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PERTURBED SUBCRITICAL DIRICHLET PROBLEMS WITH VARIABLE EXPONENTS

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Abstract. We study a class of nonhomogeneous elliptic problems with Dirichlet boundary condition and involving the $p(x)$-Laplace operator and power-type nonlinear terms with variable exponent. The main results of this articles establish sufficient conditions for the existence of nontrivial weak solutions, in relationship with the values of certain real parameters. The proofs combine the Ekeland variational principle, the mountain pass theorem and energy arguments.

1. Introduction

Let $\Omega \subset \mathbb{R}$ be a bounded domain with smooth boundary. In a pioneering paper, Ambrosetti and Rabinowitz [1] consider the subcritical elliptic problem

$$\begin{gathered} -\Delta u = |u|^{p-2}u, \quad x \in \Omega \\ u = 0, \quad x \in \partial\Omega, \end{gathered} \tag{1.1}$$

where $1 < p < 2N/(N-2)$ if $N \geq 3$ and $1 < p < \infty$ if $N \in \{1, 2\}$. Problem (1.1) illustrates in [1] the celebrated mountain pass theorem, which yields the existence of a nontrivial solution of (1.1). This result has been a rich source of valuable extensions to several classes of nonlinear elliptic equations described by various types of differential operators and involving nonlinear terms fulfilling or not the Ambrosetti-Rabinowitz growth condition. We refer, for instance, to Pucci and Rădulescu [13], where it is established that a related existence property remains true, provided that the (linear) operator in the left-hand side of (1.1) is perturbed in a suitable manner. More precisely, [13, Theorem 9] (see also Theorem 10) asserts that the perturbed problem

$$\begin{gathered} -\Delta u - \lambda u = |u|^{p-2}u, \quad x \in \Omega \\ u = 0, \quad x \in \partial\Omega \end{gathered}$$

has a nontrivial solution for all $\lambda < \lambda_1$, where λ_1 is the principal eigenvalue of the Laplace operator in $H_0^1(\Omega)$. A related problem is studied in [10] in the framework of differential operators and nonlinear terms involving variable exponents. The main result in [10] establishes the existence of nontrivial solutions in the case of small perturbations, namely with respect to the values of a suitable parameter.

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The main purpose of this article is to show that the perturbation results established in [13, Theorem 9] and [10, Theorem 2.1] can be extended to the case of variable exponents. However, the problem studied in this article has a different structure with those studied in [13] and [10] because of the several nonlinearities with nonhomogeneous behaviour.

We refer to Cencelj, Repovš and Virk [4], Repovš [16] for related results and to the monographs by Diening, Hästo, Harjulehto and Ruzicka [5] and Rădulescu and Repovš [15] for the basic functional framework used in the present paper.

2. Function spaces with variable exponent

In this section we recall some basic definitions and properties concerning the Lebesgue and Sobolev spaces with variable exponent. Consider the set

$$C_+(\overline{\Omega}) = \{h \in C(\overline{\Omega}),\ h(x) > 1 \text{ for all } x \in \overline{\Omega}\}.$$

For all $h \in C_+(\overline{\Omega})$ we define

$$h^+ = \sup_{x\in\Omega} h(x) \quad \text{and} \quad h^- = \inf_{x\in\Omega} h(x).$$

The real numbers h^+ and h^- will play a crucial role in our arguments and usually the gap between these quantities produces new results, which are no longer valid for constant exponents.

For any $p \in C_+(\overline{\Omega})$, we define the *variable exponent Lebesgue space*

$$L^{p(x)}(\Omega) = \{u : u \text{ is measurable and } \int_\Omega |u(x)|^{p(x)}\,dx < \infty\}.$$

This vector space is a Banach space if it is endowed with the *Luxemburg norm*, which is defined by

$$|u|_{p(x)} = \inf\Big\{\mu > 0;\ \int_\Omega |\frac{u(x)}{\mu}|^{p(x)}\,dx \le 1\Big\}.$$

Then $L^{p(x)}(\Omega)$ is reflexive if and only if $1 < p^- \le p^+ < \infty$ and continuous functions with compact support are dense in $L^{p(x)}(\Omega)$ if $p^+ < \infty$.

The inclusion between Lebesgue spaces with variable exponent generalizes the classical framework, namely if $0 < |\Omega| < \infty$ and p_1, p_2 are variable exponents so that $p_1 \le p_2$ in Ω then there exists the continuous embedding $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$.

Let $L^{p'(x)}(\Omega)$ be the conjugate space of $L^{p(x)}(\Omega)$, where $1/p(x) + 1/p'(x) = 1$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$ the following Hölder-type inequality holds:

$$|\int_\Omega uv\,dx| \le \Big(\frac{1}{p^-} + \frac{1}{p'^-}\Big)|u|_{p(x)}|v|_{p'(x)}\,. \tag{2.1}$$

The *modular* of $L^{p(x)}(\Omega)$ is the mapping $\rho_{p(x)} : L^{p(x)}(\Omega) \to \mathbb{R}$ defined by

$$\rho_{p(x)}(u) = \int_\Omega |u|^{p(x)}\,dx.$$

If (u_n), $u \in L^{p(x)}(\Omega)$ and $p^+ < \infty$ then the following relations are true:

$$|u|_{p(x)} > 1 \ \Rightarrow\ |u|^{p^-}_{p(x)} \le \rho_{p(x)}(u) \le |u|^{p^+}_{p(x)}, \tag{2.2}$$

$$|u|_{p(x)} < 1 \ \Rightarrow\ |u|^{p^+}_{p(x)} \le \rho_{p(x)}(u) \le |u|^{p^-}_{p(x)}, \tag{2.3}$$

$$|u_n - u|_{p(x)} \to 0 \ \Leftrightarrow\ \rho_{p(x)}(u_n - u) \to 0. \tag{2.4}$$

We define the variable exponent Sobolev space by

$$W^{1,p(x)}(\Omega)=\{u\in L^{p(x)}(\Omega):|\nabla u|\in L^{p(x)}(\Omega)\}.$$

On $W^{1,p(x)}(\Omega)$ we may consider one of the following equivalent norms

$$\|u\|_{p(x)}=|u|_{p(x)}+|\nabla u|_{p(x)}$$

or

$$\|u\|=\inf\Big\{\mu>0;\ \int_\Omega\Big(|\frac{\nabla u(x)}{\mu}|^{p(x)}+|\frac{u(x)}{\mu}|^{p(x)}\Big)\,dx\le 1\Big\}\,.$$

We define $W_0^{1,p(x)}(\Omega)$ as the closure of the set of compactly supported $W^{1,p(x)}$-functions with respect to the norm $\|u\|_{p(x)}$. When smooth functions are dense, we can also use the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$. Using the Poincaré inequality, the space $W_0^{1,p(x)}(\Omega)$ can be defined, in an equivalent manner, as the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{p(x)}=|\nabla u|_{p(x)}.$$

The space $(W_0^{1,p(x)}(\Omega),\|\cdot\|)$ is a separable and reflexive Banach space. Moreover, if $0<|\Omega|<\infty$ and p_1, p_2 are variable exponents so that $p_1\le p_2$ in Ω then there exists the continuous embedding $W_0^{1,p_2(x)}(\Omega)\hookrightarrow W_0^{1,p_1(x)}(\Omega)$. Set

$$\varrho_{p(x)}(u)=\int_\Omega|\nabla u(x)|^{p(x)}\,dx. \tag{2.5}$$

If (u_n), $u\in W_0^{1,p(x)}(\Omega)$ then the following properties are true:

$$\|u\|>1\ \Rightarrow\ \|u\|^{p^-}\le\varrho_{p(x)}(u)\le\|u\|^{p^+}\,, \tag{2.6}$$

$$\|u\|<1\ \Rightarrow\ \|u\|^{p^+}\le\varrho_{p(x)}(u)\le\|u\|^{p^-}\,, \tag{2.7}$$

$$\|u_n-u\|\to 0\ \Leftrightarrow\ \varrho_{p(x)}(u_n-u)\to 0\,. \tag{2.8}$$

Set

$$p^*(x)=\begin{cases}\frac{Np(x)}{N-p(x)} & \text{if } p(x)<N\\ +\infty & \text{if } p(x)\ge N.\end{cases}$$

We point out that if $p,q\in C_+(\overline{\Omega})$ and $q(x)<p^\star(x)$ for all $x\in\overline{\Omega}$ then the embedding $W_0^{1,p(x)}(\Omega)\hookrightarrow L^{q(x)}(\Omega)$ is compact.

For a constant function p, the variable exponent Lebesgue and Sobolev spaces coincide with the standard Lebesgue and Sobolev spaces. Cf. [15], the function spaces with variable exponent have some striking properties, such as:

(i) If $1<p^-\le p^+<\infty$ and $p:\overline{\Omega}\to[1,\infty)$ is smooth, then the formula

$$\int_\Omega|u(x)|^p dx=p\int_0^\infty t^{p-1}|\{x\in\Omega;|u(x)|>t\}|\,dt$$

has no variable exponent analogue.

(ii) Variable exponent Lebesgue spaces do not have the *mean continuity property*: if p is continuous and nonconstant in an open ball B, then there exists a function $u\in L^{p(x)}(B)$ such that $u(x+h)\notin L^{p(x)}(B)$ for all $h\in\mathbb{R}^N$ with arbitrary small norm.

(iii) An argument in the development of the theory of function spaces with variable exponent is the fact that these spaces are never translation invariant. The use of convolution is also limited, for instance the Young inequality

$$|f * g|_{p(x)} \leq C\, |f|_{p(x)}\, \|g\|_{L^1}$$

holds if and only if p is constant.

We refer to Rădulescu [14] and the monographs by Diening, Hästo, Harjulehto and Ruzicka [5] and Rădulescu and Repovš [15] for additional properties of function spaces with variable exponent and for a thorough variational analysis of these problems.

3. Main results

Let $\Delta_{p(x)}$ denote the $p(x)$-*Laplace operator*, namely

$$\Delta_{p(x)} u := \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u).$$

In this article we study of the perturbed nonhomogeneous Dirichlet problem

$$\begin{gathered} -\Delta_{p(x)} u = \lambda |u|^{p(x)-2}u + |u|^{q(x)-2}u, \quad \text{in } \Omega \\ u = 0, \quad \text{on } \partial\Omega. \end{gathered} \tag{3.1}$$

We suppose that $p, q \in C_+(\overline{\Omega})$ satisfy

$$p^+ < q^- \quad \text{and} \quad \max\{p(x), q(x)\} < p^*(x) \text{ for all } x \in \overline{\Omega}. \tag{3.2}$$

The energy functional associated with (3.1) is $\mathcal{E} : W_0^{1,p(x)}(\Omega) \to \mathbb{R}$ defined by

$$\mathcal{E}(u) = \int_\Omega \frac{1}{p(x)}(|\nabla u|^{p(x)} - \lambda |u|^{p(x)})dx - \int_\Omega \frac{|u|^{q(x)}}{q(x)} dx,$$

for all $u \in W_0^{1,p(x)}(\Omega)$.

Hypothesis (3.2) implies that $\mathcal{E}$ is well-defined and $\mathcal{E} \in C^1(W_0^{1,p(x)}(\Omega), \mathbb{R})$. We also observe that

$$\langle \mathcal{E}'(u), v\rangle = \int_\Omega \big(|\nabla u|^{p(x)-2}\nabla u \cdot \nabla v - \lambda |u|^{p(x)-2}uv\big)dx - \int_\Omega |u|^{q(x)-2}uv\, dx,$$

for all $v \in W_0^{1,p(x)}(\Omega)$.

We say that u is a *weak solution* of (3.1) if $u \in W_0^{1,p(x)}(\Omega) \setminus \{0\}$ and

$$\int_\Omega \big(|\nabla u|^{p(x)-2}\nabla u \cdot \nabla v - \lambda |u|^{p(x)-2}uv\big)dx - \int_\Omega |u|^{q(x)-2}uv\, dx = 0,$$

for all $v \in W_0^{1,p(x)}(\Omega)$.

Thus, weak solutions of (3.1) correspond to the critical points of the energy functional $\mathcal{E}$. Let

$$\lambda^* = \inf\Big\{ \int_\Omega |\nabla u|^{p(x)} dx;\ u \in W_0^{1,p(x)}(\Omega),\ |u|_{p(x)} = 1\Big\}.$$

Our first result in this paper establishes the following existence property.

Theorem 3.1. *Assume that hypothesis* (3.2) *is satisfied and* $\lambda < \lambda^*$. *Then problem* (3.1) *has at least one solution.*

We recall that the functional $\mathcal{E} : W_0^{1,p(x)}(\Omega) \to \mathbb{R}$ satisfies the Palais-Smale condition if any sequence $(u_n) \subset W_0^{1,p(x)}(\Omega)$ such that

$$\mathcal{E}(u_n) = O(1) \text{ and } \|\mathcal{E}'(u_n)\|_{W^{-1,p'(x)}} = o(1) \quad \text{as } n \to \infty, \tag{3.3}$$

is relatively compact.

An important role in the proof of Theorem 3.1 is played by the mountain pass theorem of Ambrosetti and Rabinowitz [1].

Theorem 3.2. *Assume that X is a Banach space, $\varphi \in C^1(X,\mathbb{R})$ satisfies the Palais-Smale condition, u_0, $u_1 \in X$, $\|u_1 - u_0\| > \rho > 0$*

$$\max\{\varphi(u_0), \varphi(u_1)\} < \inf[\varphi(u) : \|u - u_0\| = \rho] = m_\rho$$

and

$$c = \inf_{\gamma\in\Gamma} \max_{0\le t\le 1} \varphi(\gamma(t)) \quad \textit{with} \quad \Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = u_0, \gamma(1) = u_1\}.$$

Then $c \ge m_\rho$ and c is a critical value of φ.

As pointed out by Brezis and Browder [3], the mountain pass theorem "extends ideas already present in Poincaré and Birkhoff". More generally, this result is in fact true in Banach-Finsler manifolds.

Assumption (3.2) guarantees that the energy functional associated with (3.1) has a mountain pass geometry. We study in what follows a related perturbed problem, provided that Theorem 3.2 cannot be applied. Consider the problem

$$\begin{gathered} -\Delta_{p(x)} u = \lambda |u|^{p(x)-2}u + \mu |u|^{q(x)-2}u, \quad \text{in } \Omega \\ u = 0, \quad \text{on } \partial\Omega. \end{gathered} \tag{3.4}$$

We say that u is a *weak solution* of problem (3.4) if $u \in W_0^{1,p(x)}(\Omega) \setminus \{0\}$ and

$$\int_\Omega \big(|\nabla u|^{p(x)-2}\nabla u \cdot \nabla v - \lambda |u|^{p(x)-2}uv\big)dx - \mu \int_\Omega |u|^{q(x)-2}uv\,dx = 0,$$

for all $v \in W_0^{1,p(x)}(\Omega)$.

We assume that $p, q \in C_+(\overline{\Omega})$ satisfy

$$q^- < p^- \text{ and } \max\{p(x), q(x)\} < p^*(x) \text{ for all } x \in \overline{\Omega}. \tag{3.5}$$

Under this assumption, the functional $\mathcal{E}$ satisfies the mountain condition of the mountain pass theorem near the origin for small values of the parameter μ. However, the existence of a valley for large values of μ is not guaranteed, hence we do not have a mountain pass geometry. The second existence result of this paper extends [10, Theorem 2.1] and is stated in the following theorem.

Theorem 3.3. *Assume that hypothesis (3.5) is fulfilled and $\lambda < \lambda^*$. Then there exists $\mu^* > 0$ such that for all $\mu < \mu^*$ problem (3.4) has at least one solution.*

The key ingredient in the proof of Theorem 3.3 is the Ekeland variational principle, which asserts the existence of almost critical points of $\mathcal{E}$. The subcritical framework of the problem yields a nontrivial critical point of $\mathcal{E}$, hence a weak solution of problem (3.4). We point out that the Ekeland variational principle can be viewed as the nonlinear version of the Bishop-Phelps theorem [12].

The arguments developed in this paper show that a similar result holds if the $p(x)$-Laplace operator is replaced with other nonhomogeneous differential operators with variable exponent, for instance the *generalized mean curvature operator* defined by

$$\operatorname{div}\Big((1+|\nabla u|^2)^{[p(x)-2]/2}\nabla u\Big).$$

4. Proof of Theorem 3.1

The proof strongly relies on the mountain pass theorem in relationship with some ideas developed in [11] and [13].

We start with the verification of the geometric hypotheses of the mountain pass theorem. We observe that $\mathcal{E}(0)=0$ and we show the existence of a mountain near the origin, namely there exist positive numbers r and η such that $\mathcal{E}(u)\geq\eta$ for all $u\in W_0^{1,p(x)}(\Omega)$ with $\|u\|=r$. We first observe that the definition of λ^* combined with the fact that $\lambda<\lambda^*$ imply that there exists $\delta>0$ such that

$$\int_\Omega \frac{1}{p(x)}(|\nabla u|^{p(x)}-\lambda|u|^{p(x)})dx \geq \delta|\nabla u|_{p(x)}, \quad \text{for all } u\in W_0^{1,p(x)}(\Omega).$$

But

$$\int_\Omega \frac{|u|^{q(x)}}{q(x)}dx \leq \frac{1}{q^-}|u|_{q(x)}, \quad \text{for all } u\in W_0^{1,p(x)}(\Omega).$$

Combining these inequalities, we deduce that

$$\mathcal{E}(u)\geq \delta\left|\nabla u\right|_{p(x)}-\frac{1}{q^-}\left|u\right|_{q(x)}. \tag{4.1}$$

Fix $r\in(0,1)$ and $u\in W_0^{1,p(x)}(\Omega)$ with $\|u\|=r$. Then relations (2.7), (4.1) and the Sobolev embedding $W_0^{1,p(x)}(\Omega)\hookrightarrow L^{q(x)}(\Omega)$ yield

$$\mathcal{E}(u)\geq\delta\left\|u\right\|^{p^+}-\frac{C}{q^-}\left\|u\right\|^{q^-}.$$

Choosing eventually $r\in(0,1)$ smaller if necessary, we conclude that there exists $\eta>0$ such that $\mathcal{E}(u)\geq\eta$ for all $u\in W_0^{1,p(x)}(\Omega)$ with $\|u\|=r$.

Next, we argue the existence of a valley over the chain of mountains. For this purpose, we fix $s>1$ and $w\in W_0^{1,p(x)}(\Omega)\setminus\{0\}$. It follows that

$$\begin{aligned}\mathcal{E}(sw)&=\int_\Omega\frac{s^{p(x)}}{p(x)}\Big(|\nabla w|^{p(x)}-\lambda|w|^{q(x)}\Big)dx-\int_\Omega\frac{s^{q(x)}}{q(x)}\left|w\right|^{q(x)}dx\\&\leq A\frac{s^{p^+}}{p^-}-B\frac{s^{q^-}}{q^+},\end{aligned}\tag{4.2}$$

where

$$A=\int_\Omega\Big(|\nabla w|^{p(x)}-\lambda|w|^{q(x)}\Big)dx \quad\text{and}\quad B=\int_\Omega|w|^{q(x)}dx.$$

Using hypothesis (3.2), relation (4.2) yields $\mathcal{E}(sw)<0$ for s large enough.

To apply Theorem 3.2 to our problem (3.1) it remains to check that the energy functional $\mathcal{E}$ satisfies the Palais-Smale compactness condition. Let $(u_n)\subset W_0^{1,p(x)}(\Omega)$ be an arbitrary Palais-Smale sequence for $\mathcal{E}$, namely

$$\mathcal{E}(u_n)=O(1)\quad\text{as } n\to\infty \tag{4.3}$$

and

$$\|\mathcal{E}'(u_n)\|_{W^{-1,p'(x)}(\Omega)} = o(1) \quad \text{as } n\to\infty\,. \tag{4.4}$$

We claim that

$$\text{the sequence } (u_n) \text{ is bounded in } W_0^{1,p(x)}(\Omega). \tag{4.5}$$

Relations (4.3) and (4.4) yield

$$\int_\Omega \frac{1}{p(x)}\Big(|\nabla u_n|^{p(x)} - \lambda |u_n|^{p(x)}\Big)dx - \int_\Omega \frac{1}{q(x)}|u_n|^{q(x)}dx = O(1) \quad \text{as } n\to\infty \tag{4.6}$$

and for all $v\in W_0^{1,p(x)}(\Omega)$,

$$\begin{aligned} &\int_\Omega \Big(|\nabla u_n|^{p(x)-2}\nabla u_n\cdot\nabla v - \lambda|u_n|^{p(x)-2}u_n v\Big)\,dx - \int_\Omega |u_n|^{q(x)-2}u_n v\,dx \\ &= o(1)\|v\| \quad \text{as } n\to\infty\,. \end{aligned} \tag{4.7}$$

Choosing $v=u_n$ in (4.7) we deduce that

$$\int_\Omega \Big(|\nabla u_n|^{p(x)} - \lambda|u_n|^{p(x)}\Big)dx - \int_\Omega |u_n|^{q(x)}dx = o(1)\|u_n\| \quad \text{as } n\to\infty\,. \tag{4.8}$$

On the other hand, relation (4.6) implies

$$\begin{aligned} &O(1) + \frac{1}{p^+}\int_\Omega \big(|\nabla u_n|^{p(x)} - \lambda|u_n|^{p(x)}\big)dx \\ &\le \int_\Omega \frac{1}{q(x)}\,|u_n|^{q(x)}dx \\ &\le O(1) + \frac{1}{p^-}\int_\Omega \big(|\nabla u_n|^{p(x)} - \lambda|u_n|^{p(x)}\big)dx. \end{aligned}$$

Using now relation (4.8) we deduce that

$$\begin{aligned} &O(1) + o(1)\|u_n\| + \frac{1}{p^+}\int_\Omega |u_n|^{q(x)}dx \\ &\le \int_\Omega \frac{1}{q(x)}\,|u_n|^{q(x)}dx \\ &\le O(1) + o(1)\|u_n\| + \frac{1}{p^-}\int_\Omega |u_n|^{q(x)}dx. \end{aligned}$$

It follows that

$$\int_\Omega |u_n|^{q(x)}dx = O(1) + o(1)\|u_n\| \quad \text{as } n\to\infty\,. \tag{4.9}$$

Returning to (4.6) and using relation (4.9) we deduce that

$$\int_\Omega \frac{1}{p(x)}\big(|\nabla u_n|^{p(x)} - \lambda|u_n|^{p(x)}\big)dx = O(1) + o(1)\|u_n\|\,. \tag{4.10}$$

Taking into account the definition of λ^* and the fact that $\lambda<\lambda^*$, relation (4.10) implies that (u_n) is bounded in $W_0^{1,p(x)}(\Omega)$, hence Claim (4.5) is argued. So, up to a subsequence

$$u_n \rightharpoonup u \quad \text{in } W_0^{1,p(x)}(\Omega), \tag{4.11}$$

$$u_n \to u \quad \text{in } L^{p(x)}(\Omega). \tag{4.12}$$

We prove in what follows that

$$(u_n) \text{ contains a strongly convergent subsequence in } W_0^{1,p(x)}(\Omega). \tag{4.13}$$

We first observe that relation (4.7) yields

$$\int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n\cdot\nabla v\,dx=\int_\Omega \varrho(x,u_n)v\,dx+o(1)\|v\|\quad\text{as } n\to\infty\,, \tag{4.14}$$

for all $v\in W_0^{1,p(x)}(\Omega)$ where $\varrho(x,w)=\lambda|w|^{p(x)-2}w+|w|^{q(x)-2}w$.

We assume in what follows that $p^+<N$ (the same arguments can be developed in the contrary case). Using (4.5), relation (4.14) implies that our claim (4.13) follows as soon as we show that $\varrho(x,u_n)$ is relatively compact in $W^{-1,p'(x)}(\Omega)$. Hence, by Sobolev embeddings, it is enough to show that

$$\varrho(x,u_n)\text{ is relatively compact in } L^{Np(x)/[Np(x)-N+p(x)]}(\Omega), \tag{4.15}$$

which is the dual space of $L^{Np(x)/[N-p(x)]}(\Omega)$.

Using (4.12) in combination with the Egorov theorem, we deduce that for fixed $\eta>0$ there is a subset A of Ω of measure less than η and such that

$$u_n\to u\quad\text{uniformly on }\Omega\setminus A.$$

So, our claim (4.15) follows as soon as we show that for for all $\varepsilon>0$ and n large enough

$$\int_A |\varrho(x,u_n)-\varrho(x,u)|^{Np(x)/[Np(x)-N+p(x)]}dx\le\varepsilon\,. \tag{4.16}$$

Taking into account the subcritical growth of ϱ we have for some $C>0$

$$\int_A |\varrho(x,u)|^{Np(x)/[Np(x)-N+p(x)]}dx\le C\int_A (1+|u|)^{Np(x)/[N-p(x)]}dx,$$

which can be made sufficiently small if we choose $\eta>0$ small enough. We conclude that

$$\begin{aligned}&\int_A |\varrho(x,u_n)-\varrho(x,u)|^{Np(x)/[Np(x)-N+p(x)]}dx\\ &\le\delta\int_A |u_n-u|^{Np(x)/[N-p(x)]}dx+C_\delta|A|<\varepsilon,\end{aligned}$$

by choosing η small enough and by using Sobolev embeddings and (4.5). Our claim (4.13) is proved, which implies that $\mathcal{E}$ satisfies the Palais-Smale condition. Applying Theorem 3.2 we deduce that (3.1) has a nontrivial solution for all $\lambda<\lambda^*$.

The ideas developed in the proof of Theorem 3.1 allow to extend this result for more general perturbations. Indeed, let us consider the nonhomogeneous Dirichlet problem:

$$\begin{gathered}-\Delta_{p(x)}u+a(x)|u|^{p(x)-2}u=|u|^{q(x)-2}u,\quad\text{in }\Omega\\ u=0,\quad\text{on }\partial\Omega,\end{gathered} \tag{4.17}$$

where $a\in L^\infty(\Omega)$ and there exists $\alpha>0$ such that

$$\int_\Omega \Big(|\nabla u|^{p(x)}+a(x)|u|^{p(x)}\Big)\ge\alpha\rho_{p(x)}(\nabla u)\quad\text{for all } u\in W_0^{1,p(x)}(\Omega). \tag{4.18}$$

The following property is the counterpart of [13, Theorem 10] in the framework of operators and nonlinearities involving variable exponents.

Theorem 4.1. *Suppose that hypotheses* (3.2) *and* (4.18) *are satisfied. Then* (4.17) *has at least one nontrivial solution.*

5. Proof of Theorem 3.3

The weak solutions of (3.4) correspond to critical points of the associated energy functional $\mathcal{J} : W_0^{1,p(x)}(\Omega) \to \mathbb{R}$ defined by

$$\mathcal{J}(u) = \int_\Omega \frac{1}{p(x)}(|\nabla u|^{p(x)} - \lambda |u|^{p(x)})dx - \mu \int_\Omega \frac{|u|^{q(x)}}{q(x)} dx,$$

for all $u \in W_0^{1,p(x)}(\Omega)$.

We first establish a preliminary result, which asserts that the energy functional $\mathcal{J}$ satisfies the geometric condition around the origin, provided that $\lambda < \lambda^*$ and μ is small enough.

Lemma 5.1. *Let $\lambda < \lambda^*$. Then there exist positive numbers μ^*, r and a such that for all $\mu < \mu^*$ we have $\mathcal{J}(u) \geq a$, provided that $u \in W_0^{1,p(x)}(\Omega)$ satisfies $\|u\| = r$.*

Proof. As in the proof of Theorem 3.1, combining the definition of λ^* with the fact that $\lambda < \lambda^*$, we deduce that there is a positive number δ such that

$$\mathcal{J}(u) \geq \delta |\nabla u|_{p(x)} - \frac{\mu}{q^-} |u|_{q(x)} \quad \text{for all } u \in W_0^{1,p(x)}(\Omega). \tag{5.1}$$

Let $a_1 > 0$ denote the best constant corresponding to the continuous embedding of $W_0^{1,p(x)}(\Omega)$ into $L^{q(x)}(\Omega)$. Fix arbitrarily $r \in (0,1)$. Using (2.3) we obtain

$$\int_\Omega |u|^{q(x)} dx \leq |u|_{q(x)}^{q^-} \leq a_1^{q^-} \|u\|^{q^-} \quad \text{for all } u \in W_0^{1,p(x)}(\Omega)\ , \|u\| = \frac{r}{a_1}. \tag{5.2}$$

Next, using (2.7) we obtain

$$|\nabla u|_{p(x)} \geq \|u\|^{p^+} \quad \text{for all } u \in W_0^{1,p(x)}(\Omega), \quad \|u\| = \frac{r}{a_1}. \tag{5.3}$$

We notice that in (5.2) and (5.3) we can assume that a_1 is large enough in order to have $r/a_1 < 1$.

Relations (5.1), (5.2) and (5.3) imply that for all $u \in W_0^{1,p(x)}(\Omega)$ with $\|u\| = \frac{r}{a_1}$ we have

$$\begin{aligned} \mathcal{J}(u) &\geq \delta \|u\|^{p^+} - \frac{\mu}{q^-} a_1^{q^-} \|u\|^{q^-} \\ &= \delta r^{p^+} - \frac{\mu}{q^-} a_1^{q^-} r^{q^-} \\ &= r^{q^-} \Big(\delta r^{p^+ - q^-} - \frac{\mu}{q^-} a_1^{q^-} \Big). \end{aligned}$$

This relation shows that the lemma follows after choosing

$$\mu^* = \frac{\delta q^- r^{p^+ - q^-}}{2 a_1^{q^-}} \quad \text{and} \quad a = \frac{\delta r^{p^+}}{2}\,.$$

□

The second auxiliary result shows that $\mathcal{J}$ has a valley around the origin, hence the hypotheses of the mountain pass theorem are not fulfilled.

Lemma 5.2. *Let $\lambda < \lambda^*$. Then there exists a smooth nonnegative function $\phi \in W_0^{1,p(x)}(\Omega)$ such that $\mathcal{J}(t\phi) < 0$ for all $t > 0$ sufficiently small.*

Proof. The basic idea is to use our assumption (3.5), more exactly $q^- < p^-$. Set

$$\eta := \frac{p^- - q^-}{2}\,.$$

Let ω be a subdomain of Ω such that

$$q(x) \le q^- + \eta < p^- \quad \text{for all } x \in \omega.$$

Let $\varphi \in W_0^{1,p(x)}(\Omega)$ a smooth function with compact support such that $0 \le \phi \le 1$, $\varphi|_\omega = 1$ and $\bar\omega \subset \operatorname{supp}(\phi)$. It follows that for all $t \in (0,1)$ we have

$$\begin{aligned}
\mathcal{J}(t\phi) &= \int_\Omega \frac{t^{p(x)}}{p(x)}\big(|\nabla\phi|^{p(x)} - \lambda\phi^{p(x)}\big)dx - \mu\int_\Omega \frac{t^{q(x)}}{q(x)}\,\phi^{q(x)}dx \\
&\le \frac{t^{p^-}}{p^-}\int_\Omega \big(|\nabla\phi|^{p(x)} - \lambda\phi^{p(x)}\big)dx - \frac{\mu}{q^+}\int_\Omega t^{q(x)}\phi^{q(x)}dx \\
&\le \frac{t^{p^-}}{p^-}\int_\Omega \big(|\nabla\phi|^{p(x)} - \lambda\phi^{p(x)}\big)dx - \frac{\mu}{q^+}\int_\omega t^{q(x)}\phi^{q(x)}dx \\
&\le A\frac{t^{p^-}}{p^-} - B\frac{\mu t^{q^-+\eta}}{q^+},
\end{aligned}$$

where

$$A := \int_\Omega \Big(|\nabla\phi|^{p(x)} - \lambda\phi^{p(x)}\Big)\,dx > 0 \quad \text{and} \quad B := \int_\omega \phi^{q(x)}dx > 0.$$

We deduce that $\mathcal{J}(t\phi) < 0$, provided that $t \in (0,1)$ is small enough. □

Fix $\mu^* > 0$ as established in Lemma 5.1 and let $\mu < \mu^*$. Using Lemmata 5.1 and 5.2 we deduce that there exists $r > 0$ such that

$$\inf_{u\in\overline{B_r(0)}} \mathcal{J}(u) < 0 < \inf_{u\in\partial B_r(0)} \mathcal{J}(u),$$

where $B_r(0) := \{u \in W_0^{1,p(x)}(\Omega);\ \|u\| < r\}$.

Fix $\varepsilon > 0$ such that

$$\varepsilon < \inf_{u\in\partial B_r(0)} \mathcal{J}(u) - \inf_{u\in\overline{B_r(0)}} \mathcal{J}(u).$$

Applying Ekeland's variational principle [6] we find $u_\varepsilon \in \overline{B_r(0)}$ such that

$$\mathcal{J}(u_\varepsilon) < \inf_{u\in\overline{B_r(0)}} \mathcal{J}(u) + \varepsilon, \tag{5.4}$$

$$\mathcal{J}(u_\varepsilon) < \mathcal{J}(u) + \varepsilon\|u - u_\varepsilon\| \quad \text{for all } u \in W_0^{1,p(x)}(\Omega)\setminus\{u_\varepsilon\}. \tag{5.5}$$

We claim that $u_\varepsilon \in B_r(0)$. Indeed, relations (5.4) and (5.5) yield

$$J(u_\varepsilon) < \inf_{u\in\overline{B_r(0)}} \mathcal{J}(u) + \varepsilon < \inf_{u\in\partial B_r(0)} \mathcal{J}(u),$$

hence $u_\varepsilon \notin \partial B_r(0)$. A standard argument (see, e.g., [10, p. 2934]) shows that $\|\mathcal{J}'(u_\varepsilon)\| \le \varepsilon$.

Let $c := \inf_{u\in\overline{B_r(0)}} \mathcal{J}(u)$. It follows that u_ε is an *almost critical point* of $\mathcal{J}$ at the level c, that is,

$$\lim_{\varepsilon\to 0} \mathcal{J}(u_\varepsilon) = c \quad \text{and} \quad \lim_{\varepsilon\to 0} \|\mathcal{J}'(u_\varepsilon)\| = 0.$$

From now on, using the same argument as in [8, p. 59] (see also [7, Theorem 3.1] and [10, p. 2935]) we deduce that there exists a subsequence (v_n) of (u_ε) that converges to a nontrivial critical point u of $\mathcal{J}$, hence u is a weak solution of problem (3.4).

Perspectives and open problems

Now we raise some open problems in relationship with the study developed in this paper.

Open problem 1. Problem (3.1) has been studied in the subcritical case, namely under the basic hypothesis (3.2), namely $\max\{p(x), q(x)\} < p^*(x)$ for all $x \in \overline{\Omega}$, which is crucial for the verification of the Palais-Smale compactness condition. We consider that a very interesting research direction is to study the same problem in the almost critical setting, hence under the following weaker assumption: there exists $x_0 \in \Omega$ such that

$$q(x) < p^*(x) \quad \text{for all } x \in \overline{\Omega} \setminus \{x_0\} \text{ and } q(x_0) = p^*(x_0). \tag{5.6}$$

Of course, this hypothesis is not possible if the functions p and q are constant. We conjecture that the result stated in Theorem 3.1 remains true under assumption (5.6).

Open problem 2. [13, Theorem 11] studies with values of the parameter λ which are larger than the principal eigenvalue of the Laplace operator. In such a case it is established an existence property by using the dual variational method. This is based on the introduction of a new unknown and uses in an essential manner the linearity of the Laplace operator. We consider that a very interesting open problem is the qualitative analysis of problem (3.1) provided that $\lambda \geq \lambda^*$.

Open problem 3. We suggest to adapt the limited developments techniques introduced in the proof of Theorem 9 in [13, pp. 30-32] in order to obtain an alternative proof of Theorem 3.1 in this paper. In our framework the major difficulties arise due to the presence of variable exponents, which is not the case in [13].

Open problem 4. Inspired by the study developed in [15, Chapter 3.3], we suggest the study of problems (3.1) and (3.4) if the $p(x)$-Laplace operator is replaced with a differential operator with several variable exponents, for instance

$$\operatorname{div}\Big((|\nabla u|^{p_1(x)-2} + |\nabla u|^{p_2(x)-2})\nabla u\Big).$$

Open problem 5. We conjecture that the existence properties stated in Theorems 3.1 and 3.3 remain true if the $p(x)$-Laplace operator is replaced with a general class of Leray-Lions type operators, as defined in [15, pp. 27-28]. We refer to the pioneering paper by Leray and Lions [9] for basic properties of these operators and some relevant applications.

Open problem 6. Theorem 3.3 establishes an existence property for low perturbations, namely provided that the parameter μ is small enough. We consider that a complete bifurcation analysis for this problem, in relationship with the values of the variable exponents p and q, is a very interesting open problem. We do not have partial results on this problem even for particular variable exponents or if Ω is a ball.

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