equiv 1$ on ω_1. Multiplying the first equation of system (4.5)-(4.8) by $s^3\theta^3\chi e^{-2s\Phi} V$ and integrating over Q, we obtain
$$\begin{aligned}
\int_Q s^3\theta^3 \frac{\mu}{d} \chi e^{-2s\Phi} V^2\,dx\,dt &= \int_Q s^3\theta^3 \chi e^{-2s\Phi} U_t V\,dx\,dt \\
&\quad - \int_Q s^3\theta^3 \chi e^{-2s\Phi} (aU_x)_x V\,dx\,dt \\
&\quad - \int_Q s^3\theta^3 \frac{\lambda_1}{b_1} \chi e^{-2s\Phi} UV\,dx\,dt.
\end{aligned} \tag{4.25}$$

Integrating by parts and using the second equation in (4.5)-(4.8), we obtain

$$
\begin{aligned}
&\int_Q s^3\theta^3\chi e^{-2s\Phi}U_t V\,dx\,dt\\
&= \int_Q s^3\theta^3 a\chi e^{-2s\Phi}U_x V_x\,dx\,dt\\
&\quad + \int_Q s^3\theta^3 a(\chi e^{-2s\Phi})_x U V_x\,dx\,dt\\
&\quad - \int_Q \big[s^3\theta^3\frac{\lambda_2}{b_2} + 2s^4\theta^3\dot\theta\Psi + 3s^3\theta^2\dot\theta\big]\chi e^{-2s\Phi}UV\,dx\,dt\\
&\quad - \int_Q s^3\theta^3\frac{\mu}{d}\chi e^{-2s\Phi}U^2\,dx\,dt,
\end{aligned}
\tag{4.26}
$$

and

$$
\begin{aligned}
\int_Q s^3\theta^3\chi e^{-2s\Phi}(aU_x)_x V\,dx\,dt &= -\int_Q s^3\theta^3 a\chi e^{-2s\Phi}U_x V_x\,dx\,dt\\
&\quad + \int_Q s^3\theta^3 a(\chi e^{-2s\Phi})_x U V_x\,dx\,dt\\
&\quad + \int_Q s^3\theta^3 (a(\chi e^{-2s\Phi})_x)_x UV\,dx\,dt.
\end{aligned}
\tag{4.27}
$$

So, combining (4.25)-(4.27), we obtain

$$
\int_Q s^3\theta^3\frac{\mu}{d}\chi e^{-2s\Phi}V^2\,dx\,dt = I_1 + I_2 + I_3,
$$

where

$$
\begin{gathered}
I_1 = 2\int_Q s^3\theta^3 a\chi e^{-2s\Phi}U_x V_x\,dx\,dt,\\
I_2 = -\int_Q s^3\theta^3\frac{\mu}{d}\chi e^{-2s\Phi}U^2\,dx\,dt,\\
\begin{aligned}
I_3 = &-\int_Q \big[s^3\theta^3(\frac{\lambda_1}{b_1} + \frac{\lambda_2}{b_2}) + 2s^4\theta^3\dot\theta\Psi + 3s^3\theta^2\dot\theta\big]\chi e^{-2s\Phi}UV\,dx\,dt\\
&- \int_Q s^3\theta^3(a(\chi e^{-2s\Phi})_x)_x UV\,dx\,dt.
\end{aligned}
\end{gathered}
$$

For $\varepsilon > 0$, we have

$$
\begin{aligned}
|I_1| &= 2\int_Q (\sqrt{s\theta a}e^{s\varphi}V_x)((s\theta)^{5/2}a^{1/2}\chi e^{-s(2\Phi+\varphi)}U_x)\,dx\,dt\\
&\le \varepsilon\int_Q s\theta a e^{2s\varphi}V_x^2\,dx\,dt + \frac{1}{\varepsilon}\underbrace{\int_Q s^5\theta^5 a\chi^2 e^{-2s(2\Phi+\varphi)}U_x^2\,dx\,dt}_{L}\,.
\end{aligned}
$$

The integral L should be estimated by an integral in U^2. For this, we multiply the first equation in (4.5)-(4.8) by $s^5\theta^5\chi^2 e^{-2s(2\Phi+\varphi)}U$ and we integrate by parts, obtaining

$$
L = L_1 + L_2 + L_3 + L_4,
$$

where

$$L_1 = \frac{1}{2}\int_Q s^5(5\theta^4 - 2s\theta^5(2\Psi+\psi))\dot\theta\chi^2 e^{-2s(2\Phi+\varphi)}U^2\,dx\,dt,$$

$$L_2 = \frac{1}{2}\int_Q s^5\theta^5(a(\chi^2 e^{-2s(2\Phi+\varphi)})_x)_x U^2\,dx\,dt,$$

$$L_3 = \int_Q s^5\theta^5\chi^2\frac{\lambda_1}{b_1}e^{-2s(2\Phi+\varphi)}U^2\,dx\,dt,$$

$$L_4 = \int_Q s^5\theta^5\chi^2\frac{\mu}{d}e^{-2s(2\Phi+\varphi)}UV\,dx\,dt.$$

Since $\operatorname{supp}(\chi)\subset\omega$, we observe that the functions a, b_i, d, χ, ψ, Ψ and their derivatives are bounded on ω. Then, by the fact that $|\dot\theta|\le C\theta^2$, we deduce that, for $i\in\{1,2,3\}$

$$|L_i| \le C\int_0^T\int_\omega s^7\theta^7 e^{-2s(2\Phi+\varphi)}U^2\,dx\,dt.$$

For $i=4$ we have

$$\begin{aligned}
|L_4| &= \int_Q \big[(s\theta)^{\frac{3}{2}}\frac{(x-x_0)}{\sqrt{a}}e^{s\varphi}V\big]\big[(s\theta)^{\frac{7}{2}}\frac{\mu}{d}\chi^2\frac{\sqrt{a}}{(x-x_0)}e^{-s(4\Phi+3\varphi)}U\big]\,dx\,dt\\
&\le \varepsilon^2\int_Q s^3\theta^3\frac{(x-x_0)^2}{a}e^{2s\varphi}V^2\,dx\,dt\\
&\quad+\frac{1}{4\varepsilon^2}\int_Q s^7\theta^7\big(\frac{\mu}{d}\big)^2\chi^4\frac{a}{(x-x_0)^2}e^{-2s(4\Phi+3\varphi)}U^2\,dx\,dt\\
&\le \varepsilon^2\int_Q s^3\theta^3\frac{(x-x_0)^2}{a}e^{2s\varphi}V^2\,dx\,dt + C_\varepsilon\int_0^T\int_\omega s^7\theta^7 e^{-2s(4\Phi+3\varphi)}U^2\,dx\,dt.
\end{aligned}$$

Hence,

$$|L| \le C_\varepsilon\int_0^T\int_\omega s^7\theta^7 e^{-2s(4\Phi+3\varphi)}U^2\,dx\,dt + \varepsilon^2\int_Q s^3\theta^3\frac{(x-x_0)^2}{a}e^{2s\varphi}V^2\,dx\,dt.$$

Furthermore

$$|I_1| \le C_\varepsilon\int_0^T\int_\omega s^7\theta^7 e^{-2s(4\Phi+3\varphi)}U^2\,dx\,dt + \varepsilon J(V).$$

Using the fact that χ' and χ are supported in ω and $x_0\notin\omega$, proceeding as before, one has

$$|I_2| \le C\int_0^T\int_\omega s^3\theta^3 e^{-2s\Phi}U^2\,dx\,dt,$$

$$\begin{aligned}
|I_3| &\le C\int_Q s^5\theta^5(\chi''+\chi'+\chi)e^{-2s\Phi}UV\,dx\,dt\\
&\le C\int_Q (s^{\frac{3}{2}}\theta^{\frac{3}{2}}\frac{x-x_0}{\sqrt{a}}e^{s\varphi}V)((s\theta)^{7/2}\frac{\sqrt{a}}{x-x_0}(\chi''+\chi'+\chi)e^{-s(2\Phi+\varphi)}U)\,dx\,dt\\
&\le \varepsilon\int_Q s^3\theta^3\frac{(x-x_0)^2}{a}V^2e^{2s\varphi}\,dx\,dt + C_\varepsilon\int_0^T\int_\omega s^7\theta^7 e^{-2s(2\Phi+\varphi)}U^2\,dx\,dt.
\end{aligned}$$

So, thanks to Lemma 4.6, we have

$$e^{-2s\Phi} \le e^{-2s(2\Phi+\varphi)} \le e^{-2s(4\Phi+3\varphi)} \le 1,$$

$$\sup_{(t,x)\in Q} s^r\theta^r(t)e^{-2s(4\Phi+3\varphi)} < \infty, \quad r\in\mathbb{R}.$$

Then, for ε small enough and s large enough, we have

$$|\int_Q s^3\theta^3\frac{\mu}{d}\chi e^{-2s\Phi}V^2\,dx\,dt| \le C_\varepsilon\int_0^T\int_\omega U^2\,dx\,dt + 2\varepsilon J(V).$$

Finally, by the definition of χ and the previous inequality, it follows that

$$\begin{aligned}
\frac{\mu}{\max_{x\in\omega_1} d(x)}\int_0^T\int_{\omega_1} s^3\theta^3e^{-2s\Phi}V^2\,dx\,dt &\le |\int_0^T\int_{\bar{\omega}_1} s^3\theta^3\frac{\mu}{d}\chi e^{-2s\Phi}V^2\,dx\,dt| \\
&\le |\int_Q s^3\theta^3\frac{\mu}{d}\chi e^{-2s\Phi}V^2\,dx\,dt| \\
&\le C_\varepsilon\int_0^T\int_\omega U^2\,dx\,dt + \varepsilon J(V).
\end{aligned}$$

This completes the proof. □

5. Observability and null controllability results

In this section we prove, as a consequence of the Carleman estimates established in the above section, observability inequalities for the adjoint problem (4.1)-(4.4).

Theorem 5.1. *Let $T>0$ be given. Then there exists a positive constant C_T such that every (U,V) solution of* (4.1)-(4.4) *satisfies*

$$\int_0^1 [U^2(0,x)+V^2(0,x)]\,dx \le C_T\int_0^T\int_\omega U^2(t,x)\,dx\,dt.$$

To prove the above theorem, we need the following result.

Lemma 5.2. *Let $T>0$ be given. Then there exists a positive constant C_T such that every $(U,V)\in\mathcal{W}$ solution of* (4.5)-(4.8) *satisfies*

$$\int_0^1 \big[U^2(0,x)+V^2(0,x)\big]\,dx \le C_T\int_0^T\int_\omega U^2(t,x)dxdt.$$

Proof. Multiplying the first and the second equations in (4.5)-(4.8) respectively by U_t and V_t, integrating by parts over $(0,1)$, it is easy to see that

$$\begin{aligned}
0 &= \int_0^1 [U_t^2+V_t^2]\,dx + [a(x)\big(U_xU_t+V_xV_t\big)]_0^1 - \frac12\frac{d}{dt}\int_0^1 a[U_x^2+aV_x^2]\,dx \\
&\quad + \int_0^1 \big[\frac{\lambda_1}{b_1}UU_t+\frac{\lambda_2}{b_2}VV_t\big]\,dx + \int_0^1 \frac{\mu}{d}\big(UV_t+VU_t\big)\,dx \\
&= \int_0^1 [U_t^2+V_t^2]\,dx - \frac12\frac{d}{dt}\int_0^1 a[U_x^2+aV_x^2]\,dx \\
&\quad + \frac12\frac{d}{dt}\int_0^1 \big[\frac{\lambda_1}{b_1}U^2+\frac{\lambda_2}{b_2}V^2\big]\,dx + \mu\frac{d}{dt}\int_0^1 \frac{UV}{d}\,dx \\
&\ge -\frac12\frac{d}{dt}\int_0^1 a[U_x^2+aV_x^2]\,dx + \frac12\frac{d}{dt}\int_0^1 \big[\frac{\lambda_1}{b_1}U^2+\frac{\lambda_2}{b_2}V^2\big]\,dx + \mu\frac{d}{dt}\int_0^1 \frac{UV}{d}\,dx.
\end{aligned}$$

Hence the function

$$t\mapsto \int_0^1 a[U_x^2+V_x^2]\,dx - \int_0^1 [\frac{\lambda_1}{b_1}U^2+\frac{\lambda_2}{b_2}V^2]\,dx - 2\mu\int_0^1 \frac{UV}{d}\,dx$$

is non decreasing for all $t \in [0,T]$. In particular, using Young's inequality and by Theorem 2.8, we obtain

$$
\begin{aligned}
&\int_0^1 a(x)\big[U_x^2(0,x)+V_x^2(0,x)\big]\,dx - \int_0^1 \big[\frac{\lambda_1}{b_1}U^2(0,x)+\frac{\lambda_2}{b_2}V^2(0,x)\big]\,dx\\
&- 2\mu\int_0^1 \frac{U(0,x)V(0,x)}{d(x)}\,dx\\
&\le \int_0^1 a(x)[U_x^2(t,x)+V_x^2(t,x)]\,dx - \int_0^1 \big[\frac{\lambda_1}{b_1}U^2(t,x)+\frac{\lambda_2}{b_2}V^2(t,x)\big]\,dx\\
&\quad - 2\mu\int_0^1 \frac{U(t,x)V(t,x)}{d(x)}\,dx\\
&\le \int_0^1 a(x)[U_x^2(t,x)+V_x^2(t,x)]\,dx\\
&\quad + \lambda C^*\int_0^1 a(x)[U_x^2(t,x)+V_x^2(t,x)]\,dx\\
&\quad + \mu C_{HP}^*\int_0^1 a(x)[U_x^2(t,x)+V_x^2(t,x)]\,dx\\
&= (1+\lambda C^*+\mu C_{HP}^*)\int_0^1 a(x)[U_x^2(t,x)+V_x^2(t,x)]\,dx,
\end{aligned}
$$

where $\lambda=\max\{\lambda_1,\lambda_2\}$ and $C^*=\max\{C_1^*,C_2^*\}$.

Next, integrating the previous inequality over $[\frac{T}{4},\frac{3T}{4}]$, θ being bounded therein, and using the Carleman estimate (4.21), we find that

$$
\begin{aligned}
&\int_0^1 \big[a(x)U_x^2(0,x)-\frac{\lambda_1}{b_1}U^2(0,x)\big]\,dx + \int_0^1 \big[a(x)V_x^2(0,x)-\frac{\lambda_2}{b_2}V^2(0,x)\big]\,dx\\
&- 2\mu\int_0^1 \frac{U(0,x)V(0,x)}{d(x)}\,dx\\
&\le \frac{2}{T}(1+\lambda C^*+\mu C_{HP}^*)\int_{\frac{T}{4}}^{\frac{3T}{4}}\int_0^1 a(x)[U_x^2(t,x)+V_x^2(t,x)]\,dx\,dt\\
&\le C_T\int_{\frac{T}{4}}^{\frac{3T}{4}}\int_0^1 s\theta a(x)[U_x^2(t,x)+V_x^2(t,x)]e^{2s\varphi}\,dx\,dt\\
&\le C_T\int_0^T\int_\omega U^2(t,x)\,dx\,dt.
\end{aligned}
$$

From the previous inequality and Propositions 2.9, for $\delta>0$, we obtain

$$
\begin{aligned}
&\Lambda_1\int_0^1 a(x)U_x^2(0,x)\,dx + \Lambda_2\int_0^1 a(x)V_x^2(0,x)\,dx\\
&\le C_T\int_0^T\int_\omega U^2(t,x)\,dx\,dt + 2\mu\int_0^1 \frac{U(0,x)V(0,x)}{d(x)}\,dx\\
&\le C_T\int_0^T\int_\omega U^2(t,x)\,dx\,dt + 2\mu\delta C_{HP}^*\int_0^1 a(x)U_x^2(0,x)\,dx\\
&\quad + \mu\frac{C_{HP}^*}{2\delta}\int_0^1 a(x)V_x^2(0,x)\,dx.
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\big(\Lambda_1 - 2\mu\delta C^*_{HP}\big)\int_0^1 a(x)U_x^2(0,x)\,dx + \big(\Lambda_2 - \mu\frac{C^*_{HP}}{2\delta}\big)\int_0^1 a(x)V_x^2(0,x)\,dx \\
&\le C_T\int_0^T\int_\omega U^2(t,x)\,dx\,dt.
\end{aligned}
$$

Proceeding again as in the proof of Lemma 2.14, by (2.5) there exists $C>0$ such that

$$
\int_0^1 a(x)[U_x^2(0,x)+V_x^2(0,x)]\,dx \le C\int_0^T\int_\omega U^2(t,x)\,dx\,dt. \tag{5.1}
$$

On the other hand, by [19, Lemma 2.1], the map $x\mapsto \frac{(x-x_0)^2}{a(x)}$ is nonincreasing on $[0,x_0)$ and nondecreasing on $(x_0,1]$, then

$$
\Big(\frac{(x-x_0)^2}{a(x)}\Big)^{1/3} \le \max\big\{\Big(\frac{x_0^2}{a(0)}\Big)^{1/3}, \Big(\frac{(1-x_0)^2}{a(1)}\Big)^{1/3}\big\}.
$$

Hence, applying the Hardy-Poincaré inequality given in Theorem 1.2 and the previous inequality, one has

$$
\begin{aligned}
&\int_0^1 [U^2(0,x)+V^2(0,x)]\,dx \\
&\le C_0\int_0^1 \frac{a^{1/3}(x)}{(x-x_0)^{\frac{2}{3}}}\,\big[U^2(0,x)+V^2(0,x)\big]\,dx \\
&\le C_0\int_0^1 \frac{p}{(x-x_0)^2}\,\big[U^2(0,x)+V^2(0,x)\big]\,dx \\
&\le C_0C_{HP}\int_0^1 p\,\big[U_x^2(0,x)+V_x^2(0,x)\big]\,dx \\
&\le C_0\max\{C_1,C_2\}C_{HP}\int_0^1 a(x)[U_x^2(0,x)+V_x^2(0,x)]\,dx.
\end{aligned}
\tag{5.2}
$$

Here $p(x) = (a(x)|x-x_0|^4)^{1/3}$ if $K>4/3$ or $p(x) = \max_{[0,1]} a(x)^{1/3}|x-x_0|^{4/3}$ otherwise,

$$
\begin{aligned}
C_0 &:= \max\Big[\Big(\frac{x_0^2}{a(0)}\Big)^{1/3}, \Big(\frac{(1-x_0)^2}{a(1)}\Big)^{1/3}\Big], \\
C_1 &:= \max\Big\{\Big(\frac{x_0^2}{a(0)}\Big)^{2/3}, \Big(\frac{(1-x_0)^2}{a(1)}\Big)^{2/3}\Big\}, \\
C_2 &:= \max\Big\{\frac{x_0^{4/3}}{a(0)}, \frac{(1-x_0)^{4/3}}{a(1)}\Big\}
\end{aligned}
$$

and C_{HP} is the Hardy-Poincaré constant. Combining (5.1) and (5.2) the conclusion follows. □

The proof of Theorem 5.1 is now standard using Lemma 5.2 and proceeding as in [19, Proposition 4.1], but we give it for the reader's convenience.

Proof of Proposition 5.1. Let $(U_T,V_T)\in\mathbb{H}$ and let (U,V) be the solution of (4.1)-(4.4) associated to (U_T,V_T). Since $\mathcal{D}(\mathbb{A}^2)$ is densely defined in $\mathbb{H}$, there exists a sequence $(U_T^n,V_T^n)_n\subset\mathcal{D}(\mathbb{A}^2)$ which converge to (U_T,V_T) in $\mathbb{H}$. Now, consider the solution (U_n,V_n) associated to (U_T^n,V_T^n). Since the semigroup generated by $\mathbb{A}$ is analytic, hence $\mathbb{A}$ is closed (e.g., see [12, Theorem I.1.4]), thus, by [12, Theorem II.6.7], we obtain that $(U_n, V_n)_n$ converges to a certain (U, V) in $C([0,T];\mathbb{H})$, so that

$$\begin{aligned}
\lim_{n\to+\infty}\int_0^1 U_n^2(0,x)\,dx &= \int_0^1 U^2(0,x)\,dx,\\
\lim_{n\to+\infty}\int_0^1 V_n^2(0,x)\,dx &= \int_0^1 V^2(0,x)\,dx,\\
\lim_{n\to+\infty}\int_0^T\int_\omega U_n^2\,dx\,dt &= \int_0^T\int_\omega U^2\,dx\,dt.
\end{aligned}$$

But, by Lemma 5.2 we know that

$$\int_0^1 \big[U_n^2(0,x)+V_n^2(0,x)\big]dx \le C_T\int_0^T\int_\omega U_n^2(t,x)\,dx\,dt.$$

Thus Theorem 5.1 is now proved. □

6. Appendix

We show a Caccioppoli type inequality for linear coupled systems corresponding to our singular/degenerate situation.

Lemma 6.1 (Caccioppoli's inequality)**.** *Let ω' and ω two open subintervals of $(0,1)$ such that $\omega' \subset\subset \omega \subset (0,1)$ and $x_0 \notin \overline{\omega}$. Then, there exist two positive constants C and s_0 such that every solution $(U,V)\in\mathcal{W}$ of the adjoint problem (4.5)-(4.8) satisfies*

$$\begin{aligned}
&\int_0^T\int_{\omega'}[U_x^2(t,x)+V_x^2(t,x)]e^{-2s\Phi}dxdt\\
&\le C\int_0^T\int_\omega s^2\theta^2[U^2(t,x)+V^2(t,x)]e^{-2s\Phi}dxdt,
\end{aligned}\tag{6.1}$$

for all $s\ge s_0$.

Proof. Define a smooth cut-off function $\xi\in C^\infty(0,1)$ such that $\operatorname{supp}\xi\subset\omega$ and $\xi\equiv 1$ on ω'. Since (U,V) solves (4.5)-(4.8), We have

$$\begin{aligned}
0 &= \int_0^T \frac{d}{dt}\Big[\int_0^1 \xi^2 e^{-2s\Phi}(U^2+V^2)dx\Big]dt\\
&= -2\int_0^T\int_0^1 s\dot{\Phi}\xi^2e^{-2s\Phi}(U^2+V^2)\,dx\,dt - 2\int_0^T\int_0^1 \xi^2e^{-2s\Phi}a(x)U_x^2\,dx\,dt\\
&\quad -2\int_0^T\int_0^1(\xi^2e^{-2s\Phi})_x a(x)UU_x\,dx\,dt + 2\lambda_1\int_0^T\int_0^1\xi^2e^{-2s\Phi}\frac{U^2}{b_1}\,dx\,dt\\
&\quad -2\mu\int_0^T\int_0^1\xi^2e^{-2s\Phi}\frac{UV}{d}\,dx\,dt - 2\int_0^T\int_0^1\xi^2e^{-2s\Phi}a(x)V_x^2\,dx\,dt\\
&\quad -2\int_0^T\int_0^1(\xi^2e^{-2s\Phi})_x a(x)VV_x\,dx\,dt + 2\lambda_2\int_0^T\int_0^1\xi^2e^{-2s\Phi}\frac{V^2}{b_2}\,dx\,dt\\
&\quad -2\mu\int_0^T\int_0^1\xi^2e^{-2s\Phi}\frac{UV}{d}\,dx\,dt.
\end{aligned}$$

Then, integration by parts yields

$$
\begin{aligned}
&\int_0^T\int_0^1 \xi^2 e^{-2s\Phi} a(x)\big[U_x^2+V_x^2\big]\,dx\,dt\\
&= -\int_0^T\int_0^1 s\dot{\Phi}\xi^2 e^{-2s\Phi}(U^2+V^2)\,dx\,dt\\
&\quad -\int_0^T\int_0^1 a(x)(\xi^2 e^{-2s\Phi})_x(UU_x+VV_x)dx\,dt\\
&\quad +\int_0^T\int_0^1 \xi^2 e^{-2s\Phi}(\frac{\lambda_1}{b_1}U^2+\frac{\lambda_2}{b_2}V^2)\,dx\,dt - 2\mu\int_0^T\int_0^1 \xi^2 e^{-2s\Phi}\frac{UV}{d}\,dx\,dt\\
&= -\int_0^T\int_0^1 s\dot{\Phi}\xi^2 e^{-2s\Phi}(U^2+V^2)\,dx\,dt\\
&\quad +\frac{1}{2}\int_0^T\int_0^1 \Big(a(x)(\xi^2 e^{-2s\Phi})_x\Big)_x(U^2+V^2)dx\,dt\\
&\quad +\int_0^T\int_0^1 \xi^2 e^{-2s\Phi}(\frac{\lambda_1}{b_1}U^2+\frac{\lambda_2}{b_2}V^2)\,dx\,dt - 2\mu\int_0^T\int_0^1 \xi^2 e^{-2s\Phi}\frac{UV}{d}\,dx\,dt.
\end{aligned}
$$

Since $\min_{x\in\omega'} a(x)>0$, b_i, $d\in C^1(\bar{\omega},\mathbb{R})$, $i=1,2$, $\operatorname{supp}\xi\subset\omega$, $\xi\equiv 1$ on ω' and $|\dot{\theta}|\le c\theta^2$ then, using the Young's inequality, we obtain

$$
\begin{aligned}
&\min_{x\in\omega'} a(x)\int_0^T\int_{\omega'} e^{-2s\Phi}[U_x^2+V_x^2]\,dx\,dt\\
&\le \int_0^T\int_0^1 \xi^2 e^{-2s\Phi}a(x)[U_x^2+V_x^2]\,dx\,dt\\
&\le C\int_0^T\int_\omega (1+s^2\theta^2+s|\dot{\theta}|)[U^2+V^2]e^{-2s\Phi}\,dx\,dt\\
&\le C\int_0^T\int_\omega s^2\theta^2[U^2+V^2]e^{-2s\Phi}\,dx\,dt,
\end{aligned}
$$

and the proof is complete. □

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