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SURVEY AND NEW RESULTS ON BOUNDARY-VALUE PROBLEMS OF SINGULAR FRACTIONAL DIFFERENTIAL EQUATIONS WITH IMPULSE EFFECTS

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ABSTRACT. Firstly we prove existence and uniqueness of solutions of Cauchy problems of linear fractional differential equations (LFDEs) with two variable coefficients involving Caputo fractional derivative, Riemann-Liouville derivative, Caputo type Hadamard derivative and Riemann-Liouville type Hadamard fractional derivatives with order $q \in [n-1, n)$ by using the iterative method. Secondly we obtain exact expressions for piecewise continuous solutions of the linear fractional differential equations with a constant coefficient and a variable one. These results provide new methods to transform an impulsive fractional differential equation (IFDE) to a fractional integral equation (FIE). Thirdly, we propose four classes of boundary value problems of singular fractional differential equations with impulse effects. Sufficient conditions are given for the existence of solutions of these problems. We allow the nonlinearity $p(t)f(t,x)$ in fractional differential equations to be singular at $t = 0, 1$. Finally, we point out some incorrect formulas of solutions in cited papers. A new Banach space and the compact properties of subsets are proved. By establishing a new framework to find the solutions for impulsive fractional boundary value problems, the existence of solutions of three classes boundary value problems of impulsive fractional differential equations with multi-term fractional derivatives are established.

CONTENTS

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1. Introduction

One knows that the fractional derivatives (Riemann-Liouville fractional derivative, Caputo fractional derivative and Hadamard fractional derivative and other type see [58]) are actually nonlocal operators because integrals are nonlocal operators. Moreover, calculating time fractional derivatives of a function at some time requires all the past history and hence fractional derivatives can be used for modeling systems with memory.

Fractional order differential equations are generalizations of integer order differential equations. Using fractional order differential equations can help us to reduce the errors arising from the neglected parameters in modeling real life phenomena. Fractional differential equations have many applications see [88, Chapter 10], and books [58, 57, 88, 94].

In recent years, there have been many results obtained on the existence and uniqueness of solutions of initial value problems or boundary value problems for nonlinear fractional differential equations, see [25, 27, 74, 81, 85, 86, 93, 108, 125, 138].

Dynamics of many evolutionary processes from various fields such as population dynamics, control theory, physics, biology, and medicine. undergo abrupt changes at certain moments of time like earthquake, harvesting, shock, and so forth. These perturbations can be well approximated as instantaneous change of states or impulses.These processes are modeled by impulsive differential equations. In 1960, Milman and Myshkis introduced impulsive differential equations in their paper [82]. Based on their work, several monographs have been published by many authors like Samoilenko and Perestyuk [95], Lakshmikantham et al. [60], Bainov and Simeonov [20, 21], Bainov and Covachev [19], and Benchohra et al. [28].

Fractional differential equations were extended to impulsive fractional differential equations, since Agarwal and Benchohra published the first paper on the topic [4] in 2008. Since then many authors [16, 39, 42, 55, 72, 68, 66, 84, 93, 107, 108, 124, 73, 71] studied the existence or uniqueness of solutions of impulsive initial or boundary value problems for fractional differential equations. For examples, impulsive anti-periodic boundary value problems see [5, 6, 4, 69, 101], impulsive periodic boundary value problems see [105, 26, 115], impulsive initial value problems see [30, 38, 83, 98], two-point, three-point or multi-point impulsive boundary value problems see [12, 47, 116, 136, 106, 134], impulsive boundary value problems on infinite intervals see [131].

Feckan and Zhou [43] pointed out that the formula of solutions for impulsive fractional differential equations in [3, 11, 24, 29] is incorrect and gave their correct formula. In [116, 111], the authors established a general framework to find the solutions for impulsive fractional boundary value problems and obtained some sufficient conditions for the existence of the solutions to a kind of impulsive fractional differential equations. In [103], the authors illustrated their comprehension for the counterexample in [43] and criticized the viewpoint in [43, 116, 111]. Next, in [44], Feckan et al. expanded for the counterexample in [43] and provided further explanations in the paper.

In a fractional differential equation, there exist two cases concerning the derivatives: the firs case is $D^\alpha = D^\alpha_{0^+}$, i.e., the fractional derivative has a single start point $t = 0$. The other case is $D^\alpha = D^\alpha_{t_i^+}$, i.e., the fractional derivative has a multiple start points $t = t_i$ $(i \in N[0, m])$.

There have been many authors concerning the existence and uniqueness of solutions of boundary value problems of impulsive fractional differential equations with multiple start points $t = t_i$ $(i \in N[0, m])$.

Recently, Wang [100] consider the second case in which D^α has multiple start points, i. e., $D^\alpha = D^\alpha_{t_i^+}$. They studied the existence and uniqueness of solutions of the following initial value problem of the impulsive fractional differential equation

$$\begin{gathered} {}^CD^\alpha_{t_i^+}u(t) = f(t, u(t)), \quad t \in (t_i, t_{i+1}],\ i \in \mathbb{N}[0, p], \\ u^{(j)}(0) = u_j, \quad j \in \mathbb{N}[0, n-1], \\ \Delta u^{(j)}(t_i)] = I_{ji}(u(t_i)), \quad i \in \mathbb{N}[1, p],\ j \in \mathbb{N}[0, n-1], \end{gathered} \tag{1.1}$$

where $\alpha \in (n-1, n)$ with n being a positive integer, ${}^CD^\alpha_{t_i^+}$ represents the standard Caputo fractional derivatives of order α, $\mathbb{N}[a, b] = \{a, a+1, \dots, b\}$ with a, b being integers, $0 = t_0 < t_1 < \dots < t_p < t_{p+1} = 1$, $I_{ji} \in C(\mathbb{R}, \mathbb{R})$ $(i \in \mathbb{N}[1, p], j \in \mathbb{N}[0, n-1])$, $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

Henderson and Ouahab [50] studied the existence of solutions of the following problems

$$\begin{gathered} {}^CD^\alpha_{t_i^+}u(t) = f(t, u(t)), \quad t \in (t_i, t_{i+1}],\ i \in \mathbb{N}[0, p], \\ u^{(j)}(0) = u_j, \quad j \in \mathbb{N}[0, 1], \\ u^{(j)}(t_i)] = I_{ji}(u(t_i)), \quad i \in \mathbb{N}[1, p],\ j \in \mathbb{N}[0, 1], \end{gathered}$$

and

$$\begin{gathered} {}^CD^\alpha_{t_i^+}u(t) = f(t, u(t)), \quad t \in (t_i, t_{i+1}],\ i \in \mathbb{N}[0, p], \\ u^{(j)}(0) = u^{(j)}(b), \quad j \in \mathbb{N}[0, 1], \\ u^{(j)}(t_i)] = I_{ji}(u(t_i)), \quad i \in \mathbb{N}[1, p],\ j \in \mathbb{N}[0, 1], \end{gathered}$$

where $\alpha \in (1,2]$, $b > 0$, $0 = t_0 < t_1 < \dots < t_p < t_{p+1} = b$, $f : [0,b] \times \mathbb{R} \to \mathbb{R}$, $I_{ji} : \mathbb{R} \to \mathbb{R}$ are continuous functions. Readers should also refer [104].

Zhao and Gong [132] studied existence of positive solutions of the nonlinear impulsive fractional differential equation with generalized periodic boundary value conditions

$$\begin{gathered}
{}^C D^q_{t_i^+} u(t) = f(t, u(t)), \quad t \in (0,T] \setminus \{t_1, \dots, t_p\}, \\
\Delta u(t_i)] = I_i(u(t_i)), \quad \Delta u'(t_i)] = J_i u(t_i)), \quad i \in \mathbb{N}[1,p], \\
\alpha u(0) - \beta u(1) = 0, \quad \alpha u'(0) - \beta u'(1) = 0,
\end{gathered} \tag{1.2}$$

where $q \in (1,2)$, ${}^C D^q_{t_i^+}$ represents the standard Caputo fractional derivatives of order q, $\alpha > \beta > 0$, $0 = t_0 < t_1 < \dots < t_p < t_{p+1} = 1$, $\mathbb{N}[a,b] = \{a, a+1, \dots, b\}$ with a, b being integers,$I_i, J_i \in C([0,+\infty), [0,+\infty))$ $(i \in \mathbb{N}[1,p]$, $f : [0,1] \times [0,+\infty) \to [0,+\infty)$ is a continuous function.

Wang, Ahmad and Zhang [102] studied the existence and uniqueness of solutions of the periodic boundary value problems for nonlinear impulsive fractional differential equation

$$\begin{gathered}
{}^C D^\alpha_{t_i^+} u(t) = f(t, u(t)), \quad t \in (0,T] \setminus \{t_1, \dots, t_p\}, \\
\Delta u(t_i) = I_i(u(t_i)), \quad \Delta u'(t_i) = I_i^*(u(t_i)), \quad i \in \mathbb{N}[1,p], \\
u'(0) + (-1)^\theta u(T) = bu(T), \quad u(0) + (-1)^\theta u(T) = 0,
\end{gathered} \tag{1.3}$$

where $\alpha \in (1,2)$, ${}^C D^\alpha_{t_i^+}$ represents the standard Caputo fractional derivatives of order α, $\theta = 1,2$, $\mathbb{N}[a,b] = \{a, a+1, \dots, b\}$ with a, b being integers, $0 = t_0 < t_1 < \dots < t_p < t_{p+1} = T$, $I_i, I_i^* \in C(\mathbb{R}, \mathbb{R})$ $(i \in \mathbb{N}[1,p]$, $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

Zou and Feng, Li and Shang [13, 64, 139] studied the existence of solutions of the nonlinear boundary value problem of fractional impulsive differential equations

$$\begin{gathered}
{}^C D^\alpha_{t_i^+} x(t) = w(t) f(t, x(t), x'(t)), \quad t \in (0,1] \setminus \{t_1, \dots, t_p\}, \\
\Delta x(t_i) = I_i(x(t_i)), \quad \Delta x'(t_i) = J_i(x(t_i)), \quad i \in \mathbb{N}[1,p], \\
\alpha_1 x(0) - \beta_1 u'(0) = g_1(x), \quad \alpha_2 x(1) + \beta_2 x'(1) = g_2(x),
\end{gathered} \tag{1.4}$$

where $\alpha \in (1,2)$, ${}^C D^\alpha_{t_i^+}$ represents the standard Caputo fractional derivatives of order α, $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$ with $\alpha_1\alpha_2 + \alpha_1\beta_2 + \alpha_2\beta_1 \neq 0$, $\mathbb{N}[a,b] = \{a, a+1, \dots, b\}$ with a, b being integers, $0 = t_0 < t_1 < \dots < t_p < t_{p+1} = 1$, $I_i, J_i \in C(\mathbb{R}, \mathbb{R})$ $(i \in \mathbb{N}[1,p]$, $f : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous, $w : [0,1] \to [0,+\infty)$ is a continuous function, $g_1, g_2 : PC(0,1] \to \mathbb{R}$ are two continuous functions.

Liu and Li [71] investigated the existence and uniqueness of solutions for the nonlinear impulsive fractional differential equations

$$\begin{gathered}
{}^C D^{\alpha}_{t_i^+} u(t) = f(t, u(t), u'(t), u''(t)), \quad t \in (t_i, t_{i+1}],\ i \in \mathbb{N}[0,p], \\
u(0) = \lambda_1 u(T) + \xi_1 \int_0^T q_1(s, u(s), u'(s), u''(s))ds, \\
u'(0) = \lambda_2 u'(T) + \xi_2 \int_0^T q_2(s, u(s), u'(s), u''(s))ds, \\
u''(0) = \lambda_3 u''(T) + \xi_3 \int_0^T q_3(s, u(s), u'(s), u''(s))ds, \\
\Delta u(t_i)] = A_i(u(t_i)), \quad \Delta u'(t_i)] = B_i(u(t_i)), \quad \Delta u''(t_i)] = C_i(u(t_i)),
\end{gathered} \tag{1.5}$$

for $i \in \mathbb{N}[1,p]$, where $\alpha \in (2,3)$, ${}^C D^{\alpha}_{t_i^+}$ represents the standard Caputo fractional derivatives of order α, $\mathbb{N}[a,b] = \{a, a+1, \dots, b\}$ with a, b being integers, $0 = t_0 < t_1 < \dots < t_p < t_{p+1} = T$, $\lambda_i, \xi_i \in \mathbb{R}$ $(i = 1,2,3)$ are constants, $A_i, B_i, C_i \in C(\mathbb{R}, \mathbb{R})$ $(i \in \mathbb{N}[1,p]$, $f : [0,T] \times \mathbb{R}^3 \to \mathbb{R}$ is continuous.

Recently, in [32], to extend the problem for impulsive differential equation $u''(t) - \lambda u(t) = f(t, u(t)), u(0) = u(T) = 0, \Delta u'(t_i) = I_i(u(t)i)), i\mathbb{N}[1,p]$ to impulsive fractional differential equation, the authors studied the existence and the multiplicity of solutions for the Dirichlet's boundary value problem for impulsive fractional order differential equation

$$\begin{gathered}
{}^C D^{\alpha}_{T^-}({}^C D^{\alpha}_{0^+} x(t) + a(t)x(t) = \lambda f(t, x(t)), \quad t \in [0,T], t \neq t_i,\ i \in \mathbb{N}[1,m], \\
\Delta {}^C D^{\alpha-1}_{T^-}({}^C D^{\alpha}_{0^+} x(t_i) = \mu I_i(x(t_i^-)), \quad i \in \mathbb{N}[1,m],\ x(0) = x(T) = 0,
\end{gathered} \tag{1.6}$$

where $\alpha \in (1/2, 1]$, $\lambda, \mu > 0$ are constants, $\mathbb{N}[a,b] =: \{a, a+1, \dots, b]$ with $a \leq b$, $0 = t_0 < t_1 < \dots < t_m < t_{m+1} = T$, $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $I_i : \mathbb{R} \to \mathbb{R}(i \in \mathbb{N}[1,m])$ are continuous functions, ${}^C D^{\alpha}_{0^+}$ (or ${}^C D^{\alpha}_{T^-}$) is the standard left (or right) Caputo fractional derivative of order α, $a \in C[0,T]$ and there exist constants $a_1, a_2 > 0$ such that $a_1 \leq a(t) \leq a_2$ for all $t \in [0,T]$, $\Delta x|_{t=t_i} = \lim_{t \to t_i^+} x(t) - \lim_{t \to t_i^-} x(t) = x(t_i^+) - x(t_i^-)$ and $x(t_i^+), x(t_i^-)$ represent the right and left limits of $x(t)$ at $t = t_i$ respectively, a, b, x_0 a constant with $a + b \neq 0$. One knows that the boundary condition $ax(0) + bx(T) = x_0$ becomes $x(0) - x(T) = \frac{x_0}{a}$ when $a + b = 0$, that is so called nonhomogeneous periodic type boundary condition.

For impulsive fractional differential equations whose derivatives have single start points $t = 0$, there has been few papers published. In [91], authors presented a new method to converting the impulsive fractional differential equation (with the Caputo fractional derivative) to an equivalent integral equation and established existence and uniqueness results for some boundary value problems of impulsive fractional differential equations involving the Caputo fractional derivatives with single start point. The existence and uniqueness of solutions of the following initial or boundary value problems were discussed in [91]:

$$\begin{gathered}
{}^C D^{\alpha}_{0^+} x(t) = f(t, x(t)), \quad t \in (0,1] \setminus \{t_1, \dots, t_p\}, \\
\Delta x(t_i)] = I_i(x(t_i)), \quad \Delta x'(t_i)] = J_i(x(t_i)), \quad i \in \mathbb{N}[1,p], \\
x(0) = x_0, \quad x'(0) = x_1;
\end{gathered}$$

$$\begin{gathered}{}^{C}D_{0^+}^{\alpha}x(t) = f(t, x(t)), \quad t \in (0,1] \setminus \{t_1, \dots, t_p\}, \\ \Delta x(t_i)] = I_i(x(t_i)), \quad \Delta x'(t_i)] = J_i(x(t_i)),\ i \in \mathbb{N}[1,p], \\ x(0) + \phi(x) = x_0, \quad x'(0) = x_1;\end{gathered}$$

$$\begin{gathered}{}^{C}D_{0^+}^{\beta}x(t) = f(t, x(t)), \quad t \in (0,1] \setminus \{t_1, \dots, t_p\}, \\ \Delta x(t_i)] = I_i(x(t_i)), \quad i \in \mathbb{N}[1,p],\ ax(0) + bx(1) = 0,\end{gathered}$$

$$\begin{gathered}{}^{C}D_{0^+}^{\alpha}x(t) = f(t, x(t)), \quad t \in (0,1] \setminus \{t_1, \dots, t_p\}, \\ \Delta x(t_i)] = I_i(x(t_i)), \quad \Delta x'(t_i)] = J_i(x(t_i)),\ i \in \mathbb{N}[1,p], \\ ax(0) - bx'(0) = x_0, \quad cx(1) + dx'(1) = x_1;\end{gathered}$$

and

$$\begin{gathered}{}^{C}D_{0^+}^{\alpha}x(t) = f(t, x(t)), \quad t \in (0,1] \setminus \{t_1, \dots, t_p\}, \\ \Delta x(t_i)] = I_i(x(t_i)), \quad \Delta x'(t_i)] = J_i(x(t_i)), \quad i \in \mathbb{N}[1,p], \\ x(0) - ax(\xi) = x(1) - bx(\eta) = 0,\end{gathered}$$

where $\alpha \in (1,2], \beta \in (0,1]$, $D_{0^+}^{*}$ is the Caputo fractional derivative with order $*$ and single start point $t = 0$, $f : [0,1] \times \mathbb{R} \to \mathbb{R}$, $I_i, J_i : \mathbb{R} \to \mathbb{R}$ are continuous functions, $a, b, c, d, x_0, x_1 \in \mathbb{R}$ are constants, $\phi : PC(0,1] \to \mathbb{R}$ is a functional.

We observed that in the above-mentioned work, the authors all require that the fractional derivatives are the Caputo type derivatives, the nonlinear term f and the impulse functions are continuous. It is easy to see that these conditions are very restrictive and difficult to satisfy in applications. To the author's knowledge, there has been no paper published discussed the existence of solutions of boundary value problems of impulsive fractional differential equations involving other fractional derivatives such as the Riemann-Liouville fractional derivatives, Hadamard fractional derivatives.

In this paper, we study the existence of solutions of four classes of impulsive boundary value problems of singular fractional differential equations. The first class is the impulsive Dirichlet type integral boundary value problem

$$\begin{gathered}{}^{RL}D_{0^+}^{\beta}x(t) - \lambda x(t) = p(t)f(t, x(t)), \quad \text{a.e., } t \in (t_i, t_{i+1}],\ i \in \mathbb{N}[0,m], \\ \lim_{t \to 0^+} t^{2-\beta}x(t) = \int_0^1 \phi(s)G(s, x(s))ds, \quad x(1) = \int_0^1 \psi(s)H(s, x(s))ds, \\ \lim_{t \to t_i^+} (t - t_i)^{2-\beta}x(t) = I(t_i, x(t_i)), \quad \Delta\, {}^{RL}D_{0^+}^{\beta-1}x(t_i) = J(t_i, x(t_i)),\end{gathered} \tag{1.7}$$

for $i \in \mathbb{N}[1,m]$, where

- (1.A1) $1 < \beta < 2$, $\lambda \in \mathbb{R}$, ${}^{RL}D_{0^+}^{\beta}$ is the Riemann-Liouville fractional derivative of order β,
- (1.A2) m is a positive integer, $0 = t_0 < t_1 < t_2 < \dots < t_m < t_{m+1} = 1$, $\mathbb{N}[a,b] = \{a, a+1, a+2, \dots, a+n\}$ with a, b being integers and $a \le b$,
- (1.A3) $\phi, \psi : (0,1) \to \mathbb{R}$ are measurable functions,
- (1.A4) $p : (0,1) \to \mathbb{R}$ is continuous and there exist numbers $k > -1$ and $l \in \max\{-\beta, -2-k, 0]$ such that $|p(t)| \le t^k(1-t)^l$ for all $t \in (0,1)$,
- (1.A5) f, G, H defined on $(0,1] \times \mathbb{R}$ are impulsive II-Carathéodory functions, $I, J : \{t_i : i \in \mathbb{N}[1,m]\} \times \mathbb{R} \to \mathbb{R}$ is a discrete II-Carathéodory functions.

The second class is the impulsive mixed type integral boundary value problem

$$\begin{gathered}
{}^{C}D_{0^+}^{\beta}x(t)-\lambda x(t)=p(t)f(t,x(t)),\quad \text{a.e., } t\in(t_i,t_{i+1}],\ i\in\mathbb{N}[0,m],\\
\lim_{t\to 0^+}x(t)=\int_0^1\phi(s)G(s,x(s))ds,\quad x'(1)=\int_0^1\psi(s)H(s,x(s))ds,\\
\Delta x(t_i)=I(t_i,x(t_i)),\quad \Delta x'(t_i)=J(t_i,x(t_i)),\quad i\in\mathbb{N}[1,m],
\end{gathered}\tag{1.8}$$

where

(1.A6) $1<\beta<2$, $\lambda\in\mathbb{R}$, ${}^{C}D_{0^+}^{\beta}$ is the Caputo fractional derivative of order β, $m, t_i, \mathbb{N}[a,b]$ satisfies (1.A2), $\phi,\psi:(0,1)\to\mathbb{R}$ satisfy (1.A3),

(1.A7) $p:(0,1)\to\mathbb{R}$ is continuous and there exist numbers $k>1-\beta$ and $l\in\max\{-\beta,-\beta-k,0]$ such that $|p(t)|\le t^k(1-t)^l$ for all $t\in(0,1)$,

(1.A8) f,G,H defined on $(0,1]\times\mathbb{R}$ are impulsive I-Carathéodory functions, $I,J:\{t_i: i\in\mathbb{N}[1,m]\}\times\mathbb{R}\to\mathbb{R}$ are discrete I-Carathéodory functions.

We emphasize that much work on fractional boundary value problems involves either Riemann-Liouville or Caputo type fractional differential equations see [8, 9, 10, 6]. Another kind of fractional derivatives that appears side by side to Riemann-Liouville and Caputo derivatives in the literature is the fractional derivative due to Hadamard introduced in 1892 [48], which differs from the preceding ones in the sense that the kernel of the integral (in the definition of Hadamard derivative) contains logarithmic function of arbitrary exponent. Recent studies can be seen in [33, 34, 35].

Thirdly we study the following impulsive periodic type integral boundary value problems of singular fractional differential systems

$$\begin{gathered}
{}^{RLH}D_{1^+}^{\beta}x(t)-\lambda x(t)=p(t)f(t,x(t)),\quad \text{a.e., } t\in(t_i,t_{i+1}],\ i\in\mathbb{N}[0,m],\\
\lim_{t\to 1^+}(\log t)^{2-\beta}x(t)-x(e)=\int_1^e\phi(s)G(s,x(s))ds,\\
\lim_{t\to 1^+}{}^{RLH}D_{1^+}^{\beta-1}x(t)-{}^{RLH}D_{1^+}^{\beta-1}x(e)=\int_1^e\psi(s)H(s,x(s))ds,\\
\lim_{t\to t_i^+}(\log t-\log t_i)^{2-\beta}x(t)=I(t_i,x(t_i)),\quad \Delta^{RLH}D_{1^+}^{\beta-1}x(t_i)=J(t_i,x(t_i)),
\end{gathered}\tag{1.9}$$

for $i\in\mathbb{N}[1,m]$, where

(1.A9) $1<\beta<2$, $\lambda\in\mathbb{R}$, ${}^{RLH}D_{1^+}^{\beta}$ is the Hadamard fractional derivative of order β,

(1.A10) m is a positive integer, $1=t_0<t_1<t_2<\dots<t_m<t_{m+1}=e$, $\phi,\psi:(1,e)\to\mathbb{R}$ are measurable functions, $p:(1,e)\to\mathbb{R}$ is continuous and satisfies $|p(t)|\le(\log t)^k(1-\log t)^l)$ with $k>-1$, $l\le 0$, $2+k+l>0$, $\mathbb{N}[a,b]=\{a,a+1,a+2,\dots,a+n\}$ with a,b being integers and $a\le b$,

(1.A11) f,G,H defined on $(1,e]\times\mathbb{R}$ are impulsive III-Carathéodory functions, $I,J:\{t_i: i\in\mathbb{N}[1,m]\}\times\mathbb{R}\to\mathbb{R}$ are discrete III-Carathéodory functions.

Finally we study the following impulsive Neumann type integral boundary value problems of singular fractional differential systems

$$
\begin{gathered}
{}^{CH}D_{1^+}^{\beta}x(t)-\lambda x(t)=p(t)f(t,x(t)), \quad \text{a.e., } t\in(t_i,t_{i+1}], \quad i\in\mathbb{N}[0,m],\\
(t\frac{d}{dt})x(t)\big|_{t=1}=\int_1^e\phi(s)G(s,x(s))ds,\\
(t\frac{d}{dt})x(t)\big|_{t=e}=\int_1^e\psi(s)H(s,x(s))ds,\\
\lim_{t\to t_i^+}x(t)-x(t_i)=I(t_i,x(t_i)),\\
\lim_{t\to t_i^+}(t\frac{d}{dt})x(t)-(t\frac{d}{dt})x(t)\big|_{t=t_i}=J(t_i,x(t_i)),
\end{gathered}
\tag{1.10}
$$

for $i\in\mathbb{N}[1,m]$, where

- (1.A12) $1<\beta<2$, $\lambda\in\mathbb{R}$, ${}^{CH}D_{1^+}^{\beta}$ is the Caputo type Hadamard fractional derivative of order β, $(t\frac{d}{dt})^1x(t)=tx'(t)$,
- (1.A13) $m,t_i,\mathbb{N}[a,b]$ satisfy (1.A10), $\phi,\psi:(1,e)\to\mathbb{R}$ are measurable functions, $p:(1,e)\to\mathbb{R}$ is continuous and satisfies $|p(t)|\le(\log t)^k(1-\log t)^l)$ with $k>-1$, $l\le 0$, $\beta+k+l>0$,
- (1.A14) f,G,H defined on $(1,e]\times\mathbb{R}$ are impulsive I-Carathéodory functions, $I,J:\{t_i:i\in\mathbb{N}[1,m]\}\times\mathbb{R}\to\mathbb{R}$ are discrete I-Carathéodory functions.

A function $x:(0,1]\to\mathbb{R}$ is called a solution of BVP (1.7) (or of BVP (1.8)) if $x|_{(t_i,t_{i+1}]}(i=0,1,j\in\mathbb{N}[0,m])$ is continuous, the limits below exist

$$
\lim_{t\to t_i^+}(t-t_i)^{2-\beta}x(t), i\in\mathbb{N}[0,m], \quad (\text{or} \quad \lim_{t\to t_i^+}x(t)\ i\in\mathbb{N}[0,m])
$$

and x satisfies (1.7) (or (1.8)).

A function $x:(1,e]\to\mathbb{R}$ is called a solution of BVP (1.9) (or of BVP (1.10)) if $x|_{(t_i,t_{i+1}]}(i\in\mathbb{N}[0,m])$ is continuous, the limits below exist

$$
\lim_{t\to t_i^+}(\log\frac{t}{t_i})^{2-\beta}x(t), i\in\mathbb{N}[0,m], \quad (\text{or } \lim_{t\to t_i^+}x(t), i\in\mathbb{N}[0,m])
$$

and x satisfies (1.9) (or (1.10)).

To obtain solutions of a boundary value problem of fractional differential equations, we firstly define a Banach space X, then we transform the boundary value problem into a integral equation and define a nonlinear operator T on X by using the integral equation obtained, finally, we prove that T has fixed point in X. The fixed points are just solutions of the boundary value problem. Three difficulties occur in known papers: one is how to transform the boundary value problem into a integral equation; the other one is how to define and prove a Banach space and the completely continuous property of the nonlinear operator defined; the third one is to choose a suitable fixed point theorem and impose suitable growth conditions on functions to get the fixed points of the operator.

To the best of the authors knowledge, no one has studied the existence of strong weak or weak solutions of BVPs (1.7)–(1.10). This paper fills this gap. Another purpose of this paper is to illustrate the similarity and difference of these three kinds of fractional differential equations. We obtain results on the existence of at least one solution for BVPs (1.7)–(1.10). For simplicity we only consider the left-sided operators here. The right-sided operators can be treated similarly. For clarity and brevity, we restrict our attention to BVPs with one impulse, the difference between the theory of one or an arbitrary number of impulses is quite similar.

The remainder of this paper is organized as follows: in Section 2, we present related definitions. In Section 3 some preliminary results are given (one purpose is to establish existence and uniqueness of continuous solutions of linear fractional differential equations (Subsection 3.1), the second purpose is to get exact expression of piecewise continuous solutions of the linear fractional differential equations with a constant coefficient and a variable force term (Subsection 3.2), the third purpose is to prove preliminary results for establishing existence results of solutions of (1.7)–(1.10)in Subsections 3.3, 3.4, 3.5 and 3.6, respectively), we transform them into corresponding integral equations and define completely continuous nonlinear operators. In Sections 4, the main theorems and their proof are given (we establish existence results for solutions of BVP (1.7)–(1.10). In Section 5, we preset applications of theorems obtained in Subsection 3.2, the solvability of multi-point boundary value problem, Sturm-Liouville boundary value problem and anti-periodic boundary value problem for fractional differential equations with impulse effects are discussed, respectively. In Section 6, some mistakes happened in cited papers are showed. Corrected expressions of solutions are given. Finally, in Section 7, we survey some examples and applications of fractional differential equations in various fields: population dynamics, control theory, physics, biology, medicine.

2. Related definitions

For convenience of the readers, we firstly present the necessary definitions from the fractional calculus theory. These definitions and results can be found in [58, 88, 94].

Let the Gamma function, Beta function and the classical Mittag-Leffler special function be

$$\Gamma(\alpha)=\int_0^{+\infty}x^{\alpha-1}e^{-x}dx,\quad \mathbf{B}(p,q)=\int_0^1 x^{p-1}(1-x)^{q-1}dx,$$

$$\mathbf{E}_{\delta,\sigma}(x)=\sum_{k=0}^{+\infty}\frac{x^k}{\Gamma(\delta k+\sigma)}$$

respectively for $\alpha>0$, $p>0$, $q>0$, $\delta>0$, $\sigma>0$. We note that $\mathbf{E}_{\delta,\delta}(x)>0$ for all $x\in\mathbb{R}$ and $\mathbf{E}_{\delta,\delta}(x)$ is strictly increasing in x. Then for $x>0$ we have $\mathbf{E}_{\delta,\sigma}(-x)<\mathbf{E}_{\delta,\sigma}(0)=\frac{1}{\Gamma(\sigma)}<\mathbf{E}_{\delta,\sigma}(x)$.

Definition 2.1 ([58]). Let $c\in\mathbb{R}$. The Riemann-Liouville fractional integral of order $\alpha>0$ of a function $g:(c,\infty)\to\mathbb{R}$ is

$$I_{c^+}^{\alpha}g(t)=\frac{1}{\Gamma(\alpha)}\int_c^t(t-s)^{\alpha-1}g(s)ds,$$

provided that the right-hand side exists.

Definition 2.2 ([58]). Let $c\in\mathbb{R}$. The Riemann-Liouville fractional derivative of order $\alpha>0$ of a function $g:(c,+\infty)\to\mathbb{R}$ is

$${}^{RL}D_{c^+}^{\alpha}g(t)=\frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_c^t\frac{g(s)}{(t-s)^{\alpha-n+1}}ds,$$

where $\alpha<n<\alpha+1$, i.e., $n=\lceil\alpha\rceil$, provided that the right-hand side exists.

Definition 2.3 ([58])**.** Let $c \in \mathbb{R}$. The Caputo fractional derivative of order $\alpha > 0$ of a function $g : (c, +\infty) \to \mathbb{R}$ is

$$^{C}D_{c^+}^{\alpha} g(t) = \frac{1}{\Gamma(n-\alpha)} \int_c^t \frac{g^{(n)}(s)}{(t-s)^{\alpha-n+1}} ds,$$

where $\alpha < n < \alpha + 1$, i.e., $n = \lceil \alpha \rceil$, provided that the right-hand side exists.

Definition 2.4 ([58])**.** Let $c > 0$. The Hadamard fractional integral of order $\alpha > 0$ of a function $g : [c, +\infty) \to \mathbb{R}$ is

$$^{H}I_{c^+}^{\alpha} g(t) = \frac{1}{\Gamma(\alpha)} \int_c^t (\log \frac{t}{s})^{\alpha-1} g(s) \frac{ds}{s},$$

provided that the right-hand side exists.

Definition 2.5 ([58])**.** Let $c > 0$. The Hadamard fractional derivative of order $\alpha > 0$ of a function $g : [c, +\infty) \to \mathbb{R}$ is

$$^{RLH}D_{c^+}^{\alpha} g(t) = \frac{1}{\Gamma(n-\alpha)} (t\frac{d}{dt})^n \int_c^t (\log \frac{t}{s})^{n-\alpha-1} g(s) \frac{ds}{s},$$

where $\alpha < n < \alpha + 1$, i.e., $n = \lceil \alpha \rceil$, provided that the right-hand side exists.

Definition 2.6 ([53])**.** Let $c > 0$. The Caputo type Hadamard fractional derivative of order $\alpha > 0$ of a function $g : [c, +\infty) \to \mathbb{R}$ is

$$^{CH}D_{c^+}^{\alpha} g(t) = \frac{1}{\Gamma(n-\alpha)} \int_c^t (\log \frac{t}{s})^{n-\alpha-1} (s\frac{d}{ds})^n g(s) \frac{ds}{s},$$

where $\alpha < n \le \alpha + 1$, i.e., $n = \lceil \alpha \rceil$, provided that the right-hand side exists.

Definition 2.7. We call $F : \cup_{i=0}^m (t_i, t_{i+1}) \times \mathbb{R} \to \mathbb{R}$ an *impulsive I-Carathéodory function* if it satisfies

(i) $t \to F(t, u)$ is measurable on (t_i, t_{i+1}) $(i \in \mathbb{N}[0, m])$ for any $u \in \mathbb{R}$,
(ii) $u \to F(t, u)$ are continuous on $\mathbb{R}$ for almost all $t \in (t_i, t_{i+1})$ $(i \in \mathbb{N}[0, m])$,
(iii) for each $r > 0$ there exists $M_r > 0$ such that

$$|F(t, u)| \le M_r, t \in (t_i, t_{i+1}), |u| \le r, \quad i \in \mathbb{N}[0, m].$$

Definition 2.8. We call $F : \cup_{i=0}^m (t_i, t_{i+1}) \times \mathbb{R} \to \mathbb{R}$ an *impulsive II-Carathéodory function* if it satisfies

(i) $t \to F(t, (t-t_i)^{\beta-2}u)$ is measurable on (t_i, t_{i+1}) $(i \in \mathbb{N}[0, m])$ for any $u \in \mathbb{R}$,
(ii) $u \to F(t, (t-t_i)^{\beta-2}u)$ are continuous on $\mathbb{R}$ for almost all $t \in (t_i, t_{i+1})$ $(i \in \mathbb{N}[0, m])$,
(iii) for each $r > 0$ there exists $M_r > 0$ such that

$$|F(t, (t-t_i)^{\beta-2}u)| \le M_r, t \in (t_i, t_{i+1}), |u| \le r, \quad i \in \mathbb{N}[0, m].$$

Definition 2.9. We call $F : \cup_{i=0}^m (t_i, t_{i+1}) \times \mathbb{R} \to \mathbb{R}$ an *impulsive III-Carathéodory function* if it satisfies

(i) $t \to F(t, (\log \frac{t}{t_i})^{\beta-2}u)$ is measurable on (t_i, t_{i+1}) $(i \in \mathbb{N}[0, m])$ for any $u \in \mathbb{R}$,
(ii) $u \to F(t, (\log \frac{t}{t_i})^{\beta-2}u)$ are continuous on $\mathbb{R}$ for all $t \in (t_i, t_{i+1})$ $(i \in \mathbb{N}[0, m])$,
(iii) for each $r > 0$ there exists $M_r > 0$ such that

$$|F(t, (\log \frac{t}{t_i})^{\beta-2}u)| \le M_r, t \in (t_i, t_{i+1}), |u| \le r, \quad i \in \mathbb{N}[0, m].$$

Definition 2.10. We call $I : \{t_i : i \in \mathbb{N}[1,m]\} \times \mathbb{R} \to \mathbb{R}$ a *discrete I-Carathéodory function* if it satisfies

- (i) $u \to I(t_i, u)$ $(i \in \mathbb{N}[1,m])$ are continuous on $\mathbb{R}$,
- (ii) for each $r > 0$ there exists $M_r > 0$ such that $|I(t_i,u)| \le M_r, |u| \le r$ for $i \in \mathbb{N}[1,m]$.

Definition 2.11. We call $I : \{t_i : i \in \mathbb{N}[1,m]\} \times \mathbb{R} \to \mathbb{R}$ a *discrete II-Carathéodory function* if it satisfies

- (i) $u \to I(t_i, (t_i - t_{i-1})^{\beta-2}u)$ $(i \in \mathbb{N}[1,m])$ are continuous on $\mathbb{R}$,
- (ii) for each $r > 0$ there exists $M_r > 0$ such that $|I(t_i, (t_i - t_{i-1}^{\beta-2}u)| \le M_r, |u| \le r$ for $i \in \mathbb{N}[1,m]$.

Definition 2.12. We call $I : \{t_i : i \in \mathbb{N}[1,m]\} \times \mathbb{R} \to \mathbb{R}$ a *discrete III-Carathéodory function* if it satisfies

- (i) $u \to I(t_i, (\log t_i - \log t_{i-1})^{\beta-n}u)$ $(i \in \mathbb{N}[1,m])$ are continuous on $\mathbb{R}$,
- (ii) for each $r > 0$ there exists $M_r > 0$ such that $|I(t_1, (\log \frac{t_i}{t_{i-1}})^{\beta-n}u)| \le M_r, |u| \le r$ for $i \in \mathbb{N}[1,m]$.

Definition 2.13 ([79])**.** Let E and F be Banach spaces. A operator $T : E \to F$ is called a completely continuous operator if T is continuous and maps any bounded set into relatively compact set.

Suppose that $n-1 \le \alpha < n$. The following Banach spaces are used:

Let $a < b$ be constants. $C(a,b]$ denotes the set of continuous functions on $(a,b]$ with $\lim_{t\to a^+} x(t)$ existing, and the norm

$$\|x\| = \sup_{t\in(a,b]} |x(t)|.$$

Let $a < b$ be constants. $C_{n-\alpha}(a,b]$ the set of continuous functions on $(a,b]$ with $\lim_{t\to a^+}(t-a)^{n-\alpha}x(t)$ existing, the norm $\|x\|_{n-\alpha} = \sup_{t\in(a,b]}(t-a)^{n-\alpha}|x(t)|$.

Let $0 < a < b$. $LC_{n-\alpha}(a,b]$ denote the set of all continuous functions on $(a,b]$ with the limit $\lim_{t\to a^+}(\log \frac{t}{a})^{n-\alpha}x(t)$ existing, and the norm

$$\|x\| = \sup_{t\in(a,b]} (\log \frac{t}{a})^{n-\alpha}|x(t)|.$$

For a positive integer m let $\mathbb{N}[0,m] = \{0,1,2,\dots,m\}$, with $0 = t_0 < t_1 < \dots < t_m < t_{m+1} = 1$. The following Banach spaces are also used in this paper:

$$P_mC_{n-\alpha}(0,1] = \{x : (0,1] \to \mathbb{R} : x\big|_{(t_i,t_{i+1}]} \in C_{n-\alpha}(t_i,t_{i+1}] : i \in \mathbb{N}[0,m]\}$$

with the norm

$$\|x\| = \|x\|_{P_mC_{n-\alpha}} = \max\Big\{\sup_{t\in(t_i,t_{i+1}]} (t-t_i)^{n-\alpha}|x(t)| : i \in \mathbb{N}[0,m]\Big\}.$$

$$P_mC(0,1] = \{x : (0,1] \to \mathbb{R} : x|_{(t_i,t_{i+1}]} \in C(t_i,t_{i+1}] : i \in \mathbb{N}[0,m]\}$$

with the norm

$$\|x\| = \|x\|_{P_mC(0,1]} = \max\Big\{\sup_{t\in(t_i,t_{i+1}]} |x(t)| : i \in \mathbb{N}[0.m]\Big\}.$$

For a positive integer m let $\mathbb{N}[0,m] = \{0,1,2,\dots,m\}$, with $1 = t_0 < t_1 < \dots < t_m < t_{m+1} = e$. We also use the Banach spaces

$$LP_mC_{n-\alpha}(1,e] = \Big\{x : (1,e] \to \mathbb{R} : x\big|_{(t_i,t_{i+1}]} \in C(t_i,t_{i+1}],\ i \in \mathbb{N}[0,m],$$

$$\text{and } \lim_{t\to t_i^+}(\log\frac{t}{t_i})^{n-\alpha}x(t) \text{ exist for } i\in\mathbb{N}[0,m]\Big\}$$

with the norm

$$\|x\|=\|x\|_{LP_mC_{n-\alpha}}=\max\Big\{\sup_{t\in(t_i,t_{i+1}]}(\log\frac{t}{t_i})^{n-\alpha}|x(t)|, i\in\mathbb{N}[0,m]\Big\}.$$

$$P_mC(1,e]=\big\{x:(1,e]\to\mathbb{R}:x\big|_{(t_i,t_{i+1}]}\in C(t_i,t_{i+1}],\ i\in\mathbb{N}[0.m]\big\}$$

with the norm

$$\|x\|=\|x\|_{P_mC}=\max\Big\{\sup_{t\in(t_i,t_{i+1}]}|x(t)|, i\in\mathbb{N}[0,m]\Big\}.$$

3. Preliminaries

In this section, we present some preliminary results that can be used in next sections for obtain solutions of (1.7)–(1.10).

3.1. **Basic theory for linear fractional differential equations.** Lakshmikantham et al. [61, 62, 63, 59] investigated the basic theory of initial value problems for fractional differential equations involving Riemann-Liouville differential operators of order $q\in(0,1)$. The existence and uniqueness of solutions of the following initial value problems of fractional differential equations were discussed under the assumption that $f\in C_r[0,1]$. We will establish existence and uniqueness results for these problems under more weaker assumptions see (3.A1)–(3.A4) below.

Suppose that $n-1<\alpha<n$ and $\eta_j\in\mathbb{R}(j\in N[0,n-1])$, $F,A:(0,1)\to\mathbb{R}$ and $B,G:(1,e)\to\mathbb{R}$ are continuous functions. We consider the following four classes of initial value problems of non-homogeneous linear fractional differential equations:

$$\begin{gathered}{}^{C}D_{0^+}^{\alpha}x(t)=A(t)x(t)+F(t),\quad \text{a.e. } t\in(0,1),\\ \lim_{t\to0^+}x^{(j)}(t)=\eta_j,\quad j\in\mathbb{N}[0,n-1],\end{gathered}\tag{3.1}$$

$$\begin{gathered}{}^{RL}D_{0^+}^{\alpha}x(t)=A(t)x(t)+F(t),\quad \text{a.e. } t\in(0,1),\\ \lim_{t\to0^+}t^{n-\alpha}x(t)=\frac{\eta_n}{\Gamma(\alpha-n+1)},\\ \lim_{t\to0^+}{}^{RL}D_{0^+}^{\alpha-j}x(t)=\eta_j,\quad j\in\mathbb{N}[1,n-1],\end{gathered}\tag{3.2}$$

$$\begin{gathered}{}^{RLH}D_{0^+}^{\alpha}x(t)=B(t)x(t)+G(t),\quad \text{a.e. } t\in(1,e),\\ \lim_{t\to1^+}(\log t)^{n-\alpha}x(t)=\frac{\eta_n}{\Gamma(\alpha-n+1)},\\ \lim_{t\to1^+}{}^{RLH}D_{1^+}^{\alpha-j}x(t)=\eta_j,\quad j\in\mathbb{N}[1,n-1],\end{gathered}\tag{3.3}$$

$$\begin{gathered}{}^{CH}D_{0^+}^{\alpha}x(t)=B(t)x(t)+G(t),\quad \text{a.e. } t\in(1,e),\\ \lim_{t\to1^+}(t\frac{d}{dt})^jx(t)=\eta_j, j\in\mathbb{N}[0,n-1].\end{gathered}\tag{3.4}$$

where $(t\frac{d}{dt})^jx(t)=t\frac{d(t\frac{d}{dt})^{j-1}x(t)}{dt}$ for $j=2,3,\dots$.

To obtain solutions of (3.1), we need the following assumptions:

- (3.A1) there exist constants $k_i>-\alpha+n-1$, $l_i\le0$ with $l_i>\max\{-\alpha,-\alpha-k_i\}(i=1,2)$, $M_A\ge0$ and $M_F\ge0$ such that $|A(t)|\le M_At^{k_1}(1-t)^{l_1}$ and $|F(t)|\le M_Ft^{k_2}(1-t)^{l_2}$ for all $t\in(0,1)$.

Choose the Picard function sequence as

$$\phi_0(t)=\sum_{j=0}^{n-1}\frac{\eta_j t^j}{j!},\quad t\in[0,1],$$

$$\phi_i(t)=\sum_{j=0}^{n-1}\frac{\eta_j t^j}{j!}+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[A(s)\phi_{i-1}(s)+F(s)]ds,\quad t\in(0,1],\ i=1,2,\dots.$$

Claim 1. $\phi_i\in C[0,1]$.

Proof. One sees $\phi_0\in C[0,1]$. Then

$$\begin{aligned}
&|\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[A(s)\phi_0(s)+F(s)]ds|\\
&\le\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[M_A|\phi_0(s)|s^{k_1}(1-s)^{l_1}+M_Fs^{k_2}(1-s)^{l_2}]ds\\
&\le M_A\|\phi_0\|\int_0^t\frac{(t-s)^{\alpha+l_1-1}}{\Gamma(\alpha)}s^{k_1}ds+M_F\int_0^t\frac{(t-s)^{\alpha+l_2-1}}{\Gamma(\alpha)}s^{k_2}ds\\
&=M_A\|\phi_0\|t^{\alpha+k_1+l_1}\int_0^1\frac{(1-w)^{\alpha+l_1-1}}{\Gamma(\alpha)}w^{k_1}dw\\
&\quad+M_Ft^{\alpha+k_2+l_2}\int_0^1\frac{(1-w)^{\alpha+l_2-1}}{\Gamma(\alpha)}w^{k_2}dw\\
&=M_A\|\phi_0\|t^{\alpha+k_1+l_1}\frac{\mathbf{B}(\alpha+l_1,k_1+1)}{\Gamma(\alpha)}+M_Ft^{\alpha+k_2+l_2}\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}\\
&\to 0\quad\text{as } t\to 0^+.
\end{aligned}$$

It follows that ϕ_1 is continuous on $(0,1]$ and $\lim_{t\to0^+}\phi_1(t)$ exists. So $\phi_1\in C[0,1]$. By mathematical induction, we can prove that $\phi_i\in C[0,1]$. □

Claim 2. $\{\phi_i\}$ is convergent uniformly on $[0,1]$.

Proof. For $t\in[0,1]$ we have

$$\begin{aligned}
&|\phi_1(t)-\phi_0(t)|\\
&=\Big|\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[A(s)\phi_0(s)+F(s)]ds\Big|\\
&\le M_A\|\phi_0\|\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}s^{k_1}(1-s)^{l_1}ds+M_F\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}s^{k_2}(1-s)^{l_2}ds\\
&\le M_A\|\phi_0\|\int_0^t\frac{(t-s)^{\alpha+l_1-1}}{\Gamma(\alpha)}s^{k_1}ds+M_F\int_0^t\frac{(t-s)^{\alpha+l_2-1}}{\Gamma(\alpha)}s^{k_2}ds\\
&=M_A\|\phi_0\|t^{\alpha+k_1+l_1}\frac{\mathbf{B}(\alpha+l_1,k_1+1)}{\Gamma(\alpha)}+M_Ft^{\alpha+k_2+l_2}\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}.
\end{aligned}$$

So

$$\begin{aligned}
&|\phi_2(t)-\phi_1(t)|\\
&=|\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}A(s)[\phi_1(s)-\phi_0(s)]ds|
\end{aligned}$$

$$
\begin{aligned}
&\le \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} M_A s^{k_1}(1-s)^{l_1}(M_A\|\phi_0\| s^{\alpha+k_1+l_1}\frac{\mathbf{B}(\alpha+l_1,k_1+1)}{\Gamma(\alpha)}\\
&\quad + M_F s^{\alpha+k_2+l_2}\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)})ds\\
&\le \|\phi_0\|M_A^2\int_0^t \frac{(t-s)^{\alpha+l_1-1}}{\Gamma(\alpha)} s^{\alpha+2k_1+l_1}\frac{\mathbf{B}(\alpha+l_1,k_1+1)}{\Gamma(\alpha)}ds\\
&\quad + M_A M_F\int_0^t \frac{(t-s)^{\alpha+l_1-1}}{\Gamma(\alpha)} s^{\alpha+k_1+k_2+l_2}\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}ds\\
&= \|\phi_0\|M_A^2 t^{2\alpha+2k_1+2l_1}\frac{\mathbf{B}(\alpha+l_1,k_1+1)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_1,\alpha+2k_1+l_1+1)}{\Gamma(\alpha)}\\
&\quad + M_A M_F t^{2\alpha+k_1+k_2+l_1+l_2}\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_1,\alpha+k_1+k_2+l_2+1)}{\Gamma(\alpha)}.
\end{aligned}
$$

Now suppose that

$$
\begin{aligned}
&|\phi_j(t)-\phi_{j-1}(t)|\\
&\le \|\phi_0\|M_A^j t^{j\alpha+jk_1+jl_1}\prod_{i=0}^{j-1}\frac{\mathbf{B}(\alpha+l_1,i\alpha+(i+1)k_1+il_1+1)}{\Gamma(\alpha)}\\
&\quad + M_A^{j-1}M_F t^{j\alpha+(j-1)k_1+k_2+(j-1)l_1+l_2}\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}\\
&\quad \times\prod_{i=1}^{j-1}\frac{\mathbf{B}(\alpha+l_1,i\alpha+ik_1+k_2+(i-1)l_1+l_2+1)}{\Gamma(\alpha)}.
\end{aligned}
$$

Then we have

$$
\begin{aligned}
&|\phi_{j+1}(t)-\phi_j(t)|\\
&= \Big|\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}A(s)[\phi_j(s)-\phi_{j-1}(s)]ds\Big|\\
&\le \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}M_A\Big(\|\phi_0\|M_A^j s^{j\alpha+jk_1+jl_1}\prod_{i=0}^{j-1}\frac{\mathbf{B}(\alpha+l_1,i\alpha+(i+1)k_1+il_1+1)}{\Gamma(\alpha)}\\
&\quad + M_A^{j-1}M_F s^{j\alpha+(j-1)k_1+k_2+(j-1)l_1+l_2}\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}\\
&\quad \times\prod_{i=1}^{j-1}\frac{\mathbf{B}(\alpha+l_1,i\alpha+ik_1+k_2+(i-1)l_1+l_2+1)}{\Gamma(\alpha)})s^{k_1}(1-s)^{l_1}ds\\
&\le \int_0^t \frac{(t-s)^{\alpha+l_1-1}}{\Gamma(\alpha)}M_A\Big(\|\phi_0\|M_A^j s^{j\alpha+jk_1+jl_1}\prod_{i=0}^{j-1}\frac{\mathbf{B}(\alpha+l_1,i\alpha+(i+1)k_1+il_1+1)}{\Gamma(\alpha)}\\
&\quad + M_A^{j-1}M_F s^{j\alpha+(j-1)k_1+k_2+(j-1)l_1+l_2}\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}\\
&\quad \times\prod_{i=1}^{j-1}\frac{\mathbf{B}(\alpha+l_1,i\alpha+ik_1+k_2+(i-1)l_1+l_2+1)}{\Gamma(\alpha)}\Big)s^{k_1}ds
\end{aligned}
$$

$$\begin{aligned}
&\le \|\phi_0\| M_A^{j+1} t^{(j+1)\alpha+(j+1)k_1+(j+1)l_1} \prod_{i=0}^{j} \frac{\mathbf{B}(\alpha+l_1, i\alpha+(i+1)k_1+il_1+1)}{\Gamma(\alpha)} \\
&\quad + M_A^j M_F t^{(j+1)\alpha+jk_1+k_2+jl_1+l_2} \frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)} \\
&\quad \times \prod_{i=1}^{j} \frac{\mathbf{B}(\alpha+l_1, i\alpha+ik_1+k_2+(i-1)l_1+l_2+1)}{\Gamma(\alpha)}.
\end{aligned}$$

Using mathematical induction, for every $i=1,2,\dots$ we obtain

$$\begin{aligned}
&|\phi_{i+1}(t)-\phi_i(t)| \\
&\le \|\phi_0\| M_A^{i+1} t^{(i+1)\alpha+(i+1)k_1+(i+1)l_1} \prod_{j=0}^{i} \frac{\mathbf{B}(\alpha+l_1, j\alpha+(j+1)k_1+jl_1+1)}{\Gamma(\alpha)} \\
&\quad + M_A^i M_F t^{(i+1)\alpha+ik_1+k_2+il_1+l_2} \frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)} \\
&\quad \times \prod_{j=1}^{i} \frac{\mathbf{B}(\alpha+l_1, j\alpha+jk_1+k_2+(j-1)l_1+l_2+1)}{\Gamma(\alpha)} \\
&\le \|\phi_0\| M_A^{i+1} \prod_{j=0}^{i} \frac{\mathbf{B}(\alpha+l_1, j\alpha+(j+1)k_1+jl_1+1)}{\Gamma(\alpha)} \\
&\quad + M_A^i M_F \frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)} \prod_{j=1}^{i} \frac{\mathbf{B}(\alpha+l_1, j\alpha+jk_1+k_2+(j-1)l_1+l_2+1)}{\Gamma(\alpha)},
\end{aligned}$$

for $t \in [0,1]$. Consider

$$\begin{aligned}
\sum_{i=1}^{+\infty} u_i &= \sum_{i=1}^{+\infty} \|\phi_0\| M_A^{i+1} \prod_{i=0}^{i} \frac{\mathbf{B}(\alpha+l_1, i\alpha+(i+1)k_1+il_1+1)}{\Gamma(\alpha)}, \\
\sum_{i=1}^{+\infty} v_i &= \sum_{i=1}^{+\infty} M_A^i M_F \frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)} \\
&\quad \times \prod_{j=1}^{i} \frac{\mathbf{B}(\alpha+l_1, j\alpha+jk_1+k_2+(j-1)l_1+l_2+1)}{\Gamma(\alpha)}.
\end{aligned}$$

One sees that for sufficiently large n with $\delta \in (0,\frac{1}{2})$,

$$\begin{aligned}
\frac{u_{i+1}}{u_i} &= M_A \frac{\mathbf{B}(\alpha+l_1,(i+1)\alpha+(i+1)k_1+(i+1)l_1)}{\Gamma(\alpha)} \\
&= M_A \int_0^1 (1-x)^{\alpha+l_1-1} x^{(i+1)\alpha+(i+1)k_1+(i+1)l_1} dx \\
&\le M_A \int_0^\delta (1-x)^{\alpha+l_1-1} x^{(i+1)\alpha+(i+1)k_1+(i+1)l_1} dx + M_A \int_\delta^1 (1-x)^{\alpha+l_1-1} dx \\
&\le M_A \int_0^\delta (1-x)^{\alpha+l_1-1} dx \delta^{(i+1)\alpha+(i+1)k_1+(i+1)l_1} + \frac{M_A}{\alpha+l_1} \delta^{\alpha+l_1} \\
&\le \frac{M_A}{\alpha+l_1} \delta^{(i+1)\alpha+(i+1)k_1+(i+1)l_1} + \frac{M_A}{\alpha+l_1} \delta^{\alpha+l_1}.
\end{aligned}$$

It is easy to see that for any $\epsilon > 0$ there exists $\delta \in (0, \frac{1}{2})$ such that $\frac{M_A}{\alpha+l_1}\delta^{\alpha+l_1} < \frac{\epsilon}{2}$. For this δ, there exists an integer $N > 0$ sufficiently large such that

$$\frac{M_A}{\alpha+l_1}\delta^{(i+1)\alpha+(i+1)k_1+(i+1)l_1} < \frac{\epsilon}{2}$$

for all $i > N$. So $0 < \frac{u_{i+1}}{u_i} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$ for all $i > N$. It follows that $\lim_{i\to+\infty} u_{i+1}/u_i = 0$. Then $\sum_{i=1}^{+\infty} u_i$ converges. Similarly we obtain $\sum_{i=1}^{+\infty} v_i$ converges. Hence

$$\phi_0(t) + [\phi_1(t) - \phi_0(t)] + [\phi_2(t) - \phi_1(t)] + \dots + [\phi_i(t) - \phi_{i-1}(t)] + \dots, \quad t \in [0,1]$$

is uniformly convergent. Then $\{\phi_i(t)\}$ is convergent uniformly on $[0,1]$. □

Claim 3. $\phi(t) = \lim_{i\to+\infty} \phi_i(t)$ defined on $[0,1]$ is a unique continuous solution of the integral equation

$$x(t) = \sum_{j=0}^{n-1} \frac{\eta_j}{j!} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}[A(s)x(s) + F(s)]ds, \quad t \in [0,1]. \tag{3.5}$$

Proof. From $\phi(t) = \lim_{i\to+\infty} \phi_i(t)$ and the uniformly convergence, we see that $\phi(t)$ is continuous on $[0,1]$. From

$$\begin{aligned}
&\Big| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[A(s)\phi_{p-1}(s) + F(s)]ds - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[A(s)\phi_{q-1}(s) + F(s)]ds \Big| \\
&\le M_A\|\phi_{p-1} - \phi_{q-1}\| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} s^{k_1}(1-s)^{l_1} ds \\
&\le M_A\|\phi_{p-1} - \phi_{q-1}\| t^{\alpha+k_1+l_1} \frac{\mathbf{B}(\alpha+l_1, k_1+1)}{\Gamma(\alpha)} \\
&\le M_A\|\phi_{p-1} - \phi_{q-1}\| \frac{\mathbf{B}(\alpha+l_1, k_1+1)}{\Gamma(\alpha)} \to 0 \quad \text{as } p, q \to +\infty,
\end{aligned}$$

we have

$$\begin{aligned}
\phi(t) &= \lim_{i\to+\infty} \phi_i(t) \\
&= \lim_{i\to+\infty} \Big[\sum_{j=0}^{n-1} \frac{\eta_j t^j}{j!} + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[A(s)\phi_{i-1}(s) + F(s)]ds \Big] \\
&= \sum_{j=0}^{n-1} \frac{\eta_j t^j}{j!} + \lim_{i\to+\infty} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[A(s)\phi_{i-1}(s) + F(s)]ds \\
&= \sum_{j=0}^{n-1} \frac{\eta_j t^j}{j!} + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\big[A(s) \lim_{i\to+\infty} \phi_{i-1}(s) + F(s)\big]ds \\
&= \sum_{j=0}^{n-1} \frac{\eta_j t^j}{j!} + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[A(s)\phi(s) + F(s)]ds.
\end{aligned}$$

Then ϕ is a continuous solution of (3.5) defined on $[0,1]$.

Suppose that ψ defined on $[0,1]$ is also a solution of (3.5). Then

$$\psi(t) = \sum_{j=0}^{n-1} \frac{\eta_j}{j!} + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[A(s)\psi(s) + F(s)]ds, \quad t \in (0,1].$$

We need to prove that $\phi(t)\equiv\psi(t)$ on $[0,1]$. Then

$$
\begin{aligned}
&|\psi(t)-\phi_0(t)|\\
&=|\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|A(s)\psi_0(s)+F(s)|ds|\\
&\le\|\phi_0\|M_At^{\alpha+k_1+l_1}\frac{\mathbf{B}(\alpha+l_1,k_1+1)}{\Gamma(\alpha)}+M_Ft^{\alpha+k_2+l_2}\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}.
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
&|\psi(t)-\phi_1(t)|\\
&=|\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}A(s)[\psi(s)-\phi_0(s)]ds|\\
&\le\|\phi_0\|M_A^2t^{2\alpha+2k_1+2l_1}\frac{\mathbf{B}(\alpha+l_1,k_1+1)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_1,\alpha+2k_1+l_1+1)}{\Gamma(\alpha)}\\
&\quad+M_AM_Ft^{2\alpha+k_1+k_2+l_1+l_2}\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_1,\alpha+k_1+k_2+l_2+1)}{\Gamma(\alpha)}.
\end{aligned}
$$

Now suppose that

$$
\begin{aligned}
|\psi(t)-\phi_{j-1}(t)|&\le\|\phi_0\|M_A^jt^{j\alpha+jk_1+jl_1}\prod_{i=0}^{j-1}\frac{\mathbf{B}(\alpha+l_1,i\alpha+(i+1)k_1+il_1+1)}{\Gamma(\alpha)}\\
&\quad+M_A^{j-1}M_Ft^{j\alpha+(j-1)k_1+k_2+(j-1)l_1+l_2}\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}\\
&\quad\times\prod_{i=1}^{j-1}\frac{\mathbf{B}(\alpha+l_1,i\alpha+ik_1+k_2+(i-1)l_1+l_2+1)}{\Gamma(\alpha)}.
\end{aligned}
$$

Then

$$
\begin{aligned}
&|\psi(t)-\phi_j(t)|\\
&=\Big|\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}A(s)[\psi(s)-\phi_{j-1}(s)]ds\Big|\\
&\le\|\phi_0\|M_A^{j+1}t^{(j+1)\alpha+(j+1)k_1+(j+1)l_1}\prod_{i=0}^{j}\frac{\mathbf{B}(\alpha+l_1,i\alpha+(i+1)k_1+il_1+1)}{\Gamma(\alpha)}\\
&\quad+M_A^jM_Ft^{(j+1)\alpha+jk_1+k_2+jl_1+l_2}\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}\\
&\quad\times\prod_{i=1}^{j}\frac{\mathbf{B}(\alpha+l_1,i\alpha+ik_1+k_2+(i-1)l_1+l_2+1)}{\Gamma(\alpha)}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&|\psi(t)-\phi_i(t)|\\
&\le\|\phi_0\|M_A^{i+1}t^{(i+1)\alpha+(i+1)k_1+(i+1)l_1}\prod_{j=0}^{i}\frac{\mathbf{B}(\alpha+l_1,j\alpha+(j+1)k_1+jl_1+1)}{\Gamma(\alpha)}\\
&\quad+M_A^iM_Ft^{(i+1)\alpha+ik_1+k_2+il_1+l_2}\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}
\end{aligned}
$$

$$\begin{aligned}
&\times \prod_{j=1}^{i} \frac{\mathbf{B}(\alpha+l_1, j\alpha+jk_1+k_2+(j-1)l_1+l_2+1)}{\Gamma(\alpha)}\\
&\le \|\phi_0\| M_A^{i+1} \prod_{j=0}^{i} \frac{\mathbf{B}(\alpha+l_1, j\alpha+(j+1)k_1+jl_1+1)}{\Gamma(\alpha)}\\
&\quad + M_A^i M_F \frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)} \prod_{j=1}^{i} \frac{\mathbf{B}(\alpha+l_1, j\alpha+jk_1+k_2+(j-1)l_1+l_2+1)}{\Gamma(\alpha)}
\end{aligned}$$

for $i = 1, 2, \dots$. Similarly we have

$$\begin{gathered}
\lim_{i\to+\infty} \|\phi_0\| M_A^{i+1} \prod_{j=0}^{i} \frac{\mathbf{B}(\alpha+l_1, j\alpha+(j+1)k_1+jl_1+1)}{\Gamma(\alpha)} = 0,\\
\lim_{i\to+\infty} M_A^i M_F \frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}\\
\prod_{j=1}^{i} \frac{\mathbf{B}(\alpha+l_1, j\alpha+jk_1+k_2+(j-1)l_1+l_2+1)}{\Gamma(\alpha)} = 0.
\end{gathered}$$

Then $\lim_{i\to+\infty} \phi_i(t) = \psi(t)$ uniformly on $[0,1]$. Then $\phi(t) \equiv \psi(t)$. Then (3.5) has a unique solution ϕ. The proof is complete. □

Theorem 3.1. *Suppose that* (3.A1) *holds. Then x is a solution of IVP* (3.1) *if and only if x is a solution of the integral equation* (3.5).

Proof. Suppose that x is a solution of (3.1). Then $\lim_{t\to0^+} x(t) = \eta$ and $\|x\| = r < +\infty$. From (3.A1), we have for $t \in (0,1)$

$$\begin{aligned}
&\Big|\int_0^t \frac{(t-s)^{\alpha-n}}{\Gamma(\alpha-n+1)}[A(s)x(s)+F(s)]ds\Big|\\
&\le \|x\| \int_0^t \frac{(t-s)^{\alpha-n}}{\Gamma(\alpha-n+1)}|A(s)|ds + \int_0^t \frac{(t-s)^{\alpha-n}}{\Gamma(\alpha-n+1)}|F(s)|ds\\
&\le \int_0^t \frac{(t-s)^{\alpha-n}}{\Gamma(\alpha-n+1)}[M_A r s^{k_1}(1-s)^{l_1} + M_F s^{k_2}(1-s)^{l_2}]ds\\
&\le \int_0^t \frac{(t-s)^{\alpha-n}}{\Gamma(\alpha-n+1)}[M_A r s^{k_1}(1-t)^{l_1} + M_F s^{k_2}(1-t)^{l_2}]ds \quad \text{by } \frac{s}{t} = w\\
&= M_A r(1-t)^{l_1} t^{\alpha+k_1-n+1} \int_0^1 \frac{(1-w)^{\alpha-n}}{\Gamma(\alpha-n+1)} w^{k_1}dw\\
&\quad + M_F(1-t)^{l_2} t^{\alpha+k_2-n+1} \int_0^1 \frac{(1-w)^{\alpha-n}}{\Gamma(\alpha-n+1)} w^{k_2}dw\\
&= M_A r(1-t)^{l_1} t^{\alpha+k_1-n+1} \frac{\mathbf{B}(\alpha-n+1,k_1+1)}{\Gamma(\alpha)}\\
&\quad + M_F(1-t)^{l_2} t^{\alpha+k_2-n+1} \frac{\mathbf{B}(\alpha-n+1,k_2+1)}{\Gamma(\alpha)}.
\end{aligned}$$

So $t \to \int_0^t \frac{(t-s)^{\alpha-n}}{\Gamma(\alpha-n+1)}[A(s)x(s)+F(s)]ds$ is defined on $(0,1)$. $k_i > -\alpha+n-1$ implies that

$$\lim_{t\to 0^+}\int_0^t \frac{(t-s)^{\alpha-n}}{\Gamma(\alpha-n+1)}A(s)x(s)ds = \lim_{t\to 0^+}\int_0^t \frac{(t-s)^{\alpha-n}}{\Gamma(\alpha-n+1)}F(s)ds = 0. \tag{3.6}$$

Furthermore, for $t_1, t_2 \in (0,1)$ with $t_1 < t_2$ we have

$$\begin{aligned}
&\Big|\int_0^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}[A(s)x(s)+F(s)]ds - \int_0^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}[A(s)x(s)+F(s)]ds\Big| \\
&\le \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}|A(s)x(s)+F(s)|ds \\
&\quad + \int_0^{t_1} \frac{|(t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}|}{\Gamma(\alpha)}|A(s)x(s)+F(s)|ds \\
&\le M_A r\Big[\int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}s^{k_1}(1-s)^{l_1}ds \\
&\quad + \int_0^{t_1} \frac{(t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}s^{k_1}(1-s)^{l_1}ds\Big] \\
&\quad + M_F\Big[\int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}s^{k_2}(1-s)^{l_2}ds \\
&\quad + \int_0^{t_1} \frac{(t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}s^{k_2}(1-s)^{l_2}ds\Big] \\
&\le M_A r\Big[\int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha+l_1-1}}{\Gamma(\alpha)}s^{k_1}ds \\
&\quad + \int_0^{t_1} \frac{(t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}s^{k_1}(t_2-s)^{l_1}ds\Big] \\
&\quad + M_F\Big[\int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha+l_2-1}}{\Gamma(\alpha)}s^{k_2}ds \\
&\quad + \int_0^{t_1} \frac{(t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}s^{k_2}(t_2-s)^{l_2}ds\Big] \\
&= M_A r\Big[t_2^{\alpha+k_1+l_1}\int_{\frac{t_1}{t_2}}^1 \frac{(1-w)^{\alpha+l_1-1}}{\Gamma(\alpha)}w^{k_1}dw \\
&\quad + \int_0^{t_1} \frac{(t_1-s)^{\alpha+l_1-1}}{\Gamma(\alpha)}s^{k_1}ds - \int_0^{t_1} \frac{(t_2-s)^{\alpha+l_1-1}}{\Gamma(\alpha)}s^{k_1}ds\Big] \\
&\quad + M_F\Big[t_2^{\alpha+k_2+l_2}\int_{\frac{t_1}{t_2}}^1 \frac{(1-w)^{\alpha+l_2-1}}{\Gamma(\alpha)}w^{k_2}dw \\
&\quad + \int_0^{t_1} \frac{(t_1-s)^{\alpha+l_2-1}}{\Gamma(\alpha)}s^{k_2}ds - \int_0^{t_1} \frac{(t_2-s)^{\alpha+l_2-1}}{\Gamma(\alpha)}s^{k_2}ds\Big] \\
&= M_A r\Big[t_2^{\alpha+k_1+l_1}\int_{\frac{t_1}{t_2}}^1 \frac{(1-w)^{\alpha+l_1-1}}{\Gamma(\alpha)}w^{k_1}dw + t_1^{\alpha+k_1+l_1}\int_0^1 \frac{(1-w)^{\alpha+l_1-1}}{\Gamma(\alpha)}w^{k_1}dw \\
&\quad - t_2^{\alpha+k_1+l_1}\int_0^{\frac{t_1}{t_2}} \frac{(1-w)^{\alpha+l_1-1}}{\Gamma(\alpha)}w^{k_1}dw\Big]
\end{aligned}$$

$$
\begin{aligned}
&+ M_F\Big[t_2^{\alpha+k_2+l_2}\int_{\frac{t_1}{t_2}}^1 \frac{(1-w)^{\alpha+l_2-1}}{\Gamma(\alpha)}w^{k_2}dw\\
&+ t_1^{\alpha+k_2+l_2}\int_0^1 \frac{(1-w)^{\alpha+l_2-1}}{\Gamma(\alpha)}w^{k_2}dw - t_2^{\alpha+k_2+l_2}\int_0^{\frac{t_1}{t_2}} \frac{(1-w)^{\alpha+l_2-1}}{\Gamma(\alpha)}w^{k_2}dw\Big]\\
&= M_A r\Big[t_2^{\alpha+k_1+l_1}\int_{\frac{t_1}{t_2}}^1 \frac{(1-w)^{\alpha+l_1-1}}{\Gamma(\alpha)}w^{k_1}dw\\
&+ |t_1^{\alpha+k_1+l_1} - t_2^{\alpha+k_1+l_1}|\frac{\mathbf{B}(\alpha+l_1,k_1+1)}{\Gamma(\alpha)} - t_2^{\alpha+k_1+l_1}\int_{\frac{t_1}{t_2}}^1 \frac{(1-w)^{\alpha+l_1-1}}{\Gamma(\alpha)}w^{k_1}dw\Big]\\
&+ M_F\Big[t_2^{\alpha+k_2+l_2}\int_{\frac{t_1}{t_2}}^1 \frac{(1-w)^{\alpha+l_2-1}}{\Gamma(\alpha)}w^{k_2}dw\\
&+ |t_1^{\alpha+k_2+l_2} - t_2^{\alpha+k_2+l_2}|\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)} - t_2^{\alpha+k_2+l_2}\int_{\frac{t_1}{t_2}}^1 \frac{(1-w)^{\alpha+l_2-1}}{\Gamma(\alpha)}w^{k_2}dw\Big]\\
&\to 0 \quad \text{as } t_1 \to t_2.
\end{aligned}
$$

So $t \to \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[A(s)x(s)+F(s)]ds$ is continuous on $(0,1]$, by defining

$$
\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[A(s)x(s)+F(s)]ds\Big|_{t=0} = \lim_{t\to 0^+}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[A(s)x(s)+F(s)]ds.
$$

We have $I_{0^+}^{\alpha}\,{}^CD_{0^+}^{\alpha}x(t) = I_{0^+}^{\alpha}[A(t)x(t)+F(t)]$. So

$$
\begin{aligned}
&\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[A(s)x(s)+F(s)]ds\\
&= I_{0^+}^{\alpha}[A(t)x(t)+F(t)] = I_{0^+}^{\alpha}\,{}^CD_{0^+}^{\alpha}x(t)\\
&= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\big(\frac{1}{\Gamma(n-\alpha)}\int_0^s (s-w)^{-\alpha}x^{(n)}(w)dw\big)ds\\
&\quad \text{(interchange the order of integration)}\\
&= \frac{1}{\Gamma(\alpha)\Gamma(n-\alpha)}\int_0^t\int_w^t (t-s)^{\alpha-1}(s-w)^{n-\alpha-1}ds x^{(n)}(w)dw \quad \text{using } \frac{s-w}{t-w} = u\\
&= \frac{1}{\Gamma(\alpha)\Gamma(n-\alpha)}\int_0^t (t-u)^{n-1}\int_0^1 (1-u)^{\alpha-1}u^{n-\alpha-1}du x^{(n)}(w)dw\\
&\quad \text{(using } \mathbf{B}(\alpha,1-\alpha) = \frac{\Gamma(\alpha)\Gamma(1-\alpha)}{\Gamma(1)}\text{)}\\
&= -\frac{1}{(n-1)!}\int_0^t (t-u)^{n-1}x^{(n)}(w)dw\\
&= \frac{1}{(n-1)!}\Big[(t-u)^{n-1}x^{(n-1)}(w)\big|_0^t + (n-1)\int_0^t (t-u)^{n-2}x^{(n-1)}(w)dw\Big]\\
&= -\frac{\eta_{n-1}}{(n-1)!} + \frac{1}{(n-2)!}\int_0^t (t-u)^{n-2}x^{(n-1)}(w)dw\\
&= \dots
\end{aligned}
$$

$$
= -\sum_{j=1}^{n-1}\frac{\eta_j}{j!} + \int_0^t x'(s)ds = x(t) - \sum_{j=0}^{n-1}\frac{\eta_j}{j!}.
$$

Then $x \in C(0,1]$ is a solution of (3.5).

On the other hand, if x is a solution of (3.5). From Cases 1, 2 and 3, we have $x \in C(0,1]$ and $\lim_{t\to 0^+} x^{(j)}(t) = \eta_j (j \in N[0,n-1])$. So $x \in C[0,1]$. Furthermore, from (3.6) we have

$$
\begin{aligned}
&{}^CD_{0^+}^{\alpha}x(t)\\
&= \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}x^{(n)}(s)ds\\
&= \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}\Big(\sum_{j=0}^{n-1}\frac{\eta_j s^j}{j!}\\
&\quad + \int_0^s \frac{(s-w)^{\alpha-1}}{\Gamma(\alpha)}[A(w)x(w)+F(w)]dw\Big)^{(n)}ds\\
&= \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}\Big(\int_0^s \frac{(s-w)^{\alpha-1}}{\Gamma(\alpha)}[A(w)x(w)+F(w)]dw\Big)^{(n)}ds\\
&= \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}\frac{1}{\Gamma(\alpha-n+1)}\\
&\quad \times \Big(\int_0^s (s-w)^{\alpha-n}[A(w)x(w)+F(w)]dw\Big)'ds\\
&= \frac{1}{\Gamma(\alpha-n+1)}\frac{1}{\Gamma(n+1-\alpha)}\Big[\int_0^t (t-s)^{n-\alpha}\\
&\quad \times \Big(\int_0^s (s-w)^{\alpha-n}[A(w)x(w)+F(w)]dw\Big)'ds\Big]'\\
&= \frac{1}{\Gamma(\alpha-n+1)}\frac{1}{\Gamma(n+1-\alpha)}\Big[(t-s)^{n-\alpha}\int_0^s (s-w)^{\alpha-n}[A(w)x(w)+F(w)]dw|_0^t\\
&\quad + (n-\alpha)\int_0^t (t-s)^{n-\alpha-1}\int_0^s (s-w)^{\alpha-n}[A(w)x(w)+F(w)]\,dw\,ds\Big]'\\
&= \frac{1}{\Gamma(\alpha-n+1)}\frac{1}{\Gamma(n-\alpha)}\Big[\int_0^t (t-s)^{n-\alpha-1}\\
&\quad \times \int_0^s (s-w)^{\alpha-n}[A(w)x(w)+F(w)]\,dw\,ds\Big]' \quad \text{by (3.6)}\\
&= \frac{1}{\Gamma(\alpha-n+1)}\frac{1}{\Gamma(n-\alpha)}\frac{1}{\Gamma(\alpha)}\Big[\int_0^t\int_w^t (t-s)^{n-\alpha-1}(s-w)^{\alpha-n}ds\\
&\quad \times [A(w)x(w)+F(w)]dw\Big]'\\
&\quad \text{by changing the order of integration}\\
&= \frac{1}{\Gamma(\alpha-n+1)}\frac{1}{\Gamma(n-\alpha)}\Big[\int_0^t\int_0^1 (1-u)^{n-\alpha-1}u^{\alpha-n}du[A(w)x(w)+F(w)]dw\Big]'\\
&\quad (\text{because } \frac{s-w}{t-w} = u)
\end{aligned}
$$

$$= \Big[\int_0^t [A(w)x(w)+F(w)]dw\Big]'$$

by using $\mathbf{B}(n-\alpha,\alpha-n+1)=\Gamma(n-\alpha)\Gamma(\alpha-n+1)=A(t)x(t)+F(t)$ in the last equality. So $x\in C[0,1]$ is a solution of (3.1). The proof is complete. □

Theorem 3.2. *Suppose that* (3.A1) *holds. Then* (3.1) *has a unique solution. If there exists constants $k_2>-\alpha+n-1$, $l_2\le 0$ with $l_2>\max\{-\alpha,-\alpha-k_2\}$, $M_F\ge 0$ such that $|F(t)|\le M_F t^{k_2}(1-t)^{l_2}$ for all $t\in(0,1)$, then the following problem*

$$\begin{gathered} {}^C D_{0^+}^{\alpha} x(t)=\lambda x(t)+F(t),\quad a.e.,\ t\in(0,1],\\ \lim_{t\to 0^+} x^{(j)}(t)=\eta_j,\quad j\in\mathbb{N}[0,n-1] \end{gathered} \tag{3.7}$$

has a unique solution

$$x(t)=\sum_{j=0}^{n-1}\eta_j\mathbf{E}_{\alpha,j+1}(\lambda t^{\alpha})t^j+\int_0^t (t-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t-s)^{\alpha})F(s)ds,\quad t\in(0,1]. \tag{3.8}$$

Proof. (i) From Claims 1, 2 and 3, Theorem 3.1 implies that (3.1) has a unique solution.

(ii) From the assumption and $A(t)\equiv\lambda$, it is easy to see that (3.A1) holds with $k_1=l_1=0$ and k_2, l_2 mentioned. Thus (3.7) has a unique solution. From the Picard function sequence we have

$$\begin{aligned}
&\phi_i(t)\\
&=\sum_{j=0}^{n-1}\frac{\eta_j t^j}{j!}+\lambda\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\phi_{i-1}(s)ds+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}F(s)ds\\
&=\sum_{j=0}^{n-1}\frac{\eta_j t^j}{j!}+\lambda\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\Big(\sum_{j=0}^{n-1}\frac{\eta_j s^j}{j!}+\lambda\int_0^s\frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}\phi_{i-2}(u)du\\
&\quad+\int_0^s\frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}F(u)du\Big)ds+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}F(s)ds\\
&=\sum_{j=0}^{n-1}\frac{\eta_j t^j}{j!}+\lambda\sum_{j=0}^{n-1}\frac{\eta_j}{\Gamma(\alpha)j!}\int_0^t(t-s)^{\alpha-1}s^j ds\\
&\quad+\lambda^2\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\int_0^s\frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}\phi_{i-2}(u)\,du\,ds\\
&\quad+\lambda\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\int_0^s\frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}F(u)\,du\,ds+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}F(s)ds\\
&=\sum_{j=0}^{n-1}\frac{\eta_j t^j}{j!}+\lambda\sum_{j=0}^{n-1}\frac{\eta_j}{\Gamma(\alpha)j!}t^{\alpha+j}\int_0^1(1-w)^{\alpha-1}w^j dw\\
&\quad+\frac{\lambda^2}{\Gamma(\alpha)^2}\int_0^t\int_u^t(t-s)^{\alpha-1}(s-u)^{\alpha-1}ds\phi_{i-2}(u)du\\
&\quad+\frac{\lambda}{\Gamma(\alpha)^2}\int_0^t\int_u^t(t-s)^{\alpha-1}(s-u)^{\alpha-1}dsF(u)du\\
&\quad+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}F(s)ds
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=0}^{n-1} \frac{\eta_j t^j}{j!} + \sum_{j=0}^{n-1} \frac{\lambda \eta_j t^{\alpha+j}}{\Gamma(\alpha+j+1)} + \frac{\lambda^2}{\Gamma(2\alpha)} \int_0^t (t-u)^{2\alpha-1} \phi_{i-2}(u) du \\
&\quad + \frac{\lambda}{\Gamma(2\alpha)} \int_0^t (t-u)^{2\alpha-1} F(u) du + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} F(s) ds \\
&= \sum_{j=0}^{n-1} \eta_j t^j \big(\frac{1}{\Gamma(0\alpha+j+1)} + \frac{\lambda t^\alpha}{\Gamma(\alpha+j+1)} \big) + \frac{\lambda^2}{\Gamma(2\alpha)} \int_0^t (t-u)^{2\alpha-1} \phi_{i-2}(u) du \\
&\quad + \int_0^t (t-s)^{\alpha-1} \big(\frac{\lambda (t-s)^\alpha}{\Gamma(2\alpha)} + \frac{1}{\Gamma(\alpha)} \big) F(s) ds \\
&= \dots \\
&= \sum_{j=0}^{n-1} \eta_j t^j \big(\sum_{v=0}^{i-1} \frac{\lambda^v t^{\alpha v}}{\Gamma(v\alpha+j+1)} \big) + \frac{\lambda^i}{\Gamma(i\alpha)} \int_0^t (t-u)^{i\alpha-1} \phi_0(u) du \\
&\quad + \int_0^t (t-s)^{\alpha-1} \big(\sum_{v=0}^{i} \frac{\lambda^v (t-s)^{\alpha v}}{\Gamma((v+1)\alpha)} \big) F(s) ds \\
&= \sum_{j=0}^{n-1} \eta_j t^j \big(\sum_{v=0}^{i-1} \frac{\lambda^v t^{\alpha v}}{\Gamma(v\alpha+j+1)} \big) + \frac{\lambda^i}{\Gamma(i\alpha+j+1)} \sum_{j=0}^{n-1} \eta_j t^{\alpha n+j} \\
&\quad + \int_0^t (t-s)^{\alpha-1} \big(\sum_{v=0}^{i-1} \frac{\lambda^v (t-s)^{\alpha v}}{\Gamma((v+1)\alpha)} \big) F(s) ds \\
&= \sum_{j=0}^{n-1} \eta_j t^j \Big(\sum_{v=0}^{i} \frac{\lambda^v t^{\alpha v}}{\Gamma(v\alpha+j+1)} \Big) + \int_0^t (t-s)^{\alpha-1} \Big(\sum_{v=0}^{i-1} \frac{\lambda^v (t-s)^{\alpha v}}{\Gamma((v+1)\alpha)} \Big) F(s) ds \\
&\to \sum_{j=0}^{n-1} \eta_j t^j \mathbf{E}_{\alpha,j+1}(\lambda t^\alpha) + \int_0^t (t-s)^{\alpha-1} \mathbf{E}_{\alpha,\alpha}(\lambda (t-s)^\alpha) F(s) ds,
\end{aligned}
$$

as $m \to +\infty$. Then $x(t) = \lim_{i\to+\infty} \phi_i(t)$ is a unique solution of (3.7). So x satisfies (3.8). The proof is complete. □

To obtain solutions of (3.2), we need the following assumption:

(3.A2) There exist constants $k_i > -1$, $l_i \le 0$ with $l_1 > \max\{-\alpha, -\alpha - k_1\}$, $l_2 > \max\{-\alpha, -n - k_2\}$, $M_A \ge 0$ and $M_F \ge 0$ such that $|A(t)| \le M_A t^{k_1} (1-t)^{l_1}$ and $|F(t)| \le M_F t^{k_2} (1-t)^{l_2}$ for all $t \in (0,1)$.

We choose Picard function sequence as

$$
\phi_0(t) = \sum_{v=1}^{n} \frac{\eta_v}{\Gamma(\alpha - v + 1)} t^{\alpha - v}, \quad t \in (0,1],
$$

$$
\phi_i(t) = \sum_{v=1}^{n} \frac{\eta_v}{\Gamma(\alpha - v + 1)} t^{\alpha - v} + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} [A(s)\phi_{i-1}(s) + F(s)] ds,
$$

for $t \in (0,1]$, $i = 1, 2, \dots$.

Claim 1. $\phi_i \in C_{n-\alpha}[0,1]$.

Proof. It is easy to see that $\phi_0 \in C_{n-\alpha}[0,1]$. We have

$$\begin{aligned}
&t^{n-\alpha}\Big|\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[A(s)\phi_0(s)+F(s)]ds\Big| \\
&\le t^{n-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} s^{k_1}(1-s)^{l_1} s^{\alpha-n}[s^{n-\alpha}|\phi_{n-1}(s)|]ds \\
&\quad + M_F t^{n-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} s^{k_1}(1-s)^{l_1} \\
&\le t^{n-\alpha}\|\phi_0\|\int_0^t \frac{(t-s)^{\alpha+l_1-1}}{\Gamma(\alpha)} s^{\alpha+k_1-n}ds + M_F t^{n-\alpha}\int_0^t \frac{(t-s)^{\alpha+l_2-1}}{\Gamma(\alpha)} s^{k_2}ds \\
&= \|\phi_0\| t^{\alpha+k_1+l_1}\frac{\mathbf{B}(\alpha+l_1,\alpha+k_1-n+1)}{\Gamma(\alpha)} + M_F t^{n+k_2+l_2}\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}.
\end{aligned}$$

Then $t \to \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[A(s)\phi_{n-1}(s)+F(s)]ds$ is convergent on $(0,1]$ and

$$\lim_{t\to 0^+} t^{n-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[A(s)\phi_{n-1}(s)+F(s)]ds = 0.$$

We see that $\phi_1 \in C_{n-\alpha}[0,1]$. By mathematical induction, we can prove that $\phi_n \in C_{n-\alpha}[0,1]$. ∎

Claim 2. $\{t \to t^{n-\alpha}\phi_i(t)\}$ converges uniformly on $[0,1]$.

Proof. As in Case 1, for $t \in [0,1]$ we have

$$\begin{aligned}
&t^{n-\alpha}|\phi_1(t)-\phi_0(t)| \\
&= \Big|\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[A(s)\phi_0(s)+F(s)]ds\Big| \\
&\le \|\phi_0\| t^{\alpha+k_1+l_1}\frac{\mathbf{B}(\alpha+l_1,\alpha+k_1-n+1)}{\Gamma(\alpha)} + t^{n+k_2+l_2}\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}.
\end{aligned}$$

So

$$\begin{aligned}
&t^{n-\alpha}|\phi_2(t)-\phi_1(t)| \\
&= |\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}A(s)[\phi_1(s)-\phi_0(s)]ds| \\
&\le t^{n-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} M_A s^{k_1}(1-s)^{l_1}s^{\alpha-n} \\
&\quad \times \Big(\|\phi_0\| s^{\alpha+k_1+l_1}\frac{\mathbf{B}(\alpha+l_1,\alpha+k_1-n+1)}{\Gamma(\alpha)} + s^{n+k_2+l_2}\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}\Big)ds \\
&\le M_A\|\phi_0\| t^{n-\alpha}\int_0^t \frac{(t-s)^{\alpha+l_1-1}}{\Gamma(\alpha)} s^{2\alpha-n+2k_1+l_1}ds \frac{\mathbf{B}(\alpha+l_1,\alpha+k_1-n+1)}{\Gamma(\alpha)} \\
&\quad + M_A M_F t^{n-\alpha}\int_0^t \frac{(t-s)^{\alpha+l_1-1}}{\Gamma(\alpha)} s^{\alpha+k_1+k_2+l_2}ds \ \frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)} \\
&= M_A\|\phi_0\| t^{2\alpha+2k_1+2l_1}\frac{\mathbf{B}(\alpha+l_1,2\alpha-n+2k_1+l_1+1)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_1,\alpha+k_1-n+1)}{\Gamma(\alpha)} \\
&\quad + M_A M_F t^{\alpha+n+k_1+l_1+k_2+l_2}\frac{\mathbf{B}(\alpha+l_1,\alpha+k_1+k_2+l_2+1)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}.
\end{aligned}$$

Furthermore,

$$
\begin{aligned}
&t^{n-\alpha}|\phi_3(t)-\phi_2(t)| \\
&= \Big|\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} A(s)[\phi_2(s)-\phi_1(s)]ds\Big| \\
&\le t^{n-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} M_A s^{k_1}(1-s)^{l_1} s^{\alpha-n}\Big(M_A\|\phi_0\| s^{2\alpha+2k_1+2l_1} \\
&\quad\times \frac{\mathbf{B}(\alpha+l_1,2\alpha-n+2k_1+l_1+1)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_1,\alpha+k_1-n+1)}{\Gamma(\alpha)} \\
&\quad + M_A M_F s^{\alpha+n+k_1+l_1+k_2+l_2}\frac{\mathbf{B}(\alpha+l_1,\alpha+k_1+k_2+l_2+1)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}\Big)ds \\
&\le M_A^2\|\phi_0\| t^{n-\alpha}\int_0^t \frac{(t-s)^{\alpha+l_1-1}}{\Gamma(\alpha)} s^{3\alpha-n+3k_1+2l_1}ds \\
&\quad\times \frac{\mathbf{B}(\alpha+l_1,2\alpha-n+2k_1+l_1+1)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_1,\alpha+k_1-n+1)}{\Gamma(\alpha)} \\
&\quad + M_A^2 M_F t^{n-\alpha}\int_0^t \frac{(t-s)^{\alpha+l_1-1}}{\Gamma(\alpha)} s^{2\alpha+2k_1+l_1+k_2+l_2}ds \\
&\quad\times \frac{\mathbf{B}(\alpha+l_1,\alpha+k_1+k_2+l_2+1)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)} \\
&= M_A^2\|\phi_0\| t^{3\alpha+3k_1+3l_1}\frac{\mathbf{B}(\alpha+l_1,3\alpha-n+3k_1+2l_1+1)}{\Gamma(\alpha)} \\
&\quad\times \frac{\mathbf{B}(\alpha+l_1,2\alpha-n+2k_1+l_1+1)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_1,\alpha+k_1-n+1)}{\Gamma(\alpha)} \\
&\quad + M_A^2 M_F t^{2\alpha+n+2k_1+2l_1+k_2+l_2}\frac{\mathbf{B}(\alpha+l_1,2\alpha+2k_1+l_1+k_2+l_2+1)}{\Gamma(\alpha)} \\
&\quad\times \frac{\mathbf{B}(\alpha+l_1,\alpha+k_1+k_2+l_2+1)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}.
\end{aligned}
$$

Similarly by the mathematical induction, for every $i=1,2,\dots$ we obtain

$$
\begin{aligned}
&t^{n-\alpha}|\phi_i(t)-\phi_{i-1}(t)| \\
&\le M_A^i\|\phi_0\| t^{i\alpha+ik_1+il_1}\frac{\mathbf{B}(\alpha+l_1,\alpha+k_1-n+1)}{\Gamma(\alpha)} \\
&\quad\times\prod_{j=0}^{i-2}\frac{\mathbf{B}(\alpha+l_1,(j+1)\alpha+(j+1)k_1+jl_1-n+1)}{\Gamma(\alpha)} \\
&\quad + M_A^{m-1}M_F t^{(i-1)\alpha+n+(i-1)k_1+(i-1)l_1+k_2+l_2}\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)} \\
&\quad\times\prod_{j=0}^{i-2}\frac{\mathbf{B}(\alpha+l_1,(j+1)\alpha+(j+1)k_1+jl_1+k_2+l_2+1)}{\Gamma(\alpha)} \\
&\le M_A^i\|\phi_0\|\frac{\mathbf{B}(\alpha+l_1,\alpha+k_1-n+1)}{\Gamma(\alpha)}
\end{aligned}
$$

$$\times \prod_{j=0}^{i-2} \frac{\mathbf{B}(\alpha+l_1,(j+1)\alpha+(j+1)k_1+jl_1-n+1)}{\Gamma(\alpha)}$$
$$+ M_A^{i-1}M_F\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}$$
$$\times \prod_{j=0}^{i-2} \frac{\mathbf{B}(\alpha+l_1,(j+1)\alpha+(j+1)k_1+jl_1+k_2+l_2+1)}{\Gamma(\alpha)}, \quad t\in[0,1].$$

Similarly we can prove that both

$$\sum_{i=1}^{+\infty} u_i = \sum_{i=1}^{+\infty} M_A^i\|\phi_0\|\frac{\mathbf{B}(\alpha+l_1,\alpha+k_1-n+1)}{\Gamma(\alpha)} \times \prod_{j=0}^{i-2} \frac{\mathbf{B}(\alpha+l_1,(j+1)\alpha+(j+1)k_1+jl_1-n+1)}{\Gamma(\alpha)},$$
$$\sum_{i=1}^{+\infty} v_i = \sum_{i=1}^{+\infty} M_A^{i-1}M_F\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)} \times \prod_{j=0}^{i-2} \frac{\mathbf{B}(\alpha+l_1,(j+1)\alpha+(j+1)k_1+jl_1+k_2+l_2+1)}{\Gamma(\alpha)}$$

are convergent. Hence,

$$t^{n-\alpha}\phi_0(t)+t^{n-\alpha}[\phi_1(t)-\phi_0(t)]+t^{n-\alpha}[\phi_2(t)-\phi_1(t)]+\dots+t^{n-\alpha}[\phi_i(t)-\phi_{i-1}(t)]+\dots,$$

for $t\in[0,1]$, is uniformly convergent. Then $\{t\to t^{n-\alpha}\phi_i(t)\}$ is convergent uniformly on $(0,1]$. □

Claim 3. $\phi(t)=t^{\alpha-n}\lim_{i\to+\infty}t^{n-\alpha}\phi_i(t)$ defined on $(0,1]$ is a unique continuous solution of the integral equation

$$x(t)=\sum_{v=1}^{n}\frac{\eta_v}{\Gamma(\alpha-v+1)}t^{\alpha-v}+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[A(s)x(s)+F(s)]ds, \quad t\in(0,1]. \tag{3.9}$$

Proof. By $\lim_{i\to+\infty}t^{n-\alpha}\phi_i(t)=t^{n-\alpha}\phi(t)$ and the uniformly convergence, we see $\phi(t)$ is continuous on $(0,1]$. From

$$\begin{aligned}
&t^{n-\alpha}\Big|\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[A(s)\phi_{p-1}(s)+F(s)]ds\\
&\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[A(s)\phi_{q-1}(s)+F(s)]ds\Big|\\
&\le M_A\|\phi_{p-1}-\phi_{q-1}\|t^{n-\alpha}\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}s^{k_1}(1-s)^{l_1}s^{\alpha-n}ds\\
&\le M_A\|\phi_{p-1}-\phi_{q-1}\|t^{n-\alpha}\int_0^t\frac{(t-s)^{\alpha+l_1-1}}{\Gamma(\alpha)}s^{\alpha+k_1-n}ds\\
&\le M_A\|\phi_{p-1}-\phi_{q-1}\|t^{\alpha+k_1+l_1}\frac{\mathbf{B}(\alpha+l_1,\alpha+k_1-n+1)}{\Gamma(\alpha)}\\
&\le M_A\|\phi_{p-1}-\phi_{q-1}\|\frac{\mathbf{B}(\alpha+l_1,\alpha+k_1-n+1)}{\Gamma(\alpha)}
\end{aligned}$$

$$\to 0 \quad \text{uniformly as } p, q \to +\infty,$$

we know that

$$\begin{aligned}
\phi(t) &= t^{\alpha-n} \lim_{i\to\infty} t^{n-\alpha}\phi_i(t) \\
&= t^{\alpha-n} \lim_{i\to+\infty} \Big[t^{n-\alpha} \sum_{v=1}^{n} \frac{\eta_v}{\Gamma(\alpha-v+1)} t^{\alpha-v} \\
&\quad + t^{n-\alpha} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} [A(s)\phi_{i-1}(s) + F(s)]ds\Big] \\
&= \sum_{v=1}^{n} \frac{\eta_v}{\Gamma(\alpha-v+1)} t^{\alpha-v} + \lim_{i\to+\infty} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} [A(s)\phi_{i-1}(s) + F(s)]ds \\
&= \sum_{v=1}^{n} \frac{\eta_v}{\Gamma(\alpha-v+1)} t^{\alpha-v} + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} [A(s)\phi(s) + F(s)]ds.
\end{aligned}$$

Then ϕ is a continuous solution of (3.9) defined on $(0, 1]$.

Suppose that ψ defined on $(0, 1]$ is also a solution of (3.9). Then

$$\psi(t) = \sum_{v=1}^{n} \frac{\eta_v}{\Gamma(\alpha-v+1)} t^{\alpha-v} + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} [A(s)\psi(s) + F(s)]ds, \quad t \in [0, 1].$$

We need to prove that $\phi(t) \equiv \psi(t)$ on $(0, 1]$. Then

$$\begin{aligned}
&t^{n-\alpha}|\psi(t) - \phi_0(t)| \\
&= t^{n-\alpha}\Big|\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |A(s)\psi(s) + F(s)|ds\Big| \\
&\le \|\psi\| t^{\alpha+k_1+l_1} \frac{\mathbf{B}(\alpha+l_1, \alpha+k_1-n+1)}{\Gamma(\alpha)} + t^{n+k_2+l_2} \frac{\mathbf{B}(\alpha+l_2, k_2+1)}{\Gamma(\alpha)}.
\end{aligned}$$

Furthermore, we have

$$\begin{aligned}
&t^{n-\alpha}|\psi(t) - \phi_1(t)| \\
&= t^{n-\alpha}\Big|\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} A(s)[\psi(s) - \phi_0(s)]ds\Big| \\
&\le M_A\|\phi_0\| t^{2\alpha+2k_1+2l_1} \frac{\mathbf{B}(\alpha+l_1, 2\alpha-n+2k_1+l_1+1)}{\Gamma(\alpha)} \frac{\mathbf{B}(\alpha+l_1, \alpha+k_1-n+1)}{\Gamma(\alpha)} \\
&\quad + M_A M_F t^{\alpha+n+k_1+l_1+k_2+l_2} \frac{\mathbf{B}(\alpha+l_1, \alpha+k_1+k_2+l_2+1)}{\Gamma(\alpha)} \frac{\mathbf{B}(\alpha+l_2, k_2+1)}{\Gamma(\alpha)}.
\end{aligned}$$

Using mathematical induction, we have

$$\begin{aligned}
&t^{n-\alpha}|\psi(t) - \phi_{i-1}(t)| \\
&= t^{n-\alpha}\Big|\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} A(s)[\psi(s) - \phi_{i-2}(s)]ds\Big| \\
&\le M_A^i\|\phi_0\| t^{i\alpha+ik_1+il_1} \frac{\mathbf{B}(\alpha+l_1, \alpha+k_1-n+1)}{\Gamma(\alpha)} \\
&\quad \times \prod_{j=0}^{i-2} \frac{\mathbf{B}(\alpha+l_1, (j+1)\alpha+(j+1)k_1+jl_1-n+1)}{\Gamma(\alpha)}
\end{aligned}$$

$$
\begin{aligned}
&+ M_A^{m-1} M_F t^{(i-1)\alpha+n+(i-1)k_1+(i-1)l_1+k_2+l_2} \frac{\mathbf{B}(\alpha+l_2, k_2+1)}{\Gamma(\alpha)} \\
&\times \prod_{j=0}^{i-2} \frac{\mathbf{B}(\alpha+l_1, (j+1)\alpha+(j+1)k_1+jl_1+k_2+l_2+1)}{\Gamma(\alpha)} \\
\le\ & M_A^i \|\phi_0\| \frac{\mathbf{B}(\alpha+l_1, \alpha+k_1-n+1)}{\Gamma(\alpha)} \\
&\times \prod_{j=0}^{i-2} \frac{\mathbf{B}(\alpha+l_1, (j+1)\alpha+(j+1)k_1+jl_1-n+1)}{\Gamma(\alpha)} \\
&+ M_A^{i-1} M_F \frac{\mathbf{B}(\alpha+l_2, k_2+1)}{\Gamma(\alpha)} \\
&\times \prod_{j=0}^{i-2} \frac{\mathbf{B}(\alpha+l_1, (j+1)\alpha+(j+1)k_1+jl_1+k_2+l_2+1)}{\Gamma(\alpha)}, t \in [0,1].
\end{aligned}
$$

Hence,

$$
\begin{aligned}
t^{n-\alpha}|\psi(t)-\phi_{i-1}(t)| \le\ & M_A^i \|\phi_0\| \frac{\mathbf{B}(\alpha+l_1, \alpha+k_1-n+1)}{\Gamma(\alpha)} \\
&\times \prod_{j=0}^{i-2} \frac{\mathbf{B}(\alpha+l_1, (j+1)\alpha+(j+1)k_1+jl_1-n+1)}{\Gamma(\alpha)} \\
&+ M_A^{i-1} M_F \frac{\mathbf{B}(\alpha+l_2, k_2+1)}{\Gamma(\alpha)} \\
&\times \prod_{j=0}^{i-2} \frac{\mathbf{B}(\alpha+l_1, (j+1)\alpha+(j+1)k_1+jl_1+k_2+l_2+1)}{\Gamma(\alpha)},
\end{aligned}
$$

for $i = 1, 2, \dots$. Similarly we have $\lim_{i\to+\infty} t^{n-\alpha}\phi_i(t) = t^{n-\alpha}\psi(t)$ uniformly on $(0,1]$. Then $\phi(t) \equiv \psi(t)$ on $(0,1]$. Then (3.9) has a unique solution ϕ. The proof is complete. □

Theorem 3.3. *Suppose that* (3.A2) *holds. Then* $x \in C_{n-\alpha}(0,1]$ *is a solution of IVP* (3.2) *if and only if* $x \in C_{n-\alpha}(0,1]$ *is a solution of the integral equation* (3.9).

Proof. Suppose that $x \in C_{n-\alpha}(0,1]$ is a solution of (3.2). Then $t \to t^{n-\alpha}x(t)$is continuous on $(0,1]$ by defining $t^{n-\alpha}x(t)|_{t=0} = \lim_{t\to 0^+} t^{n-\alpha}x(t)$ and $\|x\| = r <$

$+\infty$. So from $\frac{w}{s}=u$, we obtain

$$\begin{aligned}
&\lim_{s\to 0^+}\int_0^s (s-w)^{n-\alpha-1}x(w)dw\\
&=\lim_{s\to 0^+}\int_0^s (s-w)^{n-\alpha-1}w^{\alpha-n}w^{n-\alpha}x(w)dw\\
&=\lim_{s\to 0^+}\xi^{n-\alpha}x(\xi)\int_0^s (s-w)^{n-\alpha-1}w^{\alpha-n}dw\\
&\quad\text{by mean value theorem with } \xi\in(0,s)\\
&=\lim_{s\to 0^+}\xi^{n-\alpha}x(\xi)\int_0^1 (1-u)^{n-\alpha-1}u^{\alpha-n}du\\
&=\frac{\eta_n}{\Gamma(\alpha-n+1)}\mathbf{B}(n-\alpha,\alpha-n+1).
\end{aligned}\tag{3.10}$$

From (3.A2), we have similarly to Case 1 that

$$\begin{aligned}
&t^{n-\alpha}\Big|\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[A(s)x(s)+F(s)]ds\Big|\\
&=t^{n-\alpha}\Big|\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[A(s)s^{\alpha-n}s^{n-\alpha}x(s)+F(s)]ds\Big|\\
&\le t^{n-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[M_A r s^{\alpha-n}s^{k_1}(1-s)^{l_1}+M_F s^{k_2}(1-s)^{l_2}]ds\\
&\le rM_A t^{\alpha+k_1+l_1}\frac{\mathbf{B}(\alpha+l_1,\alpha+k_1-n+1)}{\Gamma(\alpha)}+M_F t^{n+k_2+l_2}\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}.
\end{aligned}$$

So $t\to t^{n-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[A(s)x(s)+F(s)]ds$ is defined on $(0,1]$ and

$$\lim_{t\to 0^+}t^{n-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[A(s)x(s)+F(s)]ds=0. \tag{3.11}$$

Furthermore, we have similarly to Theorem 3.1 that $t\to\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[A(s)x(s)+F(s)]ds$ is continuous on $(0,1]$. So $t\to t^{n-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[A(s)x(s)+F(s)]ds$ is continuous on $[0,1]$ by defining

$$\begin{aligned}
&t^{n-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[A(s)x(s)+F(s)]ds\Big|_{t=0}\\
&=\lim_{t\to 0^+}t^{n-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[A(s)x(s)+F(s)]ds.
\end{aligned}$$

We have $I_{0^+}^{\alpha}\,{}^{RL}D_{0^+}^{\alpha}x(t)=I_{0^+}^{\alpha}[A(t)x(t)+F(t)]$. So

$$\begin{aligned}
&\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[A(s)x(s)+F(s)]ds\\
&=I_{0^+}^{\alpha}[A(t)x(t)+F(t)]=I_{0^+}^{\alpha}\,{}^{RL}D_{0^+}^{\alpha}x(t)\\
&=\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\Big[\frac{1}{\Gamma(n-\alpha)}(\int_0^s (s-w)^{n-\alpha-1}x(w)dw)^{(n)}\Big]ds\\
&=\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}d\Big(\int_0^s \frac{(s-w)^{n-\alpha-1}}{\Gamma(n-\alpha)}x(w)dw\Big)^{(n-1)}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} d\big({}^{RL}D_{0^+}^{\alpha-1} x(s)\big) \\
&= \frac{1}{\Gamma(\alpha)} (t-s)^{\alpha-1} \big({}^{RL}D_{0^+}^{\alpha-1} x(s)\big)\Big|_0^t + \frac{1}{\Gamma(n-\alpha)\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} \\
&\quad \times \Big(\int_0^s (s-w)^{n-\alpha-1} x(w) dw \Big)^{(n-1)} ds \\
&= \frac{1}{\Gamma(n-\alpha)\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} \Big(\int_0^s (s-w)^{n-\alpha-1} x(w) dw \Big)^{(n-1)} ds - \frac{\eta_1}{\Gamma(\alpha)} t^{\alpha-1} \\
&= \frac{1}{\Gamma(n-\alpha)\Gamma(\alpha-2)} \int_0^t (t-s)^{\alpha-3} \Big(\int_0^s (s-w)^{n-\alpha-1} x(w) dw \Big)^{(n-2)} ds \\
&\quad - \frac{\eta_1}{\Gamma(\alpha)} t^{\alpha-1} - \frac{\eta_2}{\Gamma(\alpha-1)} t^{\alpha-2} \\
&= \dots \\
&= \frac{1}{\Gamma(n-\alpha)\Gamma(\alpha-(n-1))} \int_0^t (t-s)^{\alpha-n} (\int_0^s (s-w)^{n-\alpha-1} x(w) dw)' ds \\
&\quad - \sum_{v=1}^{n-1} \frac{\eta_v}{\Gamma(\alpha-v+1)} t^{\alpha-v} \\
&= \frac{1}{\Gamma(n-\alpha)\Gamma(\alpha-(n-2))} \Big[\int_0^t (t-s)^{\alpha-n+1} (\int_0^s (s-w)^{n-\alpha-1} x(w) dw)' ds \Big]' \\
&\quad - \sum_{v=1}^{n-1} \frac{\eta_v}{\Gamma(\alpha-v+1)} t^{\alpha-v} \\
&= \frac{1}{\Gamma(n-\alpha)\Gamma(\alpha-(n-2))} \Big[(t-s)^{\alpha-n+1} \Big(\int_0^s (s-w)^{n-\alpha-1} x(w) dw \Big)\Big|_0^t \\
&\quad + (\alpha-n+1) \int_0^t (t-s)^{\alpha-n} \int_0^s (s-w)^{n-\alpha-1} x(w)\, dw\, ds \Big]' \\
&\quad - \sum_{v=1}^{n-1} \frac{\eta_v}{\Gamma(\alpha-v+1)} t^{\alpha-v} \quad \text{(using (3.10))} \\
&= \frac{1}{\Gamma(n-\alpha)\Gamma(\alpha-(n-1))} \Big[\int_0^t \int_u^t (t-s)^{\alpha-n} (s-w)^{n-\alpha-1} ds x(w) dw \Big]' \\
&\quad - \frac{\eta_n}{\Gamma(\alpha-n+1)} t^{\alpha-n} - \sum_{v=1}^{n-1} \frac{\eta_v}{\Gamma(\alpha-v+1)} t^{\alpha-v} \\
&= \frac{1}{\Gamma(n-\alpha)\Gamma(\alpha-(n-1))} \Big[\int_0^t \int_0^1 (1-w)^{\alpha-n} w^{n-\alpha-1} dw x(w) dw \Big]' \\
&\quad - \sum_{v=1}^{n} \frac{\eta_v}{\Gamma(\alpha-v+1)} t^{\alpha-v} t^{\alpha-n} \\
&= x(t) - \sum_{v=1}^{n} \frac{\eta_v}{\Gamma(\alpha-v+1)} t^{\alpha-v}.
\end{aligned}
$$

Then $x \in C_{n-\alpha}(0,1]$ is a solution of (3.9).

On the other hand, if $x\in C_{n-\alpha}(0,1]$ is a solution of (3.9). Then (3.10) implies $\lim_{t\to 0^+}t^{n-\alpha}x(t)=\frac{\eta_n}{\Gamma(\alpha-n+1)}$. Furthermore, we have

$$
\begin{aligned}
{}^{RL}D_{0^+}^{\alpha}x(t) &= \frac{1}{\Gamma(n-\alpha)}\Big(\int_0^t (t-s)^{n-\alpha-1}x(s)ds\Big)^{(n)}\\
&= \frac{1}{\Gamma(n-\alpha)}\Big(\int_0^t (t-s)^{n-\alpha-1}\Big(\sum_{v=1}^n \frac{\eta_v}{\Gamma(\alpha-v+1)}s^{\alpha-v}\\
&\quad + \int_0^s \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}[A(u)x(u)+F(u)]du\Big)ds\Big)^{(n)}\\
&= \frac{1}{\Gamma(n-\alpha)}\Big(\sum_{v=1}^n \frac{\eta_v}{\Gamma(\alpha-v+1)}\int_0^t (t-s)^{n-\alpha-1}s^{\alpha-v}ds\\
&\quad + \int_0^t (t-s)^{n-\alpha-1}\int_0^s \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}[A(u)x(u)+F(u)]\,du\,ds\Big)^{(n)}\\
&= \frac{1}{\Gamma(n-\alpha)}\Big(\sum_{v=1}^n \frac{\eta_v}{\Gamma(\alpha-v+1)}t^{n-v}\int_0^1 (1-w)^{n-\alpha-1}w^{\alpha-v}dw\\
&\quad + \int_0^t \int_u^t (t-s)^{n-\alpha-1}\frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}ds[A(u)x(u)+F(u)]du\Big)^{(n)}\\
&= \frac{1}{\Gamma(n-\alpha)}\Big(\sum_{v=1}^n \frac{\eta_v}{\Gamma(\alpha-v+1)}t^{n-v}\int_0^1 (1-w)^{n-\alpha-1}w^{\alpha-v}dw\\
&\quad + \int_0^t (t-u)^{n-1}\int_0^1 (1-w)^{n-\alpha-1}\frac{w^{\alpha-1}}{\Gamma(\alpha)}dw[A(u)x(u)+F(u)]du\Big)^{(n)}\\
&= A(t)x(t)+F(t).
\end{aligned}
$$

So $x\in C_{n-\alpha}(0,1]$ is a solution of IVP(3.2). The proof is complete. □

Theorem 3.4. *Suppose that* (3.A2) *holds. Then* (3.2) *has a unique solution. If* $A(t)\equiv\lambda$ *and there exists constants* $k_2>-1$, $l_2\le 0$ *with* $l_2>\max\{-\alpha,-n-k_2\}$ *and* $M_F\ge 0$ *such that* $|F(t)|\le M_F t^{k_2}(1-t)^{l_2}$ *for all* $t\in(0,1)$, *then the problem*

$$
\begin{gathered}
{}^{RL}D_{0^+}^{\alpha}x(t)=\lambda x(t)+F(t),\quad a.e.\ t\in(0,1],\\
\lim_{t\to 0^+}t^{n-\alpha}x(t)=\frac{\eta_n}{\Gamma(\alpha-n+1)},\\
\lim_{t\to 0^+}{}^{RL}D_{0^+}^{\alpha-j}x(t)=\eta_j,\quad j\in\mathbb{N}[1,n-1]
\end{gathered}
\tag{3.12}
$$

has a unique solution

$$
x(t)=\sum_{v=1}^n \eta_v t^{\alpha-v}\mathbf{E}_{\alpha,\alpha-v+1}(\lambda t^{\alpha})+\int_0^t (t-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t-s)^{\alpha})F(s)ds, \tag{3.13}
$$

for $t\in(0,1]$.

Proof. (i) From Claims 1, 2 and 3, and Theorem 3.3, we see that (3.2) has a unique solution.

(ii) From the assumption and $A(t)\equiv\lambda$, one sees that (3.A2) holds with $k_1=l_1=0$ and k_2,l_2 mentioned. Thus (3.12) has a unique solution. From the Picard function sequence we have

$\phi_i(t)$

$$
\begin{aligned}
&= \sum_{v=1}^{n} \frac{\eta_v}{\Gamma(\alpha-v+1)} t^{\alpha-v} + \lambda \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \phi_{i-1}(s)ds + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} F(s)ds \\
&= \sum_{v=1}^{n} \frac{\eta_v}{\Gamma(\alpha-v+1)} t^{\alpha-v} + \lambda \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \Big(\sum_{v=1}^{n} \frac{\eta_v}{\Gamma(\alpha-v+1)} s^{\alpha-v} \\
&\quad + \lambda \int_0^s \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)} \phi_{i-2}(u)du + \int_0^s \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)} F(u)du \Big) ds \\
&\quad + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} F(s)ds \\
&= \sum_{v=1}^{n} \frac{\eta_v}{\Gamma(\alpha-v+1)} t^{\alpha-v} + \lambda \sum_{v=1}^{n} \frac{\eta_v}{\Gamma(\alpha-v+1)} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} s^{\alpha-v} ds \\
&\quad + \lambda^2 \int_0^t \int_u^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)} ds \phi_{i-2}(u)du \\
&\quad + \lambda \int_0^t \int_u^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)} ds F(u)du + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} F(s)ds \\
&= \sum_{v=1}^{n} \frac{\eta_v}{\Gamma(\alpha-v+1)} t^{\alpha-v} + \lambda \sum_{v=1}^{n} \frac{\eta_v}{\Gamma(2\alpha-v+1)} t^{2\alpha-v} \\
&\quad + \lambda^2 \int_0^t \frac{(t-u)^{2\alpha-1}}{\Gamma(2\alpha)} \phi_{i-2}(u)du + \lambda \int_0^t \frac{(t-u)^{2\alpha-1}}{\Gamma(2\alpha)} F(u)du \\
&\quad + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} F(s)ds \\
&= \sum_{v=1}^{n} \eta_v t^{\alpha-v} (\frac{1}{\Gamma(\alpha-v+1)} + \frac{\lambda t^\alpha}{\Gamma(2\alpha-v+1)}) + \lambda^2 \int_0^t \frac{(t-u)^{2\alpha-1}}{\Gamma(2\alpha)} \phi_{i-2}(u)du \\
&\quad + \int_0^t (t-s)^{\alpha-1} (\frac{\lambda(t-s)^\alpha}{\Gamma(2\alpha)} + \frac{1}{\Gamma(\alpha)}) F(s)ds \\
&= \dots \\
&= \sum_{v=1}^{n} \eta_v t^{\alpha-v} \Big(\sum_{j=0}^{i-1} \frac{\lambda^j t^{\alpha j}}{\Gamma(j\alpha+\alpha-v+1)} \Big) + \lambda^i \int_0^t \frac{(t-u)^{i\alpha-1}}{\Gamma(m\alpha)} \phi_0(u)du \\
&\quad + \int_0^t (t-s)^{\alpha-1} \Big(\sum_{j=0}^{m-1} \frac{\lambda^j (t-s)^{\alpha j}}{\Gamma((j+1)\alpha)} \Big) F(s)ds \\
&= \sum_{v=1}^{n} \eta_v t^{\alpha-v} (\sum_{j=0}^{i} \frac{\lambda^j t^{\alpha j}}{\Gamma(j\alpha+\alpha-v+1)}) + \int_0^t (t-s)^{\alpha-1} \Big(\sum_{j=0}^{i-1} \frac{\lambda^j (t-s)^{\alpha j}}{\Gamma((j+1)\alpha)} \Big) F(s)ds \\
&\to \sum_{v=1}^{n} \eta_v t^{\alpha-v} \mathbf{E}_{\alpha,\alpha-v+1}(\lambda t^\alpha) + \int_0^t (t-s)^{\alpha-1} \mathbf{E}_{\alpha,\alpha}(\lambda(t-s)^\alpha) F(s)ds.
\end{aligned}
$$

Then we obtain $x(t) = \lim_{i\to+\infty} \phi_i(t)$ is a unique solution of (3.12). Then x satisfies (3.13). The proof is complete. □

To obtain solutions of (3.3), we need the following assumption:

(3.A3) There exist constants $k_i > -1$, $l_i \le 0$ with $l_1 > \max\{-\alpha, -\alpha - k_1\}$, $l_2 > \max\{-\alpha, -n-k_2\}$, $M_B \ge 0$ and $M_G \ge 0$ such that $|B(t)| \le M_B(\log t)^{k_1}(1-\log t)^{l_1}$ and $|G(t)| \le M_G(\log t)^{k_2}(1-\log t)^{l_2}$ for all $t \in (1, e)$.

We choose Picard function sequence as

$$\phi_0(t) = \sum_{v=1}^{n} \frac{\eta_v}{\Gamma(\alpha - v + 1)}(\log t)^{\alpha - v}, \quad t \in (1, e],$$

$$\phi_i(t) = \sum_{v=1}^{n} \frac{\eta_v}{\Gamma(\alpha - v + 1)}(\log t)^{\alpha - v} + \frac{1}{\Gamma(\alpha)} \int_1^t (\log \frac{t}{s})^{\alpha-1}[B(s)\phi_{i-1}(s) + G(s)]\frac{ds}{s},$$
$$t \in (1, e],\ i = 1, 2, \dots.$$

Claim 1. $\phi_i \in LC_{n-\alpha}(1, e]$.

Proof. We have $\phi_0 \in LC_{n-\alpha}[1, e]$ and

$$\begin{aligned}
&(\log t)^{n-\alpha}\Big|\int_1^t (\log \frac{t}{s})^{\alpha-1}[B(s)\phi_0(s) + G(s)]\frac{ds}{s}\Big| \\
&\le (\log t)^{n-\alpha}\int_1^t (\log \frac{t}{s})^{\alpha-1}\Big[M_B\|\phi_0\|(\log s)^{\alpha-n}(\log s)^{k_1}(1-\log s)^{l_1} \\
&\quad + M_G(\log s)^{k_2}(1-\log s)^{l_2}\Big]\frac{ds}{s} \\
&\le (\log t)^{n-\alpha}M_B\|\phi_0\|\int_1^t (\log \frac{t}{s})^{\alpha+l_1-1}(\log s)^{\alpha+k_1-n}\frac{ds}{s} \\
&\quad + (\log t)^{n-\alpha}M_G\int_1^t (\log \frac{t}{s})^{\alpha+l_2-1}(\log s)^{k_2}\frac{ds}{s} \\
&= M_B\|\phi_0\|(\log t)^{\alpha+k_1+l_1}\mathbf{B}(\alpha + l_1, \alpha + k_1 - n + 1) \\
&\quad + M_G(\log t)^{n+k_1+l_1}\mathbf{B}(\alpha + l_2, k_2 + 1) \to 0 \quad \text{as } t \to 0^+,
\end{aligned}$$

we know that $t \to \frac{1}{\Gamma(\alpha)}\int_1^t (\log \frac{t}{s})^{\alpha-1}[B(s)\phi_0(s) + G(s)]\frac{ds}{s}$ is continuous on $(1, e]$ and $\lim_{t\to 0^+}(\log t)^{n-\alpha}\phi_1(t)$ exists. Then $\phi_1 \in LC_{n-\alpha}[1, e]$. By mathematical induction, we can show $\phi_i \in LC_{n-\alpha}[1, e]$. □

Claim 2. $\{t \to (\log t)^{n-\alpha}\phi_i(t)\}$ is convergent uniformly on $[1, e]$.

Proof. As above, for $t \in [1, e]$ we have

$$\begin{aligned}
&(\log t)^{n-\alpha}|\phi_1(t) - \phi_0(t)| \\
&= \frac{1}{\Gamma(\alpha)}(\log t)^{n-\alpha}|\int_1^t (\log \frac{t}{s})^{\alpha-1}[B(s)\phi_0(s) + G(s)]\frac{ds}{s}| \\
&\le \frac{1}{\Gamma(\alpha)}(\log t)^{n-\alpha}\int_1^t (\log \frac{t}{s})^{\alpha-1}\Big[\|\phi_0\|M_B(\log s)^{\alpha-n+k_1}(1-\log s)^{l_1} \\
&\quad + M_G(\log s)^{k_2}(1-\log s)^{l_2}\Big]\frac{ds}{s} \\
&\le M_B\|\phi_0\|(\log t)^{\alpha+k_1+l_1}\frac{\mathbf{B}(\alpha + l_1, \alpha + k_1 - n + 1)}{\Gamma(\alpha)}
\end{aligned}$$

$$+ M_G(\log t)^{n+k_2+l_2}\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}.$$

So

$$\begin{aligned}
&(\log t)^{n-\alpha}|\phi_2(t)-\phi_1(t)|\\
&=\frac{1}{\Gamma(\alpha)}(\log t)^{n-\alpha}\Big|\int_1^t(\log\frac{t}{s})^{\alpha-1}B(s)[\phi_1(s)-\phi_0(s)]\frac{ds}{s}\Big|\\
&\le\frac{1}{\Gamma(\alpha)}(\log t)^{n-\alpha}\int_1^t(\log\frac{t}{s})^{\alpha+l_1-1}M_B(\log s)^{k_1+\alpha-n}\\
&\quad\times\Big(M_B\|\phi_0\|(\log s)^{\alpha+k_1+l_1}\frac{\mathbf{B}(\alpha+l_1,\alpha-n+k_1+1)}{\Gamma(\alpha)}\\
&\quad+M_G(\log s)^{n+k_2+l_2}\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}\Big)\frac{ds}{s}\\
&\le\|\phi_0\|M_B^2(\log t)^{2\alpha+2k_1+2l_1}\frac{\mathbf{B}(\alpha+l_1,2\alpha+2k_1+l_1-n+1)}{\Gamma(\alpha)}\\
&\quad\times\frac{\mathbf{B}(\alpha+l_1,\alpha+k_1-n+1)}{\Gamma(\alpha)}\\
&\quad+M_BM_G(\log t)^{\alpha+n+k_1+l_1+k_2+l_2}\frac{\mathbf{B}(\alpha+l_1,\alpha+k_1+k_2+l_2+1)}{\Gamma(\alpha)}\\
&\quad\times\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}.
\end{aligned}$$

Then

$$\begin{aligned}
&(\log t)^{n-\alpha}|\phi_3(t)-\phi_2(t)|\\
&=\frac{1}{\Gamma(\alpha)}(\log t)^{n-\alpha}\Big|\int_1^t(\log\frac{t}{s})^{\alpha-1}B(s)[\phi_2(s)-\phi_1(s)]ds\Big|\\
&\le\frac{1}{\Gamma(\alpha)}(\log t)^{n-\alpha}\ int_1^t(\log\frac{t}{s})^{\alpha+l_1-1}M_B(\log s)^{k_1+\alpha-n}\\
&\quad\times\Big(\|\phi_0\|M_B^2(\log s)^{\alpha+n+2k_1+2l_1}\frac{\mathbf{B}(\alpha+l_1,\alpha+2k_1+l_1+1)}{\Gamma(\alpha)}\\
&\quad\times\frac{\mathbf{B}(\alpha+l_1,\alpha+k_1-n+1)}{\Gamma(\alpha)}+M_BM_G(\log s)^{2n+k_1+l_1+k_2+l_2}\\
&\quad\times\frac{\mathbf{B}(\alpha+l_1,n+k_1+k_2+l_2+1)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}\Big)\frac{ds}{s}\\
&\le\|\phi_0\|M_B^3(\log t)^{3\alpha+3k_1+3l_1}\frac{\mathbf{B}(\alpha+l_1,3\alpha+3k_1+2l_1-n+1)}{\Gamma(\alpha)}\\
&\quad\times\frac{\mathbf{B}(\alpha+l_1,2\alpha+2k_1+l_1-n+1)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_1,\alpha+k_1-n+1)}{\Gamma(\alpha)}\\
&\quad+M_B^2M_G(\log t)^{2\alpha+n+2k_1+2l_1+k_2+l_2}\frac{\mathbf{B}(\alpha+l_1,2\alpha+2k_1+l_1+k_2+l_2+1)}{\Gamma(\alpha)}\\
&\quad\times\frac{\mathbf{B}(\alpha+l_1,\alpha+k_1+k_2+l_2+1)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}.
\end{aligned}$$

Furthermore,

$$
\begin{aligned}
&(\log t)^{n-\alpha}|\phi_4(t)-\phi_3(t)| \\
&= \frac{1}{\Gamma(\alpha)}(\log t)^{n-\alpha}\Big|\int_1^t (\log\frac{t}{s})^{\alpha-1}B(s)[\phi_3(s)-\phi_2(s)]ds\Big| \\
&\le \frac{(\log t)^{n-\alpha}}{\Gamma(\alpha)}\int_1^t (\log\frac{t}{s})^{\alpha+l_1-1}M_B(\log s)^{k_1+\alpha-n}\Big(\|\phi_0\|M_B^3(\log s)^{\alpha+2n+3k_1+3l_1} \\
&\quad\times \frac{\mathbf{B}(\alpha+l_1,\alpha+n+3k_1+2l_1+1)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_1,\alpha+2k_1+l_1+1)}{\Gamma(\alpha)} \\
&\quad\times \frac{\mathbf{B}(\alpha+l_1,\alpha+k_1-n+1)}{\Gamma(\alpha)} + M_B^2M_G(\log s)^{3n+2k_1+2l_1+k_2+l_2} \\
&\quad\times \frac{\mathbf{B}(\alpha+l_1,2n+2k_1+l_1+k_2+l_2+1)}{\Gamma(\alpha)} \\
&\quad\times \frac{\mathbf{B}(\alpha+l_1,n+k_1+k_2+l_2+1)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}\Big)\frac{ds}{s} \\
&\le \|\phi_0\|M_B^4(\log t)^{4\alpha+4k_1+4l_1}\frac{\mathbf{B}(\alpha+l_1,4\alpha+4k_1+3l_1-n+1)}{\Gamma(\alpha)} \\
&\quad\times \frac{\mathbf{B}(\alpha+l_1,3\alpha+3k_1+2l_1-n+1)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_1,2\alpha+2k_1+l_1-n+1)}{\Gamma(\alpha)} \\
&\quad\times \frac{\mathbf{B}(\alpha+l_1,\alpha+k_1-n+1)}{\Gamma(\alpha)} + M_B^3M_G(\log t)^{3\alpha+n+3k_1+3l_1+k_2+l_2} \\
&\quad\times \frac{\mathbf{B}(\alpha+l_1,3\alpha+3k_1+2l_1+k_2+l_2+1)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_1,2\alpha+2k_1+l_1+k_2+l_2+1)}{\Gamma(\alpha)} \\
&\quad\times \frac{\mathbf{B}(\alpha+l_1,\alpha+k_1+k_2+l_2+1)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}.
\end{aligned}
$$

Similarly by mathematical induction, for every $i=1,2,\dots$ we obtain

$$
\begin{aligned}
&(\log t)^{n-\alpha}|\phi_i(t)-\phi_{i-1}(t)| \\
&\le \|\phi_0\|M_B^i(\log t)^{i\alpha+ik_1+il_1}\frac{\mathbf{B}(\alpha+l_1,\alpha+k_1-n+1)}{\Gamma(\alpha)} \\
&\quad\times \prod_{j=1}^{i-1}\frac{\mathbf{B}(\alpha+l_1,(j+1)\alpha+(j+1)k_1+jl_1-n+1)}{\Gamma(\alpha)} \\
&\quad+ M_B^{i-1}M_G(\log t)^{(i-1)\alpha+n+(i-1)k_1+(i-1)l_1+k_2+l_2}\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)} \\
&\quad\times \prod_{j=1}^{i-1}\frac{\mathbf{B}(\alpha+l_1,jn+jk_1+(j-1)l_1+k_2+l_2+1)}{\Gamma(\alpha)} \\
&\le \|\phi_0\|M_B^i\frac{\mathbf{B}(\alpha+l_1,\alpha+k_1-n+1)}{\Gamma(\alpha)} \\
&\quad\times \prod_{j=1}^{i-1}\frac{\mathbf{B}(\alpha+l_1,(j+1)\alpha+(j+1)k_1+jl_1-n+1)}{\Gamma(\alpha)}
\end{aligned}
$$

$$+ M_B^{i-1} M_G \frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)} \times \prod_{j=1}^{i-1} \frac{\mathbf{B}(\alpha+l_1,jn+jk_1+(j-1)l_1+k_2+l_2+1)}{\Gamma(\alpha)}, \quad t\in(1,e].$$

Similarly we can prove that both

$$\begin{aligned}\sum_{i=1}^{+\infty} u_i &= \sum_{i=1}^{+\infty} \|\phi_0\| M_B^i \frac{\mathbf{B}(\alpha+l_1,\alpha+k_1-n+1)}{\Gamma(\alpha)} \\ &\quad\times \prod_{j=1}^{i-1} \frac{\mathbf{B}(\alpha+l_1,(j+1)\alpha+(j+1)k_1+jl_1-n+1)}{\Gamma(\alpha)}, \\ \sum_{i=1}^{+\infty} v_i &= \sum_{i=1}^{+\infty} M_B^{m-1} M_G \frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)} \\ &\quad\times \prod_{j=1}^{i-1} \frac{\mathbf{B}(\alpha+l_1,jn+jk_1+(j-1)l_1+k_2+l_2+1)}{\Gamma(\alpha)}\end{aligned}$$

converge. Hence,

$$(\log t)^{n-\alpha}\phi_0(t) + (\log t)^{n-\alpha}[\phi_1(t)-\phi_0(t)] + \dots + (\log t)^{n-\alpha}[\phi_i(t)-\phi_{i-1}(t)] + \dots,$$

for $t\in(1,e]$ converges uniformly. Then $\{t \to (\log t)^{n-\alpha}\phi_i(t)\}$ converges uniformly on $[1,e]$. □

Claim 3. $\phi(t) = (\log t)^{\alpha-n}\lim_{i\to+\infty}(\log t)^{n-\alpha}\phi_i(t)$ defined on $[1,e]$ is a unique continuous solution of the integral equation

$$x(t) = \sum_{v=1}^{n} \frac{\eta_v}{\Gamma(\alpha-v+1)}(\log t)^{\alpha-v} + \frac{1}{\Gamma(\alpha)}\int_1^t (\log\frac{t}{s})^{\alpha-1}[B(s)x(s)+G(s)]\frac{ds}{s}, \tag{3.14}$$

for $t\in(1,e]$.

Proof. From $\lim_{i\to+\infty}(\log t)^{n-\alpha}\phi_i(t) = (\log t)^{n-\alpha}\phi(t)$ and the uniformly convergence, we see that $\phi(t)$ is continuous on $[1,e]$. From

$$\begin{aligned}&(\log t)^{n-\alpha}\Big|\int_1^t (\log\frac{t}{s})^{\alpha-1}[A(s)\phi_{p-1}(s)+F(s)]\frac{ds}{s} \\ &- \int_1^t (\log\frac{t}{s})^{\alpha-1}[B(s)\phi_{q-1}(s)+G(s)]\frac{ds}{s}\Big| \\ &\le M_B\|\phi_{p-1}-\phi_{q-1}\|(\log t)^{n-\alpha}\int_1^t (\log\frac{t}{s})^{\alpha+l_1-1}(\log s)^{k_1}(\log s)^{\alpha-n}\frac{ds}{s} \\ &\le M_B\|\phi_{p-1}-\phi_{q-1}\|(\log t)^{n-\alpha}\int_1^t (\log\frac{t}{s})^{\alpha+l_1-1}(\log s)^{\alpha+k_1-n}\frac{ds}{s} \\ &\le M_B\|\phi_{p-1}-\phi_{q-1}\|(\log t)^{n+k_1+l_1}\mathbf{B}(\alpha+l_1,\alpha+k_1) \\ &\le M_B\|\phi_{p-1}-\phi_{q-1}\|\mathbf{B}(\alpha+l_1,\alpha+k_1-n+1) \\ &\to 0 \quad \text{uniformly as } p,q\to+\infty,\end{aligned}$$

we know that

$$\begin{aligned}
\phi(t) &= (\log t)^{\alpha-n} \lim_{i\to+\infty} (\log t)^{n-\alpha}\phi_i(t) \\
&= \lim_{i\to+\infty} \Big[\sum_{v=1}^{n} \frac{\eta_v}{\Gamma(\alpha-v+1)}(\log t)^{\alpha-v} \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_1^t (\log\frac{t}{s})^{\alpha-1}[B(s)\phi_{i-1}(s)+G(s)]\frac{ds}{s}\Big] \\
&= \sum_{v=1}^{n} \frac{\eta_v}{\Gamma(\alpha-v+1)}(\log t)^{\alpha-v} + \lim_{i\to+\infty}\int_1^t (\log\frac{t}{s})^{\alpha-1}[B(s)\phi_{i-1}(s)+G(s)]\frac{ds}{s} \\
&= \sum_{v=1}^{n} \frac{\eta_v}{\Gamma(\alpha-v+1)}(\log t)^{\alpha-v} + \frac{1}{\Gamma(\alpha)}\int_1^t (\log\frac{t}{s})^{\alpha-1}\,[B(s)\phi(s)+G(s)]\,\frac{ds}{s}.
\end{aligned}$$

Then ϕ is a continuous solution of (3.14) defined on $(1,e]$.

Suppose that ψ defined on $(1,e]$ is also a solution of (3.14). Then

$$\psi(t) = \sum_{v=1}^{n} \frac{\eta_v}{\Gamma(\alpha-v+1)}(\log t)^{\alpha-v} + \frac{1}{\Gamma(\alpha)}\int_1^t (\log\frac{t}{s})^{\alpha-1}[B(s)\psi(s)+G(s)]\frac{ds}{s},$$

for $t\in(1,e]$. We need to prove that $\phi(t)\equiv\psi(t)$ on $(1,e]$. Then

$$\begin{aligned}
&(\log t)^{n-\alpha}|\psi(t)-\phi_0(t)| \\
&= (\log t)^{n-\alpha}\Big|\int_1^t (\log\frac{t}{s})^{\alpha-1}|B(s)\psi(s)+G(s)|\frac{ds}{s}\Big| \\
&\le M_B\|\phi_0\|(\log t)^{\alpha+k_1+l_1}\frac{\mathbf{B}(\alpha+l_1,\alpha+k_1-n+1)}{\Gamma(\alpha)} \\
&\quad + M_G(\log t)^{n+k_2+l_2}\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
&(\log t)^{n-\alpha}|\psi(t)-\phi_1(t)| \\
&= (\log t)^{n-\alpha}\frac{1}{\Gamma(\alpha)}\Big|\int_1^t (\log\frac{t}{s})^{\alpha-1}B(s)[\psi(s)-\phi_0(s)]\frac{ds}{s}\Big| \\
&\le \|\phi_0\|M_B^2(\log t)^{2\alpha+2k_1+2l_1}\frac{\mathbf{B}(\alpha+l_1,2\alpha+2k_1+l_1-n+1)}{\Gamma(\alpha)} \\
&\quad\times \frac{\mathbf{B}(\alpha+l_1,\alpha+k_1-n+1)}{\Gamma(\alpha)} + M_BM_G(\log t)^{\alpha+n+k_1+l_1+k_2+l_2} \\
&\quad\times \frac{\mathbf{B}(\alpha+l_1,\alpha+k_1+k_2+l_2+1)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}.
\end{aligned}$$

By mathematical induction, we obtain

$$\begin{aligned}
&(\log t)^{1-\alpha}|\psi(t)-\phi_{i-1}(t)| \\
&= (\log t)^{n-\alpha}\frac{1}{\Gamma(\alpha)}\Big|\int_1^t (\log\frac{t}{s})^{\alpha-1}B(s)[\psi(s)-\phi_{i-1}(s)]ds\Big| \\
&\le \|\phi_0\|M_B^i(\log t)^{i\alpha+ik_1+il_1}\frac{\mathbf{B}(\alpha+l_1,\alpha+k_1-n+1)}{\Gamma(\alpha)}
\end{aligned}$$

$$
\begin{aligned}
&\times\prod_{j=1}^{i-1}\frac{\mathbf{B}(\alpha+l_1,(j+1)\alpha+(j+1)k_1+jl_1-n+1)}{\Gamma(\alpha)}\\
&+M_B^{i-1}M_G(\log t)^{(i-1)\alpha+n+(i-1)k_1+(i-1)l_1+k_2+l_2}\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}\\
&\times\prod_{j=1}^{i-1}\frac{\mathbf{B}(\alpha+l_1,jn+jk_1+(j-1)l_1+k_2+l_2+1)}{\Gamma(\alpha)}\\
\le{}&\|\phi_0\|M_B^i\frac{\mathbf{B}(\alpha+l_1,\alpha+k_1-n+1)}{\Gamma(\alpha)}\\
&\times\prod_{j=1}^{i-1}\frac{\mathbf{B}(\alpha+l_1,(j+1)\alpha+(j+1)k_1+jl_1-n+1)}{\Gamma(\alpha)}\\
&+M_B^{i-1}M_G\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}\prod_{j=1}^{i-1}\frac{\mathbf{B}(\alpha+l_1,jn+jk_1+(j-1)l_1+k_2+l_2+1)}{\Gamma(\alpha)},
\end{aligned}
$$

for $t\in(1,e]$, $i=1,2,\dots$. Similarly we have $\lim_{i\to+\infty}(\log t)^{n-\alpha}\phi_i(t)=(\log t)^{n-\alpha}\psi(t)$ uniformly on $(1,e]$. Then $\phi(t)\equiv\psi(t)$ on $(1,e]$. Then (3.14) has a unique solution ϕ. The proof is complete. □

Theorem 3.5. *Suppose that* (3.A3) *holds. Then x is a solution of IVP* (3.3) *if and only if $x\in LC_{n-\alpha}(1,e]$ is a solution of the integral equation* (3.14)*.*

Proof. Suppose that x is a solution of (3.3). Then $t\to(\log t)^{n-\alpha}x(t)$is continuous on $(1,e]$ by defining $(\log t)^{n-\alpha}x(t)|_{t=1}=\lim_{t\to1^+}(\log t)^{n-\alpha}x(t)$ and $\|x\|=r<+\infty$. So

$$
\begin{aligned}
&\lim_{s\to1^+}\int_1^s(\log\frac{s}{w})^{n-\alpha-1}x(w)\frac{dw}{w}\\
&=\lim_{s\to1^+}\int_1^s(\log\frac{s}{w})^{n-\alpha-1}(\log w)^{\alpha-n}(\log w)^{n-\alpha}x(w)\frac{dw}{w}\\
&=\lim_{s\to1^+}(\log\xi)^{n-\alpha}x(\xi)\int_1^s(\log\frac{s}{w})^{n-\alpha-1}(\log w)^{\alpha-n}\frac{dw}{w}\\
&\quad\text{(by the mean value theorem with }\xi\in(1,s))\\
&=\lim_{s\to1^+}(\log\xi)^{n-\alpha}x(\xi)\int_0^1(1-u)^{n-\alpha-1}u^{\alpha-n}du\quad(\text{because }\frac{\log w}{\log s}=u)\\
&=\frac{\eta_n}{\Gamma(\alpha-n+1)}\mathbf{B}(n-\alpha,\alpha-n+1).
\end{aligned}
$$

and for $v\in N[1,n-1]$ we have

$$
\begin{aligned}
&\lim_{t\to1^+}(s\frac{d}{ds})^{n-v}\Big(\int_1^s(\log\frac{s}{w})^{n-\alpha-1}x(w)\frac{dw}{w}\Big)\\
&=\Gamma(n-v-(\alpha-v))\lim_{t\to1^+}{}^{RLH}D_{0^+}^{\alpha-v}x(t)=\Gamma(n-\alpha)\eta_v.
\end{aligned}
$$

From (3.A3), we have

$$
(\log t)^{n-\alpha}\Big|\int_1^t(\log\frac{t}{s})^{\alpha-1}[B(s)x(s)+G(s)]\frac{ds}{s}\Big|
$$

$$
\begin{aligned}
&\le (\log t)^{n-\alpha}\int_1^t (\log\frac{t}{s})^{\alpha-1}[M_B r(\log s)^{\alpha-n}(\log s)^{k_1}(1-\log s)^{l_1}\\
&\quad + M_G(\log s)^{k_2}(1-\log s)^{l_2}]\frac{ds}{s}\\
&\le (\log t)^{n-\alpha}M_B r\int_1^t (\log\frac{t}{s})^{\alpha+l_1-1}(\log s)^{\alpha+k_1-n}\frac{ds}{s}\\
&\quad + (\log t)^{n-\alpha}M_G\int_1^t (\log\frac{t}{s})^{\alpha+l_2-1}(\log s)^{k_2}\frac{ds}{s}\\
&= M_B r(\log t)^{\alpha+k_1+l_1}\mathbf{B}(\alpha+l_1,k_1+\alpha)+M_G(\log t)^{n+k_1+l_1}\mathbf{B}(\alpha+l_2,k_2+1).
\end{aligned}
$$

So $t\to(\log t)^{n-\alpha}\int_1^t(\log\frac{t}{s})^{\alpha-1}[B(s)x(s)+G(s)]\frac{ds}{s}$ is defined on $(1,e]$ and

$$
\lim_{t\to 1^+}(\log t)^{n-\alpha}\int_1^t (\log\frac{t}{s})^{\alpha-1}[B(s)x(s)+G(s)]\frac{ds}{s}=0. \tag{3.15}
$$

Furthermore, similarly to Theorem 3.1 we have $t\to\int_1^t(\log\frac{t}{s})^{\alpha-1}[B(s)x(s)+G(s)]\frac{ds}{s}$ is continuous on $(1,e]$. So $t\to(\log t)^{n-\alpha}\int_1^t(\log\frac{t}{s})^{\alpha-1}[B(s)x(s)+G(s)]\frac{ds}{s}$ is continuous on $[1,e]$ by defining

$$
(\log t)^{n-\alpha}\int_1^t (\log\frac{t}{s})^{\alpha-1}[B(s)x(s)+G(s)]\frac{ds}{s}|_{t=1}=0. \tag{3.16}
$$

We have ${}^HI_{1^+}^{\alpha}\,{}^{RLH}D_{1^+}^{\alpha}x(t)={}^HI_{1^+}^{\alpha}[B(t)x(t)+G(t)]$. So

$$
\begin{aligned}
&\frac{1}{\Gamma(\alpha)}\int_1^t (\log\frac{t}{s})^{\alpha-1}[B(s)x(s)+G(s)]\frac{ds}{s}\\
&= {}^HI_{1^+}^{\alpha}[B(t)x(t)+G(t)]={}^HI_{1^+}^{\alpha}\,{}^{RLH}D_{1^+}^{\alpha}x(t)\\
&= \frac{1}{\Gamma(\alpha)}\frac{1}{\Gamma(n-\alpha)}\int_1^t (\log\frac{t}{s})^{\alpha-1}(s\frac{d}{ds})^n\Big(\int_1^s(\log\frac{s}{w})^{n-\alpha-1}x(w)\frac{dw}{w}\Big)\frac{ds}{s}\\
&= \frac{1}{\Gamma(\alpha)}\frac{1}{\Gamma(n-\alpha)}\int_1^t (\log\frac{t}{s})^{\alpha-1}d\Big[(s\frac{d}{ds})^{n-1}\Big(\int_1^s(\log\frac{s}{w})^{n-\alpha-1}x(w)\frac{dw}{w}\Big)\Big]\\
&= \frac{1}{\Gamma(\alpha)}\frac{1}{\Gamma(n-\alpha)}\Big[(\log\frac{t}{s})^{\alpha-1}(s\frac{d}{ds})^{n-1}\Big(\int_1^s(\log\frac{s}{w})^{n-\alpha-1}x(w)\frac{dw}{w}\Big)\Big|_1^t\\
&\quad + (\alpha-1)\int_1^t (\log\frac{t}{s})^{\alpha-2}(s\frac{d}{ds})^{n-1}\Big(\int_1^s(\log\frac{s}{w})^{n-\alpha-1}x(w)\frac{dw}{w}\Big)\frac{ds}{s}\Big]\\
&= -\frac{1}{\Gamma(\alpha)}\frac{1}{\Gamma(n-\alpha)}(\log t)^{\alpha-1}\lim_{t\to 1^+}(s\frac{d}{ds})^{n-1}(\int_1^s(\log\frac{s}{w})^{n-\alpha-1}x(w)\frac{dw}{w})\\
&\quad + \frac{1}{\Gamma(\alpha-1)}\frac{1}{\Gamma(n-\alpha)}\int_1^t (\log\frac{t}{s})^{\alpha-2}(s\frac{d}{ds})^{n-1}\Big(\int_1^s(\log\frac{s}{w})^{n-\alpha-1}x(w)\frac{dw}{w}\Big)\frac{ds}{s}\\
&= -\frac{\eta_1}{\Gamma(\alpha)}(\log t)^{\alpha-1}+\frac{1}{\Gamma(\alpha-1)}\frac{1}{\Gamma(n-\alpha)}\int_1^t (\log\frac{t}{s})^{\alpha-2}(s\frac{d}{ds})^{n-1}\\
&\quad \times\Big(\int_1^s(\log\frac{s}{w})^{n-\alpha-1}x(w)\frac{dw}{w}\Big)\frac{ds}{s}\\
&= \dots\\
&= -\sum_{v=1}^{n-1}\frac{\eta_v}{\Gamma(\alpha-v+1)}(\log t)^{\alpha-v}+\frac{1}{\Gamma(\alpha-n+1)}\frac{1}{\Gamma(n-\alpha)}\int_1^t (\log\frac{t}{s})^{\alpha-n}
\end{aligned}
$$

$$
\begin{aligned}
&\times \Big(\int_1^s (\log \frac{s}{w})^{-\alpha} x(w)\frac{dw}{w}\Big)' ds\\
&= -\sum_{v=1}^{n-1} \frac{\eta_v}{\Gamma(\alpha-v+1)}(\log t)^{\alpha-v} + \frac{1}{\Gamma(\alpha-n+2)}\frac{1}{\Gamma(n-\alpha)} t\\
&\times \Big[\int_1^t (\log \frac{t}{s})^{\alpha-n+1}\Big(\int_1^s (\log \frac{s}{w})^{-\alpha} x(w)\frac{dw}{w} Big)' ds\Big]'\\
&= -\sum_{v=1}^{n-1} \frac{\eta_v}{\Gamma(\alpha-v+1)}(\log t)^{\alpha-v} + \frac{1}{\Gamma(n-\alpha)\Gamma(\alpha-n+2)} t\\
&\times \Big[(\log \frac{t}{s})^{\alpha-n+1}\int_1^s (\log \frac{s}{w})^{n-\alpha-1} x(w)\frac{dw}{w}\Big|_1^t\\
&+ (\alpha-n+1)\int_1^t (\log \frac{t}{s})^{\alpha-n}\int_1^s (\log \frac{s}{w})^{n-\alpha-1} x(w)\frac{dw}{w}\frac{ds}{s}\Big]'\\
&= -\sum_{v=1}^{n-1} \frac{\eta_v}{\Gamma(\alpha-v+1)}(\log t)^{\alpha-v} + \frac{1}{\Gamma(n-\alpha)\Gamma(\alpha-n+2)} t\\
&\times \Big[\lim_{t\to 1^+}(\log t)^{\alpha-n+1}\int_1^s (\log \frac{s}{w})^{n-\alpha-1} x(w)\frac{dw}{w}\\
&+ (\alpha-n+1)\int_1^t\int_u^t (\log \frac{t}{s})^{\alpha-n}(\log \frac{s}{w})^{n-\alpha-1}\frac{ds}{s} x(w)\frac{dw}{w}\Big]'\\
&= -\sum_{v=1}^{n-1} \frac{\eta_v}{\Gamma(\alpha-v+1)}(\log t)^{\alpha-v} + \frac{1}{\Gamma(n-\alpha)\Gamma(\alpha-n+2)} t\\
&\times \Big[\frac{\eta_n}{\Gamma(\alpha-n+1)}\mathbf{B}(n-\alpha,\alpha-n+1)(\log t)^{\alpha-n+1}\\
&+ (\alpha-n+1)\int_1^t\int_0^1 (1-u)^{\alpha-n}u^{n-\alpha-1} du\, x(w)\frac{dw}{w}\Big]'\\
&= x(t) - \sum_{v=1}^{n} \frac{\eta_v}{\Gamma(\alpha-v+1)}(\log t)^{\alpha-v}.
\end{aligned}
$$

Then $x \in LC_{n-\alpha}(1,e]$ is a solution of (3.14).

On the other hand, if x is a solution of (3.14), Cases 1, 2, 3 and (3.15) imply $\lim_{t\to 1^+}(\log t)^{n-\alpha}x(t) = \frac{\eta_n}{\Gamma(\alpha-n+1)}$. Then $x \in LC_{n-\alpha}(1,e]$. Furthermore, by Definition 2.5 we have

$$
\begin{aligned}
&{}^{RLH}D_{1^+}^{\alpha} x(t)\\
&= \frac{1}{\Gamma(n-\alpha)}(t\frac{d}{dt})^n\Big(\int_1^t (\log \frac{t}{s})^{n-\alpha-1} x(s)\frac{ds}{s}\Big)\\
&= \frac{1}{\Gamma(n-\alpha)}(t\frac{d}{dt})^n\Big[\int_1^t (\log \frac{t}{s})^{n-\alpha-1}\Big(\sum_{v=1}^{n}\frac{\eta_v}{\Gamma(\alpha-v+1)}(\log s)^{\alpha-v}\\
&+ \frac{1}{\Gamma(\alpha)}\int_1^s (\log \frac{s}{w})^{\alpha-1}[A(w)x(w)+F(w)]\frac{dw}{w}\Big)\frac{ds}{s}\Big]\\
&= \frac{1}{\Gamma(n-\alpha)}\sum_{v=1}^{n}\frac{\eta_v}{\Gamma(\alpha-v+1)}(t\frac{d}{dt})^n\int_1^t (\log \frac{t}{s})^{n-\alpha-1}(\log s)^{\alpha-v}\frac{ds}{s}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(\alpha)}\frac{1}{\Gamma(n-\alpha)}(t\frac{d}{dt})^n\int_1^t(\log\frac{t}{s})^{n-\alpha-1}\int_1^s(\log\frac{s}{w})^{\alpha-1}[B(w)x(w)+G(w)]\frac{dw}{w}\frac{ds}{s}\\
&=\frac{1}{\Gamma(n-\alpha)}\sum_{v=1}^n\frac{\eta_v}{\Gamma(\alpha-v+1)}(t\frac{d}{dt})^n(\log t)^{n-v}\int_0^1(1-w)^{n-\alpha-1}w^{\alpha-v}dw\\
&\quad+\frac{1}{\Gamma(\alpha)}\frac{1}{\Gamma(n-\alpha)}(t\frac{d}{dt})^n\int_1^t(\log\frac{t}{s})^{n-\alpha-1}\int_1^s(\log\frac{s}{w})^{\alpha-1}[B(w)x(w)+G(w)]\frac{dw}{w}\frac{ds}{s}\\
&=\frac{1}{\Gamma(\alpha)}\frac{1}{\Gamma(n-\alpha)}(t\frac{d}{dt})^n\int_1^t\int_u^t(\log\frac{t}{s})^{n-\alpha-1}(\log\frac{s}{w})^{\alpha-1}\frac{ds}{s}[B(w)x(w)+G(w)]\frac{dw}{w}\\
&=\frac{1}{\Gamma(\alpha)}\frac{1}{\Gamma(n-\alpha)}(t\frac{d}{dt})^n\int_1^t(\log\frac{t}{w})^{n-1}\int_0^1(1-w)^{n-\alpha-1}w^{\alpha-1}dw\\
&\quad\times[B(w)x(w)+G(w)]\frac{dw}{w}\\
&=\frac{1}{\Gamma(n)}(t\frac{d}{dt})^n\int_1^t(\log\frac{t}{w})^{n-1}[B(w)x(w)+G(w)]\frac{dw}{w}\\
&=B(t)x(t)+G(t).
\end{aligned}
$$

So $x\in LC_{n-\alpha}(1,e]$ is a solution of IVP(3.3). The proof is complete. □

Theorem 3.6. *Suppose that* (3.A3) *holds. Then* (3.14) *has a unique solution. If* $B(t)\equiv\lambda$ *and there exists constants* $k_2>-1$, $l_2\le 0$ *with* $l_2>\max\{-\alpha,-n-k_2\}$ *and* $M_G\ge 0$ *such that* $|G(t)|\le M_Gt^{k_2}(1-t)^{l_2}$ *for all* $t\in(1,e)$, *then following problem*

$$
\begin{gathered}
{}^{RLH}D_{1^+}^{\alpha}x(t)=\lambda x(t)+G(t),\quad a.e.,\ t\in(1,e],\\
\lim_{t\to1^+}(\log t)^{n-\alpha}x(t)=\frac{\eta_n}{\Gamma(\alpha-n+1)},\\
\lim_{t\to1^+}{}^{RLH}D_{1^+}^{\alpha-j}x(t)=\eta_j,\quad j\in\mathbb{N}[1,n-1]
\end{gathered}\tag{3.17}
$$

has a unique solution

$$
\begin{aligned}
x(t)=&\sum_{v=1}^n\eta_v(\log t)^{\alpha-v}\mathbf{E}_{\alpha,\alpha-v+1}(\lambda(\log t)^{\alpha})\\
&+\int_1^t(\log\frac{t}{u})^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(\log\frac{t}{u})^{\alpha})G(s)\frac{ds}{s},\quad t\in(1,e].
\end{aligned}\tag{3.18}
$$

Proof. (i) From Claims 1, 2 and 3, (3.14) has a unique solution.

(ii) From the assumption and $B(t)\equiv\lambda$, one sees that (3.A3) holds with $k_1=l_1=0$ and k_2,l_2 mentioned in assumption. Thus (3.17) has a unique solution. From the Picard function sequence we obtain

$$
\begin{aligned}
&\phi_i(t)\\
&=\sum_{v=1}^n\frac{\eta_v}{\Gamma(\alpha-v+1)}(\log t)^{\alpha-v}+\lambda\frac{1}{\Gamma(\alpha)}\int_1^t(\log\frac{t}{s})^{\alpha-1}\phi_{i-1}(s)\frac{ds}{s}\\
&\quad+\frac{1}{\Gamma(\alpha)}\int_1^t(\log\frac{t}{s})^{\alpha-1}G(s)\frac{ds}{s}\\
&=\sum_{v=1}^n\frac{\eta_v}{\Gamma(\alpha-v+1)}(\log t)^{\alpha-v}+\lambda\sum_{v=1}^n\frac{\eta_v}{\Gamma(\alpha-v+1)}\frac{1}{\Gamma(\alpha)}\int_1^t(\log\frac{t}{s})^{\alpha-1}(\log s)^{\alpha-v}\frac{ds}{s}
\end{aligned}
$$

$$
\begin{aligned}
&+\lambda^2\frac{1}{\Gamma(\alpha)}\int_1^t(\log\frac{t}{s})^{\alpha-1}\frac{1}{\Gamma(\alpha)}\int_1^s(\log\frac{s}{u})^{\alpha-1}\phi_{i-2}(u)\frac{du}{u}\frac{ds}{s}\\
&+\frac{1}{\Gamma(\alpha)}\lambda\frac{1}{\Gamma(\alpha)}\int_1^t(\log\frac{t}{s})^{\alpha-1}\int_1^s(\log\frac{s}{u})^{\alpha-1}G(u)\frac{du}{u}\frac{ds}{s}+\frac{1}{\Gamma(\alpha)}\int_1^t(\log\frac{t}{s})^{\alpha-1}G(s)\frac{ds}{s}\\
&=\sum_{v=1}^n\frac{\eta_v}{\Gamma(\alpha-v+1)}(\log t)^{\alpha-v}+\lambda\sum_{v=1}^n\frac{\eta_v}{\Gamma(\alpha-v+1)}\frac{1}{\Gamma(\alpha)}(\log t)^{2\alpha-v}\\
&\times\int_0^1(1-w)^{\alpha-1}w^{\alpha-v}dw+\lambda^2\frac{1}{\Gamma(\alpha)}\int_1^t\int_u^t(\log\frac{t}{s})^{\alpha-1}\frac{1}{\Gamma(\alpha)}(\log\frac{s}{u})^{\alpha-1}\frac{ds}{s}\phi_{i-2}(u)\frac{du}{u}\\
&+\frac{1}{\Gamma(\alpha)}\lambda\frac{1}{\Gamma(\alpha)}\int_1^t\int_u^t(\log\frac{t}{s})^{\alpha-1}(\log\frac{s}{u})^{\alpha-1}\frac{ds}{s}G(u)\frac{du}{u}+\frac{1}{\Gamma(\alpha)}\int_1^t(\log\frac{t}{s})^{\alpha-1}G(s)\frac{ds}{s}\\
&=\sum_{v=1}^n\frac{\eta_v}{\Gamma(\alpha-v+1)}(\log t)^{\alpha-v}+\lambda\sum_{v=1}^n\frac{\eta_v}{\Gamma(2\alpha-v+1)}(\log t)^{2\alpha-v}\\
&+\frac{\lambda^2}{\Gamma(2\alpha)}\int_1^t(\log\frac{t}{u})^{2\alpha-1}\phi_{i-2}(u)\frac{du}{u}\\
&+\frac{\lambda}{\Gamma(2\alpha)}\int_1^t(\log\frac{t}{u})^{2\alpha-1}G(u)\frac{du}{u}+\frac{1}{\Gamma(\alpha)}\int_1^t(\log\frac{t}{s})^{\alpha-1}G(s)\frac{ds}{s}\\
&=\sum_{v=1}^n\eta_v(\log t)^{\alpha-v}\Big(\frac{1}{\Gamma(\alpha-v+1)}+\frac{\lambda(\log t)^{\alpha}}{\Gamma(2\alpha-v+1)}\Big)+\frac{\lambda^2}{\Gamma(2\alpha)}\int_1^t(\log\frac{t}{u})^{2\alpha-1}\phi_{i-2}(u)\frac{du}{u}\\
&+\int_1^t(\log\frac{t}{u})^{\alpha-1}\Big(\frac{\lambda}{\Gamma(2\alpha)}(\log\frac{t}{u})^{\alpha}+\frac{1}{\Gamma(\alpha)}\Big)G(s)\frac{ds}{s}\\
&=\dots\\
&=\sum_{v=1}^n\eta_v(\log t)^{\alpha-v}\Big(\sum_{j=0}^{i-1}\frac{\lambda^j(\log t)^{j\alpha}}{\Gamma(j\alpha+\alpha-v+1)}\Big)+\frac{\lambda^i}{\Gamma(i\alpha)}\int_1^t(\log\frac{t}{u})^{i\alpha-1}\phi_0(u)\frac{du}{u}\\
&+\int_1^t(\log\frac{t}{u})^{\alpha-1}\Big(\sum_{j=0}^{i-1}\frac{\lambda^j}{\Gamma((j+1)\alpha)}(\log\frac{t}{u})^{j\alpha}\Big)G(s)\frac{ds}{s}\\
&=\sum_{v=1}^n\eta_v(\log t)^{\alpha-v}\Big(\sum_{j=0}^{i}\frac{\lambda^j(\log t)^{j\alpha}}{\Gamma(j\alpha+\alpha-v+1)}\Big)\\
&+\int_1^t(\log\frac{t}{u})^{\alpha-1}\Big(\sum_{j=0}^{i-1}\frac{\lambda^j}{\Gamma((j+1)\alpha)}(\log\frac{t}{u})^{j\alpha}\Big)G(s)\frac{ds}{s}\\
&\to\sum_{v=1}^n\eta_v(\log t)^{\alpha-v}\mathbf{E}_{\alpha,\alpha-v+1}(\lambda(\log t)^{\alpha})+\int_1^t(\log\frac{t}{u})^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(\log\frac{t}{u})^{\alpha})G(s)\frac{ds}{s}.
\end{aligned}
$$

Then $x(t)=\lim_{i\to+\infty}\phi_i(t)$ is the unique solution of (3.17). x is just as in (3.18). The proof is complete. □

To obtain solutions of (3.4), we need the following assumption:

(3.A4) there exist constants $k_i>-\alpha+n-1$, $l_i\le 0$ with $l_i>\max\{-\alpha,-\alpha-k_i\}$, $M_B\ge 0$ and $M_G\ge 0$ such that $|B(t)|\le M_B(\log t)^{k_1}(1-\log t)^{l_1}$ and $|G(t)|\le M_G(\log t)^{k_2}(1-\log t)^{l_2}$ for all $t\in(1,e)$.

We choose the Picard function sequence as

$$\phi_0(t)=\sum_{j=0}^{n-1}\frac{\eta_j}{j!}(\log t)^j,\quad t\in(1,e],$$

$$\phi_i(t)=\sum_{j=0}^{n-1}\frac{\eta_j}{j!}(\log t)^j+\frac{1}{\Gamma(\alpha)}\int_1^t(\log\frac{t}{s})^{\alpha-1}[B(s)\phi_{i-1}(s)+G(s)]\frac{ds}{s},$$

for $t\in(1,e]$, $i=1,2,\dots$.

Claim 1. $\phi_i\in C(1,e]$.

Proof. On sees that $\phi_0\in C[1,e]$. From

$$\begin{aligned}
&|\int_1^t(\log\frac{t}{s})^{\alpha-1}[B(s)\phi_0(s)+G(s)]\frac{ds}{s}|\\
&\le\int_1^t(\log\frac{t}{s})^{\alpha-1}[M_B\|\phi_0\|(\log s)^{k_1}(1-\log s)^{l_1}\\
&\quad+M_G(\log s)^{k_2}(1-\log s)^{l_2}]\frac{ds}{s}\\
&\le M_B\|\phi_0\|\int_1^t(\log\frac{t}{s})^{\alpha-1}(\log s)^{k_1}(1-\log s)^{l_1}\frac{ds}{s}\\
&\quad+M_G\int_1^t(\log\frac{t}{s})^{\alpha-1}(\log s)^{k_2}(1-\log s)^{l_2}\frac{ds}{s}\\
&=M_B\|\phi_0\|(\log t)^{\alpha+k_1+l_1}\mathbf{B}(\alpha+l_1,k_1+1)+M_G(\log t)^{\alpha+k_2+l_2}\mathbf{B}(\alpha+l_2,k_2+1)\\
&\to0\quad\text{as } t\to1^+,
\end{aligned}$$

we obtain that $\lim_{t\to1^+}\phi_1(s)$ exists and ϕ_1 is continuous on $(1,e]$. Then $\phi_1\in C[1,e]$. By mathematical induction, we see that $\phi_i\in C[1,e]$. □

Claim 2. ϕ_i converges uniformly on $[1,e]$.

Proof. For $t\in[1,e]$ we have

$$\begin{aligned}
&|\phi_1(t)-\phi_0(t)|\\
&=|\frac{1}{\Gamma(\alpha)}\int_1^t(\log\frac{t}{s})^{\alpha-1}[B(s)\phi_0(s)+G(s)]\frac{ds}{s}|\\
&\le\frac{1}{\Gamma(\alpha)}\int_1^t(\log\frac{t}{s})^{\alpha-1}[M_B\|\phi_0\|(\log s)^{k_1}(1-\log s)^{l_1}+M_G(\log s)^{k_2}(1-\log s)^{l_2}]\frac{ds}{s}\\
&\le\|\phi_0\|M_B\frac{1}{\Gamma(\alpha)}\int_1^t(\log\frac{t}{s})^{\alpha+l_1-1}(\log s)^{k_1}\frac{ds}{s}+M_G\frac{1}{\Gamma(\alpha)}\int_1^t(\log\frac{t}{s})^{\alpha+l_2-1}(\log s)^{k_2}\frac{ds}{s}\\
&=\|\phi_0\|M_B(\log t)^{\alpha+k_1+l_1}\frac{\mathbf{B}(\alpha+l_1,k_1+1)}{\Gamma(\alpha)}\\
&\quad+M_G(\log t)^{\alpha+k_2+l_2}\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}.
\end{aligned}$$

So

$$|\phi_2(t)-\phi_1(t)|$$

$$
\begin{aligned}
&= \Big|\frac{1}{\Gamma(\alpha)}\int_1^t (\log\frac{t}{s})^{\alpha-1}B(s)[\phi_1(s)-\phi_0(s)]\frac{ds}{s}\Big| \\
&\le \frac{1}{\Gamma(\alpha)}\int_1^t (\log\frac{t}{s})^{\alpha-1}M_B(\log s)^{k_1}(1-\log s)^{l_1} \\
&\quad\times \Big(\|\phi_0\|M_B(\log s)^{\alpha+k_1+l_1}\frac{\mathbf{B}(\alpha+l_1,k_1+1)}{\Gamma(\alpha)} \\
&\quad+ M_G(\log s)^{\alpha+k_2+l_2}\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}\Big)\frac{ds}{s} \\
&\le \frac{1}{\Gamma(\alpha)}\|\phi_0\|M_B^2\int_1^t (\log\frac{t}{s})^{\alpha+l_1-1}(\log s)^{\alpha+2k_1+l_1}\frac{\mathbf{B}(\alpha+l_1,k_1+1)}{\Gamma(\alpha)}\frac{ds}{s} \\
&\quad+ M_BM_G \le \frac{1}{\Gamma(\alpha)}\int_1^t (\log\frac{t}{s})^{\alpha+l_1-1}(\log s)^{\alpha+k_1+k_2+l_2}\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}\frac{ds}{s} \\
&= \|\phi_0\|M_B^2(\log t)^{2k_1+2l_1+2}\frac{\mathbf{B}(\alpha+l_1,k_1+1)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_1,\alpha+2k_1+l_1+1)}{\Gamma(\alpha)} \\
&\quad+ M_BM_G(\log t)^{2\alpha+k_1+k_2+l_1+l_2}\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)} \\
&\quad\times \frac{\mathbf{B}(\alpha+l_1,\alpha+k_1+k_2+l_2+1)}{\Gamma(\alpha)},
\end{aligned}
$$

and

$$
\begin{aligned}
&|\phi_3(t)-\phi_2(t)| \\
&= \Big|\frac{1}{\Gamma(\alpha)}\int_1^t (\log\frac{t}{s})^{\alpha-1}B(s)[\phi_2(s)-\phi_1(s)]\frac{ds}{s}\Big| \\
&\le \frac{1}{\Gamma(\alpha)}\int_1^t (\log\frac{t}{s})^{\alpha-1}M_B(\log s)^{k_1}(1-\log s)^{l_1} \\
&\quad\times \Big(\|\phi_0\|M_B^2(\log s)^{2k_1+2l_1+2}\frac{\mathbf{B}(\alpha+l_1,k_1+1)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_1,\alpha+2k_1+l_1+1)}{\Gamma(\alpha)} \\
&\quad+ M_BM_G(\log s)^{2\alpha+k_1+k_2+l_1+l_2}\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)} \\
&\quad\times \frac{\mathbf{B}(\alpha+l_1,\alpha+k_1+k_2+l_2+1)}{\Gamma(\alpha)}\Big)\frac{ds}{s} \\
&\le \|\phi_0\|M_B^3\frac{1}{\Gamma(\alpha)}\int_1^t (\log\frac{t}{s})^{\alpha+l_1-1}(\log)s^{2\alpha+3k_1+2l_1}\frac{\mathbf{B}(\alpha+l_1,k_1+1)}{\Gamma(\alpha)} \\
&\quad\times \frac{\mathbf{B}(\alpha+l_1,\alpha+2k_1+l_1+1)}{\Gamma(\alpha)}\frac{ds}{s} \\
&\quad+ M_B^2M_G\frac{1}{\Gamma(\alpha)}\int_1^t (\log\frac{t}{s})^{\alpha+l_1-1}(\log s)^{2\alpha+2k_1+k_2+l_1+l_2} \\
&\quad\times \frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_1,\alpha+k_1+k_2+l_2+1)}{\Gamma(\alpha)}\frac{ds}{s} \\
&= \|\phi_0\|M_B^3(\log t)^{3\alpha+3k_1+3l_1}\frac{\mathbf{B}(\alpha+l_1,k_1+1)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_1,\alpha+2k_1+l_1+1)}{\Gamma(\alpha)}
\end{aligned}
$$

$$
\begin{aligned}
&\times \frac{\mathbf{B}(\alpha+l_1, 2\alpha+3k_1+2l_1+1)}{\Gamma(\alpha)}\\
&+ M_B^2 M_G(\log t)^{3\alpha+2k_1+k_2+2l_1+l_2}\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_1,\alpha+k_1+k_2+l_2+1)}{\Gamma(\alpha)}\\
&\times \frac{\mathbf{B}(\alpha+l_1, 2\alpha+2k_1+k_2+l_1+l_2+1)}{\Gamma(\alpha)}.
\end{aligned}
$$

$$
\begin{aligned}
&|\phi_4(t)-\phi_3(t)|\\
&= \Big|\frac{1}{\Gamma(\alpha)}\int_1^t (\log\frac{t}{s})^{\alpha-1}B(s)[\phi_3(s)-\phi_2(s)]\frac{ds}{s}\Big|\\
&\quad + \frac{\|\phi_0\|M_B^4}{\Gamma(\alpha)}\int_1^t (\log\frac{t}{s})^{\alpha+l_1-1}(\log s)^{3\alpha+4k_1+3l_1}\frac{\mathbf{B}(\alpha+l_1,k_1+1)}{\Gamma(\alpha)}\\
&\quad \times \frac{\mathbf{B}(\alpha+l_1,\alpha+2k_1+l_1+1)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_1,2\alpha+3k_1+2l_1+1)}{\Gamma(\alpha)}\frac{ds}{s}\\
&\quad + \frac{M_B^3M_G}{\Gamma(\alpha)}\int_1^t (\log\frac{t}{s})^{\alpha+l_1-1}(\log s)^{3\alpha+3k_1+k_2+2l_1+l_2}\\
&\quad \times \frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_1,\alpha+k_1+k_2+l_2+1)}{\Gamma(\alpha)}\\
&\quad \times \frac{\mathbf{B}(\alpha+l_1,2\alpha+2k_1+k_2+l_1+l_2+1)}{\Gamma(\alpha)}\frac{ds}{s}\\
&\le \|\phi_0\|M_B^4(\log t)^{4\alpha+4k_1+4l_1}\frac{\mathbf{B}(\alpha+l_1,k_1+1)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_1,\alpha+2k_1+l_1+1)}{\Gamma(\alpha)}\\
&\quad \times \frac{\mathbf{B}(\alpha+l_1,2\alpha+3k_1+2l_1+1)}{\Gamma(\alpha)}\frac{ds}{s}\frac{\mathbf{B}(\alpha+l_1,3\alpha+4k_1+3l_1+1)}{\Gamma(\alpha)}\\
&\quad + M_B^3M_G(\log t)^{4\alpha+3k_1+k_2+3l_1+l_2}\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}\\
&\quad \times \frac{\mathbf{B}(\alpha+l_1,\alpha+k_1+k_2+l_2+1)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_1,2\alpha+2k_1+k_2+l_1+l_2+1)}{\Gamma(\alpha)}\\
&\quad \times \frac{\mathbf{B}(\alpha+l_1,3\alpha+3k_1+k_2+2l_1+l_2+1)}{\Gamma(\alpha)}.
\end{aligned}
$$

Similarly by mathematical induction, for every $i = 1, 2, \dots$ we obtain

$$
\begin{aligned}
&|\phi_i(t)-\phi_{i-1}(t)|\\
&\le \|\phi_0\|M_B^i(\log t)^{i\alpha+ik_1+il_1}\prod_{j=0}^{i-1}\frac{\mathbf{B}(\alpha+l_1,j\alpha+(j+1)k_1+jl_1+1}{\Gamma(\alpha)}\\
&\quad + M_B^{i-1}M_G(\log t)^{i\alpha+(i-1)k_1+k_2+(i-1)l_1+l_2}\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}\\
&\quad \times \prod_{j=1}^{i-1}\frac{\mathbf{B}(\alpha+l,j\alpha+jk_1+k_2+(j-1)l_1+l_2+1)}{\Gamma(\alpha)}\\
&\le \|\phi_0\|M_B^i\prod_{j=0}^{i-1}\frac{\mathbf{B}(\alpha+l_1,j\alpha+(j+1)k_1+jl_1+1}{\Gamma(\alpha)}
\end{aligned}
$$

$$+ M_B^{i-1} M_G \frac{\mathbf{B}(\alpha+l_2, k_2+1)}{\Gamma(\alpha)} \prod_{j=1}^{i-1} \frac{\mathbf{B}(\alpha+l, j\alpha+jk_1+k_2+(j-1)l_1+l_2+1)}{\Gamma(\alpha)},$$

for $t \in [1, e]$. Similarly we can prove that both

$$\begin{aligned}\sum_{i=1}^{+\infty} u_i &= \sum_{i=1}^{+\infty} M_B^{i-1} M_G \frac{\mathbf{B}(\alpha+l_2, k_2+1)}{\Gamma(\alpha)}\\ &\quad\times \prod_{j=1}^{i-1} \frac{\mathbf{B}(\alpha+l, j\alpha+jk_1+k_2+(j-1)l_1+l_2+1)}{\Gamma(\alpha)},\\ \sum_{i=1}^{+\infty} v_i &= \sum_{i=1}^{+\infty} M_B^{i-1} M_G \frac{\mathbf{B}(\alpha+l_2, k_2+1)}{\Gamma(\alpha)}\\ &\quad\times \prod_{j=1}^{i-1} \frac{\mathbf{B}(\alpha+l, j\alpha+jk_1+k_2+(j-1)l_1+l_2+1)}{\Gamma(\alpha)}\end{aligned}$$

are convergent. Hence,

$$\phi_0(t) + [\phi_1(t) - \phi_0(t)] + [\phi_2(t) - \phi_1(t)] + \dots + [\phi_i(t) - \phi_{i-1}(t)] + \dots, \quad t \in [1, e]$$

is uniformly convergent. Then $\{\phi_i(t)\}$ is convergent uniformly on $[1, e]$. □

Claim 3. $\phi(t) = \lim_{i\to+\infty} \phi_i(t)$ defined on $(1, e]$ is a unique continuous solution of the integral equation

$$x(t) = \sum_{j=0}^{n-1} \frac{\eta_j}{j!} (\log t)^j + \frac{1}{\Gamma(\alpha)} \int_1^t (\log \frac{t}{s})^{\alpha-1} [B(s)x(s) + G(s)] \frac{ds}{s}.$$

Proof. From $\lim_{i\to+\infty} \phi_i(t) = \phi(t)$ and the uniformly convergence, we see that $\phi(t)$ is continuous on $[1, e]$. From

$$\begin{aligned}&\Big|\int_1^t (\log \frac{t}{s})^{\alpha-1} [B(s)\phi_{p-1}(s) + G(s)] \frac{ds}{s} - \int_1^t (\log \frac{t}{s})^{\alpha-1} [B(s)\phi_{q-1}(s) + G(s)] \frac{ds}{s}\Big|\\ &\le M_B \|\phi_{p-1} - \phi_{q-1}\| \int_1^t (\log \frac{t}{s})^{\alpha-1} (\log s)^{k_1} (1 - \log s)^{l_1} \frac{ds}{s}\\ &\le M_B \|\phi_{p-1} - \phi_{q-1}\| \int_1^t (\log \frac{t}{s})^{\alpha+l_1-1} (\log s)^{k_1} \frac{ds}{s}\\ &\le M_B \|\phi_{p-1} - \phi_{q-1}\| (\log t)^{\alpha+k_1+l_1} \frac{\mathbf{B}(\alpha+l_1, k_1+1)}{\Gamma(\alpha)}\\ &\le M_B \|\phi_{p-1} - \phi_{q-1}\| \frac{\mathbf{B}(\alpha+l_1, k_1+1)}{\Gamma(\alpha)}\\ &\to 0 \quad \text{uniformly as } p, q \to +\infty,\end{aligned}$$

we know that

$$\begin{aligned}\phi(t) &= \lim_{i\to\infty} \phi_i(t)\\ &= \lim_{i\to+\infty} \Big[\sum_{j=0}^{n-1} \frac{\eta_j}{j!} (\log t)^j + \frac{1}{\Gamma(\alpha)} \int_1^t (\log \frac{t}{s})^{\alpha-1} [B(s)\phi_{i-1}(s) + G(s)] \frac{ds}{s}\Big]\end{aligned}$$

$$= \sum_{j=0}^{n-1} \frac{\eta_j}{j!} (\log t)^j + \frac{1}{\Gamma(\alpha)} \int_1^t (\log \frac{t}{s})^{\alpha-1} [B(s)\phi(s) + G(s)] \frac{ds}{s}.$$

Then ϕ is a continuous solution of (3.1) defined on $(1, e]$.

Suppose that ψ defined on $(1, e]$ is also a solution of (3.1). Then

$$\psi(t) = \sum_{j=0}^{n-1} \frac{\eta_j}{j!} (\log t)^j + \frac{1}{\Gamma(\alpha)} \int_1^t (\log \frac{t}{s})^{\alpha-1} [B(s)\phi(s) + G(s)] \frac{ds}{s}, t \in (1, e].$$

We need to prove that $\phi(t) \equiv \psi(t)$ on $(0, 1]$. Now we have

$$\begin{aligned}
&|\psi(t) - \phi_0(t)| \\
&= \frac{1}{\Gamma(\alpha)} \Big| \int_1^t (\log \frac{t}{s})^{\alpha-1} [B(s)\phi_0(s) + G(s)] \frac{ds}{s} \Big| \\
&\le \|\phi_0\| M_B (\log t)^{\alpha+k_1+l_1} \frac{\mathbf{B}(\alpha+l_1, k_1+1)}{\Gamma(\alpha)} + M_G (\log t)^{\alpha+k_2+l_2} \frac{\mathbf{B}(\alpha+l_2, k_2+1)}{\Gamma(\alpha)}.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
&|\psi(t) - \phi_1(t)| \\
&= \frac{1}{\Gamma(\alpha)} \Big| \int_1^t (\log \frac{t}{s})^{\alpha-1} B(s) [\psi(s) - \phi_0(s)] \frac{ds}{s} \Big| \\
&\le \|\phi_0\| M_B^2 (\log t)^{2k_1+2l_1+2} \frac{\mathbf{B}(\alpha+l_1, k_1+1)}{\Gamma(\alpha)} \frac{\mathbf{B}(\alpha+l_1, \alpha+2k_1+l_1+1)}{\Gamma(\alpha)} \\
&\quad + M_B M_G (\log t)^{2\alpha+k_1+k_2+l_1+l_2} \frac{\mathbf{B}(\alpha+l_2, k_2+1)}{\Gamma(\alpha)} \frac{\mathbf{B}(\alpha+l_1, \alpha+k_1+k_2+l_2+1)}{\Gamma(\alpha)}.
\end{aligned}$$

By mathematical induction, we obtain

$$\begin{aligned}
&|\psi(t) - \phi_{i-1}(t)| \\
&= \frac{1}{\Gamma(\alpha)} \Big| \int_1^t (\log \frac{t}{s})^{\alpha-1} B(s) [\psi(s) - \phi_{m-2}(s)] \frac{ds}{s} \Big| \\
&\le \|\phi_0\| M_B^i (\log t)^{i\alpha+ik_1+il_1} \prod_{j=0}^{i-1} \frac{\mathbf{B}(\alpha+l_1, j\alpha+(j+1)k_1+jl_1+1}{\Gamma(\alpha)} \\
&\quad + M_B^{i-1} M_G (\log t)^{i\alpha+(i-1)k_1+k_2+(i-1)l_1+l_2} \frac{\mathbf{B}(\alpha+l_2, k_2+1)}{\Gamma(\alpha)} \\
&\quad \times \prod_{j=1}^{i-1} \frac{\mathbf{B}(\alpha+l, j\alpha+jk_1+k_2+(j-1)l_1+l_2+1)}{\Gamma(\alpha)} \\
&\le \|\phi_0\| M_B^i \prod_{j=0}^{i-1} \frac{\mathbf{B}(\alpha+l_1, j\alpha+(j+1)k_1+jl_1+1}{\Gamma(\alpha)} \\
&\quad + M_B^{i-1} M_G \frac{\mathbf{B}(\alpha+l_2, k_2+1)}{\Gamma(\alpha)} \prod_{j=1}^{i-1} \frac{\mathbf{B}(\alpha+l, j\alpha+jk_1+k_2+(j-1)l_1+l_2+1)}{\Gamma(\alpha)},
\end{aligned}$$

for $t \in [1, e]$. Similarly we have $\lim_{i \to +\infty} \phi_i(t) = \psi(t)$ uniformly on $[1, e]$. Then $\phi(t) \equiv \psi(t)$ on $(1, e]$. Then (3.1) has a unique solution ϕ. The proof is complete. □

Theorem 3.7. *Suppose that* (3.A4) *holds. Then* $x \in C(1,e]$ *is a solution of IVP* (3.4) *if and only if* $x \in C(1,e]$ *is a solution of the integral equation* (3.1).

Proof. Suppose that $x \in C(1,e]$ is a solution of (3.4). Then $t \to x(t)$ is continuous on $[1,e]$ by defining $x(t)|_{t=1} = \lim_{t\to 1^+} x(t)$ and $\|x\| = r < +\infty$. One can see that

$$\begin{aligned}
&\int_1^t (\log\frac{t}{s})^{\alpha-1}(\log s)^{k_1}(1-\log s)^{l_1}\frac{ds}{s}\\
&\le \int_1^t (\log\frac{t}{s})^{\alpha+l_1-1}(\log s)^{k_1}\frac{ds}{s} \quad (\text{because } \frac{\log s}{\log t} = u)\\
&= (\log t)^{\alpha+k_1+l_1}\int_0^1 (1-u)^{\alpha+l_1-1}u^{k_1}\,du\\
&\le (\log t)^{\alpha+k_1+l_1}\int_0^1 (1-u)^{\alpha+l_1-1}u^{k_1}\,du\\
&= (\log t)^{\alpha+k_1+l_1}\mathbf{B}(\alpha+l_1,k_1+1).
\end{aligned}$$

From (3.A4), for $t \in (1,e]$ we have

$$\begin{aligned}
&\Big|\int_1^t (\log\frac{t}{s})^{\alpha-1}[B(s)x(s)+G(s)]\frac{ds}{s}\Big|\\
&\le \int_1^t (\log\frac{t}{s})^{\alpha-1}[M_B r(\log s)^{k_1}(1-\log s)^{l_1} + M_G(\log s)^{k_2}(1-\log s)^{l_2}]\frac{ds}{s}\\
&\le M_B r\int_1^t (\log\frac{t}{s})^{\alpha-1}(\log s)^{k_1}(1-\log s)^{l_1}\frac{ds}{s}\\
&\quad + M_G\int_1^t (\log\frac{t}{s})^{\alpha-1}(\log s)^{k_2}(1-\log s)^{l_2}\frac{ds}{s}\\
&= M_B r(\log t)^{\alpha+k_1+l_1}\mathbf{B}(\alpha+l_1,k_1+1) + M_G(\log t)^{\alpha+k_2+l_2}\mathbf{B}(\alpha+l_2,k_2+1).
\end{aligned}$$

So $t \to \int_1^t (\log\frac{t}{s})^{\alpha-1}[B(s)x(s)+G(s)]\frac{ds}{s}$ is defined on $(1,e]$ and

$$\lim_{t\to 1^+}\int_1^t (\log\frac{t}{s})^{\alpha-1}[B(s)x(s)+G(s)]\frac{ds}{s} = 0. \tag{3.19}$$

Furthermore, similarly to Theorem 3.1 we have $t \to \int_1^t (\log\frac{t}{s})^{\alpha-1}[B(s)x(s)+G(s)]\frac{ds}{s}$ is continuous on $(1,e]$. So $t \to \int_1^t (\log\frac{t}{s})^{\alpha-1}[B(s)x(s)+G(s)]\frac{ds}{s}$ is continuous on $[1,e]$ by defining

$$\int_1^t (\log\frac{t}{s})^{\alpha-1}[B(s)x(s)+G(s)]\frac{ds}{s}\Big|_{t=1} = 0. \tag{3.20}$$

One sees that

$$\int_w^t (\log\frac{t}{s})^{\alpha-1}(\log\frac{s}{w})^{-\alpha}\frac{ds}{s} = \int_0^1 (1-u)^{\alpha-1}u^{-\alpha}du = \Gamma(1-\alpha)\Gamma(\alpha),$$

because $\frac{\log s-\log w}{\log t-\log w} = u$. By Definition 2.6 and ${}^H I_{1^+}^{\alpha}\,{}^{CH}D_{1^+}^{\alpha}x(t) = {}^H I_{1^+}^{\alpha}[B(t)x(t) + G(t)]$ We have

$$\begin{aligned}
&\frac{1}{\Gamma(\alpha)}\int_1^t (\log\frac{t}{s})^{\alpha-1}[A(s)x(s)+F(s)]\frac{ds}{s}\\
&= {}^H I_{1^+}^{\alpha}[B(t)x(t)+G(t)]
\end{aligned}$$

$$
\begin{aligned}
&= {}^{H}I_{1^+}^{\alpha}\,{}^{CH}D_{1^+}^{\alpha}x(t)\\
&= \frac{1}{\Gamma(\alpha)}\int_1^t(\log\frac{t}{s})^{\alpha-1}\Big[\frac{1}{\Gamma(n-\alpha)}\int_1^s(\log\frac{s}{w})^{n-\alpha-1}(w\frac{d}{dw})^n x(w)\frac{dw}{w}\Big]\frac{ds}{s}\\
&= \frac{1}{\Gamma(\alpha)}\frac{1}{\Gamma(n-\alpha)}\int_1^t\int_u^t(\log\frac{t}{s})^{\alpha-1}(\log\frac{s}{w})^{n-\alpha-1}\frac{ds}{s}(w\frac{d}{dw})^n x(w)\frac{dw}{w}\\
&= \frac{1}{\Gamma(\alpha)}\frac{1}{\Gamma(n-\alpha)}\int_1^t(\log\frac{t}{w})^{n-1}\int_0^1(1-u)^{\alpha-1}u^{n-\alpha-1}du(w\frac{d}{dw})^n x(w)\frac{dw}{w}\\
&= \frac{1}{(n-1)!}\int_1^t(\log\frac{t}{w})^{n-1}(w\frac{d}{dw})^n x(w)\frac{dw}{w}\\
&= \int_1^t(\log\frac{t}{w})^{n-1}d\big[(w\frac{d}{dw})^{n-1}x(w)\big]\\
&= \frac{1}{(n-1)!}(\log\frac{t}{w})^{n-1}\big[(w\frac{d}{dw})^{n-1}x(w)\big]|_1^t\\
&\quad+\frac{1}{(n-2)!}\int_1^t(\log\frac{t}{w})^{n-2}\big[(w\frac{d}{dw})^{n-1}x(w)\big]\frac{dw}{w}\\
&= -\frac{\eta_{n-1}}{(n-1)!}(\log t)^{n-1}+\frac{1}{(n-2)!}\int_1^t(\log\frac{t}{w})^{n-2}\big[(w\frac{d}{dw})^{n-1}x(w)\big]\frac{dw}{w}\\
&= \dots\\
&= -\sum_{j=1}^{n-2}\frac{\eta_{n-j}}{(n-j)!}(\log t)^{n-j}+\int_1^t x'(w)dw\\
&= x(t)-\sum_{j=0}^{n-1}\frac{\eta_j}{j!}(\log t)^j.
\end{aligned}
$$

Then $x\in C(1,e]$ is a solution of (3.1).

On the other hand, if $x\in C(1,e]$ is a solution of (3.1), then (3.19) implies $\lim_{t\to1^+}(t)=\eta_0$. Furthermore, for $t\in(1,e)$ we have

$$
\begin{aligned}
&|\int_1^t(\log\frac{t}{s})^{\alpha-n}[B(s)x(s)+G(s)]\frac{ds}{s}|\\
&\le\int_1^t(\log\frac{t}{s})^{\alpha-n}[M_Br(\log s)^{k_1}(1-\log s)^{l_1}+M_G(\log s)^{k_2}(1-\log s)^{l_2}]\frac{ds}{s}\\
&\le M_Br\int_1^t(\log\frac{t}{s})^{\alpha-n}(\log s)^{k_1}(1-\log t)^{l_1}\frac{ds}{s}\\
&\quad+M_G\int_1^t(\log\frac{t}{s})^{\alpha-n}(\log s)^{k_2}(1-\log t)^{l_2}\frac{ds}{s}\\
&= M_Br(1-\log t)^{l_1}\int_1^t(\log\frac{t}{s})^{\alpha-n}(\log s)^{k_1}\frac{ds}{s}\\
&\quad+M_G(1-\log t)^{l_2}\int_1^t(\log\frac{t}{s})^{\alpha-n}(\log s)^{k_2}\frac{ds}{s}\\
&= M_Br(1-\log t)^{l_1}(\log t)^{\alpha-n+k_1+1}\mathbf{B}(\alpha-n+1,k_1+1)\\
&\quad+M_G(1-\log t)^{l_2}(\log t)^{\alpha-n+k_2+1}\mathbf{B}(\alpha-n+1,k_2+1)\to0\quad\text{as } t\to1^+.
\end{aligned}
$$

Then

$$
\begin{aligned}
&{}^{CH}D_{1^+}^{\alpha}x(t)\\
&= \frac{1}{\Gamma(n-\alpha)}\int_1^t (\log\frac{t}{s})^{n-\alpha-1}(s\frac{d}{ds})^n x(s)\frac{ds}{s}\\
&= \frac{1}{\Gamma(n-\alpha)}\int_1^t (\log\frac{t}{s})^{n-\alpha-1}(s\frac{d}{ds})^n\Big(\sum_{j=0}^{n-1}\frac{\eta_j}{j!}(\log s)^j\\
&\quad + \frac{1}{\Gamma(\alpha)}\int_1^s (\log\frac{s}{u})^{\alpha-1}[B(u)x(u)+G(u)]\frac{du}{u}\Big)\frac{ds}{s}\\
&= \frac{1}{\Gamma(n-\alpha)}\int_1^t (\log\frac{t}{s})^{n-\alpha-1}(s\frac{d}{ds})^n\sum_{j=0}^{n-1}\frac{\eta_j}{j!}(\log s)^j\frac{ds}{s}\\
&\quad + \frac{1}{\Gamma(\alpha)}\frac{1}{\Gamma(n-\alpha)}\int_1^t (\log\frac{t}{s})^{n-\alpha-1}(s\frac{d}{ds})^n\Big(\int_1^s(\log\frac{s}{u})^{\alpha-1}[B(u)x(u)+G(u)]\frac{du}{u}\Big)\frac{ds}{s}\\
&= \frac{1}{\Gamma(\alpha)}\frac{1}{\Gamma(n-\alpha)}\int_1^t (\log\frac{t}{s})^{n-\alpha-1}(s\frac{d}{ds})^n\Big(\int_1^s(\log\frac{s}{u})^{\alpha-1}[B(u)x(u)+G(u)]\frac{du}{u}\Big)\frac{ds}{s}\\
&= \frac{1}{\Gamma(\alpha)}\frac{1}{\Gamma(n-\alpha)}\int_1^t (\log\frac{t}{s})^{n-\alpha-1}(s\frac{d}{ds})^{n-1}\Big(\int_1^s(\log\frac{s}{u})^{\alpha-2}[B(u)x(u)+G(u)]\frac{du}{u}\Big)\frac{ds}{s}\\
&= \dots\\
&= \frac{1}{\Gamma(\alpha)}\frac{1}{\Gamma(n-\alpha)}\int_1^t (\log\frac{t}{s})^{n-\alpha-1}\Big(\int_1^s(\log\frac{s}{u})^{\alpha-n}[B(u)x(u)+G(u)]\frac{du}{u}\Big)'ds\\
&= \frac{1}{\Gamma(\alpha)}\frac{1}{\Gamma(n-\alpha+1)}t\Big[\int_1^t(\log\frac{t}{s})^{n-\alpha}(\int_1^s(\log\frac{s}{u})^{\alpha-1}[B(u)x(u)+G(u)]\frac{du}{u})'ds\Big]'\\
&= \frac{1}{\Gamma(\alpha)}\frac{1}{\Gamma(n-\alpha+1)}t\Big[(\log\frac{t}{s})^{n-\alpha}\int_1^s(\log\frac{s}{w})^{\alpha-n}[B(w)x(w)+G(w)]\frac{dw}{w}\Big|_1^t\\
&\quad + (n-\alpha)\frac{1}{s}\int_1^t(\log\frac{t}{s})^{n-\alpha-1}\int_1^s(\log\frac{s}{w})^{\alpha-n}[B(w)x(w)+G(w)]\frac{dw}{w}ds\Big]'\\
&= \frac{1}{\Gamma(\alpha)}\frac{1}{\Gamma(n-\alpha)}t\Big[\int_1^t\int_u^t(\log\frac{t}{s})^{n-\alpha-1}(\log\frac{s}{w})^{\alpha-n}\frac{ds}{s}[B(w)x(w)+G(w)]\frac{dw}{w}\Big]'\\
&= \frac{1}{\Gamma(\alpha)}\frac{1}{\Gamma(n-\alpha)}t\Big[\int_1^t\int_u^t(\log\frac{t}{s})^{n-\alpha-1}(\log\frac{s}{w})^{\alpha-n}\frac{ds}{s}[B(w)x(w)+G(w)]\frac{dw}{w}\Big]'\\
&= \frac{1}{\Gamma(\alpha)}\frac{1}{\Gamma(n-\alpha)}t\Big[\int_1^t\int_0^1(1-u)^{n-\alpha-1}u^{\alpha-n}du[B(w)x(w)+G(w)]\frac{dw}{w}\Big]'\\
&= B(t)x(t)+G(t).
\end{aligned}
$$

So $x\in C(1,e]$ is a solution of (3.4). The proof is complete. □

Theorem 3.8. *Suppose that* (3.A4) *holds. Then* (3.4) *has a unique solution. If* $B(t)\equiv\lambda$ *and there exists constants* $k_2>-\alpha+n-1$, $l_2\le 0$ *with* $l_2>\max\{-\alpha,-\alpha-k_2\}$ *and* $M_G\ge 0$ *such that* $|G(t)|\le M_G t^{k_2}(1-t)^{l_2}$ *for all* $t\in(1,e)$, *then the problem*

$$
\begin{gathered}
{}^{CH}D_{0^+}^{\alpha}x(t)=\lambda x(t)+G(t),\quad a.e.\ t\in(1,e],\\
\lim_{t\to1^+}(t\frac{d}{dt})^j x(t)=\eta_j,\quad j\in\mathbb{N}[0,n-1]
\end{gathered}
\tag{3.21}
$$

has a unique solution

$$\begin{aligned} x(t) &= \sum_{j=0}^{n-1} \eta_j (\log t)^j \mathbf{E}_{\alpha,j+1}(\lambda(\log t)^\alpha) \\ &\quad + \int_1^t (\log \frac{t}{s})^{\alpha-1} \mathbf{E}_{\alpha,\alpha}(\lambda(\log t - \log s)^\alpha) G(s) \frac{ds}{s}, \quad t \in (1,e]. \end{aligned} \tag{3.22}$$

Proof. From Claims 1, 2 and 3, Theorem 3.7, (3.4) has a unique solution. From the assumption and $A(t) \equiv \lambda$, one sees that (3.A4) holds with $k_1 = l_1 = 0$ and k_2, l_2 mentioned. Thus (3.21) has a unique solution. From the Picard function sequence we obtain

$$\begin{aligned} &\phi_i(t) \\ &= \sum_{j=0}^{n-1} \frac{\eta_j}{j!} (\log t)^j + \lambda \frac{1}{\Gamma(\alpha)} \int_1^t (\log \frac{t}{s})^{\alpha-1} \phi_{i-1}(s) \frac{ds}{s} + \frac{1}{\Gamma(\alpha)} \int_1^t (\log \frac{t}{s})^{\alpha-1} G(s) \frac{ds}{s} \\ &= \sum_{j=0}^{n-1} \frac{\eta_j}{j!} (\log t)^j + \frac{\lambda}{\Gamma(\alpha)} \sum_{j=0}^{n-1} \frac{\eta_j}{j!} \int_1^t (\log \frac{t}{s})^{\alpha-1} (\log s)^j \frac{ds}{s} \\ &\quad + \frac{\lambda}{\Gamma(\alpha)} \frac{\lambda}{\Gamma(\alpha)} \int_1^t (\log \frac{t}{s})^{\alpha-1} \int_1^s (\log \frac{s}{u})^{\alpha-1} \phi_{i-2}(u) \frac{du}{u} \frac{ds}{s} \\ &\quad + \frac{1}{\Gamma(\alpha)} \frac{\lambda}{\Gamma(\alpha)} \int_1^t (\log \frac{t}{s})^{\alpha-1} \int_1^s (\log \frac{s}{u})^{\alpha-1} G(u) \frac{du}{u} \frac{ds}{s} + \frac{1}{\Gamma(\alpha)} \int_1^t (\log \frac{t}{s})^{\alpha-1} G(s) \frac{ds}{s} \\ &= \sum_{j=0}^{n-1} \frac{\eta_j}{j!} (\log t)^j + \frac{\lambda}{\Gamma(\alpha)} \sum_{j=0}^{n-1} \frac{\eta_j}{j!} (\log t)^{\alpha+j} \int_0^1 (1-w)^{\alpha-1} w^j dw \\ &\quad + \frac{\lambda}{\Gamma(\alpha)} \frac{\lambda}{\Gamma(\alpha)} \int_1^t \int_u^t (\log \frac{t}{s})^{\alpha-1} (\log \frac{s}{u})^{\alpha-1} \frac{ds}{s} \phi_{i-2}(u) \frac{du}{u} \\ &\quad + \frac{1}{\Gamma(\alpha)} \frac{\lambda}{\Gamma(\alpha)} \int_1^t \int_u^t (\log \frac{t}{s})^{\alpha-1} (\log \frac{s}{u})^{\alpha-1} \frac{ds}{s} G(u) \frac{du}{u} + \frac{1}{\Gamma(\alpha)} \int_1^t (\log \frac{t}{s})^{\alpha-1} G(s) \frac{ds}{s} \\ &= \sum_{j=0}^{n-1} \frac{\eta_j}{\Gamma(j+1)} (\log t)^j + \sum_{j=0}^{n-1} \frac{\lambda \eta_j}{\Gamma(\alpha+j+1)} (\log t)^{\alpha+j} \\ &\quad + \frac{\lambda}{\Gamma(\alpha)} \frac{\lambda}{\Gamma(\alpha)} \int_1^t (\log \frac{t}{u})^{2\alpha-1} \int_0^1 (1-w)^{\alpha-1} w^{\alpha-1} dw \phi_{i-2}(u) \frac{du}{u} \\ &\quad + \frac{1}{\Gamma(\alpha)} \frac{\lambda}{\Gamma(\alpha)} \int_1^t (\log \frac{t}{u})^{2\alpha-1} \int_0^1 (1-w)^{\alpha-1} w^{\alpha-1} dw G(u) \frac{du}{u} \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_1^t (\log \frac{t}{s})^{\alpha-1} G(s) \frac{ds}{s} \\ &= \sum_{j=0}^{n-1} \eta_j (\log t)^j \Big(\frac{1}{\Gamma(j+1)} + \frac{\lambda (\log t)^\alpha}{\Gamma(\alpha+j+1)} \Big) + \frac{\lambda^2}{\Gamma(2\alpha)} \int_1^t (\log \frac{t}{u})^{2\alpha-1} \phi_{i-2}(u) \frac{du}{u} \\ &\quad + \int_1^t (\log \frac{t}{s})^{\alpha-1} \Big(\frac{\lambda (\log t \log s)^\alpha}{\Gamma(2\alpha)} + \frac{1}{\Gamma(\alpha)} \Big) G(s) \frac{ds}{s} \\ &= \dots \end{aligned}$$

$$= \sum_{j=0}^{n-1} \eta_j (\log t)^j (\sum_{v=0}^{i-1} \frac{\lambda^v (\log t)^{v\alpha}}{\Gamma(v\alpha + j + 1)}) + \frac{\lambda^i}{\Gamma(i\alpha)} \int_1^t (\log \frac{t}{u})^{m\alpha - 1} \phi_0(u) \frac{du}{u}$$

$$+ \int_1^t (\log \frac{t}{s})^{\alpha-1} (\sum_{v=0}^{i-1} \frac{\lambda^v (\log t \log s)^{v\alpha}}{\Gamma((v+1)\alpha)}) G(s) \frac{ds}{s}$$

$$= \sum_{j=0}^{n-1} \eta_j (\log t)^j \Big(\sum_{v=0}^{i} \frac{\lambda^v (\log t)^{v\alpha}}{\Gamma(v\alpha + j + 1)} \Big) + \int_1^t (\log \frac{t}{s})^{\alpha-1} \Big(\sum_{v=0}^{i-1} \frac{\lambda^v (\log t \log s)^{v\alpha}}{\Gamma((v+1)\alpha)} \Big) G(s) \frac{ds}{s}$$

$$\to \sum_{j=0}^{n-1} \eta_j (\log t)^j \mathbf{E}_{\alpha, j+1}(\lambda (\log t)^\alpha) + \int_1^t (\log \frac{t}{s})^{\alpha-1} \mathbf{E}_{\alpha,\alpha}(\lambda (\log t - \log s)^\alpha) G(s) \frac{ds}{s}.$$

Then $x(t) = \lim_{i \to +\infty} \phi_i(t)$ is the unique solution of (3.21). x is just as in (3.22). The proof is complete. □

We list the following two fixed point theorems which will be used in Section 4.

Theorem 3.9 (Schaefer's fixed point theorem [79])**.** *Let E be a Banach spaces and $T : E \to E$ be a completely continuous operator. If the set $E(T) = \{x = \theta(Tx) :$ for some $\theta \in [0,1), x \in E\}$ is bounded, then T has at least a fixed point in E.*

Theorem 3.10 ([97])**.** *Let X be a Banach space. Assume that Ω is an open bounded subset of X with $0 \in \Omega$ and let $T : X \to X$ be a completely continuous operator such that $\|Tx\| \le \|x\|$ for all $x \in \partial\Omega$. Then T has a fixed point in Ω.*

3.2. **Exact piecewise continuous solutions of LFDEs.** In this section, we present exact piecewise continuous solutions of the linear fractional differential equations (LFDEs)

$$^{C}D_{0^+}^{\alpha} x(t) = \lambda x(t) + F(t), \quad \text{a.e., } t \in (t_i, t_{i+1}],\ i \in \mathbb{N}[0,m], \tag{3.23}$$

$$^{RL}D_{0^+}^{\alpha} x(t) = \lambda x(t) + F(t), \quad \text{a.e., } t \in (t_i, t_{i+1}],\ i \in \mathbb{N}[0,m], \tag{3.24}$$

$$^{RLH}D_{0^+}^{\alpha} x(t) = \lambda x(t) + G(t), \quad \text{a.e., } t \in (s_i, s_{i+1}],\ i \in \mathbb{N}[0,m], \tag{3.25}$$

$$^{CH}D_{0^+}^{\alpha} x(t) = \lambda x(t) + G(t), \quad \text{a.e., } t \in (s_i, s_{i+1}],\ i \in \mathbb{N}[0,m], \tag{3.26}$$

where $n-1 < \alpha < n$, $\lambda \in \mathbb{R}$, $0 = s_0 < t_1 < \dots < s_m < s_{m+1} = 1$ in (3.23) and (3.24) and $1 = t_0 < t_1 < \dots < t_m < t_{m+1} = e$ in (3.25) and (3.26).

We say that $x : (0,1] \to \mathbb{R}$ is a piecewise solution of (3.23) (or (3.24) if $x \in P_mC(0,1]$ (or $P_mC_{n-\alpha}(0,1]$ and satisfies (3.23) or (3.24). We say that $x : (1,e] \to \mathbb{R}$ is a piecewise continuous solutions of (3.25) (or (3.26)) if $x \in LP_mC_{n-\alpha}(1,e]$, (or $LP_mC(1,e]$) and x satisfies all equations in (3.25) (or (3.26)).

Theorem 3.11. *Suppose that F is continuous on $(0,1)$ and there exist constants $k > -\alpha + n - 1$ and $l \in (-\alpha, -\alpha - k, 0]$ such that $|F(t)| \le t^k(1-t)^l$ for all $t \in (0,1)$. Then x is a piecewise solution of (3.23) if and only if there exist constants $c_{iv} \in \mathbb{R}(i \in \mathbb{N}[0,m],\ v \in \mathbb{N}[0,n-1])$ such that*

$$\begin{aligned} x(t) &= \sum_{\sigma=0}^{i} \sum_{v=0}^{n-1} c_{\sigma v} \mathbf{E}_{\alpha, v+1}(\lambda (t - t_\sigma)^\alpha)(t - t_\sigma)^v \\ &\quad + \int_0^t (t-s)^{\alpha-1} \mathbf{E}_{\alpha,\alpha}(\lambda (t-s)^\alpha) F(s) ds, \quad t \in (t_i, t_{i+1}],\ i \in N[0,m]. \end{aligned} \tag{3.27}$$

Proof. Firstly, for $t\in(t_i,t_{i+1}]$ we have

$$
\begin{aligned}
&\Big|\int_0^t (t-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t-s)^\alpha)F(s)ds\Big|\\
&\le \int_0^t (t-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t-s)^\alpha)|F(s)|ds\\
&\le \int_0^t (t-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t-s)^\alpha)s^k(1-s)^l ds\\
&= \sum_{j=0}^{+\infty}\frac{\lambda^j}{\Gamma((j+1)\alpha)}\int_0^t (t-s)^{\alpha-1}(t-s)^{\alpha j}s^k(1-s)^l ds\\
&\le \sum_{j=0}^{+\infty}\frac{\lambda^j}{\Gamma((j+1)\alpha)}\int_0^t (t-s)^{\alpha+l-1}(t-s)^{\alpha j}s^k ds\\
&= \sum_{j=0}^{+\infty}\frac{\lambda^j}{\Gamma((j+1)\alpha)}t^{\alpha+\alpha j+k+l}\int_0^1 (1-w)^{\alpha+\alpha j+l-1}w^k dw\\
&\le \sum_{j=0}^{+\infty}\frac{\lambda^j t^{\alpha j}}{\Gamma((j+1)\alpha)}t^{\alpha+k+l}\int_0^1 (1-w)^{\alpha+l-1}w^k dw\\
&= t^{\alpha+k+l}\mathbf{E}_{\alpha,\alpha}(\lambda t^\alpha)\mathbf{B}(\alpha+l,k+1).
\end{aligned}
$$

Then $\int_0^t(t-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t-s)^\alpha)F(s)ds$ is convergent and is continuous on $[0,1]$. If x is a solution of (3.27), then we know that $\lim_{t\to t_i^+}x(t)$ ($i\in N_0$) exist and $x\in P_mC(0,1]$. From

$$
\begin{aligned}
\Big|\int_0^s \frac{\lambda^\tau(s-u)^{\tau\alpha+\alpha-n}}{\Gamma((\tau+1)\alpha-n+1)}F(u)du\Big| &\le \int_0^s \frac{\lambda^\tau(s-u)^{\tau\alpha+\alpha-n+l}}{\Gamma((\tau+1)\alpha-n+1)}u^k du\\
&= s^{\tau\alpha+\alpha-n+k+l+1}\int_0^1\frac{\lambda^\tau(1-w)^{\tau\alpha+\alpha-n+l}}{\Gamma((\tau+1)\alpha-n+1)}w^k dw\\
&\le s^{\alpha-n+k+l+1}\int_0^1 (1-w)^{\alpha-n+l}w^k dw\\
&= s^{\alpha-n+k+l+1}\frac{\lambda^\tau}{\Gamma((\tau+1)\alpha-n+1)},
\end{aligned}
$$

we know that

$$
\lim_{s\to0^+}\sum_{\tau=0}^{+\infty}\int_0^s\frac{\lambda^\tau(s-u)^{\tau\alpha+\alpha-n}}{\Gamma((\tau+1)\alpha-n+1)}F(u)du=0.
$$

Now we prove that x satisfies differential equation in (3.23). In fact, for $t\in(t_0,t_1]$ we have by Theorem 3.2 that ${}^CD_{0^+}^\alpha x(t)=\lambda x(t)+F(t)$. For $t\in(t_i,t_{i+1}]$($i\in N[1,m]$), we have by Definition 2.3 that

$$
\begin{aligned}
&{}^CD_{0^+}^\alpha x(t)\\
&=\frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}x^{(n)}(s)ds\\
&=\frac{1}{\Gamma(n-\alpha)}\Big[\sum_{j=0}^{i-1}\int_{t_j}^{t_{j+1}}(t-s)^{n-\alpha-1}x^{(n)}(s)ds+\int_{t_i}^t (t-s)^{n-\alpha-1}x^{(n)}(s)ds\Big]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{\Gamma(n-\alpha)}\Big[\sum_{j=0}^{i-1}\int_{t_j}^{t_{j+1}}(t-s)^{n-\alpha-1}\Big(\sum_{\sigma=0}^{j}\sum_{v=0}^{n-1}c_{\sigma v}\mathbf{E}_{\alpha,v+1}(\lambda(s-t_\sigma)^\alpha)(s-t_\sigma)^v \\
&\quad + \int_0^s (s-u)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(s-u)^\alpha)F(u)du\Big)^{(n)}ds \\
&\quad + \int_{t_i}^{t}(t-s)^{n-\alpha-1}\Big(\sum_{\sigma=0}^{i}\sum_{v=0}^{n-1}c_{\sigma v}\mathbf{E}_{\alpha,v+1}(\lambda(s-t_\sigma)^\alpha)(s-t_\sigma)^v \\
&\quad + \int_0^s (s-u)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(s-u)^\alpha)F(u)du)^{(n)}ds\Big] \\
&= \frac{1}{\Gamma(n-\alpha)}\Big[\sum_{j=0}^{i-1}\int_{t_j}^{t_{j+1}}(t-s)^{n-\alpha-1}\Big(\sum_{\sigma=0}^{j}\sum_{v=0}^{n-1}c_{\sigma v}\frac{(s-t_\sigma)^v}{\Gamma(v+1)} \\
&\quad + \sum_{\sigma=0}^{j}\sum_{v=0}^{n-1}c_{\sigma v}\sum_{\tau=1}^{+\infty}\frac{\lambda^\tau(s-t_\sigma)^{\tau\alpha+v}}{\Gamma(\tau\alpha+v+1)}\Big)^{(n)}ds \\
&\quad + \int_{t_i}^{t}(t-s)^{n-\alpha-1}\Big(\sum_{\sigma=0}^{i}\sum_{v=0}^{n-1}c_{\sigma v}\frac{(s-t_\sigma)^v}{\Gamma(v+1)} + \sum_{\sigma=1}^{i}\sum_{v=0}^{n-1}c_{\sigma v}\sum_{\tau=0}^{+\infty}\frac{\lambda^\tau(s-t_\sigma)^{\tau\alpha+v}}{\Gamma(\tau\alpha+v+1)}\Big)^{(n)}ds \\
&\quad + \int_0^t (t-s)^{n-\alpha-1}Big(\sum_{\tau=0}^{+\infty}\int_0^s \frac{\lambda^\tau(s-u)^{\tau\alpha+\alpha-1}}{\Gamma((\tau+1)\alpha)}F(u)du\Big)^{(n)}ds\Big] \\
&= \frac{1}{\Gamma(n-\alpha)}\Big[\sum_{j=0}^{i-1}\int_{t_j}^{t_{j+1}}(t-s)^{n-\alpha-1}\Big(\sum_{\sigma=0}^{j}\sum_{v=0}^{n-1}c_{\sigma v}\sum_{\tau=1}^{+\infty}\frac{\lambda^\tau(s-t_\sigma)^{\tau\alpha+v}}{\Gamma(\tau\alpha+v+1)}\Big)^{(n)}ds \\
&\quad + \int_{t_i}^{t}(t-s)^{n-\alpha-1}\Big(\sum_{\sigma=0}^{i}\sum_{v=0}^{n-1}c_{\sigma v}\sum_{\tau=0}^{+\infty}\frac{\lambda^\tau(s-t_\sigma)^{\tau\alpha+v}}{\Gamma(\tau\alpha+v+1)}\Big)^{(n)}ds \\
&\quad + \int_0^t (t-s)^{n-\alpha-1}\Big(\sum_{\tau=0}^{+\infty}\int_0^s \frac{\lambda^\tau(s-u)^{\tau\alpha+\alpha-n}}{\Gamma((\tau+1)\alpha-n+1)}F(u)du\Big)'ds\Big] \\
&= \frac{1}{\Gamma(n-\alpha)}\sum_{j=0}^{i-1}\sum_{\sigma=0}^{j}\sum_{v=0}^{n-1}c_{\sigma v}\sum_{\tau=1}^{+\infty}\frac{\lambda^\tau}{\Gamma(\tau\alpha+v+1-n)} \\
&\quad \times \int_{t_j}^{t_{j+1}}(t-s)^{n-\alpha-1}(s-t_\sigma)^{\tau\alpha+v-n}ds \\
&\quad + \frac{1}{\Gamma(n-\alpha)}\sum_{\sigma=0}^{i}\sum_{v=0}^{n-1}c_{\sigma v}\sum_{\tau=0}^{+\infty}\frac{\lambda^\tau}{\Gamma(\tau\alpha+v+1-n)} \\
&\quad \times \int_{t_i}^{t}(t-s)^{n-\alpha-1}(s-t_\sigma)^{\tau\alpha+v-n}ds \\
&\quad + \frac{1}{\Gamma(n-\alpha+1)}\Big[\int_0^t (t-s)^{n-\alpha}(\sum_{\tau=0}^{+\infty}\int_0^s \frac{\lambda^\tau(s-u)^{\tau\alpha+\alpha-n}}{\Gamma((\tau+1)\alpha-n+1)}F(u)du)'ds\Big]' \\
&= \frac{1}{\Gamma(n-\alpha)}\sum_{j=0}^{i-1}\sum_{\sigma=0}^{j}\sum_{v=0}^{n-1}c_{\sigma v}\sum_{\tau=1}^{+\infty}\frac{\lambda^\tau}{\Gamma(\tau\alpha+v+1-n)}(t-t_\sigma)^{\tau\alpha-\alpha+v}
\end{aligned}
$$

$$
\begin{aligned}
&\times \int_{\frac{t_j-t_\sigma}{t-t_\sigma}}^{\frac{t_{j+1}-t_\sigma}{t-t_\sigma}} (1-w)^{n-\alpha-1} w^{\tau\alpha+v-n} dw \\
&+ \frac{1}{\Gamma(n-\alpha)} \sum_{\sigma=0}^{i} \sum_{v=0}^{n-1} c_{\sigma v} \sum_{\tau=0}^{+\infty} \frac{\lambda^\tau}{\Gamma(\tau\alpha+v+1-n)} (t-t_\sigma)^{\tau\alpha-\alpha+v} \\
&\times \int_{\frac{t_i-t_\sigma}{t-t_\sigma}}^{1} (1-w)^{n-\alpha-1} w^{\tau\alpha+v-n} dw \\
&+ \frac{1}{\Gamma(n-\alpha+1)} \Big[(t-s)^{n-\alpha} (\sum_{\tau=0}^{+\infty} \int_0^s \frac{\lambda^\tau (s-u)^{\tau\alpha+\alpha-n}}{\Gamma((\tau+1)\alpha-n+1)} F(u) du)|_0^t \\
&+ (n-\alpha) \int_0^t (t-s)^{n-\alpha-1} (\sum_{\tau=0}^{+\infty} \int_0^s \frac{\lambda^\tau (s-u)^{\tau\alpha+\alpha-n}}{\Gamma((\tau+1)\alpha-n+1)} F(u) du) ds \Big]' \\
= &\frac{1}{\Gamma(n-\alpha)} \sum_{\sigma=0}^{i-1} \sum_{j=\sigma}^{i-1} \sum_{v=0}^{n-1} c_{\sigma v} \sum_{\tau=1}^{+\infty} \frac{\lambda^\tau}{\Gamma(\tau\alpha+v+1-n)} (t-t_\sigma)^{\tau\alpha-\alpha+v} \\
&\times \int_{\frac{t_j-t_\sigma}{t-t_\sigma}}^{\frac{t_{j+1}-t_\sigma}{t-t_\sigma}} (1-w)^{n-\alpha-1} w^{\tau\alpha+v-n} dw \\
&+ \frac{1}{\Gamma(n-\alpha)} \sum_{\sigma=0}^{i} \sum_{v=0}^{n-1} c_{\sigma v} \sum_{\tau=0}^{+\infty} \frac{\lambda^\tau}{\Gamma(\tau\alpha+v+1-n)} (t-t_\sigma)^{\tau\alpha-\alpha+v} \\
&\times \int_{\frac{t_i-t_\sigma}{t-t_\sigma}}^{1} (1-w)^{n-\alpha-1} w^{\tau\alpha+v-n} dw \\
&+ \frac{1}{\Gamma(n-\alpha)} \Big[\sum_{\tau=0}^{+\infty} \int_0^t \int_u^t (t-s)^{n-\alpha-1} \frac{\lambda^\tau (s-u)^{\tau\alpha+\alpha-n}}{\Gamma((\tau+1)\alpha-n+1)} ds F(u) du \Big]' \\
= &\frac{1}{\Gamma(n-\alpha)} \sum_{\sigma=0}^{i-1} \sum_{v=0}^{n-1} c_{\sigma v} \sum_{\tau=1}^{+\infty} \frac{\lambda^\tau}{\Gamma(\tau\alpha+v+1-n)} (t-t_\sigma)^{\tau\alpha-\alpha+v} \\
&\times \int_0^{\frac{t_i-t_\sigma}{t-t_\sigma}} (1-w)^{n-\alpha-1} w^{\tau\alpha+v-n} dw \\
&+ \frac{1}{\Gamma(n-\alpha)} \sum_{\sigma=0}^{i} \sum_{v=0}^{n-1} c_{\sigma v} \sum_{\tau=0}^{+\infty} \frac{\lambda^\tau}{\Gamma(\tau\alpha+v+1-n)} (t-t_\sigma)^{\tau\alpha-\alpha+v} \\
&\int_{\frac{t_i-t_\sigma}{t-t_\sigma}}^{1} (1-w)^{n-\alpha-1} w^{\tau\alpha+v-n} dw \\
&+ \frac{1}{\Gamma(n-\alpha)} \Big[\sum_{\tau=0}^{+\infty} \int_0^t (t-u)^{\tau\alpha} \int_0^1 (1-w)^{n-\alpha-1} \frac{\lambda^\tau w^{\tau\alpha+\alpha-n}}{\Gamma((\tau+1)\alpha-n+1)} dw F(u) du \Big]' \\
= &\sum_{\sigma=0}^{i} \sum_{v=0}^{n-1} c_{\sigma v} \sum_{\tau=1}^{+\infty} \frac{\lambda^\tau}{\Gamma((\tau-1)\alpha+v+1)} (t-t_\sigma)^{\tau\alpha-\alpha+v} \\
&+ \Big[\sum_{\tau=0}^{+\infty} \int_0^t (t-u)^{\tau\alpha} \frac{\lambda^\tau}{\Gamma(\tau\alpha+1)} F(u) du \Big]'
\end{aligned}
$$

$= \lambda x(t) + F(t).$

We have shown that x satisfies (3.23) if x satisfies (3.27).

Now, we suppose that x is a solution of (3.23). We will prove that x satisfies (3.27) by mathematical induction. Since x is continuous on $(t_i, t_{i+1}]$ and the limit $\lim_{t\to t_i^+} x(t)$ $(i \in N_0)$ exists, it follows that $x \in P_mC(0,1]$. For $t \in (t_0, t_1]$, we know from Theorem 3.2 that there exists $c_{0v} \in \mathbb{R}$ such that

$$x(t) = \sum_{v=0}^{n-1} c_{0v}\mathbf{E}_{\alpha,1}(\lambda t^\alpha)t^v + \int_0^t (t-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t-s)^\alpha)F(s)ds, \quad t \in (t_0, t_1].$$

Then (3.27) holds for $i = 0$. We suppose that (3.27) holds for all $i = 0, 1, \dots, j \le m-1$. We derive the expression of x on $(t_{j+1}, t_{j+2}]$. Suppose that

$$\begin{aligned} x(t) &= \Phi(t) + \sum_{\sigma=0}^{j}\sum_{v=0}^{n-1} c_{\sigma v}\mathbf{E}_{\alpha,v+1}(\lambda(t-t_\sigma)^\alpha)(t-t_\sigma)^v \\ &\quad + \int_0^t (t-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t-s)^\alpha)F(s)ds, \quad t \in (t_{j+1}, t_{j+2}]. \end{aligned} \tag{3.28}$$

By ${}^CD_{0^+}^\alpha x(t) - \lambda x(t) = f(t), t \in (t_{j+1}, t_{j+2}]$, we obtain

$$\begin{aligned} &F(t) + \lambda x(t) \\ &= {}^CD_{0^+}^\alpha x(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}x^{(n)}(s)ds \\ &= \sum_{\rho=0}^{j}\int_{t_\rho}^{t_{\rho+1}} \frac{(t-s)^{n-\alpha-1}}{\Gamma(n-\alpha)}\Big(\sum_{\sigma=0}^{\rho}\sum_{v=0}^{n-1} c_{\sigma v}\mathbf{E}_{\alpha,v+1}(\lambda(s-t_\sigma)^\alpha)(s-t_\sigma)^v \\ &\quad + \int_0^s (s-u)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(s-u)^\alpha)F(u)du\Big)^{(n)}ds \\ &\quad + \int_{t_{j+1}}^t \frac{(t-s)^{n-\alpha-1}}{\Gamma(n-\alpha)}(\Phi(s) + \sum_{\sigma=0}^{j}\sum_{v=0}^{n-1} c_{\sigma v}\mathbf{E}_{\alpha,v+1}(\lambda(s-t_\sigma)^\alpha)(s-t_\sigma)^v \\ &\quad + \int_0^s (s-u)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(s-u)^\alpha)F(u)du)^{(n)}ds \\ &= {}^CD_{t_{j+1}^+}^\alpha \Phi(t) + \frac{1}{\Gamma(n-\alpha)}\sum_{\rho=0}^{j}\int_{t_\rho}^{t_{\rho+1}}(t-s)^{n-\alpha-1} \\ &\quad \times \Big(\sum_{\sigma=0}^{\rho}\sum_{v=0}^{n-1} c_{\sigma v}\mathbf{E}_{\alpha,v+1}(\lambda(s-t_\sigma)^\alpha)(s-t_\sigma)^v\Big)^{(n)}ds \\ &\quad + \frac{1}{\Gamma(n-\alpha)}\int_{t_{j+1}}^t (t-s)^{n-\alpha-1}\Big(\sum_{\sigma=0}^{j}\sum_{v=0}^{n-1} c_{\sigma v}\mathbf{E}_{\alpha,v+1}(\lambda(s-t_\sigma)^\alpha)(s-t_\sigma)^v\Big)^{(n)}ds \\ &\quad + \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}(\int_0^s (s-u)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(s-u)^\alpha)F(u)du)^{(n)}ds \\ &= {}^CD_{t_{j+1}^+}^\alpha \Phi(t) + \frac{1}{\Gamma(n-\alpha)}\sum_{\rho=0}^{j}\int_{t_\rho}^{t_{\rho+1}}(t-s)^{n-\alpha-1} \end{aligned}$$

$$
\begin{aligned}
&\times \Big(\sum_{\sigma=0}^{\rho}\sum_{v=0}^{n-1} c_{\sigma v}\sum_{\tau=0}^{+\infty}\frac{\lambda^\tau (s-t_\sigma)^{\tau\alpha+v}}{\Gamma(\tau\alpha+v+1)}\Big)^{(n)} ds \\
&+ \frac{1}{\Gamma(n-\alpha)}\int_{t_{j+1}}^t (t-s)^{n-\alpha-1}\Big(\sum_{\sigma=0}^{j}\sum_{v=0}^{n-1} c_{\sigma v}\sum_{\tau=0}^{+\infty}\frac{\lambda^\tau (s-t_\sigma)^{\tau\alpha+v}}{\Gamma(\tau\alpha+v+1)}\Big)^{(n)} ds \\
&+ \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}\Big(\int_0^s (s-u)^{\alpha-1}\sum_{\tau=0}^{+\infty}\frac{\lambda^\tau (s-u)^{\tau\alpha}}{\Gamma((\tau+1)\alpha)}F(u)du\Big)^{(n)} ds \\
&= {}^C D_{t_{j+1}^+}^{\alpha}\Phi(t) + \frac{1}{\Gamma(n-\alpha)}\sum_{\rho=0}^{j}\int_{t_\rho}^{t_{\rho+1}} (t-s)^{n-\alpha-1} \\
&\times \Big(\sum_{\sigma=0}^{\rho}\sum_{v=0}^{n-1} c_{\sigma v}\sum_{\tau=1}^{+\infty}\frac{\lambda^\tau (s-t_\sigma)^{\tau\alpha+v-n}}{\Gamma(\tau\alpha+v+1-n)}\Big) ds \\
&+ \frac{1}{\Gamma(n-\alpha)}\int_{t_{j+1}}^t (t-s)^{n-\alpha-1}\Big(\sum_{\sigma=0}^{j}\sum_{v=0}^{n-1} c_{\sigma v}\sum_{\tau=1}^{+\infty}\frac{\lambda^\tau (s-t_\sigma)^{\tau\alpha+v-n}}{\Gamma(\tau\alpha+v+1-n)}\Big) ds \\
&+ \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}\Big(\sum_{\tau=0}^{+\infty}\int_0^s \frac{\lambda^\tau (s-u)^{\tau\alpha+\alpha-n}}{\Gamma((\tau+1)\alpha-n+1)}F(u)du\Big)' ds \\
&= {}^C D_{t_{j+1}^+}^{\alpha}\Phi(t) + \frac{1}{\Gamma(n-\alpha)}\sum_{\rho=0}^{j}\sum_{\sigma=0}^{\rho}\sum_{v=0}^{n-1} c_{\sigma v}\sum_{\tau=1}^{+\infty}\int_{t_\rho}^{t_{\rho+1}} (t-s)^{n-\alpha-1} \\
&\times \frac{\lambda^\tau (s-t_\sigma)^{\tau\alpha+v-n}}{\Gamma(\tau\alpha+v+1-n)} ds \\
&+ \frac{1}{\Gamma(n-\alpha)}\sum_{\sigma=0}^{j}\sum_{v=0}^{n-1} c_{\sigma v}\sum_{\tau=1}^{+\infty}\int_{t_{j+1}}^t (t-s)^{n-\alpha-1}\frac{\lambda^\tau (s-t_\sigma)^{\tau\alpha+v-n}}{\Gamma(\tau\alpha+v+1-n)} ds \\
&+ \frac{1}{\Gamma(n-\alpha+1)}\Big[\int_0^t (t-s)^{n-\alpha}(\sum_{\tau=0}^{+\infty}\int_0^s \frac{\lambda^\tau (s-u)^{\tau\alpha+\alpha-n}}{\Gamma((\tau+1)\alpha-n+1)}F(u)du)' ds\Big]'.
\end{aligned}
$$

By a similar computation, we obtain

$$
F(t)+\lambda x(t) = F(t)+\lambda x(t) + {}^c D_{t_{j+1}^+}^{\alpha}\Phi(t) - \lambda\Phi(t).
$$

It follows that ${}^C D_{t_{j+1}^+}^{\alpha}\Phi(t) - \lambda\Phi(t) = 0$ for all $t \in (t_{j+1}, t_{j+2}]$. By Theorem 3.2, we know that there exists $c_{j+1v} \in \mathbb{R}(v \in \mathbb{N}[0,n-1])$ such that $\Phi(t) = \sum_{v=0}^{n-1} c_{j+1v}\mathbf{E}_{\alpha,v+1}(\lambda(t-t_{j+1})^\alpha)(t-t_{j+1})^v$ for $t \in (t_{j+1}, t_{j+2}]$. Substituting Φ into (3.28), we obtain that (3.27) holds for $i = j+1$. Now suppose that (3.27) holds for all $j \in N_0$. By the mathematical induction, we know that x satisfies (3.27) and $x|_{(t_i,t_{i+1}]}$ is continuous and $\lim_{t\to t_i^+} x(t)$ exists. The proof is complete. □

Theorem 3.12. *Suppose that F is continuous on $(0,1)$ and there exist constants $k > -1$ and $l \in (-\alpha, -n-k, 0]$ such that $|F(t)| \le t^k(1-t)^l$ for all $t \in (0,1)$. Then x is a solution of (3.24) if and only if there exist constants $c_{\sigma v} \in \mathbb{R}(\sigma \in \mathbb{N}[0,m], v \in$*

$\mathbb{N}[1,n]$) *such that*

$$\begin{aligned} x(t) &= \sum_{\sigma=0}^{i}\sum_{v=1}^{n} c_{\sigma v}\mathbf{E}_{\alpha,\alpha-v+1}(\lambda(t-t_\sigma)^\alpha)(t-t_\sigma)^{\alpha-v} \\ &\quad + \int_0^t (t-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t-s)^\alpha)F(s)ds, \quad t\in(t_i,t_{i+1}],\ i\in\mathbb{N}[0,m]. \end{aligned} \tag{3.29}$$

Proof. For $t\in(t_j,t_{j+1}](j\in\mathbb{N}_0^m)$, similarly to the beginning of the proof of Theorem 3.11 we know that

$$\begin{aligned} & t^{n-\alpha}\Big|\int_0^t (t-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t-s)^\alpha)F(s)ds\Big| \\ & \le \int_0^t (t-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t-s)^\alpha)|F(s)|ds \\ & \le t^{n-\alpha}\int_0^t (t-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t-s)^\alpha)s^k(1-s)^l ds \\ & = t^{n-\alpha}\sum_{j=0}^{+\infty}\frac{\lambda^j}{\Gamma((j+1)\alpha)}\int_0^t (t-s)^{\alpha-1}(t-s)^{\alpha j}s^k(1-s)^l ds \\ & \le t^{n-\alpha}\sum_{j=0}^{+\infty}\frac{\lambda^j}{\Gamma((j+1)\alpha)}\int_0^t (t-s)^{\alpha+l-1}(t-s)^{\alpha j}s^k ds \\ & = t^{n-\alpha}\sum_{j=0}^{+\infty}\frac{\lambda^j}{\Gamma((j+1)\alpha)}t^{\alpha+\alpha j+k+l}\int_0^1 (1-w)^{\alpha+\alpha j+l-1}w^k dw \\ & \le t^{n-\alpha}\sum_{j=0}^{+\infty}\frac{\lambda^j t^{\alpha j}}{\Gamma((j+1)\alpha)}t^{\alpha+k+l}\int_0^1 (1-w)^{\alpha+l-1}w^k dw \\ & = t^{n+k+l}\mathbf{E}_{\alpha,\alpha}(\lambda t^\alpha)\mathbf{B}(\alpha+l,k+1). \end{aligned}$$

So $t^{n-\alpha}\int_0^t (t-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t-s)^\alpha)F(s)ds$ is convergent and is continuous on $[0,1]$.

If x is a solution of (3.29), we have $x\in P_mC_{1-\alpha}(0,1]$. It follows for $t\in(t_i,t_{i+1}]$ and from Definition 2.2 that

$$\begin{aligned} & {}^{RL}D_{0^+}^\alpha x(t) \\ & = \frac{1}{\Gamma(n-\alpha)}\Big[\int_0^t (t-s)^{n-\alpha-1}x(s)ds\Big]^{(n)} \\ & = \frac{1}{\Gamma(n-\alpha)}\Big[\sum_{j=0}^{i-1}\int_{t_j}^{t_{j+1}}(t-s)^{n-\alpha-1} \\ & \quad\times\sum_{\sigma=0}^{j}\sum_{v=1}^{n} c_{\sigma v}\mathbf{E}_{\alpha,\alpha-v+1}(\lambda(s-t_\sigma)^\alpha)(s-t_\sigma)^{\alpha-v}ds\Big]^{(n)} \\ & \quad + \frac{1}{\Gamma(n-\alpha)}\Big[\int_{t_i}^t (t-s)^{n-\alpha-1}\sum_{\sigma=0}^{i}\sum_{v=1}^{n} c_{\sigma v}\mathbf{E}_{\alpha,\alpha-v+1}(\lambda(s-t_\sigma)^\alpha)(s-t_\sigma)^{\alpha-v}\Big]^{(n)} \\ & \quad + \frac{1}{\Gamma(n-\alpha)}\Big[\int_0^t (t-s)^{n-\alpha-1}\int_0^s (t-u)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(s-u)^\alpha)F(u)\,du\,ds\Big]^{(n)} \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\Gamma(n-\alpha)}\Big[\sum_{j=0}^{i-1}\int_{t_j}^{t_{j+1}}(t-s)^{n-\alpha-1}\sum_{\sigma=0}^{j}\sum_{v=1}^{n}c_{\sigma v}\sum_{\tau=0}^{+\infty}\frac{\lambda^\tau(s-t_\sigma)^{\tau\alpha+\alpha-v}}{\Gamma(\tau\alpha+\alpha-v+1)}ds\Big]^{(n)}\\
&\quad+\frac{1}{\Gamma(n-\alpha)}\Big[\int_{t_i}^{t}(t-s)^{n-\alpha-1}\sum_{\sigma=0}^{i}\sum_{v=1}^{n}c_{\sigma v}\sum_{\tau=0}^{+\infty}\frac{\lambda^\tau(s-t_\sigma)^{\tau\alpha+\alpha-v}}{\Gamma(\tau\alpha+\alpha-v+1)}ds\Big]^{(n)}\\
&\quad+\frac{1}{\Gamma(n-\alpha)}\Big[\int_0^t(t-s)^{n-\alpha-1}\int_0^s(s-u)^{\alpha-1}\sum_{\tau=0}^{+\infty}\frac{\lambda^\tau(s-u)^{\tau\alpha}}{\Gamma((\tau+1)\alpha)}F(u)\,du\,ds\Big]^{(n)}\\
&= \frac{1}{\Gamma(n-\alpha)}\Big[\sum_{j=0}^{i-1}\sum_{\sigma=0}^{j}\sum_{v=1}^{n}c_{\sigma v}\sum_{\tau=0}^{+\infty}\\
&\quad\times\int_{t_j}^{t_{j+1}}(t-s)^{n-\alpha-1}\frac{\lambda^\tau(s-t_\sigma)^{\tau\alpha+\alpha-v}}{\Gamma(\tau\alpha+\alpha-v+1)}ds\Big]^{(n)}\\
&\quad+\frac{1}{\Gamma(n-\alpha)}\Big[\sum_{\sigma=0}^{i}\sum_{v=1}^{n}c_{\sigma v}\sum_{\tau=0}^{+\infty}\\
&\quad\times\int_{t_i}^{t}(t-s)^{n-\alpha-1}\frac{\lambda^\tau(s-t_\sigma)^{\tau\alpha+\alpha-v}}{\Gamma(\tau\alpha+\alpha-v+1)}ds\Big]^{(n)}\\
&\quad+\frac{1}{\Gamma(n-\alpha)}\Big[\sum_{\tau=0}^{+\infty}\int_0^t\int_u^t(t-s)^{n-\alpha-1}(s-u)^{\alpha-1}\frac{\lambda^\tau(s-u)^{\tau\alpha}}{\Gamma((\tau+1)\alpha)}dsF(u)du\Big]^{(n)}\\
&= \frac{1}{\Gamma(n-\alpha)}\Big[\sum_{j=0}^{i-1}\sum_{\sigma=0}^{j}\sum_{v=1}^{n}c_{\sigma v}\sum_{\tau=0}^{+\infty}(t-t_\sigma)^{\tau\alpha+n-v}\\
&\quad\times\int_{\frac{t_j-t_\sigma}{t-t_\sigma}}^{\frac{t_{j+1}-t_\sigma}{t-t_\sigma}}(1-w)^{n-\alpha-1}\frac{\lambda^\tau w^{\tau\alpha+\alpha-v}}{\Gamma(\tau\alpha+\alpha-v+1)}dw\Big]^{(n)}\\
&\quad+\frac{1}{\Gamma(n-\alpha)}\Big[\sum_{\sigma=0}^{i}\sum_{v=1}^{n}c_{\sigma v}\sum_{\tau=0}^{+\infty}(t-t_\sigma)^{\tau\alpha+n-v}\\
&\quad\times\int_{\frac{t_i-t_\sigma}{t-t_\sigma}}^{1}(1-w)^{n-\alpha-1}\frac{\lambda^\tau w^{\tau\alpha+\alpha-v}}{\Gamma(\tau\alpha+\alpha-v+1)}dw\Big]^{(n)}\\
&\quad+\frac{1}{\Gamma(n-\alpha)}\Big[\sum_{\tau=0}^{+\infty}\int_0^t(t-u)^{\tau\alpha+n-1}\int_0^1(1-w)^{n-\alpha-1}\frac{\lambda^\tau w^{\tau\alpha+\alpha-1}}{\Gamma((\tau+1)\alpha)}dwF(u)du\Big]^{(n)}\\
&= \frac{1}{\Gamma(n-\alpha)}\Big[\sum_{\sigma=0}^{i-1}\sum_{j=\sigma}^{i-1}\sum_{v=1}^{n}c_{\sigma v}\sum_{\tau=0}^{+\infty}(t-t_\sigma)^{\tau\alpha+n-v}\\
&\quad\times\int_{\frac{t_j-t_\sigma}{t-t_\sigma}}^{\frac{t_{j+1}-t_\sigma}{t-t_\sigma}}(1-w)^{n-\alpha-1}\frac{\lambda^\tau w^{\tau\alpha+\alpha-v}}{\Gamma(\tau\alpha+\alpha-v+1)}dw\Big]^{(n)}\\
&\quad+\frac{1}{\Gamma(n-\alpha)}\Big[\sum_{\sigma=0}^{i}\sum_{v=1}^{n}c_{\sigma v}\sum_{\tau=0}^{+\infty}(t-t_\sigma)^{\tau\alpha+n-v}\\
&\quad\times\int_{\frac{t_i-t_\sigma}{t-t_\sigma}}^{1}(1-w)^{n-\alpha-1}\frac{\lambda^\tau w^{\tau\alpha+\alpha-v}}{\Gamma(\tau\alpha+\alpha-v+1)}dw\Big]^{(n)}
\end{aligned}
$$

$$
\begin{aligned}
&\quad + \Big[\sum_{\tau=0}^{+\infty}\int_0^t (t-u)^{\tau\alpha+n-1}\frac{\lambda^\tau}{\Gamma(\tau\alpha+n)}F(u)du\Big]^{(n)}\\
&= \frac{1}{\Gamma(n-\alpha)}\Big[\sum_{\sigma=0}^{i}\sum_{v=1}^{n}c_{\sigma v}\sum_{\tau=0}^{+\infty}(t-t_\sigma)^{\tau\alpha+n-v}\\
&\quad\times\int_0^1(1-w)^{n-\alpha-1}\frac{\lambda^\tau w^{\tau\alpha+\alpha-v}}{\Gamma(\tau\alpha+\alpha-v+1)}dw\Big]^{(n)}\\
&\quad + \Big[\sum_{\tau=0}^{+\infty}\int_0^t (t-u)^{\tau\alpha+n-1}\frac{\lambda^\tau}{\Gamma(\tau\alpha+n)}F(u)du\Big]^{(n)}\\
&= \Big[\sum_{\sigma=0}^{i}\sum_{v=1}^{n}c_{\sigma v}\sum_{\tau=0}^{+\infty}(t-t_\sigma)^{\tau\alpha+n-v}\frac{\lambda^\tau}{\Gamma(\tau\alpha+n-v+1)}\Big]^{(n)}\\
&\quad + \Big[\sum_{\tau=0}^{+\infty}\int_0^t (t-u)^{\tau\alpha+n-1}\frac{\lambda^\tau}{\Gamma(\tau\alpha+n)}F(u)du\Big]^{(n)}\\
&= \Big[\sum_{\sigma=0}^{i}\sum_{v=1}^{n}c_{\sigma v}\sum_{\tau=1}^{+\infty}(t-t_\sigma)^{\tau\alpha-v}\frac{\lambda^\tau}{\Gamma(\tau\alpha-v+1)}\Big]\\
&\quad + \Big[\sum_{\tau=1}^{+\infty}\int_0^t (t-u)^{\tau\alpha-1}\frac{\lambda^\tau}{\Gamma(\tau\alpha)}F(u)du\Big]^{(n)} + F(t)\\
&= \lambda x(t) + F(t), t\in(t_i,t_{i+1}].
\end{aligned}
$$

It follows that x is a solution of (3.24).

Now we prove that if x is a solution of (3.24), then x satisfies (3.29) and $x \in P_mC_{1-\alpha}(0,1]$ by mathematical induction. By Theorem 3.4, we know that there exists a constant $c_{0v}\in\mathbb{R}(v\in N[1,n]$ such that

$$
x(t)=\sum_{v=1}^{n}c_{0v}t^{\alpha-v}\mathbf{E}_{\alpha,\alpha-v+1}(\lambda t^\alpha)+\int_0^t(t-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t-s)^\alpha)F(s)ds,\quad t\in(t_0,t_1].
$$

Hence, (3.29) holds for $i=0$. Assume that (3.29) holds for $i=0,1,2,\dots,j\le m-1$, we will prove that (3.29) holds for $i=j+1$. Suppose that

$$
\begin{aligned}
x(t) &= \Phi(t)+\sum_{\sigma=0}^{j}\sum_{v=1}^{n}c_{\sigma v}\mathbf{E}_{\alpha,\alpha-v+1}(\lambda(t-t_\sigma)^\alpha)(t-t_\sigma)^{\alpha-v}\\
&\quad + \int_0^t(t-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t-s)^\alpha)F(s)ds,\quad t\in(t_{j+1},t_{j+2}].
\end{aligned}
$$

Then for $t\in(t_{j+1},t_{j+2}]$ we have

$$
\begin{aligned}
&F(t)+\lambda x(t)\\
&= {}^{RL}D_{0^+}^{\alpha}x(t) = \frac{1}{\Gamma(n-\alpha)}\Big[\sum_{\rho=0}^{j}\int_{t_\rho}^{t_{\rho+1}}(t-s)^{-\alpha}x(s)ds+\int_{t_{j+1}}^{t}(t-s)^{-\alpha}x(s)ds\Big]^{(n)}\\
&= \frac{1}{\Gamma(n-\alpha)}\Big[\sum_{\rho=0}^{j}\int_{t_\rho}^{t_{\rho+1}}(t-s)^{-\alpha}\Big(\sum_{\sigma=0}^{\rho}\sum_{v=1}^{n}c_{\sigma v}\mathbf{E}_{\alpha,\alpha-v+1}(\lambda(s-t_\sigma)^\alpha)(s-t_\sigma)^{\alpha-v}
\end{aligned}
$$

$$+\int_0^s (s-u)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(s-u)^\alpha)F(u)du\Big)ds\Big]^{(n)}$$
$$+\frac{1}{\Gamma(n-\alpha)}\Big[\int_{t_{j+1}}^t (t-s)^{-\alpha}\Big(\Phi(s)+\sum_{\sigma=0}^{j}\sum_{v=1}^{n}c_{\sigma v}\mathbf{E}_{\alpha,\alpha-v+1}(\lambda(s-t_\sigma)^\alpha)(s-t_\sigma)^{\alpha-v}$$
$$+\int_0^s (s-u)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(s-u)^\alpha)F(u)du\Big)ds\Big]^{(n)}$$
$$=\frac{1}{\Gamma(n-\alpha)}\Big[\sum_{\rho=0}^{j}\int_{t_\rho}^{t_{\rho+1}}(t-s)^{-\alpha}\sum_{\sigma=0}^{\rho}\sum_{v=1}^{n}c_{\sigma v}\mathbf{E}_{\alpha,\alpha-v+1}(\lambda(s-t_\sigma)^\alpha)(s-t_\sigma)^{\alpha-v}ds\Big]^{(n)}$$
$$+\frac{1}{\Gamma(n-\alpha)}\Big[\int_{t_{j+1}}^t (t-s)^{-\alpha}\Phi(s)$$
$$+\sum_{\sigma=0}^{j}\sum_{v=1}^{n}c_{\sigma v}\mathbf{E}_{\alpha,\alpha-v+1}(\lambda(s-t_\sigma)^\alpha)(s-t_\sigma)^{\alpha-v}ds\Big]^{(n)}$$
$$+\frac{1}{\Gamma(n-\alpha)}\Big[\int_0^t (t-s)^{-\alpha}\int_0^s (s-u)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(s-u)^\alpha)F(u)\,du\,ds\Big]^{(n)}.$$

As in the above discussion, we obtain

$$F(t)+\lambda x(t)={}^{RL}D_{0^+}^\alpha x(t)=F(t)+\lambda x(t)+{}^{RL}D_{t_{j+1}^+}^\alpha\Phi(t)-\lambda\Phi(t).$$

So ${}^{RL}D_{t_{j+1}^+}^\alpha\Phi(t)-\lambda\Phi(t)=0$ on $(t_{j+1},t_{j+2}]$. Then Theorem 3.4 implies that there exists a constant $c_{j+1v}\in\mathbb{R}$ such that $\Phi(t)=\sum_{v=1}^n c_{j+1v}(t-t_{i+1})^{\alpha-v}\mathbf{E}_{\alpha,\alpha-v+1}(\lambda(t-t_{i+1})^\alpha)$ on $(t_{j+1},t_{j+2}]$. Hence,

$$x(t)=\sum_{\rho=0}^{j+1}\sum_{v=1}^{n}c_{\rho v}(t-t_\rho)^{\alpha-v}\mathbf{E}_{\alpha,\alpha-v+1}(\lambda(t-t_\rho)^\alpha)$$
$$+\int_1^t (t-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t-s)^\alpha)f(s)ds,\quad t\in(t_{j+1},t_{j+2}].$$

By mathematical induction, we know that (3.29) holds for $j\in\mathbb{N}[0,m]$. The proof is complete. □

Theorem 3.13. *Suppose that G is continuous on $(1,e)$ and there exist constants $k>-1$ and $l\in(-\alpha,-n-k,0]$ such that $|G(t)|\le(\log t)^k(1-\log t)^l$ for all $t\in(1,e)$. Then x is a solution of (3.25) if and only if there exist constants $c_{jv}\in\mathbb{R}(j\in\mathbb{N}[0,m],\ v\in\mathbb{N}[1,n])$ such that*

$$x(t)=\sum_{j=0}^{i}\sum_{v=1}^{n}c_{jv}(\log\frac{t}{t_j})^{\alpha-v}\mathbf{E}_{\alpha,\alpha-v+1}\big(\lambda(\log\frac{t}{t_j})^\alpha\big)$$
$$+\int_1^t(\log\frac{t}{s})^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(\log\frac{t}{s})^\alpha)G(s)\frac{ds}{s},\quad t\in(t_i,t_{i+1}],\ i\in\mathbb{N}[0,m]. \tag{3.30}$$

Proof. For $t\in(t_i,t_{i+1}](i\in\mathbb{N}[0,m])$, similarly to the beginning of the proof of Theorem 3.12 we know that

$$(\log t)^{n-\alpha}\Big|\int_1^t(\log\frac{t}{s})^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(\log\frac{t}{s})^\alpha)G(s)\frac{ds}{s}\Big|$$

$$
\begin{aligned}
&\le (\log t)^{n-\alpha}\int_1^t (\log\frac{t}{s})^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(\log\frac{t}{s})^{\alpha})(\log s)^k(1-\log s)^l\frac{ds}{s}\\
&\le (\log t)^{n-\alpha}\sum_{\iota=0}^{+\infty}\frac{\lambda^\iota}{\Gamma(\alpha(\iota+1))}\int_1^t(\log\frac{t}{s})^{\alpha\iota+\alpha+l-1}(\log s)^k\frac{ds}{s}\\
&\quad\text{(because } \frac{\log s}{\log t}=w)\\
&= (\log t)^{n-\alpha}\sum_{\iota=0}^{+\infty}\frac{\lambda^\iota}{\Gamma(\alpha(\iota+1))}(\log t)^{\alpha\iota+\alpha+k+l}\int_0^1(1-w)^{\alpha\iota+\alpha+l-1}w^k dw\\
&\le (\log t)^{n+k+l}\mathbf{E}_{\alpha,\alpha}(\lambda(\log\frac{t}{s})^{\alpha})\int_0^1(1-w)^{\alpha+l-1}w^k dw\\
&= (\log t)^{n+k+l}\mathbf{E}_{\alpha,\alpha}\big(\lambda(\log\frac{t}{s})^{\alpha}\big)\mathbf{B}(\alpha+l,k+1).
\end{aligned}
$$

So $\int_1^t(\log\frac{t}{s})^{\alpha-1}\mathbf{E}_{\alpha,\alpha}\big(\lambda(\log\frac{t}{s})^{\alpha}\big)G(s)\frac{ds}{s}$ is convergent for all $t\in(1,e]$ and

$$
\lim_{t\to 1^+}(\log t)^{n-\alpha}\int_1^t(\log\frac{t}{s})^{\alpha-1}\mathbf{E}_{\alpha,\alpha}\big(\lambda(\log\frac{t}{s})^{\alpha}\big)G(s)\frac{ds}{s}
$$

exists.

If x is a solution of (3.30), we have $x\in LP_mC_{n-\alpha}(1,e]$. By using Definition 2.5, it follows for $t\in(t_i,t_{i+1}]$ that

$$
\begin{aligned}
&{}^{RLH}D_{1^+}^{\alpha}x(t)\\
&=\frac{1}{\Gamma(n-\alpha)}(t\frac{d}{dt})^n\Big[\int_1^t(\log\frac{t}{s})^{n-\alpha-1}x(s)\frac{ds}{s}\Big]\\
&=\frac{1}{\Gamma(n-\alpha)}(t\frac{d}{dt})^n\Big[\sum_{\sigma=0}^{i-1}\int_{t_\sigma}^{t_{\sigma+1}}(\log\frac{t}{s})^{n-\alpha-1}\\
&\quad\times\Big(\sum_{j=0}^{\sigma}\sum_{v=1}^{n}c_{jv}(\log\frac{s}{t_j})^{\alpha-v}\mathbf{E}_{\alpha,\alpha-v+1}(\lambda(\log\frac{s}{t_j})^{\alpha})\\
&\quad+\int_1^s(\log\frac{s}{u})^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(\log\frac{s}{u})^{\alpha})G(u)\frac{du}{u}\Big)\frac{ds}{s}\\
&\quad+\int_{t_i}^t(\log\frac{t}{s})^{n-\alpha-1}\Big(\sum_{j=0}^{i}\sum_{v=1}^{n}c_{jv}(\log\frac{s}{t_j})^{\alpha-v}\mathbf{E}_{\alpha,\alpha-v+1}(\lambda(\log\frac{s}{t_j})^{\alpha})\\
&\quad+\int_1^s(\log\frac{s}{u})^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(\log\frac{s}{u})^{\alpha})G(u)\frac{du}{u}\Big)\frac{ds}{s}\Big]\\
&=\frac{1}{\Gamma(n-\alpha)}(t\frac{d}{dt})^n\Big[\sum_{\sigma=0}^{i-1}\sum_{j=0}^{\sigma}\sum_{v=1}^{n}c_{jv}\int_{t_\sigma}^{t_{\sigma+1}}(\log\frac{t}{s})^{n-\alpha-1}(\log\frac{s}{t_j})^{\alpha-v}\\
&\quad\times\mathbf{E}_{\alpha,\alpha-v+1}(\lambda(\log\frac{s}{t_j})^{\alpha})\frac{ds}{s}\Big]\\
&\quad+(t\frac{d}{dt})^n\Big[\sum_{j=0}^{i}\sum_{v=1}^{n}c_{jv}\int_{t_i}^t(\log\frac{t}{s})^{n-\alpha-1}(\log\frac{s}{t_j})^{\alpha-v}\mathbf{E}_{\alpha,\alpha-v+1}(\lambda(\log\frac{s}{t_j})^{\alpha})\frac{ds}{s}\Big]
\end{aligned}
$$

$$+\frac{1}{\Gamma(n-\alpha)}(t\frac{d}{dt})^n\Big[\int_1^t(\log\frac{t}{s})^{n-\alpha-1}\int_1^s(\log\frac{s}{u})^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(\log\frac{s}{u})^{\alpha})G(u)\frac{du}{u}\frac{ds}{s}\Big].$$

One sees that

$$\begin{aligned}
&\int_{t_i}^t(\log\frac{t}{s})^{n-\alpha-1}(\log\frac{s}{t_v})^{\alpha-v}\mathbf{E}_{\alpha,\alpha-v+1}(\lambda(\log\frac{s}{t_v})^{\alpha})\frac{ds}{s}\\
&=\sum_{\kappa=0}^{+\infty}\frac{\lambda^\kappa}{\Gamma(\kappa\alpha+\alpha-v+1))}\int_{t_i}^t(\log\frac{t}{s})^{n-\alpha-1}(\log\frac{s}{t_v})^{\alpha\kappa+\alpha-v}\frac{ds}{s}\\
&\quad(\text{because }\frac{\log s-\log t_v}{\log t-\log t_v}=w)\\
&=\sum_{\kappa=0}^{+\infty}\frac{\lambda^\kappa}{\Gamma(\kappa\alpha+\alpha-v+1)}(\log\frac{t}{t_v})^{\alpha\kappa+n-v}\int_{\frac{\log t_i-\log t_v}{\log t-\log t_v}}^1(1-w)^{n-\alpha-1}w^{\alpha\kappa+\alpha-v}dw
\end{aligned}$$

and

$$\begin{aligned}
&\int_{t_i}^{t_{i+1}}(\log\frac{t}{s})^{n-\alpha-1}(\log\frac{s}{t_v})^{\alpha-v}\mathbf{E}_{\alpha,\alpha-v+1}\big(\lambda(\log\frac{s}{t_v})^{\alpha}\big)\frac{ds}{s}\\
&=\sum_{\kappa=0}^{+\infty}\frac{\lambda^\kappa}{\Gamma(\kappa\alpha+\alpha-v+1)}(\log\frac{t}{t_v})^{\alpha\kappa+n-v}\int_{\frac{\log t_i-\log t_v}{\log t-\log t_v}}^{\frac{\log t_{i+1}-\log t_v}{\log t-\log t_v}}(1-w)^{n-\alpha-1}w^{\alpha\kappa+\alpha-v}dw.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\int_1^t(\log\frac{t}{s})^{n-\alpha-1}\int_1^s(\log\frac{s}{u})^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(\log\frac{s}{u})^{\alpha})G(u)\frac{du}{u}\frac{ds}{s}\\
&=\int_1^t\int_u^t(\log\frac{t}{s})^{n-\alpha-1}(\log\frac{s}{u})^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(\log\frac{s}{u})^{\alpha})\frac{ds}{s}G(u)\frac{du}{u}\\
&=\int_1^t\int_u^t(\log\frac{t}{s})^{n-\alpha-1}(\log\frac{s}{u})^{\alpha-1}\sum_{\kappa=0}^{+\infty}\frac{\lambda^\kappa(\log\frac{s}{u})^{\kappa\alpha}}{\Gamma((\kappa+1)\alpha)}\frac{ds}{s}G(u)\frac{du}{u}\\
&=\sum_{\kappa=0}^{+\infty}\int_1^t(\log\frac{t}{u})^{\alpha\kappa+n-1}\frac{\lambda^\kappa}{\Gamma(\alpha(\kappa+1))}\int_0^1(1-w)^{n-\alpha-1}w^{\alpha\kappa+\alpha-1}dwG(u)\frac{du}{u}.
\end{aligned}$$

So

$$\begin{aligned}
&{}^{RLH}D_{1^+}^{\alpha}x(t)\\
&=\frac{1}{\Gamma(n-\alpha)}(t\frac{d}{dt})^n\Big[\sum_{\sigma=0}^{i-1}\sum_{j=0}^{\sigma}\sum_{v=1}^{n}c_{jv}\sum_{\kappa=0}^{+\infty}\frac{\lambda^\kappa(\log\frac{t}{t_j})^{\alpha\kappa+n-v}}{\Gamma(\kappa\alpha+\alpha-v+1)}\\
&\quad\times\int_{\frac{\log t_\sigma-\log t_j}{\log t-\log t_j}}^{\frac{\log t_{\sigma+1}-\log t_j}{\log t-\log t_j}}(1-w)^{n-\alpha-1}w^{\alpha\kappa+\alpha-v}dw\Big]\\
&\quad+(t\frac{d}{dt})^n\Big[\sum_{j=0}^{i}\sum_{v=1}^{n}c_{jv}\sum_{\kappa=0}^{+\infty}\frac{\lambda^\kappa}{\Gamma(\kappa\alpha+\alpha-v+1)}(\log\frac{t}{t_j})^{\alpha\kappa+n-v}\\
&\quad\times\int_{\frac{\log t_i-\log t_j}{\log t-\log t_j}}^1(1-w)^{n-\alpha-1}w^{\alpha\kappa+\alpha-v}dw\Big]
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(n-\alpha)}(t\frac{d}{dt})^n\Big[\sum_{\kappa=0}^{+\infty}\int_1^t(\log\frac{t}{u})^{\alpha\kappa+n-1}\frac{\lambda^\kappa}{\Gamma(\alpha(\kappa+1))}\\
&\times\int_0^1(1-w)^{n-\alpha-1}w^{\alpha\kappa+\alpha-1}dwG(u)\frac{du}{u}\Big]\\
&=\frac{1}{\Gamma(n-\alpha)}(t\frac{d}{dt})^n\Big[\sum_{j=0}^{i-1}\sum_{v=1}^{n}c_{jv}\sum_{\kappa=0}^{+\infty}\frac{\lambda^\kappa(\log\frac{t}{t_j})^{\alpha\kappa+n-v}}{\Gamma(\kappa\alpha+\alpha-v+1)}\\
&\times\int_0^{\frac{\log t_i-\log t_j}{\log t-\log t_j}}(1-w)^{n-\alpha-1}w^{\alpha\kappa+\alpha-v}dw\Big]\\
&+(t\frac{d}{dt})^n\Big[\sum_{j=0}^{i}\sum_{v=1}^{n}c_{jv}\sum_{\kappa=0}^{+\infty}\frac{\lambda^\kappa}{\Gamma(\kappa\alpha+\alpha-v+1)}(\log\frac{t}{t_j})^{\alpha\kappa+n-v}\\
&\times\int_{\frac{\log t_i-\log t_j}{\log t-\log t_j}}^1(1-w)^{n-\alpha-1}w^{\alpha\kappa+\alpha-v}dw\Big]\\
&+\frac{1}{\Gamma(n-\alpha)}(t\frac{d}{dt})^n\Big[\sum_{\kappa=0}^{+\infty}\int_1^t(\log\frac{t}{u})^{\alpha\kappa+n-1}\frac{\lambda^\kappa}{\Gamma(\alpha(\kappa+1))}\\
&\times\int_0^1(1-w)^{n-\alpha-1}w^{\alpha\kappa+\alpha-1}dwG(u)\frac{du}{u}\Big]\\
&=(t\frac{d}{dt})^n\Big[\sum_{j=0}^{i}\sum_{v=1}^{n}c_{jv}\sum_{\kappa=0}^{+\infty}\frac{\lambda^\kappa(\log\frac{t}{t_j})^{\alpha\kappa+n-v}}{\Gamma(\kappa\alpha+n-v+1)}\Big]\\
&+(t\frac{d}{dt})^n\Big[\sum_{\kappa=0}^{+\infty}\int_1^t(\log\frac{t}{u})^{\alpha\kappa+n-1}\frac{\lambda^\kappa}{\Gamma(\alpha\kappa+n)}G(u)\frac{du}{u}\Big]\\
&=(t\frac{d}{dt})^{n-1}\Big[\sum_{j=0}^{i}\sum_{v=1}^{n}c_{jv}\sum_{\kappa=0}^{+\infty}\frac{\lambda^\kappa(\log\frac{t}{t_j})^{\alpha\kappa+n-v-1}}{\Gamma(\kappa\alpha+n-v)}\Big]\\
&+(t\frac{d}{dt})^{n-1}\Big[\sum_{\kappa=0}^{+\infty}\int_1^t(\log\frac{t}{u})^{\alpha\kappa+n-2}\frac{\lambda^\kappa}{\Gamma(\alpha\kappa+n-1)}G(u)\frac{du}{u}\Big]\\
&=\dots\\
&=\sum_{j=0}^{i}\sum_{v=1}^{n}c_{jv}\sum_{\kappa=0}^{+\infty}\frac{\lambda^\kappa(\log\frac{t}{t_j})^{\alpha\kappa-v}}{\Gamma(\kappa\alpha-v+1)}+\sum_{\kappa=0}^{+\infty}\int_1^t(\log\frac{t}{u})^{\alpha\kappa-1}\frac{\lambda^\kappa}{\Gamma(\alpha\kappa)}G(u)\frac{du}{u}\\
&=\lambda x(t)+F(t),t\in(t_i,t_{i+1}].
\end{aligned}
$$

It follows that x is a solution of (3.25).

Now we prove that if x is a solution of (3.25), then x satisfies (3.30) and $x\in LP_mC_{n-\alpha}(1,e]$ by mathematical induction. By Theorem 3.6, we know that there exists a constant $c_{0v}\in\mathbb{R}(v\in N[1,n])$ such that

$$
x(t)=\sum_{v=0}^{n}c_{0v}(\log t)^{\alpha-v}\mathbf{E}_{\alpha,\alpha}(\lambda(\log t)^\alpha)+\int_1^t(\log\frac{t}{s})^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(\frac{t}{s})^\alpha)F(s)ds,
$$

for $t\in(t_0,t_1]$. Hence (3.30) holds for $i=0$. Assume that (3.30) holds for $i=0,1,2,\dots,j\le m-1$, we will prove that (3.30) holds for $i=j+1$. Suppose that

$$\begin{aligned} x(t)=\Phi(t)&+\sum_{\sigma=0}^{j}\sum_{v=1}^{n}c_{\sigma v}(\log\frac{t}{t_\sigma})^{\alpha-v}\mathbf{E}_{\alpha,\alpha-v+1}\big(\lambda(\log\frac{t}{t_\sigma})^{\alpha}\big)\\ &+\int_1^t(\log\frac{t}{s})^{\alpha-1}\mathbf{E}_{\alpha,\alpha}\big(\lambda(\log\frac{t}{s})^{\alpha}\big)G(s)\frac{ds}{s},\quad t\in(t_{j+1},t_{j+2}]. \end{aligned}$$

Then for $t\in(t_{j+1},t_{j+2}]$ we have

$$\begin{aligned} &F(t)+\lambda x(t)\\ &={}^{RLH}D_{1^+}^{\alpha}x(t)\\ &=\frac{1}{\Gamma(n-\alpha)}(t\frac{d}{dt})^n\Big[\sum_{\rho=0}^{j}\int_{t_\rho}^{t_{\rho+1}}(\log\frac{t}{s})^{n-\alpha-1}x(s)\frac{ds}{s}+\int_{t_{j+1}}^t(\log\frac{t}{s})^{n-\alpha-1}x(s)\frac{ds}{s}\Big]\\ &=\frac{1}{\Gamma(n-\alpha)}(t\frac{d}{dt})^n\Big[\sum_{\rho=0}^{j}\int_{t_\rho}^{t_{\rho+1}}(\log\frac{t}{s})^{n-\alpha-1}\\ &\quad\times\Big(\sum_{\sigma=0}^{\rho}\sum_{v=1}^{n}c_{\sigma v}(\log\frac{s}{t_\sigma})^{\alpha-v}\mathbf{E}_{\alpha,\alpha-v+1}\big(\lambda(\log\frac{s}{t_\sigma})^{\alpha}\big)\\ &\quad+\int_1^s(\log\frac{s}{u})^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(\log\frac{s}{u})^{\alpha})G(u)\frac{du}{u}\Big)\frac{ds}{s}\\ &\quad+\int_{t_{j+1}}^t(\log\frac{t}{s})^{n-\alpha-1}(\Phi(s)+\sum_{\sigma=0}^{j}\sum_{v=1}^{n}c_{\sigma v}(\log\frac{s}{t_\sigma})^{\alpha-v}\mathbf{E}_{\alpha,\alpha-v+1}(\lambda(\log\frac{s}{t_\sigma})^{\alpha})\\ &\quad+\int_1^s(\log\frac{s}{u})^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(\log\frac{s}{u})^{\alpha})G(u)\frac{du}{u})\frac{ds}{s}\Big]\\ &={}^{RLH}D_{t_{j+1}^+}^{\alpha}\Phi(t)+\frac{1}{\Gamma(n-\alpha)}(t\frac{d}{dt})^n\Big[\sum_{\rho=0}^{j}\int_{t_\rho}^{t_{\rho+1}}(\log\frac{t}{s})^{n-\alpha-1}\\ &\quad\times\sum_{\sigma=0}^{\rho}\sum_{v=1}^{n}c_{\sigma v}(\log\frac{s}{t_\sigma})^{\alpha-v}\mathbf{E}_{\alpha,\alpha-v+1}\big(\lambda(\log\frac{s}{t_\sigma})^{\alpha}\big)\Big]+\frac{1}{\Gamma(n-\alpha)}(t\frac{d}{dt})^n\\ &\quad\times\Big[\int_{t_{j+1}}^t(\log\frac{t}{s})^{n-\alpha-1}\sum_{\sigma=0}^{j}\sum_{v=1}^{n}c_{\sigma v}(\log\frac{s}{t_\sigma})^{\alpha-v}\mathbf{E}_{\alpha,\alpha-v+1}(\lambda(\log\frac{s}{t_\sigma})^{\alpha})\Big]\\ &\quad+\frac{1}{\Gamma(n-\alpha)}(t\frac{d}{dt})^n\Big[\int_1^t(\log\frac{t}{s})^{n-\alpha-1}\int_1^s(\log\frac{s}{u})^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(\log\frac{s}{u})^{\alpha})G(u)\frac{du}{u}\frac{ds}{s}\Big]. \end{aligned}$$

As above, we can obtain

$$F(t)+\lambda x(t)={}^{RLH}D_{1^+}^{\alpha}x(t)=F(t)+\lambda x(t)+{}^{RLH}D_{t_{j+1}^+}^{\alpha}\Phi(t)-\lambda\Phi(t).$$

So ${}^{RLH}D_{t_{j+1}^+}^{\alpha}\Phi(t)-\lambda\Phi(t)=0$ on $(t_{j+1},t_{j+2}]$. Then Theorem 3.6 implies that there exists a constant $c_{j+1v}\in\mathbb{R}(v\in N[1,n])$ such that

$$\Phi(t)=\sum_{v=1}^{n}c_{j+1v}(\log\frac{t}{t_{j+1}})^{\alpha-v}\mathbf{E}_{\alpha,\alpha}(\lambda(\frac{t}{t_{j+1}})^{\alpha})$$

on $(t_{j+1},t_{j+2}]$. Hence

$$\begin{aligned} x(t)=&\sum_{\rho=0}^{j+1}\sum_{v=1}^{n}c_{\rho v}(\log\frac{t}{t_\rho})^{\alpha-v}\mathbf{E}_{\alpha,\alpha-v+1}(\lambda(\log\frac{t}{t_\rho})^\alpha)\\ &+\int_1^t(\log\frac{t}{s})^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(\log\frac{t}{s})^\alpha)G(s)\frac{ds}{s},\quad t\in(t_{j+1},t_{j+2}]. \end{aligned}$$

By mathematical induction, we know that (3.30) holds for $j\in N_0$. The proof is complete. □

Theorem 3.14. *Suppose that G is continuous on $(1,e)$ and there exist constants $k>-\alpha+n-1$ and $l\in(-\alpha,-\alpha+k,0]$ such that $|G(t)|\le(\log t)^k(1-\log t)^l$ for all $t\in(1,e)$. Then x is a piecewise solution of (3.26) if and only if there exist constants $c_{jv}\in\mathbb{R}(j\in N[0,m],v\in N[1,n])$ such that*

$$\begin{aligned} x(t)=&\sum_{\rho=0}^{j}\sum_{v=0}^{n-1}c_{\rho v}(\log\frac{t}{t_\rho})^{v}\mathbf{E}_{\alpha,v+1}(\lambda(\log\frac{t}{t_\rho})^\alpha)\\ &+\int_1^t(\log\frac{t}{s})^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(\log\frac{t}{s})^\alpha)G(s)\frac{ds}{s},\quad t\in(t_j,t_{j+1}],\ j\in N[0,m]. \end{aligned} \tag{3.31}$$

Proof. For $t\in(t_j,t_{j+1}](j\in N[0,m])$, similarly to the beginning of the proof of Theorem 3.13 we know that

$$\begin{aligned} &\Big|\int_1^t(\log\frac{t}{s})^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(\log\frac{t}{s})^\alpha)G(s)\frac{ds}{s}\Big|\\ &\le(\log t)^{\alpha+k+l}\mathbf{E}_{\alpha,\alpha}(\lambda(\log\frac{t}{s})^\alpha)\mathbf{B}(\alpha+l,k+1). \end{aligned}$$

So $\int_1^t(\log\frac{t}{s})^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(\log\frac{t}{s})^\alpha)G(s)\frac{ds}{s}$ is convergent for all $t\in(1,e]$ and

$$\lim_{t\to1^+}\int_1^t(\log\frac{t}{s})^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(\log\frac{t}{s})^\alpha)G(s)\frac{ds}{s}$$

exists. We have

$$\begin{aligned} &\sum_{\tau=0}^{+\infty}\frac{\lambda^\tau}{\Gamma(\alpha(\tau+1)-n+1)}\int_1^s(\log\frac{s}{u})^{\tau\alpha+\alpha-n}|G(u)|\frac{du}{u}\\ &\le\sum_{\tau=0}^{+\infty}\frac{\lambda^\tau}{\Gamma(\alpha(\tau+1)-n+1)}\int_1^s(\log\frac{s}{u})^{\tau\alpha+\alpha-n}(\log\frac{e}{u})^l(\log u)^k\frac{du}{u}\\ &\le\sum_{\tau=0}^{+\infty}\frac{\lambda^\tau}{\Gamma(\alpha(\tau+1)-n+1)}\int_1^s(\log\frac{s}{u})^{\tau\alpha+\alpha+l-n}(\log u)^k\frac{du}{u}\\ &=\sum_{\tau=0}^{+\infty}\frac{\lambda^\tau}{\Gamma(\alpha(\tau+1)-n+1)}(\log s)^{\tau\alpha+\alpha+k+l-n+1}\int_0^1(1-w)^{\alpha\tau+\alpha+l-n}w^kdw\\ &\le(\log s)^{\alpha+k+l-n+1}\mathbf{E}_{\alpha,\alpha-v+1}(\lambda(\log s)^\alpha)\int_0^1(1-w)^{\alpha+l-n}w^kdw\\ &\le\mathbf{E}_{\alpha,\alpha-v+1}(\lambda(\log s)^\alpha)\mathbf{B}(\alpha+l-n+1,k+1). \end{aligned}$$

So

$$\lim_{s\to 1^+}\sum_{\tau=0}^{+\infty}\frac{\lambda^\tau}{\Gamma(\alpha(\tau+1)-n+1)}\int_1^s(\log\frac{s}{u})^{\tau\alpha+\alpha-n}|G(u)|\frac{du}{u}=0.$$

If x is a solution of (3.31), we have $x\in LP_mC(1,e]$. By using Definition 2.6, it follows for $t\in(t_i,t_{i+1}]$ that

$$\begin{aligned}
&{}^{CH}D_{1^+}^\alpha x(t)\\
&=\frac{1}{\Gamma(n-\alpha)}\int_1^t(\log\frac{t}{s})^{n-\alpha-1}(s\frac{d}{ds})^n x(s)\frac{ds}{s}\\
&=\frac{1}{\Gamma(n-\alpha)}\Big[\sum_{\sigma=0}^{i-1}\int_{t_\sigma}^{t_{\sigma+1}}(\log\frac{t}{s})^{n-\alpha-1}(s\frac{d}{ds})^n\Big(\sum_{\rho=0}^{\sigma}\sum_{v=0}^{n-1}c_{\rho v}(\log\frac{s}{t_\rho})^v\mathbf{E}_{\alpha,v+1}(\lambda(\log\frac{s}{t_\rho})^\alpha)\\
&\quad+\int_1^s(\log\frac{s}{u})^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(\log\frac{s}{u})^\alpha)G(u)\frac{du}{u}\Big)\frac{ds}{s}\\
&\quad+\int_{t_i}^t(\log\frac{t}{s})^{n-\alpha-1}(s\frac{d}{ds})^n\Big(\sum_{\rho=0}^{i}\sum_{v=0}^{n-1}c_{\rho v}(\log\frac{s}{t_\rho})^v\mathbf{E}_{\alpha,v+1}(\lambda(\log\frac{s}{t_\rho})^\alpha)\\
&\quad+\int_1^s(\log\frac{s}{u})^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(\log\frac{s}{u})^\alpha)G(u)\frac{du}{u}\Big)\frac{ds}{s}\Big]\\
&=\frac{1}{\Gamma(n-\alpha)}\sum_{\sigma=0}^{i-1}\sum_{\rho=0}^{\sigma}\sum_{v=0}^{n-1}c_{\rho v}\int_{t_\sigma}^{t_{\sigma+1}}(\log\frac{t}{s})^{n-\alpha-1}(s\frac{d}{ds})^n(\log\frac{s}{t_\rho})^v\mathbf{E}_{\alpha,v+1}\big(\lambda(\log\frac{s}{t_\rho})^\alpha\big)\frac{ds}{s}\\
&\quad+\frac{1}{\Gamma(n-\alpha)}\sum_{\rho=0}^{i}\sum_{v=0}^{n-1}c_{\rho v}\int_{t_i}^t(\log\frac{t}{s})^{n-\alpha-1}(s\frac{d}{ds})^n(\log\frac{s}{t_\rho})^v\mathbf{E}_{\alpha,v+1}(\lambda(\log\frac{s}{t_\rho})^\alpha)\frac{ds}{s}\\
&\quad\times\frac{1}{\Gamma(n-\alpha)}\int_1^t(\log\frac{t}{s})^{n-\alpha-1}(s\frac{d}{ds})^n\int_1^s(\log\frac{s}{u})^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(\log\frac{s}{u})^\alpha)G(u)\frac{du}{u}\frac{ds}{s}.
\end{aligned}$$

One sees that

$$\begin{aligned}
&\int_{t_i}^t(\log\frac{t}{s})^{n-\alpha-1}(s\frac{d}{ds})^n\Big((\log\frac{s}{t_\rho})^v\mathbf{E}_{\alpha,v+1}\big(\lambda(\log\frac{s}{t_v})^\alpha\big)\Big)\frac{ds}{s}\\
&=\int_{t_i}^t(\log\frac{t}{s})^{n-\alpha-1}(s\frac{d}{ds})^n\Big(\sum_{\tau=0}^{+\infty}\frac{\lambda^\tau(\log\frac{s}{t_\rho})^{\tau\alpha+v}}{\Gamma(\tau\alpha+v+1)}\Big)\frac{ds}{s}\\
&=\int_{t_i}^t(\log\frac{t}{s})^{n-\alpha-1}\Big(\sum_{\tau=1}^{+\infty}\frac{\lambda^\tau(\log\frac{s}{t_\rho})^{\tau\alpha+v-n}}{\Gamma(\tau\alpha+v-n+1)}\Big)\frac{ds}{s}\\
&=\sum_{\tau=1}^{+\infty}\frac{\lambda^\tau}{\Gamma(\tau\alpha+v-n+1)}(\log\frac{t}{t_\rho})^{\alpha(\tau-1)+v}\int_{\frac{\log t_i-\log t_\rho}{\log t-\log t_\rho}}^1(1-w)^{n-\alpha-1}w^{\tau\alpha+v-n}dw
\end{aligned}$$

and

$$\begin{aligned}
&\int_{t_\sigma}^{t_{\sigma+1}}(\log\frac{t}{s})^{n-\alpha-1}(s\frac{d}{ds})^n(\log\frac{s}{t_\rho})^v\mathbf{E}_{\alpha,v+1}\big(\lambda(\log\frac{s}{t_\rho})^\alpha\big)\frac{ds}{s}\\
&=\sum_{\tau=1}^{+\infty}\frac{\lambda^\tau}{\Gamma(\tau\alpha+v-n+1)}(\log\frac{t}{t_\rho})^{\alpha(\tau-1)+v}\int_{\frac{\log t_\sigma-\log t_\rho}{\log t-\log t_\rho}}^{\frac{\log t_{\sigma+1}-\log t_\rho}{\log t-\log t_\rho}}(1-w)^{n-\alpha-1}w^{\alpha\tau+v-n}dw.
\end{aligned}$$

Similarly,

$$
\begin{aligned}
&\int_1^t (\log\frac{t}{s})^{n-\alpha-1}(s\frac{d}{ds})^n\int_1^s(\log\frac{s}{u})^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(\log\frac{s}{u})^{\alpha})G(u)\frac{du}{u}\frac{ds}{s}\\
&=\int_1^t (\log\frac{t}{s})^{n-\alpha-1}(s\frac{d}{ds})^n\sum_{\tau=0}^{+\infty}\frac{\lambda^\tau}{\Gamma(\alpha(\tau+1))}\int_1^s(\log\frac{s}{u})^{\tau\alpha+\alpha-1}G(u)\frac{du}{u}\frac{ds}{s}\\
&=\int_1^t (\log\frac{t}{s})^{n-\alpha-1}(s\frac{d}{ds})\sum_{\tau=0}^{+\infty}\frac{\lambda^\tau}{\Gamma(\alpha(\tau+1)-n+1)}\int_1^s(\log\frac{s}{u})^{\tau\alpha+\alpha-n}G(u)\frac{du}{u}\frac{ds}{s}\\
&=\frac{t}{n-\alpha}\Big[\int_1^t (\log\frac{t}{s})^{n-\alpha}(\sum_{\tau=0}^{+\infty}\frac{\lambda^\tau}{\Gamma(\alpha(\tau+1)-n+1)}\int_1^s(\log\frac{s}{u})^{\tau\alpha+\alpha-n}G(u)\frac{du}{u})'ds\Big]'\\
&=\frac{t}{n-\alpha}\Big[(\log\frac{t}{s})^{n-\alpha}(\sum_{\tau=0}^{+\infty}\frac{\lambda^\tau}{\Gamma(\alpha(\tau+1)-n+1)}\int_1^s(\log\frac{s}{u})^{\tau\alpha+\alpha-n}G(u)\frac{du}{u})|_1^t\\
&\quad+(n-\alpha)\int_1^t (\log\frac{t}{s})^{n-\alpha-1}\sum_{\tau=0}^{+\infty}\frac{\lambda^\tau}{\Gamma(\alpha(\tau+1)-n+1)}\int_1^s(\log\frac{s}{u})^{\tau\alpha+\alpha-n}G(u)\frac{du}{u}\frac{ds}{s}\Big]'\\
&=t\Big[\int_1^t (\log\frac{t}{s})^{n-\alpha-1}\sum_{\tau=0}^{+\infty}\frac{\lambda^\tau}{\Gamma(\alpha(\tau+1)-n+1)}\int_1^s(\log\frac{s}{u})^{\tau\alpha+\alpha-n}G(u)\frac{du}{u}\frac{ds}{s}\Big]'\\
&=t\Big[\sum_{\tau=0}^{+\infty}\frac{\lambda^\tau}{\Gamma(\alpha(\tau+1)-n+1)}\int_1^t\int_u^t(\log\frac{t}{s})^{n-\alpha-1}(\log\frac{s}{u})^{\tau\alpha+\alpha-n}\frac{ds}{s}G(u)\frac{du}{u}\Big]'\\
&=t\left[\sum_{\tau=0}^{+\infty}\frac{\lambda^\tau}{\Gamma(\alpha(\tau+1)-n+1)}\int_1^t(\log\frac{t}{u})^{\alpha\tau}\int_0^1(1-w)^{n-\alpha-1}w^{\tau\alpha+\alpha-n}dwG(u)\frac{du}{u}\right]'\\
&=G(t)\frac{\mathbf{B}(n-\alpha,\alpha-n+1)}{\Gamma(\alpha-n+1)}+\sum_{\tau=1}^{+\infty}\frac{(\alpha\tau)\lambda^\tau\mathbf{B}(n-\alpha,\alpha\tau+\alpha-n+1)}{\Gamma(\alpha(\tau+1)-n+1)}\\
&\quad\times\int_1^t(\log\frac{t}{u})^{\alpha\tau-1}G(u)\frac{du}{u}.
\end{aligned}
$$

So

$$
\begin{aligned}
&{}^{CH}D_{1^+}^{\alpha}x(t)\\
&=\frac{1}{\Gamma(n-\alpha)}\sum_{\sigma=0}^{i-1}\sum_{\rho=0}^{\sigma}\sum_{v=0}^{n-1}c_{\rho v}\sum_{\tau=1}^{+\infty}\frac{\lambda^\tau}{\Gamma(\tau\alpha+v-n+1)}(\log\frac{t}{t_\rho})^{\alpha(\tau-1)+v}\\
&\quad\times\int_{\frac{\log t_\sigma-\log t_\rho}{\log t-\log t_\rho}}^{\frac{\log t_{\sigma+1}-\log t_\rho}{\log t-\log t_\rho}}(1-w)^{n-\alpha-1}w^{\alpha\tau+v-n}dw\\
&\quad+\frac{1}{\Gamma(n-\alpha)}\sum_{\rho=0}^{i}\sum_{v=0}^{n-1}c_{\rho v}\sum_{\tau=1}^{+\infty}\frac{\lambda^\tau}{\Gamma(\tau\alpha+v-n+1)}(\log\frac{t}{t_\rho})^{\alpha(\tau-1)+v}\\
&\quad\times\int_{\frac{\log t_i-\log t_\rho}{\log t-\log t_\rho}}^{1}(1-w)^{n-\alpha-1}w^{\tau\alpha+v-n}dw
\end{aligned}
$$

$$
\begin{aligned}
&+ G(t) + \sum_{\tau=1}^{+\infty} \frac{(\alpha\tau)\lambda^\tau}{\Gamma(\alpha\tau+1)} \int_1^t (\log\frac{t}{u})^{\alpha\tau-1} G(u)\frac{du}{u} \\
&= \frac{1}{\Gamma(n-\alpha)} \sum_{\rho=0}^{i-1}\sum_{v=0}^{n-1} c_{\rho v} \sum_{\tau=1}^{+\infty} \frac{\lambda^\tau}{\Gamma(\tau\alpha+v-n+1)} (\log\frac{t}{t_\rho})^{\alpha(\tau-1)+v} \\
&\quad \times \sum_{\sigma=\rho}^{i-1} \int_{\frac{\log t_\sigma-\log t_\rho}{\log t-\log t_\rho}}^{\frac{\log t_{\sigma+1}-\log t_\rho}{\log t-\log t_\rho}} (1-w)^{n-\alpha-1} w^{\alpha\tau+v-n} dw \\
&\quad + \frac{1}{\Gamma(n-\alpha)} \sum_{\rho=0}^{i}\sum_{v=0}^{n-1} c_{\rho v} \sum_{\tau=1}^{+\infty} \frac{\lambda^\tau}{\Gamma(\tau\alpha+v-n+1)} (\log\frac{t}{t_\rho})^{\alpha(\tau-1)+v} \\
&\quad \times \int_{\frac{\log t_i-\log t_\rho}{\log t-\log t_\rho}}^{1} (1-w)^{n-\alpha-1} w^{\tau\alpha+v-n} dw \\
&\quad + G(t) + \sum_{\tau=1}^{+\infty} \frac{(\alpha\tau)\lambda^\tau}{\Gamma(\alpha\tau+1)} \int_1^t (\log\frac{t}{u})^{\alpha\tau-1} G(u)\frac{du}{u} \\
&= \frac{1}{\Gamma(n-\alpha)} \sum_{\rho=0}^{i-1}\sum_{v=0}^{n-1} c_{\rho v} \sum_{\tau=1}^{+\infty} \frac{\lambda^\tau}{\Gamma(\tau\alpha+v-n+1)} (\log\frac{t}{t_\rho})^{\alpha(\tau-1)+v} \\
&\quad \times \int_0^{\frac{\log t_{i+1}-\log t_\rho}{\log t-\log t_\rho}} (1-w)^{n-\alpha-1} w^{\alpha\tau+v-n} dw \\
&\quad + \frac{1}{\Gamma(n-\alpha)} \sum_{\rho=0}^{i}\sum_{v=0}^{n-1} c_{\rho v} \sum_{\tau=1}^{+\infty} \frac{\lambda^\tau}{\Gamma(\tau\alpha+v-n+1)} (\log\frac{t}{t_\rho})^{\alpha(\tau-1)+v} \\
&\quad \times \int_{\frac{\log t_i-\log t_\rho}{\log t-\log t_\rho}}^{1} (1-w)^{n-\alpha-1} w^{\tau\alpha+v-n} dw \\
&\quad + G(t) \\
&\quad + \sum_{\tau=1}^{+\infty} \frac{(\alpha\tau)\lambda^\tau}{\Gamma(\alpha\tau+1)} \int_1^t (\log\frac{t}{u})^{\alpha\tau-1} G(u)\frac{du}{u} \\
&= \frac{1}{\Gamma(n-\alpha)} \sum_{\rho=0}^{i}\sum_{v=0}^{n-1} c_{\rho v} \sum_{\tau=1}^{+\infty} \frac{\lambda^\tau}{\Gamma(\tau\alpha+v-n+1)} (\log\frac{t}{t_\rho})^{\alpha(\tau-1)+v} \\
&\quad \times \int_0^1 (1-w)^{n-\alpha-1} w^{\alpha\tau+v-n} dw + G(t) \\
&\quad + \sum_{\tau=1}^{+\infty} \frac{(\alpha\tau)\lambda^\tau}{\Gamma(\alpha\tau+1)} \int_1^t (\log\frac{t}{u})^{\alpha\tau-1} G(u)\frac{du}{u} \\
&= \sum_{\rho=0}^{i}\sum_{v=0}^{n-1} c_{\rho v} \sum_{\tau=1}^{+\infty} \frac{\lambda^\tau}{\Gamma(\tau\alpha-\alpha+v+1)} (\log\frac{t}{t_\rho})^{\alpha(\tau-1)+v} + G(t) \\
&\quad + \sum_{\tau=1}^{+\infty} \frac{(\alpha\tau)\lambda^\tau}{\Gamma(\alpha\tau+1)} \int_1^t (\log\frac{t}{u})^{\alpha\tau-1} G(u)\frac{du}{u} \\
&= \lambda x(t) + G(t), t \in (t_i, t_{i+1}].
\end{aligned}
$$

It follows that x is a solution of (3.26).

Now we prove that if x is a solution of (3.26), then x satisfies (3.31) and $x \in LP_mC(1,e]$ by mathematical induction. By Theorem 3.8, we know that there exists a constant $c_{0v} \in \mathbb{R} (v \in N[0,n-1])$ such that

$$x(t)=\sum_{v=0}^{n-1} c_{0v}(\log t)^v \mathbf{E}_{\alpha,v+1}(\lambda(\log t)^\alpha)+\int_1^t (\log \frac{t}{s})^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(\frac{t}{s})^\alpha)F(s)ds,$$

for $t \in (t_0,t_1]$. Hence (3.31) holds for $j=0$. Assume that (3.31) holds for $j=0,1,2,\dots,i\le m-1$, we will prove that (3.31) holds for $j=i+1$. Suppose that

$$\begin{aligned} x(t)&=\Phi(t)+\sum_{\sigma=0}^{i}\sum_{v=0}^{n-1} c_{\sigma v}(\log\frac{t}{t_\sigma})^v \mathbf{E}_{\alpha,v+1}\big(\lambda(\log\frac{t}{t_\sigma})^\alpha\big)\\ &\quad+\int_1^t (\log\frac{t}{s})^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(\log\frac{t}{s})^\alpha)G(s)\frac{ds}{s},\quad t\in(t_{i+1},t_{i+2}]. \end{aligned}$$

Then for $t \in (t_{i+1},t_{i+2}]$ we have

$$\begin{aligned} &F(t)+\lambda x(t)\\ &= {}^{CH}D_{1^+}^\alpha x(t)\\ &=\frac{1}{\Gamma(n-\alpha)}\Big[\sum_{j=0}^{i}\int_{t_j}^{t_{j+1}}(\log\frac{t}{s})^{n-\alpha-1}(s\frac{d}{ds})^n x(s)\frac{ds}{s}\\ &\quad+\int_{t_{i+1}}^t(\log\frac{t}{s})^{n-\alpha-1}(s\frac{d}{ds})^n x(s)\frac{ds}{s}\Big]\\ &=\frac{1}{\Gamma(n-\alpha)}\sum_{j=0}^{i}\int_{t_j}^{t_{j+1}}(\log\frac{t}{s})^{n-\alpha-1}(s\frac{d}{ds})^n\Big(\sum_{\sigma=0}^{j}\sum_{v=0}^{n-1}c_{\sigma v}(\log\frac{s}{t_\sigma})^v\mathbf{E}_{\alpha,v+1}\big(\lambda(\log\frac{s}{t_\sigma})^\alpha\big)\\ &\quad+\int_1^s(\log\frac{s}{u})^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(\log\frac{s}{u})^\alpha)G(u)\frac{du}{u}\Big)\frac{ds}{s}+\frac{1}{\Gamma(n-\alpha)}\int_{t_{i+1}}^t(\log\frac{t}{s})^{n-\alpha-1}\\ &\quad\times(s\frac{d}{ds})^n\Big(\Phi(s)+\sum_{\sigma=0}^{i}\sum_{v=0}^{n-1}c_{\sigma v}(\log\frac{s}{t_\sigma})^v\mathbf{E}_{\alpha,v+1}(\lambda(\log\frac{s}{t_\sigma})^\alpha)\\ &\quad+\int_1^s(\log\frac{s}{u})^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(\log\frac{s}{u})^\alpha)G(u)\frac{du}{u}\Big)\frac{ds}{s}\\ &={}^{CH}D_{t_{i+1}^+}^\alpha\Phi(t)+\frac{1}{\Gamma(n-\alpha)}\sum_{j=0}^{i}\int_{t_j}^{t_{j+1}}(\log\frac{t}{s})^{n-\alpha-1}(s\frac{d}{ds})^n\\ &\quad\times\sum_{\sigma=0}^{j}\sum_{v=0}^{n-1}c_{\sigma v}(\log\frac{s}{t_\sigma})^v\mathbf{E}_{\alpha,v+1}(\lambda(\log\frac{s}{t_\sigma})^\alpha)\frac{ds}{s}\\ &\quad+\frac{1}{\Gamma(n-\alpha)}\int_{t_{i+1}}^t(\log\frac{t}{s})^{n-\alpha-1}(s\frac{d}{ds})^n\sum_{\sigma=0}^{i}\sum_{v=0}^{n-1}c_{\sigma v}(\log\frac{s}{t_\sigma})^v\mathbf{E}_{\alpha,v+1}\big(\lambda(\log\frac{s}{t_\sigma})^\alpha\big)\frac{ds}{s}\\ &\quad+\frac{1}{\Gamma(n-\alpha)}\int_1^t(\log\frac{t}{s})^{n-\alpha-1}(s\frac{d}{ds})^n\int_1^s(\log\frac{s}{u})^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(\log\frac{s}{u})^\alpha)G(u)\frac{du}{u}\frac{ds}{s}. \end{aligned}$$

as above discussion we obtain

$$F(t)+\lambda x(t)={}^{CH}D_{1^+}^{\alpha}x(t)=F(t)+\lambda x(t)+{}^{CH}D_{t_{i+1}^+}^{\alpha}\Phi(t)-\lambda\Phi(t).$$

So ${}^{CH}D_{t_{i+1}^+}^{\alpha}\Phi(t)-\lambda\Phi(t)=0$ for $t\in(t_{i+1},t_{i+2}]$. Then Theorem 3.8 implies that there exists a constant $c_{i+1v}\in\mathbb{R}(v\in N[0,n-1])$ such that

$$\Phi(t)=\sum_{v=0}^{n-1}c_{i+1v}(\log\frac{t}{t_{i+1}})^v\mathbf{E}_{\alpha,v+1}(\lambda(\frac{t}{t_{i+1}})^{\alpha}),\quad t\in(t_{i+1},t_{i+2}].$$

Hence

$$\begin{aligned}x(t)&=\sum_{j=0}^{i+1}\sum_{v=0}^{n-1}c_{jv}(\log\frac{t}{t_j})^v\mathbf{E}_{\alpha,v+1}\big(\lambda(\log\frac{t}{t_j})^{\alpha}\big)\\&\quad+\int_1^t(\log\frac{t}{s})^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(\log\frac{t}{s})^{\alpha})G(s)\frac{ds}{s},t\in(t_{i+1},t_{i+2}].\end{aligned}$$

By mathematical induction, we know that (3.31) holds for $j\in N_0$. The proof is complete. □

3.3. **Preliminaries for BVP** (1.7)**.** In this section, we present some preliminary results that can be used in next sections for obtaining solutions to (1.7).

Lemma 3.15. *Suppose that $\sigma:(0,1)\to\mathbb{R}$ is continuous and satisfies that there exist numbers $k>-1$ and $\max\{-\beta,-2-k\}<l\le 0$ such that $|\sigma(t)|\le t^k(1-t)^l$ for all $t\in(0,1)$. The x is a solutions of*

$$\begin{gathered}{}^{RL}D_{0^+}^{\beta}x(t)-\lambda x(t)=\sigma(t),\quad t\in(t_i,t_{i+1}],\ i\in\mathbb{N}[0,m],\\ \lim_{t\to0^+}t^{2-\beta}x(t)=a,\quad x(1)=b,\\ \lim_{t\to t_i^+}(t-t_i)^{2-\beta}x(t)=I_i,\quad i\in\mathbb{N}[1,m],\\ \Delta^{RL}D_{0^+}^{\beta-1}x(t_i)=J_i,\quad i\in\mathbb{N}[1,m],\end{gathered}\tag{3.32}$$

if and only if $x\in P_1C_{1-\alpha}(0,1]$ and

$$\begin{aligned}x(t)&=\frac{t^{\beta-1}\mathbf{E}_{\beta,\beta}(\lambda t^{\beta})}{\mathbf{E}_{\beta,\beta}(\lambda)}\Big[b-a\mathbf{E}_{\beta,\beta-1}(\lambda)-\int_0^1(1-s)^{\beta-1}\mathbf{E}_{\beta,\beta}(\lambda(1-s)^{\beta})\sigma(s)ds\\&\quad-\sum_{\sigma=1}^{m}((1-t_\sigma)^{\beta-1}\mathbf{E}_{\beta,\beta}(\lambda(1-t_\sigma)^{\beta})J_\sigma+(1-t_\sigma)^{\beta-2}\mathbf{E}_{\beta,\beta-1}(\lambda(1-t_\sigma)^{\beta})I_\sigma)\Big]\\&\quad+at^{\beta-2}\mathbf{E}_{\beta,\beta-1}(\lambda t^{\alpha})\\&\quad+\sum_{\sigma=1}^{i}\big[(t-t_\sigma)^{\beta-1}\mathbf{E}_{\beta,\beta}(\lambda(t-t_\sigma)^{\beta})J_\sigma+(t-t_\sigma)^{\beta-2}\mathbf{E}_{\beta,\beta-1}(\lambda(t-t_\sigma)^{\beta})I_\sigma\big]\\&\quad+\int_0^t(t-s)^{\beta-1}\mathbf{E}_{\beta,\beta}(\lambda(t-s)^{\beta})\sigma(s)ds,\quad t\in(t_i,t_{i+1}],\ i\in\mathbb{N}[0,m].\end{aligned}\tag{3.33}$$

Proof. Let x be a solution of (3.32). By Theorem 3.12, we know that there exist numbers $c_{\sigma 0}, c_{\sigma 1} \in \mathbb{R}(\sigma \in \mathbb{N}[1,n])$ such that

$$\begin{aligned} x(t) = \sum_{\sigma=0}^{i} \big[c_{\sigma 1}(t-t_\sigma)^{\beta-1}\mathbf{E}_{\beta,\beta}(\lambda(t-t_\sigma)^\beta) + c_{\sigma 2}(t-t_\sigma)^{\beta-2}\mathbf{E}_{\beta,\beta-1}(\lambda(t-t_\sigma)^\beta)\big] \\ + \int_0^t (t-s)^{\beta-1}\mathbf{E}_{\beta,\beta}(\lambda(t-s)^\beta)\sigma(s)ds, \quad t \in (t_i, t_{i+1}],\ i \in \mathbb{N}[0,m]. \end{aligned} \tag{3.34}$$

One has

$$\begin{aligned} &{}^{RL}D_{0^+}^{\beta-1}[(t-t_\sigma)^{\beta-1}\mathbf{E}_{\beta,\beta}(\lambda(t-t_\sigma)^\beta)] \\ &= \frac{1}{\Gamma(2-\beta)}\Big[\sum_{\tau=0}^{+\infty}\frac{\lambda^\tau}{\Gamma((\tau+1)\beta)}\int_0^t (t-s)^{1-\beta}(s-t_\sigma)^{\tau\beta+\beta-1}ds\Big]' \\ &= \frac{1}{\Gamma(2-\beta)}\Big[\sum_{\tau=0}^{+\infty}\frac{\lambda^\tau}{\Gamma((\tau+1)\beta)}(t-t_\sigma)^{\tau\beta+1}\int_0^1 (1-w)^{1-\beta}w^{\tau\beta+\beta-1}dw\Big]' \\ &= \mathbf{E}_{\beta,1}(\lambda(t-t_\sigma)^\beta), \end{aligned}$$

$${}^{RL}D_{0^+}^{\beta-1}\big[(t-t_\sigma)^{\beta-2}\mathbf{E}_{\beta,\beta-1}(\lambda(t-t_\sigma)^\beta)\big] = \lambda(t-t_\sigma)^\beta\mathbf{E}_{\beta,\beta+1}(\lambda(t-t_\sigma)^\beta),$$

$${}^{RL}D_{0^+}^{\beta-1}\big[(t-s)^{\beta-1}\mathbf{E}_{\beta,\beta}(\lambda(t-s)^\beta)\big] = \mathbf{E}_{\beta,1}(\lambda(t-s)^\beta).$$

It follows that

$$\begin{aligned} &{}^{RL}D_{0^+}^{\beta-1}x(t) \\ &= \sum_{\sigma=0}^{i}\big[c_{\sigma 1}\mathbf{E}_{\beta,1}(\lambda(t-t_\sigma)^\beta) + c_{\sigma 2}\lambda(t-t_\sigma)^\beta\mathbf{E}_{\beta,\beta+1}(\lambda(t-t_\sigma)^\beta)\big] \\ &\quad + \int_0^t \mathbf{E}_{\beta,1}(\lambda(t-s)^\beta)\sigma(s)ds, \quad t \in (t_i,t_{i+1}],\ i \in \mathbb{N}[0,m]. \end{aligned} \tag{3.35}$$

It follows from the boundary conditions and the impulse assumption in (3.32) that $c_{02} = a$, $c_{\sigma 2} = I_\sigma(\sigma \in \mathbb{N}[1,m])$, $c_{\sigma 1} = J_\sigma(\sigma \in \mathbb{N}[1,m])$ and

$$\begin{aligned} &\sum_{\sigma=0}^{m}\big[c_{\sigma 1}(1-t_\sigma)^{\beta-1}\mathbf{E}_{\beta,\beta}(\lambda(1-t_\sigma)^\beta) + c_{\sigma 2}(1-t_\sigma)^{\beta-2}\mathbf{E}_{\beta,\beta-1}(\lambda(1-t_\sigma)^\beta)\big] \\ &+ \int_0^1 (1-s)^{\beta-1}\mathbf{E}_{\beta,\beta}(\lambda(1-s)^\beta)\sigma(s)ds = b. \end{aligned}$$

Then

$$\begin{aligned} c_{0,1} &= \frac{1}{\mathbf{E}_{\beta,\beta}(\lambda)}\Big[b - a\mathbf{E}_{\beta,\beta-1}(\lambda) - \int_0^1 (1-s)^{\beta-1}\mathbf{E}_{\beta,\beta}(\lambda(1-s)^\beta)\sigma(s)ds \\ &\quad - \sum_{\sigma=1}^{m}((1-t_\sigma)^{\beta-1}\mathbf{E}_{\beta,\beta}(\lambda(1-t_\sigma)^\beta)J_\sigma \\ &\quad + (1-t_\sigma)^{\beta-2}\mathbf{E}_{\beta,\beta-1}(\lambda(1-t_\sigma)^\beta)I_\sigma)\Big]. \end{aligned} \tag{3.36}$$

Substituting $c_{\sigma 1}, c_{\sigma 2}(\sigma \in N[0,m])$ into (3.25), we obtain (3.33) obviously.

On the other hand, if x satisfies (3.33), then $x|_{(t_i,t_{i+1}]}(i \in \mathbb{N}[0,m])$ are continuous and the limits $\lim_{t\to t_i^+}(t-t_i)^{2-\beta}x(t)$ $(i \in \mathbb{N}[0,m])$ exist. So $x \in P_mC_{2-\beta}(0,1]$.

Using (3.36) and $c_{02}=a$, $c_{\sigma 2}=I_\sigma(\sigma\in N[1,m])$, $c_{\sigma 1}=J_\sigma(\sigma\in\mathbb{N}[1,m])$, we rewrite x by

$$\begin{aligned}x(t)&=\sum_{\sigma=0}^{i}\big[c_{\sigma 1}(t-t_\sigma)^{\beta-1}\mathbf{E}_{\beta,\beta}(\lambda(t-t_\sigma)^{\beta})+c_{\sigma 2}(t-t_\sigma)^{\beta-2}\mathbf{E}_{\beta,\beta-1}(\lambda(t-t_\sigma)^{\beta})\big]\\&\quad+\int_0^t(t-s)^{\beta-1}\mathbf{E}_{\beta,\beta}(\lambda(t-s)^{\alpha})\sigma(s)ds,\quad t\in(t_i,t_{i+1}],\ i\in\mathbb{N}[0,m].\end{aligned}$$

Since σ is continuous on $(0,1)$ and $|\sigma(t)|\le t^k(1-t)^l$, one can show easily that x is continuous on $(t_i,t_{i+1}](i=0,1)$ and using the method at the beginning of the proof of this lemma, we know that both limits $\lim_{t\to t_i^+}(t-t_i)^{2-\beta}x(t)$ $(i\in\mathbb{N}[0,m])$ exist. So $x\in P_mC_{2-\beta}(0,1]$. Furthermore, by direct computation, we have $x(1)=b$, and $\lim_{t\to 0^+}t^{1-\beta}x(t)=a$. One have from Theorem 3.12 easily for $t\in(t_0,t_1]$ that $D_{0^+}^{\beta}x(t)=\lambda x(t)+\sigma(t)$ and for $t\in(t_j,t_{j+1}]$ that

$$\begin{aligned}&{}^{RL}D_{0^+}^{\beta}x(t)\\&=\frac{1}{\Gamma(2-\beta)}\Big[\int_0^t(t-s)^{1-\beta}x(s)ds\Big]''\\&=\frac{1}{\Gamma(2-\beta)}\Big[\sum_{i=0}^{j-1}\int_{t_i}^{t_{i+1}}(t-s)^{1-\beta}\Big(\sum_{\sigma=0}^{i}\sum_{v=1}^{2}c_{\sigma v}(s-t_\sigma)^{\beta-v}\mathbf{E}_{\beta,\beta-v+1}(\lambda(s-t_\sigma)^{\beta})\\&\quad+\int_0^s(s-u)^{\beta-1}\mathbf{E}_{\beta,\beta}(\lambda(s-u)^{\beta})\sigma(u)du\Big)ds\Big]''\\&\quad+\frac{1}{\Gamma(2-\beta)}\Big[\int_{t_j}^{t}(t-s)^{1-\beta}\Big(\sum_{\sigma=0}^{j}\sum_{v=1}^{2}c_{\sigma v}(s-t_\sigma)^{\beta-v}\mathbf{E}_{\beta,\beta-v+1}(\lambda(s-t_\sigma)^{\beta})\\&\quad+\int_0^s(s-u)^{\beta-1}\mathbf{E}_{\beta,\beta}(\lambda(s-u)^{\beta})\sigma(u)du\Big)ds\Big]''\\&=\frac{1}{\Gamma(2-\beta)}\Big[\sum_{i=0}^{j-1}\int_{t_i}^{t_{i+1}}(t-s)^{1-\beta}\sum_{\sigma=0}^{i}\sum_{v=1}^{2}c_{\sigma v}(s-t_\sigma)^{\beta-v}\mathbf{E}_{\beta,\beta-v+1}(\lambda(s-t_\sigma)^{\beta})ds\Big]''\\&\quad+\frac{1}{\Gamma(2-\beta)}\Big[\int_{t_j}^{t}(t-s)^{1-\beta}\sum_{\sigma=0}^{j}\sum_{v=1}^{2}c_{\sigma v}(s-t_\sigma)^{\beta-v}\mathbf{E}_{\beta,\beta-v+1}(\lambda(s-t_\sigma)^{\beta})ds\Big]''\\&\quad+\frac{1}{\Gamma(2-\beta)}\Big[\int_0^t(t-s)^{1-\beta}\int_0^s(s-u)^{\beta-1}\mathbf{E}_{\beta,\beta}(\lambda(s-u)^{\beta})\sigma(u)\,du\,ds\Big]''\\&=\frac{1}{\Gamma(2-\beta)}\Big[\sum_{i=0}^{j-1}\sum_{\sigma=0}^{i}\sum_{v=1}^{2}c_{\sigma v}\sum_{\tau=0}^{+\infty}\frac{\lambda^{\tau}}{\Gamma((\tau+1)\beta-v+1)}\\&\quad\times\int_{t_i}^{t_{i+1}}(t-s)^{1-\beta}(s-t_\sigma)^{\tau\beta+\beta-v}ds\Big]''\\&\quad+\frac{1}{\Gamma(2-\beta)}\Big[\sum_{\sigma=0}^{j}\sum_{v=1}^{2}c_{\sigma v}\sum_{\tau=0}^{+\infty}\frac{\lambda^{\tau}}{\Gamma((\tau+1)\beta-v+1)}\int_{t_j}^{t}(t-s)^{1-\beta}(s-t_\sigma)^{\tau\beta+\beta-v}ds\Big]''\\&\quad+\frac{1}{\Gamma(2-\beta)}\Big[\sum_{\tau=0}^{+\infty}\frac{\lambda^{\tau}}{\Gamma((\tau+1)\beta)}\int_0^t\int_u^t(t-s)^{1-\beta}(s-u)^{\tau\beta+\beta-1}ds\sigma(u)du\Big]''\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\Gamma(2-\beta)}\Big[\sum_{\sigma=0}^{j-1}\sum_{i=\sigma}^{j-1}\sum_{v=1}^{2} c_{\sigma v}\sum_{\tau=0}^{+\infty}\frac{\lambda^\tau}{\Gamma((\tau+1)\beta-v+1)}(t-t_\sigma)^{\tau\beta-v+2}\\
&\quad\times\int_{\frac{t_i-t_\sigma}{t-t_\sigma}}^{\frac{t_{i+1}-t_\sigma}{t-t_\sigma}}(1-w)^{1-\beta}w^{\tau\beta+\beta-v}dw\Big]''\\
&\quad+\frac{1}{\Gamma(2-\beta)}\Big[\sum_{\sigma=0}^{j}\sum_{v=1}^{2} c_{\sigma v}\sum_{\tau=0}^{+\infty}\frac{\lambda^\tau}{\Gamma((\tau+1)\beta-v+1)}(t-t_\sigma)^{\tau\beta-v+2}\\
&\quad\times\int_{\frac{t_j-t_\sigma}{t-t_\sigma}}^{1}(1-w)^{1-\beta}w^{\tau\beta+\beta-v}dw\Big]''\\
&\quad+\frac{1}{\Gamma(2-\beta)}\Big[\sum_{\tau=0}^{+\infty}\frac{\lambda^\tau}{\Gamma((\tau+1)\beta)}\int_0^t(t-u)^{\tau\beta+1}\int_0^1(1-w)^{1-\beta}w^{\tau\beta+\beta-1}dw\sigma(u)du\Big]''\\
&= \frac{1}{\Gamma(2-\beta)}\Big[\sum_{\sigma=0}^{j-1}\sum_{v=1}^{2} c_{\sigma v}\sum_{\tau=0}^{+\infty}\frac{\lambda^\tau}{\Gamma((\tau+1)\beta-v+1)}(t-t_\sigma)^{\tau\beta-v+2}\\
&\quad\times\int_0^{\frac{t_j-t_\sigma}{t-t_\sigma}}(1-w)^{1-\beta}w^{\tau\beta+\beta-v}dw\Big]''\\
&\quad+\frac{1}{\Gamma(2-\beta)}\Big[\sum_{\sigma=0}^{j}\sum_{v=1}^{2} c_{\sigma v}\sum_{\tau=0}^{+\infty}\frac{\lambda^\tau}{\Gamma((\tau+1)\beta-v+1)}(t-t_\sigma)^{\tau\beta-v+2}\\
&\quad\times\int_{\frac{t_j-t_\sigma}{t-t_\sigma}}^{1}(1-w)^{1-\beta}w^{\tau\beta+\beta-v}dw\Big]''\Big[\sum_{\tau=0}^{+\infty}\frac{\lambda^\tau}{\Gamma(\tau\beta+2)}\int_0^t(t-u)^{\tau\beta+1}\sigma(u)du\Big]''\\
&= \frac{1}{\Gamma(2-\beta)}\Big[\sum_{\sigma=0}^{j}\sum_{v=1}^{2} c_{\sigma v}\sum_{\tau=0}^{+\infty}\frac{\lambda^\tau}{\Gamma((\tau+1)\beta-v+1)}(t-t_\sigma)^{\tau\beta-v+2}\\
&\quad\int_0^1(1-w)^{1-\beta}w^{\tau\beta+\beta-v}dw\Big]''+\sigma(t)+\sum_{\tau=1}^{+\infty}\frac{\lambda^\tau}{\Gamma(\tau\beta)}\int_0^t(t-u)^{\tau\beta-1}\sigma(u)du\\
&= \Big[\sum_{\sigma=0}^{j}\sum_{v=1}^{2} c_{\sigma v}\sum_{\tau=0}^{+\infty}\frac{\lambda^\tau}{\Gamma(\tau\beta-v+3)}(t-t_\sigma)^{\tau\beta-v+2}\Big]''+\sigma(t)+\sum_{\tau=1}^{+\infty}\frac{\lambda^\tau}{\Gamma(\tau\beta)}\\
&\quad\times\int_0^t(t-u)^{\tau\beta-1}\sigma(u)du\\
&= \sum_{\sigma=0}^{j}\sum_{v=1}^{2} c_{\sigma v}\sum_{\tau=1}^{+\infty}\frac{\lambda^\tau}{\Gamma(\tau\beta-v+1)}(t-t_\sigma)^{\tau\beta-v}+\sigma(t)\\
&\quad+\sum_{\tau=1}^{+\infty}\frac{\lambda^\tau}{\Gamma(\tau\beta)}\int_0^t(t-u)^{\tau\beta-1}\sigma(u)du\\
&= \lambda x(t)+\sigma(t).
\end{aligned}
$$

So x is a solution of (3.32). The proof is complete. □

Define the nonlinear operator T on $P_mC_{2-\beta}(0,1]$ for $x \in P_mC_{2-\beta}(0,1]$ by

$$
\begin{aligned}
(Tx)(t) = \frac{t^{\beta-1}\mathbf{E}_{\beta,\beta}(\lambda t^\beta)}{\mathbf{E}_{\beta,\beta}(\lambda)}\Big[&\int_0^1 \psi(s)H(s,x(s))ds - \mathbf{E}_{\beta,\beta-1}(\lambda)\int_0^1 \phi(s)G(s,x(s))ds \\
&- \int_0^1 (1-s)^{\beta-1}\mathbf{E}_{\beta,\beta}(\lambda(1-s)^\beta)p(s)f(s,x(s))ds \\
&- \sum_{\sigma=1}^m ((1-t_\sigma)^{\beta-1}\mathbf{E}_{\beta,\beta}(\lambda(1-t_\sigma)^\beta)J(t_\sigma,x(t_\sigma)) \\
&+ (1-t_\sigma)^{\beta-2}\mathbf{E}_{\beta,\beta-1}(\lambda(1-t_\sigma)^\beta)I(t_\sigma,x(t_\sigma)))\Big] \\
&+ t^{\beta-2}\mathbf{E}_{\beta,\beta-1}(\lambda t^\alpha)\int_0^1 \phi(s)G(s,x(s))ds \\
&+ \sum_{\sigma=1}^i \Big[(t-t_\sigma)^{\beta-1}\mathbf{E}_{\beta,\beta}(\lambda(t-t_\sigma)^\beta)J(t_\sigma,x(t_\sigma)) \\
&+ (t-t_\sigma)^{\beta-2}\mathbf{E}_{\beta,\beta-1}(\lambda(t-t_\sigma)^\beta)I(t_\sigma,x(t_\sigma))\Big] \\
&+ \int_0^t (t-s)^{\beta-1}\mathbf{E}_{\beta,\beta}(\lambda(t-s)^\beta)p(s)f(s,x(s))ds,
\end{aligned}
$$

for $t \in (t_i,t_{i+1}]$, $i \in \mathbb{N}[0,m]$.

Lemma 3.16. *Suppose that* (1.A1)–(1.A5) *hold, and that* f,G,H *are impulsive II-Carathéodory functions,* I,J *discrete II-Carathódory functions. Then* $T : P_mC_{2-\beta}(0,1] \to P_mC_{2-\beta}(0,1]$ *is well defined and is completely continuous,* $x \in P_mC_{2-\beta}(0,1]$ *is a solution of* (1.7) *if and only if* $x \in P_mC_{2-\beta}(0,1]$ *is a fixed point of* T*.*

Proof. **Step (i)** $T : P_mC_{2-\beta}(0,1] \to P_mC_{2-\beta}(0,1]$ is well defined. It comes from the method in Theorem 3.12 that $Tx|_{(t_i,t_{i+1}]}$ ($i \in \mathbb{N}[0,m]$) are continuous and the limits $\lim_{t\to t_i^+}(t-t_i)^{2-\beta}(Tx)(t)$ ($i \in \mathbb{N}[0,m]$) exist. We see from Lemma 3.15 that $x \in P_mC_{2-\beta}(0,1]$ is a solution of BVP(1.7) if and only if $x \in P_mC_{2-\beta}(0,1]$ is a fixed point of T in $P_mC_{2-\beta}(0,1]$.

Step (ii) T is continuous. Let $x_n \in P_mC_{2-\beta}(0,1]$ with $x_n \to x_0$ as $n \to +\infty$. We can show that $Tx_n \to Tx_0$ as $n \to +\infty$ by using the dominant convergence theorem. We refer the readers to the papers [120, 114, 92].

Step (iii) T is compact, i.e., $T(\overline{\Omega})$ is relatively compact for every bounded subset $\Omega \subset P_1C_{1-\alpha}(0,1]$. Let Ω be a bounded open nonempty subset of $P_mC_{2-\beta}(0,1]$. We have

$$
\|x\| = \max\{\sup_{t\in(t_i,t_{i+1}]}(t-t_i)^{2-\beta}|x(t)| : i \in \mathbb{N}[0,m]\} \le r < +\infty, \tag{3.37}
$$

for $(x,y) \in \overline{\Omega}$. Since f,G,H are impulsive II-Carathéodory functions, I,J are discrete II-Carathéodory functions, then there exists constants $M_f, M_I, M_J, M_G, M_H \ge$

0 such that

$$\begin{gathered}|f(t,x(t)))|=|f(t,(t-t_i)^{\alpha-2}(t-t_i)^{2-\alpha}x(t))|\le M_f,\quad t\in(t_i,t_{i+1}],\ i\in\mathbb{N}[0,m],\\ |G(t,x(t))|\le M_G, t\in(t_i,t_{i+1}],\quad i\in\mathbb{N}[0,m],\\ |H(t,x(t))|\le M_H, t\in(t_i,t_{i+1}],\quad i\in\mathbb{N}[0,m],\\ |I(t_i,x(t_i))|=|I(t_i,(t_i-t_{i-1})^{\beta-2}(t_i-t_{i-1})^{2-\beta}x(t_i))|\le M_I,\quad i\in\mathbb{N}[1,m],\\ |J(t_i,x(t_i))|=|I(t_i,(t_i-t_{i-1})^{\beta-2}(t_i-t_{i-1})^{2-\beta}x(t_i))|\le M_J,\quad i\in\mathbb{N}[1,m].\end{gathered}\tag{3.38}$$

This step is done in three sub-steps:

Sub-step (iii1) $T(\overline{\Omega})$ is uniformly bounded. Using (3.33) and (3.38), we have for $t\in(t_i,t_{i+1}]$ that

$$\begin{aligned}
&(t-t_i)^{2-\beta}|(Tx)(t)|\\
&\le\frac{(t-t_i)^{2-\beta}t^{\beta-1}\mathbf{E}_{\beta,\beta}(\lambda t^\beta)}{\mathbf{E}_{\beta,\beta}(\lambda)}\Big[\|\psi\|_1M_H+\mathbf{E}_{\beta,\beta-1}(\lambda)\|\phi\|_1M_G\\
&\quad+\int_0^1(1-s)^{\beta-1}\mathbf{E}_{\beta,\beta}(\lambda(1-s)^\beta)s^k(1-s)^l dsM_f\\
&\quad+\sum_{\sigma=1}^m((1-t_\sigma)^{\beta-1}\mathbf{E}_{\beta,\beta}(\lambda(1-t_\sigma)^\beta)M_J+(1-t_\sigma)^{\beta-2}\mathbf{E}_{\beta,\beta-1}(\lambda(1-t_\sigma)^\beta)M_I)\Big]\\
&\quad+(t-t_i)^{2-\beta}t^{\beta-2}\mathbf{E}_{\beta,\beta-1}(\lambda t^\alpha)\|\phi\|_1M_G\\
&\quad+(t-t_i)^{2-\beta}\sum_{\sigma=1}^i\Big[(t-t_\sigma)^{\beta-1}\mathbf{E}_{\beta,\beta}(\lambda(t-t_\sigma)^\beta)M_J\\
&\quad+(t-t_\sigma)^{\beta-2}\mathbf{E}_{\beta,\beta-1}(\lambda(t-t_\sigma)^\beta)M_I\Big]\\
&\quad+\int_0^t(t-s)^{\beta-1}\mathbf{E}_{\beta,\beta}(\lambda(t-s)^\beta)s^k(1-s)^l dsM_f\\
&\le\frac{\mathbf{E}_{\beta,\beta}(|\lambda|)}{\mathbf{E}_{\beta,\beta}(\lambda)}\Big[\|\psi\|_1M_H+\mathbf{E}_{\beta,\beta-1}(\lambda)\|\phi\|_1M_G\\
&\quad+\sum_{\tau=0}^{+\infty}\frac{\lambda^\tau}{\Gamma((\tau+1)\beta)}\int_0^1(1-s)^{\tau\beta+\beta+l-1}s^k dsM_f\\
&\quad+\sum_{\sigma=1}^m(\mathbf{E}_{\beta,\beta}(|\lambda|)M_J+(1-t_\sigma)^{\beta-2}\mathbf{E}_{\beta,\beta-1}(|\lambda|)M_I)\Big]\\
&\quad+\mathbf{E}_{\beta,\beta-1}(|\lambda|)\|\phi\|_1M_G+\sum_{\sigma=1}^m\Big[\mathbf{E}_{\beta,\beta}(|\lambda|)M_J+\mathbf{E}_{\beta,\beta-1}(|\lambda|)M_I\Big]\\
&\quad+\sum_{\tau=0}^{+\infty}\frac{\lambda^\tau}{\Gamma((\tau+1)\beta)}\int_0^t(t-s)^{\tau\beta+\beta+l-1}s^k dsM_f\\
&\le\Big(\frac{\mathbf{E}_{\beta,\beta}(|\lambda|)\mathbf{E}_{\beta,\beta-1}(\lambda)\|\phi\|_1}{\mathbf{E}_{\beta,\beta}(\lambda)}+\mathbf{E}_{\beta,\beta-1}(|\lambda|)\|\phi\|_1\Big)M_G+\frac{\mathbf{E}_{\beta,\beta}(|\lambda|)\|\psi\|_1}{\mathbf{E}_{\beta,\beta}(\lambda)}M_H\\
&\quad+\Big(\frac{\mathbf{E}_{\beta,\beta}(|\lambda|)\mathbf{E}_{\beta,\beta-1}(|\lambda|)}{\mathbf{E}_{\beta,\beta}(\lambda)}\sum_{\sigma=1}^m(1-t_\sigma)^{\beta-2}+m\mathbf{E}_{\beta,\beta-1}(|\lambda|)\Big)M_I
\end{aligned}$$

$$
\begin{aligned}
&+\Big(\frac{m\mathbf{E}_{\beta,\beta}(|\lambda|)\mathbf{E}_{\beta,\beta}(|\lambda|)}{\mathbf{E}_{\beta,\beta}(\lambda)}+m\mathbf{E}_{\beta,\beta}(|\lambda|)\Big)M_J\\
&+\Big(\frac{\mathbf{E}_{\beta,\beta}(|\lambda|)}{\mathbf{E}_{\beta,\beta}(\lambda)}\mathbf{E}_{\beta,\beta}(|\lambda|)+\mathbf{E}_{\beta,\beta}(|\lambda|)\Big)\mathbf{B}(\beta+l,k+1)M_f.
\end{aligned}
$$

From the above discussion, we obtain

$$
\begin{aligned}
\|Tx\|\le&\Big(\frac{\mathbf{E}_{\beta,\beta}(|\lambda|)\mathbf{E}_{\beta,\beta-1}(\lambda)\|\phi\|_1}{\mathbf{E}_{\beta,\beta}(\lambda)}+\mathbf{E}_{\beta,\beta-1}(|\lambda|)\|\phi\|_1\Big)M_G+\frac{\mathbf{E}_{\beta,\beta}(|\lambda|)\|\psi\|_1}{\mathbf{E}_{\beta,\beta}(\lambda)}M_H\\
&+\Big(\frac{\mathbf{E}_{\beta,\beta}(|\lambda|)\mathbf{E}_{\beta,\beta-1}(|\lambda|)}{\mathbf{E}_{\beta,\beta}(\lambda)}\sum_{\sigma=1}^{m}(1-t_\sigma)^{\beta-2}+m\mathbf{E}_{\beta,\beta-1}(|\lambda|)\Big)M_I\\
&+\Big(\frac{m\mathbf{E}_{\beta,\beta}(|\lambda|)\mathbf{E}_{\beta,\beta}(|\lambda|)}{\mathbf{E}_{\beta,\beta}(\lambda)}+m\mathbf{E}_{\beta,\beta}(|\lambda|)\Big)M_J\\
&+(\frac{\mathbf{E}_{\beta,\beta}(|\lambda|)}{\mathbf{E}_{\beta,\beta}(\lambda)}\mathbf{E}_{\beta,\beta}(|\lambda|)+\mathbf{E}_{\beta,\beta}(|\lambda|))\mathbf{B}(\beta+l,k+1)M_f.
\end{aligned}
\tag{3.39}
$$

From the above discussion, $T(\overline{\Omega})$ is uniformly bounded.

Sub-step (iii2) Prove that $t\to(t-t_i)^{2-\beta}T(\overline{\Omega})$ is equi-continuous on $(t_i,t_{i+1}](i\in\mathbb{N}[0,m])$. Let

$$
(t-t_i)^{2-\beta}\overline{(Tx)}(t)=\begin{cases}(t-t_i)^{2-\beta}(Tx)(t), & t\in(t_i,t_{i+1}],\\ \lim_{t\to t_i^+}(t-t_i)^{2-\beta}(Tx)(t), & t=t_i.\end{cases}
$$

Then $t\to(t-t_i)^{2-\beta}\overline{(Tx)}(t)$ is continuous on $[t_i,t_{i+1}]$. Let $s_2\le s_1$ and $s_1,s_2\in[t_i,t_{i+1}]$. By Ascoli-CArzela theorem on the closed interval, we have

$$
\big|(s_1-t_i)^{2-\beta}\overline{(Tx)}(s_1)-(s_2-t_i)^{2-\beta}\overline{(Tx)}(s_2)\big|\to0\quad\text{uniformly as } s_1\to s_2.
$$

Then $t\to(t-t_i)^{2-\beta}T(\overline{\Omega})$ is equi-continuous on $(t_i,t_{i+1}](i\in N[0,m])$. So $T(\overline{\Omega})$ is relatively compact. Then T is completely continuous. The proof is complete. □

3.4. **Preliminaries for BVP** (1.8)**.** In this section, we present some preliminary results that can be used in next sections for get solutions of (1.8).

Lemma 3.17. *Suppose that $\sigma:(0,1)\to\mathbb{R}$ is continuous and satisfies that there exist numbers $k>1-\beta$ and $l\le0$ with $l>\max\{-\beta,-\beta-k\}$ such that $|\sigma(t)|\le t^k(1-t)^l$ for all $t\in(0,1)$. The x is a solutions of*

$$
\begin{gathered}
{}^CD_{0^+}^{\beta}x(t)-\lambda x(t)=\sigma(t),\quad t\in(t_i,t_{i+1}],\ i\in\mathbb{N}[0,m],\\
x(0)=a,\quad x'(1)=b,\\
\Delta x(t_i)=I_i,\quad \Delta x'(t_i)=J_i,\quad i\in\mathbb{N}[1,m]
\end{gathered}
\tag{3.40}
$$

if and only if

$$\begin{aligned} x(t) = a + \frac{t}{\mathbf{E}_{\beta,1}(\lambda)} \Big[& b - \lambda \mathbf{E}_{\beta,\beta}(\lambda) a - \sum_{\sigma=1}^{m} \Big(\lambda \mathbf{E}_{\beta,\beta}(\lambda(1-t_\sigma)^\beta)(1-t_\sigma)^{\beta-1} I_\sigma \\ & + \mathbf{E}_{\beta,1}(\lambda(1-t_\sigma)^\beta) J_\sigma \Big) - \int_0^1 (1-s)^{\beta-1} \mathbf{E}_{\beta,\beta-1}(\lambda(1-s)^\beta)\sigma(s) ds \Big] \\ & + \sum_{j=1}^{i} [\mathbf{E}_{\beta,1}(\lambda(t-t_\sigma)^\beta) I_\sigma + (t-t_\sigma)\mathbf{E}_{\beta,2}(\lambda(t-t_\sigma)^\beta) J_\sigma] \\ & + \int_0^t (t-s)^{\beta-1} \mathbf{E}_{\beta,\beta}(\lambda(t-s)^\beta)\sigma(s) ds, \quad t \in (t_i, t_{i+1}],\ i \in \mathbb{N}[0,m] \end{aligned} \tag{3.41}$$

Proof. Let x be a solution of (3.40). We know by Theorem 3.11 that there exist numbers $c_{\sigma 0}, c_{\sigma 1} \in \mathbb{R}(\sigma \in \mathbb{N}[0,m])$ such that

$$\begin{aligned} x(t) = & \sum_{\sigma=0}^{i} \big[c_{\sigma 0} \mathbf{E}_{\beta,1}(\lambda(t-t_\sigma)^\beta) + c_{\sigma 1}(t-t_\sigma)\mathbf{E}_{\beta,2}(\lambda(t-t_\sigma)^\beta) \big] \\ & + \int_0^t (t-s)^{\beta-1} \mathbf{E}_{\beta,\beta}(\lambda(t-s)^\beta)\sigma(s) ds, \quad t \in (t_i, t_{i+1}],\ i \in \mathbb{N}[0,m]. \end{aligned} \tag{3.42}$$

It is easy to show that

$$\begin{gathered} \big[\mathbf{E}_{\beta,1}(\lambda(t-t_\sigma)^\beta)\big]' = \lambda \mathbf{E}_{\beta,\beta}(\lambda(t-t_\sigma)^\beta)(t-t_\sigma)^{\beta-1}, \\ \big[(t-t_\sigma)\mathbf{E}_{\beta,2}(\lambda(t-t_\sigma)^\beta)\big]' = \mathbf{E}_{\beta,1}(\lambda(t-t_\sigma)^\beta), \\ \big[(t-s)^{\beta-1}\mathbf{E}_{\beta,\beta}(\lambda(t-s)^\beta)\big]' = (t-s)^{\beta-2}\mathbf{E}_{\beta,\beta-1}(\lambda(t-s)^\beta). \end{gathered}$$

So

$$\begin{aligned} x'(t) = & \sum_{\sigma=0}^{i} \big[\lambda c_{\sigma 0} \mathbf{E}_{\beta,\beta}(\lambda(t-t_\sigma)^\beta)(t-t_\sigma)^{\beta-1} + c_{\sigma 1}\mathbf{E}_{\beta,1}(\lambda(t-t_\sigma)^\beta) \big] \\ & + \int_0^t (t-s)^{\beta-2} \mathbf{E}_{\beta,\beta-1}(\lambda(t-s)^\beta)\sigma(s) ds, \quad t \in (t_i, t_{i+1}],\ i \in \mathbb{N}[0,m]. \end{aligned} \tag{3.43}$$

Note $\mathbf{E}_{\beta,1}(0) = 1$, $\mathbf{E}_{\beta,2}(0) = 1$ and $\mathbf{E}_{\beta,\beta}(0) = \frac{1}{\Gamma(\beta)}$ and

$$\begin{aligned} & \Big| \int_0^t (t-s)^{\beta-1} \mathbf{E}_{\beta,\beta}(\lambda(t-s)^\beta)\sigma(s) ds \Big| \\ & \le \int_0^t (t-s)^{\beta-1} \mathbf{E}_{\beta,\beta}(\lambda(t-s)^\beta) s^k (1-s)^l ds \\ & \le \sum_{\tau=0}^{+\infty} \frac{\lambda^\tau}{\Gamma((\tau+1)\beta)} \int_0^t (t-s)^{\tau\beta+\beta+l-1} s^k ds \\ & = \sum_{\tau=0}^{+\infty} \frac{\lambda^\tau t^{\tau\beta+\beta+k+l}}{\Gamma((\tau+1)\beta)} \int_0^1 (1-w)^{\tau\beta+\beta+l-1} w^k dw \\ & \le t^{\beta+k+l} \mathbf{E}_{\beta,\beta}(\lambda t^\beta) \mathbf{B}(\beta+l, k+1) \to 0 \quad \text{as } t \to 0^+. \end{aligned}$$

It follows from (3.42), (3.43), the boundary conditions and the impulse assumption in (3.40) that $c_{00} = a$, $c_{\sigma 0} = I_\sigma$, $c_{\sigma 1} = J_\sigma$, $\sigma \in \mathbb{N}[1, m]$, and

$$\begin{aligned} &\sum_{\sigma=0}^{m} \big[\lambda c_{\sigma 0} \mathbf{E}_{\beta,\beta}(\lambda(1-t_\sigma)^\beta)(1-t_\sigma)^{\beta-1} + c_{\sigma 1}\mathbf{E}_{\beta,1}(\lambda(1-t_\sigma)^\beta)\big] \\ &+ \int_0^1 (1-s)^{\beta-2}\mathbf{E}_{\beta,\beta-1}(\lambda(1-s)^\beta)\sigma(s)ds = b. \end{aligned}$$

Then

$$\begin{aligned} c_{01} = \frac{1}{\mathbf{E}_{\beta,1}(\lambda)} \Big[b - &\int_0^1 (1-s)^{\beta-2}\mathbf{E}_{\beta,\beta-1}(\lambda(1-s)^\beta)\sigma(s)ds \\ \times &\sum_{\sigma=1}^{m} (\lambda \mathbf{E}_{\beta,\beta}(\lambda(1-t_\sigma)^\beta)(1-t_\sigma)^{\beta-1} I_\sigma + \mathbf{E}_{\beta,1}(\lambda(1-t_\sigma)^\beta) J_\sigma) - \lambda \mathbf{E}_{\beta,\beta}(\lambda) a \Big]. \end{aligned} \tag{3.44}$$

Substituting $c_{\sigma 0}, c_{\sigma 1} (\sigma \in N[0, m])$ into (3.42), we obtain (3.33).

On the other hand, if x satisfies (3.41), then $x|_{(t_i,t_{i+1}]} (i \in \mathbb{N}[0, m])$ are continuous and the limits $\lim_{t\to t_i^+} x(t)$ $(i \in \mathbb{N}[0, m])$ exist. So $x \in P_mC(0, 1]$. Using (3.44) and $c_{00} = a$, $c_{\sigma 0} = I_\sigma$, $c_{\sigma 1} = J_\sigma$, $\sigma \in \mathbb{N}[1, m]$, we rewrite x by

$$\begin{aligned} x(t) = &\sum_{\sigma=0}^{i} \big[c_{\sigma 0}\mathbf{E}_{\beta,1}(\lambda(t-t_\sigma)^\beta) + c_{\sigma 1}(t-t_\sigma)\mathbf{E}_{\beta,2}(\lambda(t-t_\sigma)^\beta)\big] \\ &+ \int_0^t (t-s)^{\beta-1}\mathbf{E}_{\beta,\beta}(\lambda(t-s)^\beta)\sigma(s)ds, \quad t \in (t_i, t_{i+1}],\ i \in \mathbb{N}[0, 1]. \end{aligned}$$

Since σ is continuous on $(0, 1)$ and $|\sigma(t)| \le t^k(1-t)^l$, one can show easily that x is continuous on $(t_i, t_{i+1}] (i = 0, 1)$ and using the method at the beginning of the proof of this lemma, we know that the limits $\lim_{t\to t_i^+} x(t)$ $(i \in \mathbb{N}[0, m])$ exist. So $x \in P_mC(0, 1]$. Next, by direct computation, we have $x(0) = a, x'(1) = b$, $\lim_{t\to t_i^+} x(t) - x(t_i) = I_i$ and $\lim_{t\to t_i^+} x(t) - x(t_i) = J_i$. Furthermore, we have

$$\begin{aligned} &\Big| \sum_{\tau=0}^{+\infty} \frac{\lambda^\tau}{\Gamma((\tau+1)\beta-1)} \int_0^s (s-u)^{\tau\beta+\beta-2}\sigma(u)du \Big| \\ &\le \sum_{\tau=0}^{+\infty} \frac{\lambda^\tau}{\Gamma((\tau+1)\beta-1)} \int_0^s (s-u)^{\tau\beta+\beta-2}u^k(1-u)^l du \\ &\le \sum_{\tau=0}^{+\infty} \frac{\lambda^\tau}{\Gamma((\tau+1)\beta-1)} \int_0^s (s-u)^{\tau\beta+\beta+l-2}u^k du \\ &= \sum_{\tau=0}^{+\infty} \frac{\lambda^\tau}{\Gamma((\tau+1)\beta-1)} s^{\tau\beta+\beta+k+l-1} \int_0^1 (1-w)^{\tau\beta+\beta+l-2}w^k dw \\ &\le \sum_{\tau=0}^{+\infty} \frac{\lambda^\tau s^{\tau\beta}}{\Gamma((\tau+1)\beta-1)} s^{\beta+k+l-1} \int_0^1 (1-w)^{\beta+l-2}w^k dw \\ &= \mathbf{E}_{\alpha,\beta-1}(\lambda s^\beta)s^{\beta+k+l-1}\mathbf{B}(\beta+l-1, k+1) \to 0 \text{ as } s \to 0^+. \end{aligned}$$

From Theorem 3.11 for $t \in (t_0, t_1]$ we have easily that ${}^CD^{\beta}_{0+}x(t) = \lambda x(t) + \sigma(t)$, and for $t \in (t_i, t_{i+1}]$ that

$$
\begin{aligned}
&{}^CD^{\beta}_{0+}x(t)\\
&= \frac{1}{\Gamma(2-\beta)}\int_0^t (t-s)^{1-\beta}x''(s)ds\\
&= \frac{1}{\Gamma(2-\beta)}\sum_{j=0}^{i-1}\int_{t_j}^{t_{j+1}}(t-s)^{1-\beta}\Big(\sum_{\sigma=0}^{j}\sum_{v=0}^{1}c_{\sigma v}(s-t_\sigma)^v\mathbf{E}_{\beta,v+1}(\lambda(s-t_\sigma)^\beta)\\
&\quad+\int_0^s (s-u)^{\beta-1}\mathbf{E}_{\beta,\beta}(\lambda(s-u)^\beta)\sigma(u)du\Big)''ds\\
&\quad+\frac{1}{\Gamma(2-\beta)}\int_{t_i}^t (t-s)^{1-\beta}\Big(\sum_{\sigma=0}^{i}\sum_{v=0}^{1}c_{\sigma v}(s-t_\sigma)^v\mathbf{E}_{\beta,v+1}(\lambda(s-t_\sigma)^\beta)\\
&\quad+\int_0^s (s-u)^{\beta-1}\mathbf{E}_{\beta,\beta}(\lambda(s-u)^\beta)\sigma(u)du\Big)''ds\\
&= \frac{1}{\Gamma(2-\beta)}\sum_{j=0}^{i-1}\int_{t_j}^{t_{j+1}}(t-s)^{1-\beta}(\sum_{\sigma=0}^{j}\sum_{v=0}^{1}c_{\sigma v}\sum_{\tau=0}^{+\infty}\frac{\lambda^\tau}{\Gamma(\tau\beta+v+1)}(s-t_\sigma)^{\beta\tau+v})''ds\\
&\quad+\frac{1}{\Gamma(2-\beta)}\int_{t_i}^t (t-s)^{1-\beta}Big(\sum_{\sigma=0}^{i}\sum_{v=0}^{1}c_{\sigma v}\sum_{\tau=0}^{+\infty}\frac{\lambda^\tau}{\Gamma(\tau\beta+v+1)}(s-t_\sigma)^{\beta\tau+v}\Big)''ds\\
&\quad+\frac{1}{\Gamma(2-\beta)}\int_0^t (t-s)^{1-\beta}\Big(\sum_{\tau=0}^{+\infty}\frac{\lambda^\tau}{\Gamma((\tau+1)\beta)}\int_0^s (s-u)^{\tau\beta+\beta-1}\sigma(u)du\Big)''ds\\
&= \frac{1}{\Gamma(2-\beta)}\sum_{j=0}^{i-1}\int_{t_j}^{t_{j+1}}(t-s)^{1-\beta}\Big(\sum_{\sigma=0}^{j}\sum_{v=0}^{1}c_{\sigma v}\sum_{\tau=1}^{+\infty}\frac{\lambda^\tau}{\Gamma(\tau\beta+v-1)}(s-t_\sigma)^{\beta\tau+v-2}\Big)ds\\
&\quad+\frac{1}{\Gamma(2-\beta)}\int_{t_i}^t (t-s)^{1-\beta}\Big(\sum_{\sigma=0}^{i}\sum_{v=0}^{1}c_{\sigma v}\sum_{\tau=1}^{+\infty}\frac{\lambda^\tau}{\Gamma(\tau\beta+v-1)}(s-t_\sigma)^{\beta\tau+v-2}\Big)ds\\
&\quad+\frac{1}{\Gamma(2-\beta)}\int_0^t (t-s)^{1-\beta}(\sum_{\tau=0}^{+\infty}\frac{\lambda^\tau}{\Gamma((\tau+1)\beta-1)}\int_0^s (s-u)^{\tau\beta+\beta-2}\sigma(u)du)'ds\\
&= \frac{1}{\Gamma(2-\beta)}\sum_{j=0}^{i-1}\sum_{\sigma=0}^{j}\sum_{v=0}^{1}c_{\sigma v}\sum_{\tau=1}^{+\infty}\frac{\lambda^\tau}{\Gamma(\tau\beta+v-1)}\int_{t_j}^{t_{j+1}}(t-s)^{1-\beta}(s-t_\sigma)^{\beta\tau+v-2}ds\\
&\quad+\frac{1}{\Gamma(2-\beta)}\sum_{\sigma=0}^{i}\sum_{v=0}^{1}c_{\sigma v}\sum_{\tau=1}^{+\infty}\frac{\lambda^\tau}{\Gamma(\tau\beta+v-1)}\int_{t_i}^t (t-s)^{1-\beta}(s-t_\sigma)^{\beta\tau+v-2}ds\\
&\quad+\frac{1}{\Gamma(3-\beta)}\Big[(t-s)^{2-\beta}(\sum_{\tau=0}^{+\infty}\frac{\lambda^\tau}{\Gamma((\tau+1)\beta-1)}\int_0^s (s-u)^{\tau\beta+\beta-2}\sigma(u)du)|_0^t\\
&\quad+((2-\beta)\int_0^t (t-s)^{1-\beta}(\sum_{\tau=0}^{+\infty}\frac{\lambda^\tau}{\Gamma((\tau+1)\beta-1)}\int_0^s (s-u)^{\tau\beta+\beta-2}\sigma(u)du)ds\Big]'\\
&= \frac{1}{\Gamma(2-\beta)}\sum_{j=0}^{i-1}\sum_{\sigma=0}^{j}\sum_{v=0}^{1}c_{\sigma v}\sum_{\tau=1}^{+\infty}\frac{\lambda^\tau}{\Gamma(\tau\beta+v-1)}(t-t_\sigma)^{\beta\tau-\beta+v}
\end{aligned}
$$

$$
\begin{aligned}
&\times \int_{\frac{t_j-t_\sigma}{t-t_\sigma}}^{\frac{t_{j+1}-t_\sigma}{t-t_\sigma}} (1-w)^{1-\beta} w^{\beta\tau+v-2}dw \\
&+ \frac{1}{\Gamma(2-\beta)} \sum_{\sigma=0}^{i}\sum_{v=0}^{1} c_{\sigma v} \sum_{\tau=1}^{+\infty} \frac{\lambda^\tau}{\Gamma(\tau\beta+v-1)}(t-t_\sigma)^{\beta\tau-\beta+v} \\
&\times \int_{\frac{t_i-t_\sigma}{t-t_\sigma}}^{1} (1-w)^{1-\beta} w^{\beta\tau+v-2}dw \\
&+ \frac{1}{\Gamma(2-\beta)}\Big[\sum_{\tau=0}^{+\infty} \frac{\lambda^\tau}{\Gamma((\tau+1)\beta-1)} \int_0^t\int_u^t (t-s)^{1-\beta}(s-u)^{\tau\beta+\beta-2}ds\sigma(u)du\Big]' \\
&= \frac{1}{\Gamma(2-\beta)} \sum_{\sigma=0}^{i-1}\sum_{j=\sigma}^{i-1}\sum_{v=0}^{1} c_{\sigma v} \sum_{\tau=1}^{+\infty} \frac{\lambda^\tau}{\Gamma(\tau\beta+v-1)}(t-t_\sigma)^{\beta\tau-\beta+v} \\
&\times \int_{\frac{t_j-t_\sigma}{t-t_\sigma}}^{\frac{t_{j+1}-t_\sigma}{t-t_\sigma}} (1-w)^{1-\beta} w^{\beta\tau+v-2}dw \\
&+ \frac{1}{\Gamma(2-\beta)} \sum_{\sigma=0}^{i}\sum_{v=0}^{1} c_{\sigma v} \sum_{\tau=1}^{+\infty} \frac{\lambda^\tau}{\Gamma(\tau\beta+v-1)}(t-t_\sigma)^{\beta\tau-\beta+v} \\
&\times \int_{\frac{t_i-t_\sigma}{t-t_\sigma}}^{1} (1-w)^{1-\beta} w^{\beta\tau+v-2}dw + \Big[\sum_{\tau=0}^{+\infty} \frac{\lambda^\tau}{\Gamma(\tau\beta+1)} \int_0^t (t-u)^{\tau\beta}\sigma(u)du\Big]' \\
&= \frac{1}{\Gamma(2-\beta)} \sum_{\sigma=0}^{i-1}\sum_{v=0}^{1} c_{\sigma v} \sum_{\tau=1}^{+\infty} \frac{\lambda^\tau}{\Gamma(\tau\beta+v-1)}(t-t_\sigma)^{\beta\tau-\beta+v} \\
&\times \int_0^{\frac{t_i-t_\sigma}{t-t_\sigma}} (1-w)^{1-\beta} w^{\beta\tau+v-2}dw \\
&+ \frac{1}{\Gamma(2-\beta)} \sum_{\sigma=0}^{i}\sum_{v=0}^{1} c_{\sigma v} \sum_{\tau=1}^{+\infty} \frac{\lambda^\tau}{\Gamma(\tau\beta+v-1)}(t-t_\sigma)^{\beta\tau-\beta+v} \\
&\times \int_{\frac{t_i-t_\sigma}{t-t_\sigma}}^{1} (1-w)^{1-\beta} w^{\beta\tau+v-2}dw + \Big[\sum_{\tau=0}^{+\infty} \frac{\lambda^\tau}{\Gamma(\tau\beta+1)} \int_0^t (t-u)^{\tau\beta}\sigma(u)du\Big]' \\
&= \frac{1}{\Gamma(2-\beta)} \sum_{\sigma=0}^{i}\sum_{v=0}^{1} c_{\sigma v} \sum_{\tau=1}^{+\infty} \frac{\lambda^\tau}{\Gamma(\tau\beta+v-1)}(t-t_\sigma)^{\beta\tau-\beta+v} \\
&\times \int_0^1 (1-w)^{1-\beta} w^{\beta\tau+v-2}dw + \sigma(t) + \sum_{\tau=1}^{+\infty} \frac{\lambda^\tau}{\Gamma(\tau\beta)} \int_0^t (t-u)^{\tau\beta-1}\sigma(u)du \\
&= \sum_{\sigma=0}^{i}\sum_{v=0}^{1} c_{\sigma v} \sum_{\tau=1}^{+\infty} \frac{\lambda^\tau}{\Gamma((\tau-1)\beta+v+1)}(t-t_\sigma)^{\beta\tau-\beta+v} + \sigma(t) \\
&+ \sum_{\tau=1}^{+\infty} \frac{\lambda^\tau}{\Gamma(\tau\beta)} \int_0^t (t-u)^{\tau\beta-1}\sigma(u)du \\
&= \lambda x(t) + \sigma(t).
\end{aligned}
$$

So x is a solution of (3.40). The proof is complete. □

Define the nonlinear operator Q on $P_mC(0,1]$ by Qx for $x \in P_mC(0,1]$ with

$$\begin{aligned}
&(Qx)(t)\\
&= \int_0^1 \phi(s)G(s,x(s))ds + \frac{t}{\mathbf{E}_{\beta,1}(\lambda)}\Big[\int_0^1 \psi(s)H(s,x(s))ds\\
&\quad - \lambda\mathbf{E}_{\beta,\beta}(\lambda)\int_0^1 \phi(s)G(s,x(s))ds\\
&\quad - \sum_{\sigma=1}^{m}(\lambda\mathbf{E}_{\beta,\beta}(\lambda(1-t_\sigma)^\beta)(1-t_\sigma)^{\beta-1}I(t_\sigma,x(t_\sigma)) + \mathbf{E}_{\beta,1}(\lambda(1-t_\sigma)^\beta)J(t_\sigma,x(t_\sigma)))\\
&\quad - \int_0^1 (1-s)^{\beta-1}\mathbf{E}_{\beta,\beta-1}(\lambda(1-s)^\beta)p(s)f(s,x(s))ds\Big]\\
&\quad + \sum_{j=1}^{i}\big[\mathbf{E}_{\beta,1}(\lambda(t-t_\sigma)^\beta)I(t_\sigma,x(t_\sigma)) + (t-t_\sigma)\mathbf{E}_{\beta,2}(\lambda(t-t_\sigma)^\beta)J(t_\sigma,x(t_\sigma))\big]\\
&\quad + \int_0^t (t-s)^{\beta-1}\mathbf{E}_{\beta,\beta}(\lambda(t-s)^\beta)p(s)f(s,x(s))ds, \quad t\in(t_i,t_{i+1}],\ i\in\mathbb{N}[0,m]
\end{aligned}$$

Lemma 3.18. *Suppose that (1.A2)–(1.A3), (1.A6)–(1.A8) in the introduction hold, and f,G,H are impulsive I-Carathéodory functions, I,J discrete I-Carathódory functions. Then $Q: P_mC(0,1]\to P_mC(0,1]$ is well defined and is completely continuous, $x\in P_mC(0,1]$ is a solution of BVP(1.8) if and only if $x\in P_mC(0,1]$ is a fixed point of Q.*

The proof of the above lemma is similar to that of Lemma 3.16 and is omitted.

3.5. **Preliminaries for BVP** (1.9)**.** In this section, we present some preliminary results that can be used in next sections for get solutions of (1.9). Denote

$$\Delta = \lambda\mathbf{E}_{\beta,\beta}(\lambda)^2 - \Big(\frac{1}{\Gamma(\beta-1)} - \mathbf{E}_{\beta,\beta-1}(\lambda)\Big)\Big(\frac{1}{\Gamma(\beta)} - \mathbf{E}_{\beta,\beta}(\lambda)\Big).$$

Lemma 3.19. *Suppose that $\Delta\neq 0$, $\sigma:(0,1)\to\mathbb{R}$ is continuous and that there exist numbers $k>-1$ and $\max\{-\beta,-2-k\}<l\le 0$ such that $|\sigma(t)|\le(\log t)^k(1-\log t)^l$ for all $t\in(1,e)$. Then x is a solution of*

$$\begin{gathered}
{}^{RLH}D_{1^+}^{\beta}x(t) - \lambda x(t) = \sigma(t), \quad t\in(t_i,t_{i+1}],\ i\in\mathbb{N}[0,m],\\
\lim_{t\to1^+}(\log t)^{2-\beta}x(t) - x(e) = a,\\
\lim_{t\to1^+}{}^{RLH}D_{1^+}^{\beta-1}x(t) - {}^{RLH}D_{1^+}^{\beta-1}x(e) = b,\\
\lim_{t\to t_i^+}(\log t-\log t_i)^{2-\beta}x(t) = I_i, i\in\mathbb{N}[1,m],\\
\Delta\,{}^{RL}D_{1^+}^{\beta-1}x(t_i) = J_i, i\in\mathbb{N}[1,m],
\end{gathered}\tag{3.45}$$

if and only if $x \in LP_mC_{2-\alpha}(1,e]$ *and*

$$\begin{aligned} x(t) &= M_1(\log t)^{\beta-1}\mathbf{E}_{\beta,\beta}\big(\lambda(\log t)^{\beta}\big) - M_2(\log t)^{\beta-2}\mathbf{E}_{\beta,\beta-1}\big(\lambda(\log t)^{\beta}\big) \\ &\quad + \sum_{\sigma=1}^{i}(\log\frac{t}{t_\sigma})^{\beta-1}\mathbf{E}_{\beta,\beta}\big(\lambda(\log\frac{t}{t_\sigma})^{\beta}\big)\Gamma(\beta)J_\sigma \\ &\quad + \sum_{\sigma=1}^{i}(\log\frac{t}{t_\sigma})^{\beta-2}\mathbf{E}_{\beta,\beta-1}\big(\lambda(\log\frac{t}{t_\sigma})^{\beta}\big)\Gamma(\beta-1)I_\sigma \\ &\quad + \int_1^t(\log\frac{t}{s})^{\beta-1}\mathbf{E}_{\beta,\beta}\big(\lambda(\log\frac{t}{s})^{\beta}\big)\sigma(s)\frac{ds}{s}, \quad t\in(t_i,t_{i+1}],\ i\in\mathbb{N}[0,m]. \end{aligned} \tag{3.46}$$

where

$$\begin{aligned} M_1 &= \frac{\lambda\mathbf{E}_{\beta,\beta}(\lambda)}{\Delta}a + \frac{\frac{1}{\Gamma(\beta-1)} - \mathbf{E}_{\beta,\beta-1}(\lambda)}{\Delta}b \\ &\quad + \sum_{\sigma=1}^{m}\Big(\frac{\frac{1}{\Gamma(\beta-1)} - \mathbf{E}_{\beta,\beta-1}(\lambda)}{\Delta}\mathbf{E}_{\beta,\beta}\big(\lambda(\log\frac{e}{t_\sigma})^{\beta}\big) \\ &\quad + \frac{\lambda\mathbf{E}_{\beta,\beta}(\lambda)}{\Delta}(\log\frac{e}{t_\sigma})^{\beta-1}\mathbf{E}_{\beta,\beta}\big(\lambda(\log\frac{e}{t_\sigma})^{\beta}\big)\Big)\Gamma(\beta)J_\sigma \\ &\quad + \sum_{\sigma=1}^{m}\Big(\lambda\frac{\frac{1}{\Gamma(\beta-1)} - \mathbf{E}_{\beta,\beta-1}(\lambda)}{\Delta}(\log\frac{e}{t_\sigma})^{\beta-1}\mathbf{E}_{\beta,\beta}\big(\lambda(\log\frac{e}{t_\sigma})^{\beta}\big) \\ &\quad + \frac{\lambda\mathbf{E}_{\beta,\beta}(\lambda)}{\Delta}(\log\frac{e}{t_\sigma})^{\beta-2}\mathbf{E}_{\beta,\beta-1}\big(\lambda(\log\frac{e}{t_\sigma})^{\beta}\big)\Big)\Gamma(\beta-1)I_\sigma \\ &\quad + \int_1^e\Big(\lambda\frac{\frac{1}{\Gamma(\beta-1)} - \mathbf{E}_{\beta,\beta-1}(\lambda)}{\Delta}\mathbf{E}_{\beta,1}\big(\lambda(\log\frac{e}{s})^{\beta}\big) \\ &\quad + \frac{\lambda\mathbf{E}_{\beta,\beta}(\lambda)}{\Delta}(\log\frac{e}{s})^{\beta-1}\mathbf{E}_{\beta,\beta}\big(\lambda(\log\frac{e}{s})^{\beta}\big)\Big)\sigma(s)\frac{ds}{s}, \end{aligned}$$

and

$$\begin{aligned} M_2 &= \frac{\frac{1}{\Gamma(\beta)} - \mathbf{E}_{\beta,\beta}(\lambda)}{\Delta}a + \frac{\mathbf{E}_{\beta,\beta}(\lambda)}{\Delta}b + \sum_{\sigma=1}^{m}\Big(\frac{\frac{1}{\Gamma(\beta)} - \mathbf{E}_{\beta,\beta}(\lambda)}{\Delta}(\log\frac{e}{t_\sigma})^{\beta-1} \\ &\quad \times \mathbf{E}_{\beta,\beta}\big(\lambda(\log\frac{e}{t_\sigma})^{\beta}\big) + \frac{\mathbf{E}_{\beta,\beta}(\lambda)}{\Delta}\mathbf{E}_{\beta,\beta}\big(\lambda(\log\frac{e}{t_\sigma})^{\beta}\big)\Big)\Gamma(\beta)J_\sigma \\ &\quad + \sum_{\sigma=1}^{m}\Big(\frac{\frac{1}{\Gamma(\beta)} - \mathbf{E}_{\beta,\beta}(\lambda)}{\Delta}(\log\frac{e}{t_\sigma})^{\beta-2}\mathbf{E}_{\beta,\beta-1}\big(\lambda(\log\frac{e}{t_\sigma})^{\beta}\big) \\ &\quad + \frac{\lambda\mathbf{E}_{\beta,\beta}(\lambda)}{\Delta}(\log\frac{e}{t_\sigma})^{\beta-1}\mathbf{E}_{\beta,\beta}\big(\lambda(\log\frac{e}{t_\sigma})^{\beta}\big)\Big)\Gamma(\beta-1)I_\sigma \\ &\quad + \int_1^e\Big(\frac{\frac{1}{\Gamma(\beta)} - \mathbf{E}_{\beta,\beta}(\lambda)}{\Delta}(\log\frac{e}{s})^{\beta-1}\mathbf{E}_{\beta,\beta}\big(\lambda(\log\frac{e}{s})^{\beta}\big) \\ &\quad + \lambda\frac{\mathbf{E}_{\beta,\beta}(9(\lambda)}{\Delta}\mathbf{E}_{\beta,1}\big(\lambda(\log\frac{e}{s})^{\beta}\big)\Big)\sigma(s)\frac{ds}{s}. \end{aligned}$$

Proof. Let x be a solution of (3.45). We know from Theorem 3.6 that there exist numbers $c_{\sigma 1}, c_{\sigma 2} \in \mathbb{R}$ such that

$$\begin{aligned} x(t) &= \sum_{\sigma=0}^{i} \big(c_{\sigma 1}(\log \frac{t}{t_\sigma})^{\beta-1}\mathbf{E}_{\beta,\beta}\big(\lambda(\log \frac{t}{t_\sigma})^{\beta}\big) \\ &\quad + c_{\sigma 2}(\log \frac{t}{t_\sigma})^{\beta-2}\mathbf{E}_{\beta,\beta-1}\big(\lambda(\log \frac{t}{t_\sigma})^{\beta}\big)\big) \\ &\quad + \int_1^t (\log \frac{t}{s})^{\beta-1}\mathbf{E}_{\beta,\beta}\big(\lambda(\log \frac{t}{s})^{\beta}\big)\sigma(s)\frac{ds}{s}, \end{aligned} \tag{3.47}$$

for $t \in (t_i, t_{i+1}]$, $i \in \mathbb{N}[0, m]$. From

$$\begin{aligned} &{}^{RLH}D_{1^+}^{\beta-1}(\log \frac{t}{t_\sigma})^{\beta-v}\mathbf{E}_{\beta,\beta-v+1}\big(\lambda(\log \frac{t}{t_\sigma})^{\beta}\big) \\ &= \frac{1}{\Gamma(2-\beta)}(t\frac{d}{dt})\int_1^t (\log \frac{t}{s})^{1-\beta}(\log \frac{s}{t_\sigma})^{\beta-v}\mathbf{E}_{\beta,\beta-v+1}\big(\lambda(\log \frac{s}{t_\sigma})^{\beta}\big)\frac{ds}{s} \\ &= \frac{1}{\Gamma(2-\beta)}(t\frac{d}{dt})\sum_{\tau=0}^{+\infty}\frac{\lambda^\tau}{\Gamma(\tau\beta+\beta-v+1)}\int_1^t (\log \frac{t}{s})^{1-\beta}(\log \frac{s}{t_\sigma})^{\tau\beta+\beta-v}\frac{ds}{s} \\ &= \frac{1}{\Gamma(2-\beta)}(t\frac{d}{dt})\sum_{\tau=0}^{+\infty}\frac{\lambda^\tau}{\Gamma(\tau\beta+\beta-v+1)}(\log \frac{t}{t_\sigma})^{\tau\beta-v+2} \\ &\quad \times \int_0^1 (1-w)^{1-\beta}w^{\tau\beta+\beta-v}dw \\ &= (t\frac{d}{dt})\sum_{\tau=0}^{+\infty}\frac{\lambda^\tau}{\Gamma(\tau\beta-v+3)}(\log \frac{t}{t_\sigma})^{\tau\beta-v+2} \\ &= \begin{cases} 1+\sum_{\tau=1}^{+\infty}\frac{\lambda^\tau}{\Gamma(\tau\beta)}(\log \frac{t}{t_\sigma})^{\tau\beta} = \mathbf{E}_{\beta,\beta}(\lambda(\log \frac{t}{t_\sigma})^{\beta}), & v=1, \\ \sum_{\tau=1}^{+\infty}\frac{\lambda^\tau}{\Gamma(\tau\beta)}(\log \frac{t}{t_\sigma})^{\tau\beta-1} = \lambda\mathbf{E}_{\beta,\beta}(\lambda(\log \frac{t}{t_\sigma})^{\beta})(\log \frac{t}{t_\sigma})^{\beta-1}, & v=2, \end{cases} \end{aligned}$$

and

$$\begin{aligned} &{}^{RLH}D_{1^+}^{\beta-1}(\log \frac{t}{s})^{\beta-1}\mathbf{E}_{\beta,\beta}\big(\lambda(\log \frac{t}{s})^{\beta}\big) \\ &= \frac{1}{\Gamma(2-\beta)}(t\frac{d}{dt})\int_1^t (\log \frac{t}{u})^{1-\beta}(\log \frac{u}{s})^{\beta-1}\mathbf{E}_{\beta,\beta}(\lambda(\log \frac{u}{s})^{\beta})\frac{du}{u} \\ &= \frac{1}{\Gamma(2-\beta)}(t\frac{d}{dt})\sum_{\tau=0}^{+\infty}\frac{\lambda^\tau}{\Gamma((\tau+1)\beta)}\int_1^t (\log \frac{t}{u})^{1-\beta}(\log \frac{u}{s})^{\tau\beta+\beta-1}\frac{du}{u} \\ &= \frac{1}{\Gamma(2-\beta)}(t\frac{d}{dt})\sum_{\tau=0}^{+\infty}\frac{\lambda^\tau}{\Gamma((\tau+1)\beta)}(\log \frac{t}{s})^{\tau\beta+1}\int_0^1 (1-w)^{1-\beta}w^{\tau\beta+\beta-1}dw \\ &= (t\frac{d}{dt})\sum_{\tau=0}^{+\infty}\frac{\lambda^\tau}{\Gamma(\tau\beta+2)}(\log \frac{t}{s})^{\tau\beta+1} \\ &= \sum_{\tau=0}^{+\infty}\frac{\lambda^\tau}{\Gamma(\tau\beta+1)}(\log \frac{t}{s})^{\tau\beta} = \mathbf{E}_{\beta,1}(\lambda(\log \frac{t}{s})^{\beta}), \end{aligned}$$

we have

$$\begin{aligned} &{}^{RLH}D_{1^+}^{\beta-1}x(t)\\ &= \sum_{\sigma=0}^{i}\Big[c_{\sigma 1}\mathbf{E}_{\beta,\beta}(\lambda(\log\frac{t}{t_\sigma})^\beta) + \lambda c_{\sigma 2}\mathbf{E}_{\beta,\beta}(\lambda(\log\frac{t}{t_\sigma})^\beta)(\log\frac{t}{t_\sigma})^{\beta-1}\Big] \\ &\quad + \lambda\int_1^t \mathbf{E}_{\beta,1}\big(\lambda(\log\frac{t}{s})^\beta\big)\sigma(s)\frac{ds}{s}, \quad t\in(t_i,t_{i+1}],\ i\in\mathbb{N}[0,m]. \end{aligned} \tag{3.48}$$

One sees that

$$\begin{aligned} &(\log t)^{2-\beta}\Big|\int_1^t(\log\frac{t}{s})^{\beta-1}\mathbf{E}_{\beta,\beta}\big(\lambda(\log\frac{t}{s})^\beta\big)\sigma(s)\frac{ds}{s}\Big| \\ &\le (\log t)^{2-\beta}\int_1^t(\log\frac{t}{s})^{\beta-1}\mathbf{E}_{\beta,\beta}\big(\lambda(\log\frac{t}{s})^\beta\big)(\log s)^k(\log\frac{e}{s})^l\frac{ds}{s} \\ &\le (\log t)^{2-\beta}\sum_{\tau=0}^{+\infty}\frac{\lambda^\tau}{\Gamma((\tau+1)\beta)}\int_1^t(\log\frac{t}{s})^{\tau\beta+\beta+l-1}(\log s)^k\frac{ds}{s} \\ &= (\log t)^{2-\beta}\sum_{\tau=0}^{+\infty}\frac{\lambda^\tau}{\Gamma((\tau+1)\beta)}(\log t)^{\tau\beta+\beta+k+l}\int_0^1(1-w)^{\tau\beta+\beta+l-1}w^k dw \\ &\le (\log t)^{2-\beta}\sum_{\tau=0}^{+\infty}\frac{\lambda^\tau}{\Gamma((\tau+1)\beta)}(\log t)^{\tau\beta+\beta+k+l}\int_0^1(1-w)^{\beta+l-1}w^k dw \\ &= (\log t)^{2+k+l}\mathbf{E}_{\beta,\beta}(\lambda(\log t)^\beta)\mathbf{B}(\beta+l,k+1)\to 0 \text{ as } t\to 1^+. \end{aligned}$$

Similarly we have

$$\begin{aligned} &\Big|\int_1^t\mathbf{E}_{\beta,1}\big(\lambda(\log\frac{t}{s})^\beta\big)\sigma(s)\frac{ds}{s}\Big| \\ &\le \int_1^t\mathbf{E}_{\beta,1}\big(\lambda(\log\frac{t}{s})^\beta\big)(\log s)^k(\log\frac{e}{s})^l\frac{ds}{s} \\ &\le \sum_{\tau=0}^{+\infty}\frac{\lambda^\tau}{\Gamma(\tau\beta+1)}\int_1^t(\log\frac{t}{s})^{\tau\beta+l}(\log s)^k\frac{ds}{s} \\ &= \sum_{\tau=0}^{+\infty}\frac{\lambda^\tau}{\Gamma(\tau\beta+1)}(\log t)^{\tau\beta+k+l+1}\int_0^1(1-w)^{\tau\beta+l}w^k dw \\ &\le \sum_{\tau=0}^{+\infty}\frac{\lambda^\tau}{\Gamma(\tau\beta+1)}(\log t)^{\tau\beta+k+l+1}\int_0^1(1-w)^{l}w^k dw \\ &= (\log t)^{k+l+1}\mathbf{E}_{\beta,1}\big(\lambda(\log t)^\beta\big)\mathbf{B}(l+1,k+1)\to 0 \quad \text{as } t\to 1^+. \end{aligned}$$

It follows from (3.47), (3.48), the boundary conditions and the impulse assumption in (3.45) that

$$\begin{aligned} &\frac{1}{\Gamma(\beta-1)}c_{02} - \Big[\sum_{\sigma=0}^{m}\Big(c_{\sigma 1}(\log\frac{e}{t_\sigma})^{\beta-1}\mathbf{E}_{\beta,\beta}\big(\lambda(\log\frac{e}{t_\sigma})^\beta\big) \\ &+ c_{\sigma 2}(\log\frac{e}{t_\sigma})^{\beta-2}\mathbf{E}_{\beta,\beta-1}\big(\lambda(\log\frac{e}{t_\sigma})^\beta\big)\Big) + \int_1^e(\log\frac{e}{s})^{\beta-1}\mathbf{E}_{\beta,\beta}\big(\lambda(\log\frac{e}{s})^\beta\big)\sigma(s)\frac{ds}{s}\Big] = a, \end{aligned}$$

$$\begin{aligned}
&\frac{1}{\Gamma(\beta)}c_{01} - \Big[\sum_{\sigma=0}^{m}\Big(c_{\sigma 1}\mathbf{E}_{\beta,\beta}\big(\lambda(\log\frac{e}{t_\sigma})^\beta\big) \\
&+ \lambda c_{\sigma 2}\mathbf{E}_{\beta,\beta}\big(\lambda(\log\frac{e}{t_\sigma})^\beta\big)(\log\frac{e}{t_\sigma})^{\beta-1}\Big) + \lambda\int_1^e \mathbf{E}_{\beta,1}\big(\lambda(\log\frac{e}{s})^\beta\big)\sigma(s)\frac{ds}{s}\Big] = b
\end{aligned}$$

and $c_{i2} = \Gamma(\beta-1)I_i (i \in \mathbb{N}[1,m])$, $c_{i1} = \Gamma(\beta)J_i (i \in \mathbb{N}[1,m])$. Then

$$\begin{aligned}
c_{01} &= \frac{\lambda\mathbf{E}_{\beta,\beta}(\lambda)}{\Delta}a + \frac{\frac{1}{\Gamma(\beta-1)} - \mathbf{E}_{\beta,\beta-1}(\lambda)}{\Delta}b \\
&+ \sum_{\sigma=1}^{m}\Big[\frac{\frac{1}{\Gamma(\beta-1)} - \mathbf{E}_{\beta,\beta-1}(\lambda)}{\Delta}\mathbf{E}_{\beta,\beta}\big(\lambda(\log\frac{e}{t_\sigma})^\beta\big) \\
&+ \frac{\lambda\mathbf{E}_{\beta,\beta}(\lambda)}{\Delta}(\log\frac{e}{t_\sigma})^{\beta-1}\mathbf{E}_{\beta,\beta}\big(\lambda(\log\frac{e}{t_\sigma})^\beta\big)\Big]\Gamma(\beta)J_\sigma \\
&+ \sum_{\sigma=1}^{m}\Big[\lambda\frac{\frac{1}{\Gamma(\beta-1)} - \mathbf{E}_{\beta,\beta-1}(\lambda)}{\Delta}(\log\frac{e}{t_\sigma})^{\beta-1}\mathbf{E}_{\beta,\beta}\big(\lambda(\log\frac{e}{t_\sigma})^\beta\big) \\
&+ \frac{\lambda\mathbf{E}_{\beta,\beta}(\lambda)}{\Delta}(\log\frac{e}{t_\sigma})^{\beta-2}\mathbf{E}_{\beta,\beta-1}\big(\lambda(\log\frac{e}{t_\sigma})^\beta\big)\Big]\Gamma(\beta-1)I_\sigma \\
&+ \int_1^e\Big[\lambda\frac{\frac{1}{\Gamma(\beta-1)} - \mathbf{E}_{\beta,\beta-1}(\lambda)}{\Delta}\mathbf{E}_{\beta,1}\big(\lambda(\log\frac{e}{s})^\beta\big) \\
&+ \frac{\lambda\mathbf{E}_{\beta,\beta}(\lambda)}{\Delta}(\log\frac{e}{s})^{\beta-1}\mathbf{E}_{\beta,\beta}\big(\lambda(\log\frac{e}{s})^\beta\big)\Big]\sigma(s)\frac{ds}{s} =: M_1,
\end{aligned}$$

$$\begin{aligned}
c_{02} &= -\frac{\frac{1}{\Gamma(\beta)} - \mathbf{E}_{\beta,\beta}(\lambda)}{\Delta}a - \frac{\mathbf{E}_{\beta,\beta}(\lambda)}{\Delta}b \\
&- \sum_{\sigma=1}^{m}\Big[\frac{\frac{1}{\Gamma(\beta)} - \mathbf{E}_{\beta,\beta}(\lambda)}{\Delta}(\log\frac{e}{t_\sigma})^{\beta-1}\mathbf{E}_{\beta,\beta}\big(\lambda(\log\frac{e}{t_\sigma})^\beta\big) \\
&+ \frac{\mathbf{E}_{\beta,\beta}(\lambda)}{\Delta}\mathbf{E}_{\beta,\beta}\big(\lambda(\log\frac{e}{t_\sigma})^\beta\big)\Big]\Gamma(\beta)J_\sigma \\
&- \sum_{\sigma=1}^{m}\Big[\frac{\frac{1}{\Gamma(\beta)} - \mathbf{E}_{\beta,\beta}(\lambda)}{\Delta}(\log\frac{e}{t_\sigma})^{\beta-2}\mathbf{E}_{\beta,\beta-1}\big(\lambda(\log\frac{e}{t_\sigma})^\beta\big) \qquad (3.49)\\
&+ \frac{\lambda\mathbf{E}_{\beta,\beta}(\lambda)}{\Delta}(\log\frac{e}{t_\sigma})^{\beta-1}\mathbf{E}_{\beta,\beta}\big(\lambda(\log\frac{e}{t_\sigma})^\beta\big)\Big]\Gamma(\beta-1)I_\sigma \\
&- \int_1^e\Big[\frac{\frac{1}{\Gamma(\beta)} - \mathbf{E}_{\beta,\beta}(\lambda)}{\Delta}(\log\frac{e}{s})^{\beta-1}\mathbf{E}_{\beta,\beta}\big(\lambda(\log\frac{e}{s})^\beta\big) \\
&+ \lambda\frac{\mathbf{E}_{\beta,\beta}(\lambda)}{\Delta}\mathbf{E}_{\beta,1}\big(\lambda(\log\frac{e}{s})^\beta\big)\Big]\sigma(s)\frac{ds}{s} =: M_2.
\end{aligned}$$

Substituting $c_{\sigma v}(\sigma \in N[0,m], v \in N[1,2]$ into (3.47) and (3.48), we obtain (3.46).

On the other hand, if x satisfies (3.46), then $x|_{(t_i,t_{i+1}]}$ $(i \in \mathbb{N}[0,m])$ are continuous and the limits $\lim_{t\to t_i^+}(\log t - \log t_i)^{2-\beta}x(t)$ $(i \in \mathbb{N}[0,m]$ exist. So $x \in LP_mC_{2-\beta}(1,e]$. Furthermore, by direct computation, we have $\lim_{t\to 1^+}(\log t)^{2-\beta}x(t) - x(e) = a$, $\lim_{t\to 1^+} {}^{RLH}D_{1^+}^{\beta-1}x(t) - {}^{RLH}D_{1^+}^{\beta-1}x(e) = b$, $\lim_{t\to t_i^+}(\log t - \log t_i)^{2-\beta}x(t) = I_i$, $i \in \mathbb{N}[1,m]$ and $\Delta {}^{RL}D_{1^+}^{\beta-1}x(t_i) = J_i$, $i \in \mathbb{N}[1,m]$.

Using (3.49) and $c_{i2} = \Gamma(\beta-1)I_i (i \in \mathbb{N}[1,m])$, $c_{i1} = \Gamma(\beta)J_i (i \in \mathbb{N}[1,m])$, we rewrite x by (3.47). From Theorem 3.13, for $t \in (t_0,t_1]$ easily one has ${}^{RLH}D_{0^+}^{\beta}x(t) = \lambda x(t) + \sigma(t)$ for $t \in (t_0,t_1]$. For $t \in (t_j,t_{j+1}]$, by Definition 2.5 we have

$$\begin{aligned}
&{}^{RLH}D_{1^+}^{\beta}x(t)\\
&= \frac{1}{\Gamma(2-\beta)}(t\frac{d}{dt})^2 \int_1^t (\log\frac{t}{s})^{1-\beta}x(s)\frac{ds}{s}\\
&= \frac{1}{\Gamma(2-\beta)}(t\frac{d}{dt})^2 \Big[\sum_{i=0}^{j-1}\int_{t_i}^{t_{i+1}}(\log\frac{t}{s})^{1-\beta}\Big(\sum_{\sigma=0}^{i}\sum_{v=1}^{2}c_{\sigma v}(\log\frac{s}{t_\sigma})^{\beta-v}\\
&\quad\times \mathbf{E}_{\beta,\beta-v+1}\big(\lambda(\log\frac{s}{t_\sigma})^{\beta}\big) + \int_1^s (\log\frac{s}{u})^{\beta-1}\mathbf{E}_{\beta,\beta}\big(\lambda(\log\frac{s}{u})^{\beta}\big)\sigma(u)\frac{du}{u}\Big)\frac{ds}{s}\\
&\quad+ \int_{t_j}^{t}(\log\frac{t}{s})^{1-\beta}\Big(\sum_{\sigma=0}^{j}\sum_{v=1}^{2}c_{\sigma v}(\log\frac{s}{t_\sigma})^{\beta-v}\mathbf{E}_{\beta,\beta-v+1}\big(\lambda(\log\frac{s}{t_\sigma})^{\beta}\big)\\
&\quad+ \int_1^s (\log\frac{s}{u})^{\beta-1}\mathbf{E}_{\beta,\beta}\big(\lambda(\log\frac{s}{u})^{\beta}\big)\sigma(u)\frac{du}{u}\Big)\frac{ds}{s}\Big]\\
&= \frac{1}{\Gamma(2-\beta)}(t\frac{d}{dt})^2 \Big[\sum_{i=0}^{j-1}\int_{t_i}^{t_{i+1}}(\log\frac{t}{s})^{1-\beta}\sum_{\sigma=0}^{i}\sum_{v=1}^{2}c_{\sigma v}(\log\frac{s}{t_\sigma})^{\beta-v}\\
&\quad\times \mathbf{E}_{\beta,\beta-v+1}\big(\lambda(\log\frac{s}{t_\sigma})^{\beta}\big)\frac{ds}{s}\Big] + \frac{1}{\Gamma(2-\beta)}(t\frac{d}{dt})^2\\
&\quad\times \Big[\int_{t_j}^{t}(\log\frac{t}{s})^{1-\beta}\sum_{\sigma=0}^{j}\sum_{v=1}^{2}c_{\sigma v}(\log\frac{s}{t_\sigma})^{\beta-v}\mathbf{E}_{\beta,\beta-v+1}\big(\lambda(\log\frac{s}{t_\sigma})^{\beta}\big)\frac{ds}{s}\Big]\\
&\quad+ \frac{1}{\Gamma(2-\beta)}(t\frac{d}{dt})^2\Big[\int_1^t(\log\frac{t}{s})^{1-\beta}\int_1^s(\log\frac{s}{u})^{\beta-1}\mathbf{E}_{\beta,\beta}\big(\lambda(\log\frac{s}{u})^{\beta}\big)\sigma(u)\frac{du}{u}\frac{ds}{s}\Big]
\end{aligned}$$

By using the method in the proof of Theorem 3.13, we have

$$ {}^{RLH}D_{1^+}^{\beta}x(t) = \frac{1}{\Gamma(2-\beta)}(t\frac{d}{dt})^2 \int_1^t (\log\frac{t}{s})^{1-\beta}x(s)\frac{ds}{s} = \lambda x(t) + \sigma(t). $$

So x is a solution of (3.45). The proof is complete. □

Define the nonlinear operator R on $LP_mC_{2-\beta}(1,e]$ for $x \in LP_mC_{2-\alpha}(1,e]$ by

$$\begin{aligned}
(Rx)(t) &= M_{1x}(\log t)^{\beta-1}\mathbf{E}_{\beta,\beta}\big(\lambda(\log t)^{\beta}\big) - M_{2x}(\log t)^{\beta-2}\mathbf{E}_{\beta,\beta-1}\big(\lambda(\log t)^{\beta}\big)\\
&\quad+ \sum_{\sigma=1}^{i}\Big[(\log\frac{t}{t_\sigma})^{\beta-1}\mathbf{E}_{\beta,\beta}\big(\lambda(\log\frac{t}{t_\sigma})^{\beta}\big)\Gamma(\beta)J(t_\sigma,x(t_\sigma))\\
&\quad+ (\log\frac{t}{t_\sigma})^{\beta-2}\mathbf{E}_{\beta,\beta-1}\big(\lambda(\log\frac{t}{t_\sigma})^{\beta}\big)\Gamma(\beta-1)I(t_\sigma,x(t_\sigma))\Big]\\
&\quad+ \int_1^t (\log\frac{t}{s})^{\beta-1}\mathbf{E}_{\beta,\beta}\big(\lambda(\log\frac{t}{s})^{\beta}\big)p(s)f(s,x(s))\frac{ds}{s},
\end{aligned}$$

for $t \in (t_i,t_{i+1}]$ and $i \in \mathbb{N}[0,m]$, where

$$ M_{1x} = \frac{\lambda\mathbf{E}_{\beta,\beta}(\lambda)}{\Delta}\int_0^1 \phi(s)G(s,x(s))ds + \frac{\frac{1}{\Gamma(\beta-1)} - \mathbf{E}_{\beta,\beta-1}(\lambda)}{\Delta}\int_0^1 \psi(s)H(s,x(s))ds $$

$$
\begin{aligned}
&+\sum_{\sigma=1}^{m}\Big(\frac{\frac{1}{\Gamma(\beta-1)}-\mathbf{E}_{\beta,\beta-1}(\lambda)}{\Delta}\mathbf{E}_{\beta,\beta}\big(\lambda(\log\frac{e}{t_\sigma})^\beta\big)\\
&+\frac{\lambda\mathbf{E}_{\beta,\beta}(\lambda)}{\Delta}(\log\frac{e}{t_\sigma})^{\beta-1}\mathbf{E}_{\beta,\beta}\big(\lambda(\log\frac{e}{t_\sigma})^\beta\big)\Big)\Gamma(\beta)J(t_\sigma,x(t_\sigma))\\
&+\sum_{\sigma=1}^{m}\Big(\lambda\frac{\frac{1}{\Gamma(\beta-1)}-\mathbf{E}_{\beta,\beta-1}(\lambda)}{\Delta}(\log\frac{e}{t_\sigma})^{\beta-1}\mathbf{E}_{\beta,\beta}\big(\lambda(\log\frac{e}{t_\sigma})^\beta\big)\\
&+\frac{\lambda\mathbf{E}_{\beta,\beta}(\lambda)}{\Delta}(\log\frac{e}{t_\sigma})^{\beta-2}\mathbf{E}_{\beta,\beta-1}\big(\lambda(\log\frac{e}{t_\sigma})^\beta\big)\Big)\Gamma(\beta-1)I(t_\sigma,x(t_\sigma))\\
&+\int_1^e\Big(\lambda\frac{\frac{1}{\Gamma(\beta-1)}-\mathbf{E}_{\beta,\beta-1}(\lambda)}{\Delta}\mathbf{E}_{\beta,1}\big(\lambda(\log\frac{e}{s})^\beta\big)\\
&+\frac{\lambda\mathbf{E}_{\beta,\beta}(\lambda)}{\Delta}(\log\frac{e}{s})^{\beta-1}\mathbf{E}_{\beta,\beta}\big(\lambda(\log\frac{e}{s})^\beta\big)\Big)p(s)f(s,x(s))\frac{ds}{s}
\end{aligned}
$$

and

$$
\begin{aligned}
M_{2x}=&\frac{\frac{1}{\Gamma(\beta)}-\mathbf{E}_{\beta,\beta}(\lambda)}{\Delta}\int_0^1\phi(s)G(s,x(s))ds+\frac{\mathbf{E}_{\beta,\beta}(\lambda)}{\Delta}\int_0^1\psi(s)H(s,x(s))ds\\
&+\sum_{\sigma=1}^{m}\Big(\frac{\frac{1}{\Gamma(\beta)}-\mathbf{E}_{\beta,\beta}(\lambda)}{\Delta}(\log\frac{e}{t_\sigma})^{\beta-1}\mathbf{E}_{\beta,\beta}\big(\lambda(\log\frac{e}{t_\sigma})^\beta\big)\\
&+\frac{\mathbf{E}_{\beta,\beta}(\lambda)}{\Delta}\mathbf{E}_{\beta,\beta}\big(\lambda(\log\frac{e}{t_\sigma})^\beta\big)\Big)\Gamma(\beta)J(t_\sigma,x(t_\sigma))\\
&+\sum_{\sigma=1}^{m}\Big(\frac{\frac{1}{\Gamma(\beta)}-\mathbf{E}_{\beta,\beta}(\lambda)}{\Delta}(\log\frac{e}{t_\sigma})^{\beta-2}\mathbf{E}_{\beta,\beta-1}\big(\lambda(\log\frac{e}{t_\sigma})^\beta\big)\\
&+\frac{\lambda\mathbf{E}_{\beta,\beta}(\lambda)}{\Delta}(\log\frac{e}{t_\sigma})^{\beta-1}\mathbf{E}_{\beta,\beta}\big(\lambda(\log\frac{e}{t_\sigma})^\beta\big)\Big)\Gamma(\beta-1)I(t_\sigma,x(t_\sigma))\\
&+\int_1^e\Big(\frac{\frac{1}{\Gamma(\beta)}-\mathbf{E}_{\beta,\beta}(\lambda)}{\Delta}(\log\frac{e}{s})^{\beta-1}\mathbf{E}_{\beta,\beta}\big(\lambda(\log\frac{e}{s})^\beta\big)\\
&+\lambda\frac{\mathbf{E}_{\beta,\beta}(\lambda)}{\Delta}\mathbf{E}_{\beta,1}\big(\lambda(\log\frac{e}{s})^\beta\big)\Big)p(s)f(s,x(s))\frac{ds}{s}.
\end{aligned}
$$

Lemma 3.20. *Suppose that* (d), (i), (j), (k) *in the intoruction hold,* $\Delta\neq 0$*, and* f,G,H *are impulsive III-Carathéodory functions,* I,J *are discrete III-Carathódory functions. Then* $R: LP_mC_{2-\beta}(1,e]\to LP_mC_{2-\alpha}(1,e]$ *is well defined and is completely continuous,* x *is a solution of* $BVP(1.9)$ *if and only if* x *is a fixed point of* R *in* $LP_mC_{2-\alpha}(1,e]$.

The proof of the above lemma is similar to that of Lemma 3.16 and is omitted.

3.6. **Preliminaries for BVP** (1.10)**.** In this section, we present some preliminary results that can be used in next sections for obtain solutions of (1.10).

Lemma 3.21. *Suppose that* $\lambda\neq 0$*,* $\sigma:(0,1)\to\mathbb{R}$ *is continuous and satisfies that there exist numbers* $k>1-\beta$ *and* $l\leq 0$ *with* $l>\max\{-\beta,-\beta-k\}$ *such that*

$|\sigma(t)| \le (\log t)^k(1-\log t)^l$ *for all* $t\in(0,1)$. *The* x *is a solutions of*

$$\begin{gathered}
{}^{CH}D_{1^+}^{\beta}x(t)-\lambda x(t)=\sigma(t),\quad t\in(t_i,t_{i+1}],\ i\in\mathbb{N}[0,m],\\
(t\frac{d}{dt})x(t)\big|_{t=1}=a,\quad (t\frac{d}{dt})x(t)\big|_{t=e}=b,\quad \lim_{t\to t_i^+}x(t)-x(t_i)=I_i,\\
\lim_{t\to t_i^+}(t\frac{d}{dt})x(t)-(t\frac{d}{dt})x(t)\big|_{t=t_i}=J_i,\quad i\in\mathbb{N}[1,m],
\end{gathered}\tag{3.50}$$

if and only if $x\in LP_mC(1,e]$ *and*

$$\begin{aligned}
x(t)&=\frac{\mathbf{E}_{\beta,1}\big(\lambda(\log t)^{\beta}\big)}{\lambda\mathbf{E}_{\beta,\beta}(\lambda)}b+\big[(\log t)\mathbf{E}_{\beta,2}(\lambda(\log t)^{\beta})-\frac{\mathbf{E}_{\beta,1}\big(\lambda(\log t)^{\beta}\big)}{\lambda\mathbf{E}_{\beta,\beta}(\lambda)}\mathbf{E}_{\beta,1}(\lambda)\big]a\\
&\quad-\frac{\mathbf{E}_{\beta,1}\big(\lambda(\log t)^{\beta}\big)}{\lambda\mathbf{E}_{\beta,\beta}(\lambda)}\sum_{\sigma=1}^{m}\Big(\lambda(\log\frac{e}{t_\sigma})^{\beta-1}\mathbf{E}_{\beta,\beta}\big(\lambda(\log\frac{e}{t_\sigma})^{\beta}\big)I_\sigma\\
&\quad+\mathbf{E}_{\beta,1}(\lambda(\log\frac{e}{t_\sigma})^{\beta})J_\sigma\Big)\\
&\quad-\frac{\mathbf{E}_{\beta,1}\big(\lambda(\log t)^{\beta}\big)}{\lambda\mathbf{E}_{\beta,\beta}(\lambda)}\int_1^e(\log\frac{e}{s})^{\beta-2}\mathbf{E}_{\beta,\beta-1}\big(\lambda(\log\frac{e}{s})^{\beta}\big)\sigma(s)\frac{ds}{s}\\
&\quad+\sum_{\sigma=1}^{i}\Big(\mathbf{E}_{\beta,1}\big(\lambda(\log\frac{t}{t_\sigma})^{\beta}\big)I_\sigma+(\log\frac{t}{t_\sigma})\mathbf{E}_{\beta,2}\big(\lambda(\log\frac{t}{t_\sigma})^{\beta}\big)J_\sigma\Big)\\
&\quad+\int_1^t(\log\frac{t}{s})^{\beta-1}\mathbf{E}_{\beta,\beta}\big(\lambda(\log\frac{t}{s})^{\beta}\big)\sigma(s)\frac{ds}{s},
\end{aligned}\tag{3.51}$$

for $t\in(t_i,t_{i+1}]$ *and* $i\in\mathbb{N}[0,m]$.

Proof. Let x be a solution of (3.50). We know by Theorem 3.14 that there exist numbers $c_{\sigma0},c_{\sigma1}\in\mathbb{R}(\sigma\in\mathbb{N}[0,n-1])$ such that

$$\begin{aligned}
x(t)&=\sum_{\sigma=0}^{i}(c_{\sigma0}\mathbf{E}_{\beta,1}\big(\lambda(\log\frac{t}{t_\sigma})^{\beta}\big)+c_{\sigma1}(\log\frac{t}{t_\sigma})\mathbf{E}_{\beta,2}\big(\lambda(\log\frac{t}{t_\sigma})^{\beta}\big))\\
&\quad+\int_1^t(\log\frac{t}{s})^{\beta-1}\mathbf{E}_{\beta,\beta}\big(\lambda(\log\frac{t}{s})^{\beta}\big)\sigma(s)\frac{ds}{s},
\end{aligned}\tag{3.52}$$

for $t\in(t_i,t_{i+1}]$ and $i\in\mathbb{N}[0,m]$. One has

$$\begin{aligned}
(t\frac{d}{dt})[\mathbf{E}_{\beta,1}(\lambda(\log t-\log t_\sigma)^{\beta})]&=(t\frac{d}{dt})\Big[\sum_{\tau=0}^{+\infty}\frac{\lambda^\tau(\log t-\log t_\sigma)^{\tau\beta}}{\Gamma(\tau\beta+1)}\Big]\\
&=\sum_{\tau=1}^{+\infty}\frac{\lambda^\tau(\log t-\log t_\sigma)^{\tau\beta-1}}{\Gamma(\tau\beta)}\\
&=\lambda(\log\frac{t}{t_\sigma})^{\beta-1}\mathbf{E}_{\beta,\beta}\big(\lambda(\log\frac{t}{t_\sigma})^{\beta}\big),
\end{aligned}$$

$$\begin{aligned}
(t\frac{d}{dt})\big[(\log t-\log t_\sigma)\mathbf{E}_{\beta,2}\big(\lambda(\log t-\log t_\sigma)^{\beta}\big)\big]'&=\Big[\sum_{\tau=0}^{+\infty}\frac{\lambda^\tau(\log t-\log t_\sigma)^{\tau\beta+1}}{\Gamma(\beta\tau+2)}\Big]\\
&=\sum_{\tau=0}^{+\infty}\frac{\lambda^\tau(\log t-\log t_\sigma)^{\tau\beta}}{\Gamma(\beta\tau+1)}
\end{aligned}$$

$$= \mathbf{E}_{\beta,1}(\lambda(\log\frac{t}{t_\sigma})^\beta),$$

$$\begin{aligned}
&(t\frac{d}{dt})\big[(\log\frac{t}{s})^{\beta-1}\mathbf{E}_{\beta,\beta}\big(\lambda(\log\frac{t}{s})^\beta\big)\big]\\
&=(t\frac{d}{dt})\Big[\sum_{\tau=0}^{+\infty}\frac{\lambda^\tau(\log t-\log s)^{\tau\beta+\beta-1}}{\Gamma((\tau+1)\beta)}\Big]\\
&=\sum_{\tau=0}^{+\infty}\frac{\lambda^\tau(\log t-\log s)^{\tau\beta+\beta-2}}{\Gamma((\tau+1)\beta-1)}\\
&=(\log\frac{t}{s})^{\beta-2}\mathbf{E}_{\beta,\beta-1}(\lambda(\log\frac{t}{s})^\beta).
\end{aligned}$$

It follows that

$$\begin{aligned}
(t\frac{d}{dt})x(t) &= \sum_{\sigma=0}^{i}\Big(\lambda(\log\frac{t}{t_\sigma})^{\beta-1}\mathbf{E}_{\beta,\beta}\big(\lambda(\log\frac{t}{t_\sigma})^\beta\big)c_{\sigma 0}+\mathbf{E}_{\beta,1}(\lambda(\log\frac{t}{t_\sigma})^\beta)c_{\sigma 1}\Big)\\
&\quad+\int_1^t(\log\frac{t}{s})^{\beta-2}\mathbf{E}_{\beta,\beta-1}(\lambda(\log\frac{t}{s})^\beta)\sigma(s)\frac{ds}{s},
\end{aligned} \tag{3.53}$$

for $t\in(t_i,t_{i+1}]$ and $i\in N[0,m]$. From (3.52), (3.53), the boundary conditions and the impulse assumption in (3.50) it follows that $c_{01}=a$, and

$$\begin{gathered}
\sum_{\sigma=0}^{m}\Big(\lambda(\log\frac{e}{t_\sigma})^{\beta-1}\mathbf{E}_{\beta,\beta}\big(\lambda(\log\frac{e}{t_\sigma})^\beta\big)c_{\sigma 0}+\mathbf{E}_{\beta,1}(\lambda(\log\frac{e}{t_\sigma})^\beta)c_{\sigma 1}\Big)\\
+\int_1^e(\log\frac{e}{s})^{\beta-2}\mathbf{E}_{\beta,\beta-1}\big(\lambda(\log\frac{e}{s})^\beta\big)\sigma(s)\frac{ds}{s}=b,\\
c_{\sigma 0}=I_\sigma,\quad c_{\sigma 1}=J_\sigma,\quad \sigma\in\mathbb{N}[1,m].
\end{gathered}$$

Then

$$\begin{aligned}
c_{00}=\frac{1}{\lambda\mathbf{E}_{\beta,\beta}(\lambda)}\Big[&b-\mathbf{E}_{\beta,1}(\lambda)a-\sum_{\sigma=1}^{m}\Big(\lambda(\log\frac{e}{t_\sigma})^{\beta-1}\mathbf{E}_{\beta,\beta}\big(\lambda(\log\frac{e}{t_\sigma})^\beta\big)I_\sigma\\
&+\mathbf{E}_{\beta,1}(\lambda(\log\frac{e}{t_\sigma})^\beta)J_\sigma\Big)-\int_1^e(\log\frac{e}{s})^{\beta-2}\mathbf{E}_{\beta,\beta-1}\big(\lambda(\log\frac{e}{s})^\beta\big)\sigma(s)\frac{ds}{s}\Big].
\end{aligned} \tag{3.54}$$

Substituting $c_{\sigma 0}, c_{\sigma 1}(\sigma\in N[0,m])$ into (3.52) and (3.53), we obtain (3.51).

On the other hand, if x satisfies (3.51), then $x|_{(t_i,t_{i+1}]}(i\in\mathbb{N}[0,m])$ are continuous and the limits $\lim_{t\to t_i^+}x(t)$. So $x\in LP_mC(1,e]$. Using (3.54), $c_{01}=a$ and $c_{\sigma 0}=I_\sigma$, $c_{\sigma 1}=J_\sigma$, $\sigma\in\mathbb{N}[1,m]$, we rewrite x by (3.52). One have from Theorem 3.14 easily for $t\in(t_0,t_1]$ that ${}^{CH}D_{1^+}^{\beta}x(t)=\lambda x(t)+\sigma(t)$ and for $t\in(t_i,t_{i+1}]$ similarly to the proof of Theorem 3.14 that

$$\begin{aligned}
&{}^{CH}D_{1^+}^{\beta}x(t)\\
&=\frac{1}{\Gamma(2-\beta)}\int_1^t(\log\frac{t}{s})^{1-\alpha}(s\frac{d}{ds})^2x(s)\frac{ds}{s}\\
&=\frac{1}{\Gamma(2-\beta)}\sum_{i=0}^{j-1}\int_{t_i}^{t_{i+1}}(\log\frac{t}{s})^{1-\alpha}(s\frac{d}{ds})^2\Big[\sum_{\sigma=0}^{i}\Big(c_{\sigma 0}\mathbf{E}_{\beta,1}\big(\lambda(\log\frac{s}{t_\sigma})^\beta\big)
\end{aligned}$$

$$
\begin{aligned}
&+ c_{\sigma 1}(\log \frac{s}{t_\sigma})\mathbf{E}_{\beta,2}\big(\lambda(\log \frac{s}{t_\sigma})^\beta\big)\Big)\\
&+ \int_1^s (\log \frac{s}{u})^{\beta-1}\mathbf{E}_{\beta,\beta}\big(\lambda(\log \frac{s}{u})^\beta\big)\sigma(u)\frac{du}{u}\Big]\frac{ds}{s}\\
&+ \frac{1}{\Gamma(2-\beta)}\int_{t_j}^t (\log \frac{t}{s})^{1-\alpha}(s\frac{d}{ds})^2\Big[\sum_{\sigma=0}^{j}\Big(c_{\sigma 0}\mathbf{E}_{\beta,1}\big(\lambda(\log \frac{s}{t_\sigma})^\beta\big)\\
&+ c_{\sigma 1}(\log \frac{s}{t_\sigma})\mathbf{E}_{\beta,2}\big(\lambda(\log \frac{s}{t_\sigma})^\beta\big)\Big)\\
&+ \int_1^s (\log \frac{s}{u})^{\beta-1}\mathbf{E}_{\beta,\beta}\big(\lambda(\log \frac{s}{u})^\beta\big)\sigma(u)\frac{du}{u}\Big]\frac{ds}{s}\\
&= \frac{1}{\Gamma(2-\beta)}\sum_{i=0}^{j-1}\int_{t_i}^{t_{i+1}} (\log \frac{t}{s})^{1-\alpha}(s\frac{d}{ds})^2\sum_{\sigma=0}^{i}\Big[c_{\sigma 0}\mathbf{E}_{\beta,1}\big(\lambda(\log \frac{s}{t_\sigma})^\beta\big)\\
&+ c_{\sigma 1}(\log \frac{s}{t_\sigma})\mathbf{E}_{\beta,2}\big(\lambda(\log \frac{s}{t_\sigma})^\beta\big)\Big]\frac{ds}{s}\\
&+ \frac{1}{\Gamma(2-\beta)}\int_{t_j}^t (\log \frac{t}{s})^{1-\alpha}(s\frac{d}{ds})^2\sum_{\sigma=0}^{j}\Big[c_{\sigma 0}\mathbf{E}_{\beta,1}\big(\lambda(\log \frac{s}{t_\sigma})^\beta\big)\\
&+ c_{\sigma 1}(\log \frac{s}{t_\sigma})\mathbf{E}_{\beta,2}\big(\lambda(\log \frac{s}{t_\sigma})^\beta\big)\Big]\frac{ds}{s}\\
&+ \frac{1}{\Gamma(2-\beta)}\int_1^t (\log \frac{t}{s})^{1-\alpha}(s\frac{d}{ds})^2\int_1^s (\log \frac{s}{u})^{\beta-1}\mathbf{E}_{\beta,\beta}\big(\lambda(\log \frac{s}{u})^\beta\big)\sigma(u)\frac{du}{u}\frac{ds}{s}\\
&= \lambda x(t) + \sigma(t).
\end{aligned}
$$

So x is a solution of (3.50). The proof is complete. □

Define the nonlinear operator J on $LP_mC(1,e]$ by (Jx) by

$$
\begin{aligned}
&(Jx)(t)\\
&= \frac{\mathbf{E}_{\beta,1}\big(\lambda(\log t)^\beta\big)}{\lambda\mathbf{E}_{\beta,\beta}(\lambda)}\int_0^1 \psi(s)H(s,x(s))ds\\
&\quad + \Big[(\log t)\mathbf{E}_{\beta,2}\big(\lambda(\log t)^\beta\big) - \frac{\mathbf{E}_{\beta,1}\big(\lambda(\log t)^\beta\big)}{\lambda\mathbf{E}_{\beta,\beta}(\lambda)}\mathbf{E}_{\beta,1}(\lambda)\Big]\int_0^1 \phi(s)G(s,x(s))ds\\
&\quad - \frac{\mathbf{E}_{\beta,1}\big(\lambda(\log t)^\beta\big)}{\lambda\mathbf{E}_{\beta,\beta}(\lambda)}\sum_{\sigma=1}^{m}(\lambda(\log \frac{e}{t_\sigma})^{\beta-1}\mathbf{E}_{\beta,\beta}\big(\lambda(\log \frac{e}{t_\sigma})^\beta\big)I(t_\sigma,x(t_\sigma))\\
&\quad + \mathbf{E}_{\beta,1}(\lambda(\log \frac{e}{t_\sigma})^\beta)J(t_\sigma,x(t_\sigma)))\\
&\quad - \frac{\mathbf{E}_{\beta,1}\big(\lambda(\log t)^\beta\big)}{\lambda\mathbf{E}_{\beta,\beta}(\lambda)}\int_1^e (\log \frac{e}{s})^{\beta-2}\mathbf{E}_{\beta,\beta-1}\big(\lambda(\log \frac{e}{s})^\beta\big)p(s)f(s,x(s))\frac{ds}{s}\\
&\quad + \sum_{\sigma=1}^{i}(\mathbf{E}_{\beta,1}\big(\lambda(\log \frac{t}{t_\sigma})^\beta\big)I(t_\sigma,x(t_\sigma)) + (\log \frac{t}{t_\sigma})\mathbf{E}_{\beta,2}\big(\lambda(\log \frac{t}{t_\sigma})^\beta\big)J(t_\sigma,x(t_\sigma)))\\
&\quad + \int_1^t (\log \frac{t}{s})^{\beta-1}\mathbf{E}_{\beta,\beta}\big(\lambda(\log \frac{t}{s})^\beta\big)p(s)f(s,x(s))\frac{ds}{s}, \quad t \in (t_i,t_{i+1}],\ i \in \mathbb{N}[0,m].
\end{aligned}
$$

Lemma 3.22. *Suppose that* (1.A7), (1.A12), (1.A13, (1.A14) *hold,* $\lambda \neq 0$*, and* f, G, H *are impulsive I-Carathéodory functions, and* I, J *are discrete I-Carathódory functions. Then* $R : LP_mC(1,e] \to LP_mC(1,e]$ *is well defined and is completely continuous,* x *is a solution of BVP* (1.10) *if and only if* x *is a fixed point of* J *in* $LP_mC(1,e]$.

The proof of the above lemma is similar to that of the proof of Lemma 3.16 and is omitted.

4. Solvability of BVPs (1.7)–(1.10)

Now, we prove that main theorems in this article by using the Schaefer's fixed point theorem, i.e., [79, Lemma 3.1.9].

Theorem 4.1. *Suppose that* (1.A1)–(1.A5) *are satisfied and*

(4.A1) *There exist nondecreasing functions* M_f, M_g, M_h, M_I, M_J *from* $[0,+\infty)$ *to* $[0,+\infty)$ *such that*

$$
\begin{gathered}
|f(t,(t-t_i)^{\beta-2}x)| \leq M_f(|x|), \quad t \in (t_i, t_{i+1}],\ i \in \mathbb{N}[0,m], x \in \mathbb{R},\\
|G(t,(t-t_i)^{\beta-2}x)| \leq M_G(|x|), \quad t \in (t_i, t_{i1}],\ i \in \mathbb{N}[0,m], x \in \mathbb{R},\\
|H(t,(t-t_i)^{\beta-2}x)| \leq M_H(|x|), \quad t \in (t_i, t_{i1}],\ i \in \mathbb{N}[0,m], x \in \mathbb{R},\\
|I(t_i,(t_i-t_{i-1})^{\beta-2}x)| \leq M_I(|x|), \quad i \in N[1,m],\ x \in \mathbb{R},\\
|J(t_i,(t_i-t_{i-1})^{\beta-2}x)| \leq M_J(|x|), \quad i \in N[1,m],\ x \in \mathbb{R}.
\end{gathered}
$$

Then (1.7) *has at least one solution if there exists a* $r_0 > 0$ *such that*

$$
\begin{aligned}
&\Big[\frac{\mathbf{E}_{\beta,\beta}(|\lambda|)\mathbf{E}_{\beta,\beta-1}(\lambda)\|\phi\|_1}{\mathbf{E}_{\beta,\beta}(\lambda)} + \mathbf{E}_{\beta,\beta-1}(|\lambda|)\|\phi\|_1\Big]M_G(r_0)\\
&+ \frac{\mathbf{E}_{\beta,\beta}(|\lambda|)\|\psi\|_1}{\mathbf{E}_{\beta,\beta}(\lambda)}M_H(r_0) + \Big[\frac{m\mathbf{E}_{\beta,\beta}(|\lambda|)^2}{\mathbf{E}_{\beta,\beta}(\lambda)} + m\mathbf{E}_{\beta,\beta}(|\lambda|)\Big]M_J(r_0)\\
&+ \Big[\frac{\mathbf{E}_{\beta,\beta}(|\lambda|)\mathbf{E}_{\beta,\beta-1}(|\lambda|)}{\mathbf{E}_{\beta,\beta}(\lambda)(1-t_\sigma)^{2-\beta}} + m\mathbf{E}_{\beta,\beta-1}(|\lambda|)\Big]M_I(r_0)\\
&+ \Big[\frac{\mathbf{E}_{\beta,\beta}(|\lambda|)\mathbf{E}_{\beta,\beta}(|\lambda|)\mathbf{B}(\beta+l,k+1)}{\mathbf{E}_{\beta,\beta}(\lambda)} + \mathbf{E}_{\beta,\beta}(\lambda|)\mathbf{B}(\beta+l,k+1)\Big]M_f(r_0)\\
&< r_0.
\end{aligned}
\tag{4.1}
$$

Proof. From Lemmas 3.15 and 3.16, and the definition of T, it follows that $x \in P_mC_{2-\beta}(0,1]$ is a solution of (1.7) if and only if $x \in P_mC_{2-\beta}(0,1]$ is a fixed point of T in $P_mC_{2-\beta}(0,1]$. Lemma 3.16 implies that T is a completely continuous operator. From (4.A1), we have for $x \in P_1C(0,1]$ that

$$
\begin{aligned}
|f(t,x(t))| &= |f(t,(t-t_i)^{\beta-2}(t-t_i)^{2-\beta}x(t))|\\
&\leq M_f(|(t-t_i)^{2-\beta}x(t)|)\\
&\leq M_f(\|x\|), \quad t \in (t_i, t_{i+1}],\ i \in \mathbb{N}[0,m],\\
|G(t,x(t))| &\leq M_G(\|x\|), \quad t \in (t_i, t_{i+1}],\ i \in \mathbb{N}[0,m],\\
|H(t,x(t))| &\leq M_H(\|x\|), \quad t \in (t_i, t_{i+1}],\ i \in \mathbb{N}[0,m],
\end{aligned}
$$

$$
\begin{aligned}
|I(t_i,x(t_i))| &= |I(t_i,(t_i-t_{i-1})^{\beta-2}(t_i-t_{i-1})^{2-\beta}x(t_i))|\\
&\le M_I((t_i-t_{i-1})^{2-\beta}|x(t_i)|)\\
&\le M_I(\|x\|),\quad i\in\mathbb{N}[1,m],\\
|J(t_i,x(t_i))| &\le M_J(\|x\|),\quad i\in\mathbb{N}[1,m].
\end{aligned}
$$

We consider the set $\Omega=\{x\in P_mC_{2-\beta}(0,1]: x=\lambda(Tx),\ \text{for some } \lambda\in[0,1]\}$. For $x\in\Omega$ and $t\in(t_i,t_{i+1}]$ we have

$$
\begin{aligned}
&(t-t_i)^{2-\beta}|(Tx)(t)|\\
&\le \frac{(t-t_i)^{2-\beta}t^{\beta-1}\mathbf{E}_{\beta,\beta}(\lambda t^\beta)}{\mathbf{E}_{\beta,\beta}(\lambda)}\Big[\|\psi\|_1M_H(\|x\|)+\mathbf{E}_{\beta,\beta-1}(\lambda)\|\phi\|_1M_G(\|x\|)\\
&\quad+\sum_{\tau=0}^{+\infty}\frac{\lambda^\tau}{\Gamma((\tau+1)\beta)}\int_0^1(1-s)^{\tau\beta+\beta-1}s^k(1-s)^l ds M_f(\|x\|)\\
&\quad+\sum_{\sigma=1}^{m}\Big((1-t_\sigma)^{\beta-1}\mathbf{E}_{\beta,\beta}(\lambda(1-t_\sigma)^\beta)M_J(\|x\|)\\
&\quad+(1-t_\sigma)^{\beta-2}\mathbf{E}_{\beta,\beta-1}(\lambda(1-t_\sigma)^\beta)M_I(\|x\|)\Big)\Big]\\
&\quad+(t-t_i)^{2-\beta}t^{\beta-2}\mathbf{E}_{\beta,\beta-1}(|\lambda|)\|\phi\|_1M_G(\|x\|)\\
&\quad+(t-t_i)^{2-\beta}\sum_{\sigma=1}^{i}\Big[(t-t_\sigma)^{\beta-1}\mathbf{E}_{\beta,\beta}(\lambda(t-t_\sigma)^\beta)M_J(\|x\|)\\
&\quad+(t-t_\sigma)^{\beta-2}\mathbf{E}_{\beta,\beta-1}(\lambda(t-t_\sigma)^\beta)M_I(\|x\|)\Big]\\
&\quad+(t-t_i)^{2-\beta}\sum_{\tau=0}^{+\infty}\frac{\lambda^\tau}{\Gamma((\tau+1)\beta)}\int_0^t(t-s)^{\tau\beta+\beta-1}s^k(1-s)^l ds M_f(\|x\|)\\
&\le \frac{\mathbf{E}_{\beta,\beta}(|\lambda|)}{\mathbf{E}_{\beta,\beta}(\lambda)}\Big[\|\psi\|_1M_H(\|x\|)+\mathbf{E}_{\beta,\beta-1}(\lambda)\|\phi\|_1M_G(\|x\|)\\
&\quad+\sum_{\tau=0}^{+\infty}\frac{\lambda^\tau}{\Gamma((\tau+1)\beta)}\int_0^1(1-s)^{\beta+l-1}s^k ds M_f(\|x\|)+m\mathbf{E}_{\beta,\beta}(|\lambda|)M_J(\|x\|)\\
&\quad+(1-t_\sigma)^{\beta-2}\mathbf{E}_{\beta,\beta-1}(|\lambda|)M_I(\|x\|)\Big]+\mathbf{E}_{\beta,\beta-1}(|\lambda|)\|\phi\|_1M_G(\|x\|)\\
&\quad+\sum_{\sigma=1}^{i}\big[\mathbf{E}_{\beta,\beta}(|\lambda|)M_J(\|x\|)+\mathbf{E}_{\beta,\beta-1}(|\lambda|)M_I(\|x\|)\big]\\
&\quad+(t-t_i)^{2-\beta}\sum_{\tau=0}^{+\infty}\frac{\lambda^\tau t^{\tau\beta+\beta+k+l}}{\Gamma((\tau+1)\beta)}\int_0^1(1-w)^{\tau\beta+\beta+l-1}w^k dw M_f(\|x\|)\\
&\le \frac{\mathbf{E}_{\beta,\beta}(|\lambda|)\|\psi\|_1}{\mathbf{E}_{\beta,\beta}(\lambda)}M_H(\|x\|)+\frac{\mathbf{E}_{\beta,\beta}(|\lambda|)\mathbf{E}_{\beta,\beta-1}(\lambda)\|\phi\|_1}{\mathbf{E}_{\beta,\beta}(\lambda)}M_G(\|x\|)\\
&\quad+\frac{\mathbf{E}_{\beta,\beta}(|\lambda|)\mathbf{E}_{\beta,\beta}(|\lambda|)\mathbf{B}(\beta+l,k+1)}{\mathbf{E}_{\beta,\beta}(\lambda)}M_f(\|x\|)+\frac{m\mathbf{E}_{\beta,\beta}(|\lambda|)^2}{\mathbf{E}_{\beta,\beta}(\lambda)}M_J(\|x\|)\\
&\quad+\frac{\mathbf{E}_{\beta,\beta}(|\lambda|)\mathbf{E}_{\beta,\beta-1}(|\lambda|)}{\mathbf{E}_{\beta,\beta}(\lambda)(1-t_\sigma)^{2-\beta}}M_I(\|x\|)+\mathbf{E}_{\beta,\beta-1}(|\lambda|)\|\phi\|_1M_G(\|x\|)\\
&\quad+m\mathbf{E}_{\beta,\beta}(|\lambda|)M_J(\|x\|)+m\mathbf{E}_{\beta,\beta-1}(|\lambda|)M_I(\|x\|)
\end{aligned}
$$

$$
\begin{aligned}
&+ \mathbf{E}_{\beta,\beta}(\lambda|)\mathbf{B}(\beta+l,k+1)M_f(\|x\|)\\
&= \big[\frac{\mathbf{E}_{\beta,\beta}(|\lambda|)\mathbf{E}_{\beta,\beta-1}(\lambda)\|\phi\|_1}{\mathbf{E}_{\beta,\beta}(\lambda)} + \mathbf{E}_{\beta,\beta-1}(|\lambda|)\|\phi\|_1\big]M_G(\|x\|)\\
&+ \frac{\mathbf{E}_{\beta,\beta}(|\lambda|)\|\psi\|_1}{\mathbf{E}_{\beta,\beta}(\lambda)}M_H(\|x\|) + \big[\frac{m\mathbf{E}_{\beta,\beta}(|\lambda|)^2}{\mathbf{E}_{\beta,\beta}(\lambda)} + m\mathbf{E}_{\beta,\beta}(|\lambda|)\big]M_J(\|x\|)\\
&+ \big[\frac{\mathbf{E}_{\beta,\beta}(|\lambda|)\mathbf{E}_{\beta,\beta-1}(|\lambda|)}{\mathbf{E}_{\beta,\beta}(\lambda)(1-t_\sigma)^{2-\beta}} + m\mathbf{E}_{\beta,\beta-1}(|\lambda|)\big]M_I(\|x\|)\\
&+ \big[\frac{\mathbf{E}_{\beta,\beta}(|\lambda|)\mathbf{E}_{\beta,\beta}(|\lambda|)\mathbf{B}(\beta+l,k+1)}{\mathbf{E}_{\beta,\beta}(\lambda)} + \mathbf{E}_{\beta,\beta}(\lambda|)\mathbf{B}(\beta+l,k+1)\big]M_f(\|x\|).
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\|x\| &= \lambda\|Tx\| \le \|Tx\|\\
&\le \big[\frac{\mathbf{E}_{\beta,\beta}(|\lambda|)\mathbf{E}_{\beta,\beta-1}(\lambda)\|\phi\|_1}{\mathbf{E}_{\beta,\beta}(\lambda)} + \mathbf{E}_{\beta,\beta-1}(|\lambda|)\|\phi\|_1\big]M_G(\|x\|)\\
&\quad+ \frac{\mathbf{E}_{\beta,\beta}(|\lambda|)\|\psi\|_1}{\mathbf{E}_{\beta,\beta}(\lambda)}M_H(\|x\|)\\
&\quad+ \big[\frac{m\mathbf{E}_{\beta,\beta}(|\lambda|)^2}{\mathbf{E}_{\beta,\beta}(\lambda)} + m\mathbf{E}_{\beta,\beta}(|\lambda|)\big]M_J(\|x\|) + \big[\frac{\mathbf{E}_{\beta,\beta}(|\lambda|)\mathbf{E}_{\beta,\beta-1}(|\lambda|)}{\mathbf{E}_{\beta,\beta}(\lambda)(1-t_\sigma)^{2-\beta}}\\
&\quad+ m\mathbf{E}_{\beta,\beta-1}(|\lambda|)\big]M_I(\|x\|) + \big[\frac{\mathbf{E}_{\beta,\beta}(|\lambda|)\mathbf{E}_{\beta,\beta}(|\lambda|)\mathbf{B}(\beta+l,k+1)}{\mathbf{E}_{\beta,\beta}(\lambda)}\\
&\quad+ \mathbf{E}_{\beta,\beta}(\lambda|)\mathbf{B}(\beta+l,k+1)\big]M_f(\|x\|).
\end{aligned}
$$

From (4.1), we choose $\Omega = \{x \in P_1C_{2-\beta}(0,1] : \|x\| \le r_0\}$. For $x \in \Omega$, we obtain $x \ne \lambda(Tx)$ for any $\lambda \in [0,1]$ and $x \in \partial\Omega$. In fact, if $x = \lambda(Tx)$ for some $\lambda \in [0,1]$ and $x \in \partial\Omega$, then

$$
\begin{aligned}
&r_0 = \|x\|\\
&\le \big[\frac{\mathbf{E}_{\beta,\beta}(|\lambda|)\mathbf{E}_{\beta,\beta-1}(\lambda)\|\phi\|_1}{\mathbf{E}_{\beta,\beta}(\lambda)} + \mathbf{E}_{\beta,\beta-1}(|\lambda|)\|\phi\|_1\big]M_G(r_0)\\
&\quad+ \frac{\mathbf{E}_{\beta,\beta}(|\lambda|)\|\psi\|_1}{\mathbf{E}_{\beta,\beta}(\lambda)}M_H(r_0) + \big[\frac{m\mathbf{E}_{\beta,\beta}(|\lambda|)^2}{\mathbf{E}_{\beta,\beta}(\lambda)} + m\mathbf{E}_{\beta,\beta}(|\lambda|)\big]M_J(r_0)\\
&\quad+ \big[\frac{\mathbf{E}_{\beta,\beta}(|\lambda|)\mathbf{E}_{\beta,\beta-1}(|\lambda|)}{\mathbf{E}_{\beta,\beta}(\lambda)(1-t_\sigma)^{2-\beta}} + m\mathbf{E}_{\beta,\beta-1}(|\lambda|)\big]M_I(r_0)\\
&\quad+ \big[\frac{\mathbf{E}_{\beta,\beta}(|\lambda|)\mathbf{E}_{\beta,\beta}(|\lambda|)\mathbf{B}(\beta+l,k+1)}{\mathbf{E}_{\beta,\beta}(\lambda)} + \mathbf{E}_{\beta,\beta}(\lambda|)\mathbf{B}(\beta+l,k+1)\big]M_f(r_0) < r_0,
\end{aligned}
$$

which is a contradiction. As a consequence of Schaefer's fixed point theorem, we deduce that T has a fixed point which is a solution of the problem BVP(1.7). The proof is complete. □

Corollary 4.2. *Suppose that* (1.A1)–(1.A5) *and* (4.A1) *hold. Then* (1.7) *has at least one solution if*

$$
\begin{aligned}
&\inf_{r\in(0,+\infty)} \frac{1}{r}\Big[\big(\frac{\mathbf{E}_{\beta,\beta}(|\lambda|)\mathbf{E}_{\beta,\beta-1}(\lambda)\|\phi\|_1}{\mathbf{E}_{\beta,\beta}(\lambda)} + \mathbf{E}_{\beta,\beta-1}(|\lambda|)\|\phi\|_1)M_G(r)\\
&+ \frac{\mathbf{E}_{\beta,\beta}(|\lambda|)\|\psi\|_1}{\mathbf{E}_{\beta,\beta}(\lambda)}M_H(r) + \big[\frac{m\mathbf{E}_{\beta,\beta}(|\lambda|)^2}{\mathbf{E}_{\beta,\beta}(\lambda)} + m\mathbf{E}_{\beta,\beta}(|\lambda|)\big]M_J(r)
\end{aligned}
$$

$$+\big[\frac{\mathbf{E}_{\beta,\beta}(|\lambda|)\mathbf{E}_{\beta,\beta-1}(|\lambda|)}{\mathbf{E}_{\beta,\beta}(\lambda)(1-t_\sigma)^{2-\beta}}+m\mathbf{E}_{\beta,\beta-1}(|\lambda|)\big]M_I(r)$$
$$+\big[\frac{\mathbf{E}_{\beta,\beta}(|\lambda|)\mathbf{E}_{\beta,\beta}(|\lambda|)\mathbf{B}(\beta+l,k+1)}{\mathbf{E}_{\beta,\beta}(\lambda)}+\mathbf{E}_{\beta,\beta}(\lambda|)\mathbf{B}(\beta+l,k+1)\big]M_f(r)\Big]<1.$$

Proof. Form the assumption, we know that there exists $r_0>0$ such that (4.1) holds. By Theorem 4.1, (1.7) has at least one solution. The proof is omitted. □

Theorem 4.3. *Suppose that* (1.A2), (1.A3) (1.A6)–(1.A8) *hold, and*

(4.A2) *there exist nondecreasing functions* $M_f, M_g, M_h, M_I, M_J : [0,+\infty)\to[0,+\infty)$ *such that*

$$\begin{gathered}
|f(t,x)|\le M_f(|x|),\quad t\in(0,1),\ x\in\mathbb{R},\\
|G(t,x)|\le M_G(|x|),\quad t\in(0,1),\ x\in\mathbb{R},\\
|H(t,x)|\le M_H(|x|),\quad t\in(0,1),\ x\in\mathbb{R},\\
|I(t_i,x)|\le M_I(|x|),\quad i\in\mathbb{N}[1,m],\ x\in\mathbb{R},\\
|J(t_i,x)|\le M_J(|x|),\quad i\in\mathbb{N}[1,m],\ x\in\mathbb{R}.
\end{gathered}$$

Then (1.8) *has at least one solution if there exists* $r_0>0$ *such that*

$$\begin{aligned}
&\big[\|\phi\|_1+\frac{|\lambda|\mathbf{E}_{\beta,\beta}(\lambda)\|\phi\|_1}{\mathbf{E}_{\beta,1}(\lambda)}\big]M_G(r_0)+\frac{\|\psi\|_1}{\mathbf{E}_{\beta,1}(\lambda)}M_H(r_0)\\
&+\big[\frac{m|\lambda|\mathbf{E}_{\beta,\beta}(|\lambda|)}{\mathbf{E}_{\beta,1}(\lambda)}+m\mathbf{E}_{\beta,1}(|\lambda|)\big]M_I(r_0)+\big[m\mathbf{E}_{\beta,2}(|\lambda|)\\
&+\frac{m\mathbf{E}_{\beta,1}(|\lambda|)}{\mathbf{E}_{\beta,1}(\lambda)}\big]M_J(r_0)+\big[\frac{\mathbf{E}_{\beta,\beta-1}(|\lambda|)\mathbf{B}(\beta+l,k+1)}{\mathbf{E}_{\beta,1}(\lambda)}\\
&+\mathbf{E}_{\beta,\beta}(|\lambda|)\mathbf{B}(\beta+l,k+1)\big]M_f(r_0)<r_0.
\end{aligned}\tag{4.2}$$

Proof. From Lemmas 3.17 and 3.18, and the definition of Q, it follows that $x\in P_mC(0,1]$ is a solution of (1.8) if and only if $x\in P_mC(0,1]$ is a fixed point of Q. Lemma 3.18 implies that Q is a completely continuous operator. From (4.A2), for $x\in P_mC(0,1]$ we have

$$\begin{gathered}
|f(t,x(t))|\le M_f(|x(t)|)\le M_f(\|x\|),\quad t\in(0,1),\\
|G(t,x(t))|\le M_G(\|x\|), t\in(0,1),\\
|H(t,x(t))|\le M_H(\|x\|), t\in(0,1),\\
|I(t_i,x(t_i))|\le M_I(\|x\|), i\in\mathbb{N}[1,m],\\
|J(t_i,x(t_i))|\le M_J(\|x\|), i\in\mathbb{N}[1,m].
\end{gathered}$$

We consider the set $\Omega=\{x\in P_mC(0,1]: x=\lambda(Tx),\ \text{for some }\lambda\in[0,1]\}$. For $x\in\Omega$, and $t\in(t_i,t_{i+1}]$ we have

$$\begin{aligned}
&|(Qx)(t)|\\
&\le\|\phi\|_1M_G(\|x\|)+\frac{1}{\mathbf{E}_{\beta,1}(\lambda)}\Big[\|\psi\|_1M_H(\|x\|)+|\lambda|\mathbf{E}_{\beta,\beta}(\lambda)\|\phi\|_1M_G(\|x\|)\\
&\quad+\sum_{\sigma=1}^{m}(|\lambda|\mathbf{E}_{\beta,\beta}(|\lambda|)M_I(\|x\|)+\mathbf{E}_{\beta,1}(|\lambda|)M_J(\|x\|))
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{\tau=0}^{+\infty}\frac{\lambda^\tau}{\Gamma(\tau\beta+\beta-1)}\int_0^1(1-s)^{\tau\beta+\beta+l-1}s^k dsM_f(\|x\|)\Big]\\
&+\sum_{j=1}^{i}[\mathbf{E}_{\beta,1}(|\lambda|)M_I(\|x\|)+\mathbf{E}_{\beta,2}(|\lambda|)M_J(\|x\|)]\\
&+\sum_{\tau=0}^{+\infty}\frac{\lambda^\tau}{\Gamma((\tau+1)\beta)}\int_0^t(t-s)^{\tau\beta+\beta-1}s^k(1-s)^l dsM_f(\|x\|)\\
\le{}& \|\phi\|_1M_G(\|x\|)+\frac{1}{\mathbf{E}_{\beta,1}(\lambda)}\Big[\|\psi\|_1M_H(\|x\|)+|\lambda|\mathbf{E}_{\beta,\beta}(\lambda)\|\phi\|_1M_G(\|x\|)\\
&+m|\lambda|\mathbf{E}_{\beta,\beta}(|\lambda|)M_I(\|x\|)+m\mathbf{E}_{\beta,1}(|\lambda|)M_J(\|x\|)\\
&+\sum_{\tau=0}^{+\infty}\frac{\lambda^\tau}{\Gamma(\tau\beta+\beta-1)}\int_0^1(1-s)^{\beta+l-1}s^k dsM_f(\|x\|)\Big]\\
&+m\mathbf{E}_{\beta,1}(|\lambda|)M_I(\|x\|)+m\mathbf{E}_{\beta,2}(|\lambda|)M_J(\|x\|)\\
&+\sum_{\tau=0}^{+\infty}\frac{\lambda^\tau t^{\tau\beta+\beta+k+l}}{\Gamma((\tau+1)\beta)}\int_0^1(1-w)^{\tau\beta+\beta+l-1}w^k dwM_f(\|x\|)\\
\le{}& [\|\phi\|_1+\frac{|\lambda|\mathbf{E}_{\beta,\beta}(\lambda)\|\phi\|_1}{\mathbf{E}_{\beta,1}(\lambda)}]M_G(\|x\|)+\frac{\|\psi\|_1}{\mathbf{E}_{\beta,1}(\lambda)}M_H(\|x\|)+[\frac{m|\lambda|\mathbf{E}_{\beta,\beta}(|\lambda|)}{\mathbf{E}_{\beta,1}(\lambda)}\\
&+m\mathbf{E}_{\beta,1}(|\lambda|)]M_I(\|x\|)+[m\mathbf{E}_{\beta,2}(|\lambda|)+\frac{m\mathbf{E}_{\beta,1}(|\lambda|)}{\mathbf{E}_{\beta,1}(\lambda)}]M_J(\|x\|)\\
&+\big[\frac{\mathbf{E}_{\beta,\beta-1}(|\lambda|)\mathbf{B}(\beta+l,k+1)}{\mathbf{E}_{\beta,1}(\lambda)}+\mathbf{E}_{\beta,\beta}(|\lambda|)\mathbf{B}(\beta+l,k+1)\big]M_f(\|x\|).
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\|x\|&=\lambda\|Tx\|\le\|Tx\|\\
&\le\big[\|\phi\|_1+\frac{|\lambda|\mathbf{E}_{\beta,\beta}(\lambda)\|\phi\|_1}{\mathbf{E}_{\beta,1}(\lambda)}\big]M_G(\|x\|)+\frac{\|\psi\|_1}{\mathbf{E}_{\beta,1}(\lambda)}M_H(\|x\|)\\
&\quad+\big[\frac{m|\lambda|\mathbf{E}_{\beta,\beta}(|\lambda|)}{\mathbf{E}_{\beta,1}(\lambda)}+m\mathbf{E}_{\beta,1}(|\lambda|)\big]M_I(\|x\|)\\
&\quad+\big[m\mathbf{E}_{\beta,2}(|\lambda|)+\frac{m\mathbf{E}_{\beta,1}(|\lambda|)}{\mathbf{E}_{\beta,1}(\lambda)}\big]M_J(\|x\|)+\big[\frac{\mathbf{E}_{\beta,\beta-1}(|\lambda|)\mathbf{B}(\beta+l,k+1)}{\mathbf{E}_{\beta,1}(\lambda)}\\
&\quad+\mathbf{E}_{\beta,\beta}(|\lambda|)\mathbf{B}(\beta+l,k+1)\big]M_f(\|x\|).
\end{aligned}
$$

From (4.2), we choose $\Omega=\{x\in P_mC(0,1]:\|x\|\le r_0\}$. For $x\in\Omega$, we obtain $x\neq\lambda(Tx)$ for any $\lambda\in[0,1]$ and $x\in\partial\Omega$.

As a consequence of Schaefer's fixed point theorem, we deduce that Q has a fixed point which is a solution of problem (1.8). The proof is complete. □

Theorem 4.4. *Suppose that* (1.A4), (1.A4) (1.A9)–(1.A11) *hold,* $\Delta\neq 0$*, and*

(4.A3) *there exist nondecreasing functions* $M_f,M_g,M_h,M_I,M_J:[0,+\infty)\to[0,+\infty)$ *such that*

$$|f(t,(\log\frac{t}{t_i})^{\beta-2}x)|\le M_f(|x|),\quad t\in(t_i,t_{i+1}],\ i\in\mathbb{N}[0,m],x\in\mathbb{R},$$

$$
\begin{aligned}
&|G(t,(\log\frac{t}{t_i})^{\beta-2}x)|\le M_f(|x|),\quad t\in(t_i,t_{i+1}],\ i\in\mathbb{N}[0,m],x\in\mathbb{R},\\
&|H(t,(\log\frac{t}{t_i})^{\beta-2}x)|\le M_f(|x|),\quad t\in(t_i,t_{i+1}],\ i\in\mathbb{N}[0,m],x\in\mathbb{R},\\
&|I(t_i,(\log\frac{t_i}{t_{i-1}})^{\beta-2}x)|\le M_I(|x|),\quad i\in\mathbb{N}[1,m],\ x\in\mathbb{R},\\
&|J(t_i,(\log\frac{t_i}{t_{i-1}})^{\beta-2}x)|\le M_I(|x|),\quad i\in\mathbb{N}[1,m],\ x\in\mathbb{R}.
\end{aligned}
$$

Then (1.9) *has at least one solution if there exists a constant* $r_0>0$ *such that*

$$
B_{1G}M_G(r_0)+B_{2H}M_H(r_0)+B_{3J}M_J(r_0)+B_{4I}M_I(r_0)+B_{5f}M_f(r_0)<r_0, \tag{4.3}
$$

where

$$
\begin{aligned}
B_{1G}&=\frac{|\lambda|\mathbf{E}_{\beta,\beta}(\lambda)\mathbf{E}_{\beta,\beta}(|\lambda|)\|\phi\|_1}{|\Delta|}+\frac{|\frac{1}{\Gamma(\beta)}-\mathbf{E}_{\beta,\beta}(\lambda)|\mathbf{E}_{\beta,\beta-1}(|\lambda|)\|\phi\|_1}{|\Delta|},\\
B_{2H}&=\frac{\mathbf{E}_{\beta,\beta}(\lambda)\mathbf{E}_{\beta,\beta-1}(|\lambda|)\|\psi\|_1}{|\Delta|}+\frac{|\frac{1}{\Gamma(\beta-1)}-\mathbf{E}_{\beta,\beta-1}(\lambda)|\mathbf{E}_{\beta,\beta}(|\lambda|)\|\psi\|_1}{|\Delta|},\\
B_{3J}&=\frac{m\Gamma(\beta)|\frac{1}{\Gamma(\beta-1)}-\mathbf{E}_{\beta,\beta-1}(\lambda)|\mathbf{E}_{\beta,\beta}(|\lambda|)^2}{|\Delta|}+\frac{m|\lambda|\Gamma(\beta)\mathbf{E}_{\beta,\beta}(\lambda)\mathbf{E}_{\beta,\beta}(|\lambda|)^2}{|\Delta|}\\
&\quad+\frac{m\Gamma(\beta)|\frac{1}{\Gamma(\beta)}-\mathbf{E}_{\beta,\beta}(\lambda)|\mathbf{E}_{\beta,\beta}(|\lambda|)\mathbf{E}_{\beta,\beta-1}(|\lambda|)}{|\Delta|}\\
&\quad+\frac{m\Gamma(\beta)\mathbf{E}_{\beta,\beta}(\lambda)\mathbf{E}_{\beta,\beta}(|\lambda|)\mathbf{E}_{\beta,\beta-1}(|\lambda|)}{|\Delta|}+m\mathbf{E}_{\beta,\beta}(|\lambda|)\Gamma(\beta),\\
B_{4I}&=\frac{m|\lambda|\Gamma(\beta-1)|\frac{1}{\Gamma(\beta-1)}-\mathbf{E}_{\beta,\beta-1}(\lambda)|\mathbf{E}_{\beta,\beta}(|\lambda|)^2}{|\Delta|}\\
&\quad+\frac{|\lambda|\Gamma(\beta-1)\mathbf{E}_{\beta,\beta}(\lambda)\mathbf{E}_{\beta,\beta-1}(|\lambda|)\mathbf{E}_{\beta,\beta}(|\lambda|)}{|\Delta|}\sum_{\sigma=1}^{m}(\log\frac{e}{t_\sigma})^{\beta-2}\\
&\quad+\Big(\Gamma(\beta-1)\mathbf{E}_{\beta,\beta-1}(|\lambda|)(|\frac{1}{\Gamma(\beta)}-\mathbf{E}_{\beta,\beta}(\lambda)|\mathbf{E}_{\beta,\beta-1}(|\lambda|)\sum_{\sigma=1}^{m}(\log\frac{e}{t_\sigma})^{\beta-2}\\
&\quad+m|\lambda|\mathbf{E}_{\beta,\beta}(\lambda)\mathbf{E}_{\beta,\beta}(|\lambda|))\Big)/|\Delta|+m\mathbf{E}_{\beta,\beta-1}(|\lambda|)\Gamma(\beta-1),\\
B_{5f}&=\frac{|\frac{1}{\Gamma(\beta-1)}-\mathbf{E}_{\beta,\beta-1}(\lambda)|\big[|\lambda|\|\mathbf{E}_{\beta,1}(|\lambda|)\mathbf{E}_{\beta,\beta}(|\lambda|)+\mathbf{E}_{\beta,\beta}(|\lambda|)\mathbf{E}_{\beta,\beta-1}(|\lambda|)\big]}{|\Delta|}\\
&\quad\times\mathbf{B}(l+1,k+1)+\frac{|\lambda|\mathbf{E}_{\beta,\beta}(\lambda)[\mathbf{E}_{\beta,\beta}(|\lambda|)^2+\mathbf{E}_{\beta,1}(|\lambda|)\mathbf{E}_{\beta,\beta-1}(|\lambda|)]}{|\Delta|}\\
&\quad\times\mathbf{B}(l+1,k+1)+\mathbf{E}_{\beta,\beta}(|\lambda|)\mathbf{B}(\beta+l,k+1).
\end{aligned}
$$

Proof. From Lemmas 3.19 and 3.20, and the definition of R, $x\in LP_mC_{2-\beta}(1,e]$ is a solution of BVP(1.9) if and only if $x\in LP_mC_{2-\beta}(1,e]$ is a fixed point of R. Lemma 3.18 implies that R is a completely continuous operator.

From (4.A3), for $x\in LP_mC_{2-\beta}(1,e]$ we have

$$\begin{aligned}
|f(t,x(t))| &= \Big|f\big(t,(\log\frac{t}{t_i})^{\beta-2}(\log\frac{t}{t_i})^{2-\beta}x(t)\big)\Big|\\
&\le M_f\Big(|(\log\frac{t}{t_i})^{2-\beta}x(t)|\Big)\\
&\le M_f(\|x\|),\quad t\in(t_i,t_{i+1}],\ i\in\mathbb{N}[0,m],\\
|G(t,x(t))| &\le M_G(\|x\|),\quad t\in(t_i,t_{i+1}],\ i\in\mathbb{N}[0,m],\\
|H(t,x(t))| &\le M_H(\|x\|),\quad t\in(t_i,t_{i+1}],\ i\in\mathbb{N}[0,m],\\
|I(t_i,x(t_i))| &= |I(t_i,(\log\frac{t_i}{t_{i-1}})^{\beta-2}(\log\frac{t_i}{t_{i-1}})^{2-\beta}x(t))|\\
&\le M_f(|(\log\frac{t_i}{t_{i-1}})^{2-\beta}x(t)|)\le M_G(\|x\|), i\in\mathbb{N}[1,m],\\
&|I(t_i,x(t_i))|\le M_H(\|x\|), i\in\mathbb{N}[1,m].
\end{aligned}$$

We consider the set $\Omega=\{x\in LP_mC_{2-\beta}(1,e]: x=\lambda(Rx),\ \text{for some}\ \lambda\in[0,1]\}$. For $x\in\Omega$ and $t\in(t_i,t_{i+1}]$ we have

$$\begin{aligned}
&(\log\frac{t}{t_i})^{2-\beta}|(Rx)(t)|\\
&\le \mathbf{E}_{\beta,\beta}(|\lambda|)\Big[\frac{|\lambda|\mathbf{E}_{\beta,\beta}(\lambda)}{|\Delta|}\|\phi\|_1M_G(\|x\|)+\frac{|\frac{1}{\Gamma(\beta-1)}-\mathbf{E}_{\beta,\beta-1}(\lambda)|}{|\Delta|}\|\psi\|_1M_H(\|x)\\
&\quad+\Gamma(\beta)\sum_{\sigma=1}^m(\frac{|\frac{1}{\Gamma(\beta-1)}-\mathbf{E}_{\beta,\beta-1}(\lambda)|}{|\Delta|}\mathbf{E}_{\beta,\beta}(|\lambda|)+\frac{\lambda\mathbf{E}_{\beta,\beta}(\lambda)}{|\Delta|}\mathbf{E}_{\beta,\beta}(|\lambda|))M_J(\|x\|)\\
&\quad+\Gamma(\beta-1)\sum_{\sigma=1}^m\Big(|\lambda|\frac{|\frac{1}{\Gamma(\beta-1)}-\mathbf{E}_{\beta,\beta-1}(\lambda)|}{|\Delta|}\mathbf{E}_{\beta,\beta}(|\lambda|)\\
&\quad+\frac{\lambda\mathbf{E}_{\beta,\beta}(\lambda)}{|\Delta|}(\log\frac{e}{t_\sigma})^{\beta-2}\mathbf{E}_{\beta,\beta-1}(|\lambda|)\Big)M_I(\|x\|)\\
&\quad+\Big(|\lambda|\frac{|\frac{1}{\Gamma(\beta-1)}-\mathbf{E}_{\beta,\beta-1}(\lambda)|}{|\Delta|}\mathbf{E}_{\beta,1}(|\lambda|)+\frac{|\lambda|\mathbf{E}_{\beta,\beta}(\lambda)}{|\Delta|}\mathbf{E}_{\beta,\beta}(|\lambda|)\Big)\\
&\quad\times\int_1^e(\log s)^k(1-\log s)^l\frac{ds}{s}M_f(\|x\|)\Big]\\
&\quad+\mathbf{E}_{\beta,\beta-1}(|\lambda|)\Big[\frac{|\frac{1}{\Gamma(\beta)}-\mathbf{E}_{\beta,\beta}(\lambda)|}{|\Delta|}\|\phi\|_1M_G(\|x\|)+\frac{\mathbf{E}_{\beta,\beta}(\lambda)}{|\Delta|}\|\psi\|_1M_H(\|x\|)\\
&\quad+\Gamma(\beta)\sum_{\sigma=1}^m\Big(\frac{|\frac{1}{\Gamma(\beta)}-\mathbf{E}_{\beta,\beta}(\lambda)|}{|\Delta|}\mathbf{E}_{\beta,\beta}(|\lambda|)+\frac{\mathbf{E}_{\beta,\beta}(\lambda)}{|\Delta|}\mathbf{E}_{\beta,\beta}(|\lambda|)\Big)M_J(\|x\|)\\
&\quad+\Gamma(\beta-1)\sum_{\sigma=1}^m\Big(\frac{|\frac{1}{\Gamma(\beta)}-\mathbf{E}_{\beta,\beta}(\lambda)|}{|\Delta|}(\log\frac{e}{t_\sigma})^{\beta-2}\mathbf{E}_{\beta,\beta-1}(|\lambda|)\\
&\quad+\frac{|\lambda|\mathbf{E}_{\beta,\beta}(\lambda)}{|\Delta|}\mathbf{E}_{\beta,\beta}(|\lambda|)\Big)M_I(\|x\|)\\
&\quad+\Big(\frac{|\frac{1}{\Gamma(\beta)}-\mathbf{E}_{\beta,\beta}(\lambda)|}{|\Delta|}\mathbf{E}_{\beta,\beta}(|\lambda|)+|\lambda|\frac{\mathbf{E}_{\beta,\beta}(\lambda)}{|\Delta|}\mathbf{E}_{\beta,1}(|\lambda|)\Big)
\end{aligned}$$

$$
\begin{aligned}
&\times \int_1^e (\log s)^k (1-\log s)^l \frac{ds}{s} M_f(\|x\|)\Big] \\
&+ \sum_{\sigma=1}^{i} \big[\mathbf{E}_{\beta,\beta}(|\lambda|)\Gamma(\beta)M_J(\|x\|) + \mathbf{E}_{\beta,\beta-1}(|\lambda|)\Gamma(\beta-1)M_I(\|x\|)\big] \\
&+ (\log\frac{t}{t_i})^{2-\beta} \sum_{\tau=0}^{+\infty} \frac{\lambda^\tau}{\Gamma((\tau+1)\beta)} \int_1^t (\log\frac{t}{s})^{\tau\beta+\beta+l-1} (\log s)^k \frac{ds}{s} M_f(\|x\|) \\
\le{}& \frac{|\lambda|\mathbf{E}_{\beta,\beta}(\lambda)\mathbf{E}_{\beta,\beta}(|\lambda|)\|\phi\|_1}{|\Delta|} M_G(\|x\|) + \frac{|\frac{1}{\Gamma(\beta-1)} - \mathbf{E}_{\beta,\beta-1}(\lambda)|\mathbf{E}_{\beta,\beta}(|\lambda|)\|\psi\|_1}{|\Delta|} M_H(\|x)) \\
&+ m\Gamma(\beta)\Big(\frac{|\frac{1}{\Gamma(\beta-1)} - \mathbf{E}_{\beta,\beta-1}(\lambda)|\mathbf{E}_{\beta,\beta}(|\lambda|)^2}{|\Delta|} + \frac{\lambda\mathbf{E}_{\beta,\beta}(\lambda)\mathbf{E}_{\beta,\beta}(|\lambda|)^2}{|\Delta|}\Big) M_J(\|x\|) \\
&+ \Gamma(\beta-1)\Big(\frac{m|\lambda||\frac{1}{\Gamma(\beta-1)} - \mathbf{E}_{\beta,\beta-1}(\lambda)|\mathbf{E}_{\beta,\beta}(|\lambda|)^2}{|\Delta|} \\
&+ \frac{\lambda\mathbf{E}_{\beta,\beta}(\lambda)\mathbf{E}_{\beta,\beta-1}(|\lambda|)\mathbf{E}_{\beta,\beta}(|\lambda|)}{|\Delta|} \sum_{\sigma=1}^{m} (\log\frac{e}{t_\sigma})^{\beta-2}\Big) M_I(\|x\|) \\
&+ \Big(\frac{|\lambda||\frac{1}{\Gamma(\beta-1)} - \mathbf{E}_{\beta,\beta-1}(\lambda)|\mathbf{E}_{\beta,1}(|\lambda|)\mathbf{E}_{\beta,\beta}(|\lambda|)}{|\Delta|} \\
&+ \frac{|\lambda|\mathbf{E}_{\beta,\beta}(\lambda)\mathbf{E}_{\beta,\beta}(|\lambda|)^2}{|\Delta|}\Big)\mathbf{B}(l+1,k+1)M_f(\|x\|) \\
&+ \frac{|\frac{1}{\Gamma(\beta)} - \mathbf{E}_{\beta,\beta}(\lambda)|\mathbf{E}_{\beta,\beta-1}(|\lambda|)\|\phi\|_1}{|\Delta|} M_G(\|x\|) + \frac{\mathbf{E}_{\beta,\beta}(\lambda)\mathbf{E}_{\beta,\beta-1}(|\lambda|)\|\psi\|_1}{|\Delta|} M_H(\|x\|) \\
&+ m\Gamma(\beta)\Big(\frac{|\frac{1}{\Gamma(\beta)} - \mathbf{E}_{\beta,\beta}(\lambda)|\mathbf{E}_{\beta,\beta}(|\lambda|)\mathbf{E}_{\beta,\beta-1}(|\lambda|)}{|\Delta|} \\
&+ \frac{\mathbf{E}_{\beta,\beta}(\lambda)\mathbf{E}_{\beta,\beta}(|\lambda|)\mathbf{E}_{\beta,\beta-1}(|\lambda|)}{|\Delta|}\Big) M_J(\|x\|) \\
&+ \Gamma(\beta-1)\Big(\frac{|\frac{1}{\Gamma(\beta)} - \mathbf{E}_{\beta,\beta}(\lambda)|\mathbf{E}_{\beta,\beta-1}(|\lambda|)^2}{|\Delta|} \sum_{\sigma=1}^{m} (\log\frac{e}{t_\sigma})^{\beta-2} \\
&+ m\frac{|\lambda|\mathbf{E}_{\beta,\beta}(\lambda)\mathbf{E}_{\beta,\beta}(|\lambda|)\mathbf{E}_{\beta,\beta-1}(|\lambda|)}{|\Delta|}\Big) M_I(\|x\|) \\
&+ \Big(\frac{|\frac{1}{\Gamma(\beta)} - \mathbf{E}_{\beta,\beta}(\lambda)|\mathbf{E}_{\beta,\beta}(|\lambda|)\mathbf{E}_{\beta,\beta-1}(|\lambda|)}{|\Delta|} + \frac{|\lambda|\mathbf{E}_{\beta,\beta}(\lambda)\mathbf{E}_{\beta,1}(|\lambda|)\mathbf{E}_{\beta,\beta-1}(|\lambda|)}{|\Delta|}\Big) \\
&\times \mathbf{B}(l+1,k+1)M_f(\|x\|) + m\mathbf{E}_{\beta,\beta}(|\lambda|)\Gamma(\beta)M_J(\|x\|) \\
&+ m\mathbf{E}_{\beta,\beta-1}(|\lambda|)\Gamma(\beta-1)M_I(\|x\|) + \mathbf{E}_{\beta,\beta}(|\lambda|)\mathbf{B}(\beta+l,k+1)M_f(\|x\|) \\
={}& \Big[\frac{|\lambda|\mathbf{E}_{\beta,\beta}(\lambda)\mathbf{E}_{\beta,\beta}(|\lambda|)\|\phi\|_1}{|\Delta|} + \frac{|\frac{1}{\Gamma(\beta)} - \mathbf{E}_{\beta,\beta}(\lambda)|\mathbf{E}_{\beta,\beta-1}(|\lambda|)\|\phi\|_1}{|\Delta|}\Big] M_G(\|x\|) \\
&+ \Big[\frac{\mathbf{E}_{\beta,\beta}(\lambda)\mathbf{E}_{\beta,\beta-1}(|\lambda|)\|\psi\|_1}{|\Delta|} + \frac{|\frac{1}{\Gamma(\beta-1)} - \mathbf{E}_{\beta,\beta-1}(\lambda)|\mathbf{E}_{\beta,\beta}(|\lambda|)\|\psi\|_1}{|\Delta|}\Big] M_H(\|x)) \\
&+ \Big[\frac{m\Gamma(\beta)|\frac{1}{\Gamma(\beta-1)} - \mathbf{E}_{\beta,\beta-1}(\lambda)|\mathbf{E}_{\beta,\beta}(|\lambda|)^2}{|\Delta|} + \frac{m|\lambda|\Gamma(\beta)\mathbf{E}_{\beta,\beta}(\lambda)\mathbf{E}_{\beta,\beta}(|\lambda|)^2}{|\Delta|}
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{m\Gamma(\beta)|\frac{1}{\Gamma(\beta)} - \mathbf{E}_{\beta,\beta}(\lambda)|\mathbf{E}_{\beta,\beta}(|\lambda|)\mathbf{E}_{\beta,\beta-1}(|\lambda|)}{|\Delta|} \\
&+ \frac{m\Gamma(\beta)\mathbf{E}_{\beta,\beta}(\lambda)\mathbf{E}_{\beta,\beta}(|\lambda|)\mathbf{E}_{\beta,\beta-1}(|\lambda|)}{|\Delta|} + m\mathbf{E}_{\beta,\beta}(|\lambda|)\Gamma(\beta)\Big]M_J(\|x\|) \\
&+ \Big[\frac{m|\lambda|\Gamma(\beta-1)|\frac{1}{\Gamma(\beta-1)} - \mathbf{E}_{\beta,\beta-1}(\lambda)|\mathbf{E}_{\beta,\beta}(|\lambda|)^2}{|\Delta|} \\
&+ \frac{|\lambda|\Gamma(\beta-1)\mathbf{E}_{\beta,\beta}(\lambda)\mathbf{E}_{\beta,\beta-1}(|\lambda|)\mathbf{E}_{\beta,\beta}(|\lambda|)}{|\Delta|}\sum_{\sigma=1}^{m}(\log\frac{e}{t_\sigma})^{\beta-2} \\
&+ \Big(\Gamma(\beta-1)\mathbf{E}_{\beta,\beta-1}(|\lambda|)(|\frac{1}{\Gamma(\beta)} - \mathbf{E}_{\beta,\beta}(\lambda)|\mathbf{E}_{\beta,\beta-1}(|\lambda|)\sum_{\sigma=1}^{m}(\log\frac{e}{t_\sigma})^{\beta-2} \\
&+ m|\lambda|\mathbf{E}_{\beta,\beta}(\lambda)\mathbf{E}_{\beta,\beta}(|\lambda|))\Big)/|\Delta| + m\mathbf{E}_{\beta,\beta-1}(|\lambda|)\Gamma(\beta-1)\Big]M_I(\|x\|) \\
&+ \Big[\frac{|\frac{1}{\Gamma(\beta-1)} - \mathbf{E}_{\beta,\beta-1}(\lambda)|\big[|\lambda\|\mathbf{E}_{\beta,1}(|\lambda|)\mathbf{E}_{\beta,\beta}(|\lambda|) + \mathbf{E}_{\beta,\beta}(|\lambda|)\mathbf{E}_{\beta,\beta-1}(|\lambda|)\big]}{|\Delta|} \\
&\times \mathbf{B}(l+1,k+1) + \frac{|\lambda|\mathbf{E}_{\beta,\beta}(\lambda)[\mathbf{E}_{\beta,\beta}(|\lambda|)^2 + \mathbf{E}_{\beta,1}(|\lambda|)\mathbf{E}_{\beta,\beta-1}(|\lambda|)]}{|\Delta|}\mathbf{B}(l+1,k+1) \\
&+ \mathbf{E}_{\beta,\beta}(|\lambda|)\mathbf{B}(\beta+l,k+1)\Big]M_f(\|x\|) \\
&= B_{1G}M_G(\|x\|) + B_{2H}M_H(\|x\|) + B_{3J}M_J(\|x\|) + B_{4I}M_I(\|x\|) + B_{5f}M_f(\|x\|).
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&\|x\| = \lambda\|Rx\| \le \|Rx\| \\
&\le B_{1G}M_G(\|x\|) + B_{2H}M_H(\|x\|) + B_{3J}M_J(\|x\|) + B_{4I}M_I(\|x\|) + B_{5f}M_f(\|x\|).
\end{aligned}
$$

From (4.3), we choose $\Omega = \{x \in LP_mC_{2-\beta}(1,e] : \|x\| \le r_0\}$. For $x \in \partial\Omega$, we obtain $x \ne \lambda(Rx)$ for any $\lambda \in [0,1]$. In fact, if there exists $x \in \partial\Omega$ such that $x = \lambda(Rx)$ for some $\lambda \in [0,1]$. Then $r_0 = \|x\| = \lambda\|Rx\| \le \|Rx\| \le B_{1G}M_G(r_0) + B_{2H}M_H(r_0) + B_{3J}M_J(r_0) + B_{4I}M_I(r_0) + B_{5f}M_f(r_0) < r_0$, which is a contradiction.

As a consequence of Schaefer's fixed point theorem, we deduce that R has a fixed point which is a solution of problem (1.9). The proof is complete. □

Theorem 4.5. *Suppose that* (1.A7), (1.A12)–(1.A14) *hold,* $\lambda \ne 0$, *and*

(4.A4) *there exist nondecreasing functions* $M_f, M_g, M_h, M_I, M_J : [0,+\infty) \to [0,+\infty)$ *such that*

$$
\begin{gathered}
|f(t,x)| \le M_f(|x|), \quad t \in (t_i, t_{i+1}],\ i \in \mathbb{N}[0,m], x \in \mathbb{R}, \\
|G(t,x)| \le M_f(|x|), \quad t \in (t_i, t_{i+1}],\ i \in \mathbb{N}[0,m], x \in \mathbb{R}, \\
|H(t,x)| \le M_f(|x|), \quad t \in (t_i, t_{i+1}],\ i \in \mathbb{N}[0,m], x \in \mathbb{R}, \\
|I(t_i,x)| \le M_I(|x|), \quad i \in \mathbb{N}[1,m],\ x \in \mathbb{R}, \\
|J(t_i,x)| \le M_J(|x|), \quad i \in \mathbb{N}[1,m],\ x \in \mathbb{R}.
\end{gathered}
$$

Then BVP (1.10) *has at least one solution if there exists a constant* $r_0 > 0$ *such that*

$$\begin{aligned}&\frac{\mathbf{E}_{\beta,1}(|\lambda|)\|\psi\|_1}{\lambda\mathbf{E}_{\beta,\beta}(\lambda)}M_H(r_0)+\big[\mathbf{E}_{\beta,2}(|\lambda|)+\frac{\mathbf{E}_{\beta,1}(|\lambda|)}{|\lambda|\mathbf{E}_{\beta,\beta}(\lambda)}\mathbf{E}_{\beta,1}(|\lambda|)\big]\|\phi\|_1M_G(r_0)\\&+\big[\frac{\mathbf{E}_{\beta,1}(|\lambda|)}{|\lambda|\mathbf{E}_{\beta,\beta}(\lambda)}m|\lambda|\mathbf{E}_{\beta,\beta}(|\lambda|)+m\mathbf{E}_{\beta,1}(|\lambda|)\big]M_I(r_0)+[m\mathbf{E}_{\beta,1}(|\lambda|)\\&+m\mathbf{E}_{\beta,2}(|\lambda|)]M_J(r_0)+\big[\frac{\mathbf{E}_{\beta,1}(|\lambda|)}{|\lambda|\mathbf{E}_{\beta,\beta}(\lambda)}\mathbf{E}_{\beta,\beta}(|\lambda|)\mathbf{B}(\beta+l-1,k+1)\\&+\mathbf{E}_{\beta,\beta}(|\lambda|)\mathbf{B}(\beta+l,k+1)\big]M_f(r_0)<r_0.\end{aligned}\tag{4.4}$$

Proof. From Lemmas 3.21 and 3.22, and the definition of J, it follows that $x \in LP_mC(1,e]$ is a solution of (1.10) if and only if $x \in LP_mC(1,e]$ is a fixed point of R. Lemma 3.22 implies that J is a completely continuous operator.

From (4.A4), for $x \in LP_mC(1,e]$ we have

$$\begin{gathered}|f(t,x(t))|\le M_f(\|x\|),\quad t\in(t_i,t_{i+1}],\ i\in\mathbb{N}[0,m],\\|G(t,x(t))|\le M_f(\|x\|),\quad t\in(t_i,t_{i+1}],\ i\in\mathbb{N}[0,m],\\|H(t,x(t))|\le M_f(\|x\|),\quad t\in(t_i,t_{i+1}],\ i\in\mathbb{N}[0,m],\\|I(t_i,x(t_i))|\le M_I(\|x\|),\quad i\in\mathbb{N}[1,m],\\|J(t_i,x(t_i))|\le M_J(\|x\|),\quad i\in\mathbb{N}[1,m].\end{gathered}$$

We consider the set $\Omega = \{x \in LP_mC(1,e] : x = \lambda(Jx),\ \text{for some } \lambda \in [0,1]\}$. For $x \in \Omega$ and $t \in (t_i,t_{i+1}]$, we have

$$\begin{aligned}&|(Jx)(t)|\\&\le\frac{\mathbf{E}_{\beta,1}(|\lambda|)\|\psi\|_1}{\lambda\mathbf{E}_{\beta,\beta}(\lambda)}M_H(\|x\|)+\big[\mathbf{E}_{\beta,2}(|\lambda|)+\frac{\mathbf{E}_{\beta,1}(|\lambda|)}{|\lambda|\mathbf{E}_{\beta,\beta}(\lambda)}\mathbf{E}_{\beta,1}(|\lambda|)\big]\|\phi\|_1M_G(\|x\|)\\&\quad+\big[\frac{\mathbf{E}_{\beta,1}(|\lambda|)}{|\lambda|\mathbf{E}_{\beta,\beta}(\lambda)}m|\lambda|\mathbf{E}_{\beta,\beta}(|\lambda|)+m\mathbf{E}_{\beta,1}(|\lambda|)\big]M_I(\|x\|)\\&\quad+[m\mathbf{E}_{\beta,1}(|\lambda|)+m\mathbf{E}_{\beta,2}(|\lambda|)]M_J(\|x\|)+\Big[\frac{\mathbf{E}_{\beta,1}(|\lambda|)}{|\lambda|\mathbf{E}_{\beta,\beta}(\lambda)}\mathbf{E}_{\beta,\beta}(|\lambda|)\mathbf{B}(\beta+l-1,k+1)\\&\quad+\mathbf{E}_{\beta,\beta}(|\lambda|)\mathbf{B}(\beta+l,k+1)\Big]M_f(\|x\|).\end{aligned}$$

It follows that

$$\begin{aligned}\|x\|&=\lambda\|Rx\|\le\|Rx\|\\&\le\frac{\mathbf{E}_{\beta,1}(|\lambda|)\|\psi\|_1}{\lambda\mathbf{E}_{\beta,\beta}(\lambda)}M_H(\|x\|)+\big[\mathbf{E}_{\beta,2}(|\lambda|)+\frac{\mathbf{E}_{\beta,1}(|\lambda|)}{|\lambda|\mathbf{E}_{\beta,\beta}(\lambda)}\mathbf{E}_{\beta,1}(|\lambda|)\big]\|\phi\|_1M_G(\|x\|)\\&\quad+\big[\frac{\mathbf{E}_{\beta,1}(|\lambda|)}{|\lambda|\mathbf{E}_{\beta,\beta}(\lambda)}m|\lambda|\mathbf{E}_{\beta,\beta}(|\lambda|)+m\mathbf{E}_{\beta,1}(|\lambda|)\big]M_I(\|x\|)\\&\quad+[m\mathbf{E}_{\beta,1}(|\lambda|)+m\mathbf{E}_{\beta,2}(|\lambda|)]M_J(\|x\|)+\Big[\frac{\mathbf{E}_{\beta,1}(|\lambda|)}{|\lambda|\mathbf{E}_{\beta,\beta}(\lambda)}\mathbf{E}_{\beta,\beta}(|\lambda|)\\&\quad\times\mathbf{B}(\beta+l-1,k+1)+\mathbf{E}_{\beta,\beta}(|\lambda|)\mathbf{B}(\beta+l,k+1)\Big]M_f(\|x\|).\end{aligned}$$

From (4.4), we choose $\Omega = \{x \in LP_mC(1,e] : \|x\| \le r_0\}$. For $x \in \partial\Omega$, we obtain $x \ne \lambda(Jx)$ for any $\lambda \in [0,1]$. In fact, if there exists $x \in \partial\Omega$ such that $x = \lambda(Jx)$ for some $\lambda \in [0,1]$. Then

$$\begin{aligned} r_0 &= \|x\| = \lambda \|Jx\| \le \|Jx\| \\ &\le \frac{\mathbf{E}_{\beta,1}(|\lambda|)\|\psi\|_1}{\lambda \mathbf{E}_{\beta,\beta}(\lambda)} M_H(r_0) + [\mathbf{E}_{\beta,2}(|\lambda|) + \frac{\mathbf{E}_{\beta,1}(|\lambda|)}{|\lambda| \mathbf{E}_{\beta,\beta}(\lambda)} \mathbf{E}_{\beta,1}(|\lambda|)]\|\phi\|_1 M_G(r_0) \\ &\quad + \big[\frac{\mathbf{E}_{\beta,1}(|\lambda|)}{|\lambda| \mathbf{E}_{\beta,\beta}(\lambda)} m|\lambda| \mathbf{E}_{\beta,\beta}(|\lambda|) + m\mathbf{E}_{\beta,1}(|\lambda|)\big] M_I(r_0) + [m\mathbf{E}_{\beta,1}(|\lambda|) \\ &\quad + m\mathbf{E}_{\beta,2}(|\lambda|)] M_J(r_0) + \big[\frac{\mathbf{E}_{\beta,1}(|\lambda|)}{|\lambda| \mathbf{E}_{\beta,\beta}(\lambda)} \mathbf{E}_{\beta,\beta}(|\lambda|) \mathbf{B}(\beta + l - 1, k+1) \\ &\quad + \mathbf{E}_{\beta,\beta}(|\lambda|) \mathbf{B}(\beta + l, k+1)\big] M_f(r_0) < r_0, \end{aligned}$$

which is a contradiction.

As a consequence of Schaefer's fixed point theorem, we deduce that J has a fixed point which is a solution of (1.10). The proof of is complete. □

5. Applications of main results

In this section, we present firstly applications of the results obtained in Section 3.2. We also point out some mistakes occurred in cited papers. Finally we establish sufficient conditions for the existence of solutions of three classes of boundary value problems of impulsive fractional differential equations.

Applying the results obtained in Section 3.2, choose $\lambda = 0$, by Theorems 3.11–3.14, we obtain the exact piecewise continuous solutions of the following fractional differential equations (see Corollaries 5.1, 5.3, 5.5 and 5.6 below)

$$ {}^{C}D_{0^+}^{\alpha} x(t) = F(t), \quad t \in (t_i, t_{i+1}],\ i \in \mathbb{N}[0,m], \tag{5.1}$$

$$ {}^{RL}D_{0^+}^{\alpha} x(t) = F(t), \quad t \in (t_i, t_{i+1}],\ i \in \mathbb{N}[0,m], \tag{5.2}$$

$$ {}^{RLH}D_{0^+}^{\alpha} x(t) = G(t), \quad t \in (s_i, s_{i+1}],\ i \in \mathbb{N}[0,m], \tag{5.3}$$

$$ {}^{CH}D_{0^+}^{\alpha} x(t) = G(t), \quad t \in (s_i, s_{i+1}],\ i \in \mathbb{N}[0,m], \tag{5.4}$$

where $n-1 \le \alpha < n$, $0 = t_0 < t_1 < \dots < t_m < t_{m+1} = 1$ in (5.1) and (5.2) and $1 = t_0 < t_1 < \dots < t_m < t_{m+1} = e$ in (5.3) and (5.4).

Corollary 5.1. *Suppose that F is continuous on $(0,1)$ and there exist constants $k > -\alpha + n - 1$ and $l \in (-\alpha, -\alpha - k, 0]$ such that $|F(t)| \le t^k(1-t)^l$ for all $t \in (0,1)$. Then x is a piecewise solution of (5.1) if and only if there exist constants $c_{iv}(i \in N[0,m], v \in \mathbb{N}[0,n-1]) \in \mathbb{R}$ such that*

$$x(t) = \sum_{\sigma=0}^{i} \sum_{v=0}^{n-1} \frac{c_{\sigma v}}{\Gamma(v+1)} (t - t_\sigma)^v + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} F(s) ds, \tag{5.5}$$

for $t \in (t_i, t_{i+1}]$ and $i \in \mathbb{N}[0,m]$.

The above corollary follows from Theorem 3.11, with $\lambda = 0$.

Remark 5.2. We note that $(t - t_\sigma)^v = \sum_{\tau=0}^{v} (-1)^\tau \binom{v}{\tau} t_\sigma^\tau t^{v-\tau}$. By Corollary 5.1, we can transform (5.5) into

$$x(t) = \sum_{v=0}^{n-1} \frac{d_{iv}}{\Gamma(v+1)} t^v + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} F(s) ds, \tag{5.6}$$

for $t \in (t_i, t_{i+1}]$ and $i \in \mathbb{N}[0, m]$. Here d_{iv} are constants. Certainly (5.6) can be transformed into

$$x(t) = \sum_{v=0}^{n-1} \frac{e_{iv}}{\Gamma(v+1)}(t - t_i)^v + \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} F(s) ds, \tag{5.7}$$

for $t \in (t_i, t_{i+1}]$ and $i \in \mathbb{N}[0, m]$.

In [71], the authors actually used (5.6) to construct the nonlinear operator for getting solutions of (1.5), but they did not prove the equivalence between the boundary value problem and the integral equation, see [71, Lemma 2.6].

Rehman and Eloe [91] proved that (5.1) is equivalent to (5.6) under the assumption that $F \in PC[0, 1]$. So Corollary generalizes [91, Lemma 2.4]. Furthermore, it is difficult to convert a BVP for impulsive fractional differential equation to a integral equation by using (5.6), while it is easy to do this job by using (5.5). See the examples in Section 6.

Corollary 5.3. *Suppose that F is continuous on $(0, 1)$ and there exist constants $k > -1$ and $l \in (-\alpha, -n - k, 0]$ such that $|F(t)| \le t^k (1 - t)^l$ for all $t \in (0, 1)$. Then x is a solution of (5.2) if and only if there exist constants $c_{\sigma v}(\sigma \in \mathbb{N}[0, m], v \in \mathbb{N}[1, n]) \in \mathbb{R}$ such that*

$$x(t) = \sum_{\sigma=0}^{i} \sum_{v=1}^{n} \frac{c_{\sigma v}}{\Gamma(\alpha - v + 1)}(t - t_\sigma)^{\alpha-v} + \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} F(s) ds, \tag{5.8}$$

for $t \in (t_i, t_{i+1}]$, $i \in \mathbb{N}[0, m]$.

Remark 5.4. When α is not an integer, (5.8) is not equivalent to the equation

$$x(t) = \sum_{v=1}^{n} \frac{c_{iv}}{\Gamma(\alpha - v + 1)}(t - t_i)^{\alpha-v} + \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} F(s) ds,$$

for $t \in (t_i, t_{i+1}]$, $i \in \mathbb{N}[0, m]$.

Corollary 5.5. *Suppose that G is continuous on $(1, e)$ and there exist constants $k > -1$ and $l \in (-\alpha, -n - k, 0]$ such that $|G(t)| \le (\log t)^k (1 - \log t)^l$ for all $t \in (1, e)$. Then x is a solution of (5.3) if and only if there exist constants $c_{jv} \in \mathbb{R}(j \in \mathbb{N}[0, m], v \in \mathbb{N}[1, n])$ such that*

$$x(t) = \sum_{j=0}^{i} \sum_{v=1}^{n} \frac{c_{jv}}{\Gamma(\alpha - v + 1)}(\log \frac{t}{t_j})^{\alpha-v} + \frac{1}{\Gamma(\alpha)} \int_1^t (\log \frac{t}{s})^{\alpha-1} G(s) \frac{ds}{s}, \tag{5.9}$$

for $t \in (t_i, t_{i+1}]$, $i \in \mathbb{N}[0, m]$.

In [57], the same result was proved under the assumption that $G \in L(1, e)$. So Corollary 5.5 improve the results in [57].

Corollary 5.6. *Suppose that G is continuous on $(1, e)$ and there exist constants $k > -\alpha + n - 1$ and $l \in (-\alpha, -\alpha + k, 0]$ such that $|G(t)| \le (\log t)^k (1 - \log t)^l$ for all $t \in (1, e)$. Then x is a piecewise solution of (5.4) if and only if there exist constants $c_{jv} \in \mathbb{R}(j \in \mathbb{N}[0, m], i \in \mathbb{N}[0, n - 1])$ such that*

$$x(t) = \sum_{\rho=0}^{j} \sum_{v=0}^{n-1} \frac{c_{\rho v}}{\Gamma(v + 1)}(\log \frac{t}{t_\rho})^v + \frac{1}{\Gamma(\alpha)} \int_1^t (\log \frac{t}{s})^{\alpha-1} G(s) \frac{ds}{s}, \tag{5.10}$$

for $t \in (t_j, t_{j+1}]$, $j \in \mathbb{N}[0, m]$.

We now construct a Banach space X and prove the compact criterion for subsets of X. X will be used in next three subsections. Choose the set of functions

$$X=\Big\{x:x|_{(t_i,t_{i+1}]},\ D_{0^+}^{\beta}x|_{(t_i,t_{i+1}]} \text{ are continuous, } i\in\mathbb{N}[0,m],\ \lim_{t\to t_i^+}(t-t_i)^{2-\alpha}x(t),\\ \lim_{t\to t_i^+}(t-t_i)^{2+\beta-\alpha}D_{0^+}^{\beta}x(t) \text{ exist, } i\in\mathbb{N}[0,m]\Big\}$$

For $x\in X$ define the norm

$$\|x\|=\max\big\{\sup_{t\in(t_i,t_{i+1}]}(t-t_i)^{2-\alpha}|x(t)|,\ \sup_{t\in(t_i,t_{i+1}]}(t-t_i)^{2+\beta-\alpha}|D_{0^+}^{\beta}x(t)|:i\in\mathbb{N}[0,m]\big\}.$$

Lemma 5.7. *X is a Banach space with the norm defined above.*

Proof. It is easy to see that X is a normed linear space. Let $\{x_u\}$ be a Cauchy sequence in X. Then $\|x_u-x_v\|\to 0$, $u,v\to+\infty$. It follows that

$$\lim_{t\to t_i^+}(t-t_i)^{2-\alpha}x_u(t),\quad \lim_{t\to t_i^+}(t-t_i)^{2+\beta-\alpha}D_{0^+}^{\beta}x_u(t)\quad \text{exist for } i\in\mathbb{N}[0,m]. \tag{5.11}$$

Let

$$\overline{x}_{u,i}(t)=\begin{cases}\lim_{t\to t_i^+}(t-t_i)^{2-\alpha}x_u(t), & t=t_i,\\ (t-t_i)^{2-\alpha}x_u(t), & t\in(t_i,t_{i+1}],\end{cases}$$

$$\overline{Dx}_{u,i}(t)=\begin{cases}\lim_{t\to t_i^+}(t-t_i)^{2+\beta-\alpha}D_{0^+}^{\beta}x_u(t), & t=t_i,\\ (t-t_i)^{2+\beta\alpha}D_{0^+}^{\beta}x_u(t), & t\in(t_i,t_{i+1}].\end{cases}$$

Then both $\overline{x}_{u,i}$ and $\overline{Dx}_{u,i}$ are continuous on $[t_i,t_{i+1}]$. Hence there exist two continuous function $x_{0,i},y_{0,i}$ defined on $[t_i,t_{i+1}]$ such that $\max_{t\in[t_i,t_{i+1}]}|\overline{x}_{u,i}(t)-x_{0,i}(t)|\to 0$ as $u\to+\infty$ and $\max_{t\in[t_i,t_{i+1}]}|\overline{Dx}_{u,i}(t)-y_{0,i}(t)|\to 0$ as $u\to+\infty$.

Denote $x_0(t)=(t-t_i)^{\alpha-2}x_{0,i}(t)$ and $y_0(t)=(t-t_i)^{\alpha-\beta-2}y_{0,i}(t)$ for $t\in(t_i,t_{i+1}]$. One sees that x_0 and y_0 defined on $(0,1]$ such that

$$\lim_{t\to t_i^+}(t-t_i)^{2-\alpha}x_0(t),\quad \lim_{t\to t_i^+}(t-t_i)^{2+\beta-\alpha}D_{0^+}^{\beta}y_0(t)\quad \text{exist for } i\in\mathbb{N}[0,m],$$

$$\lim_{u\to+\infty}x_u(t)=x_0(t),\quad \lim_{u\to+\infty}D_{0^+}^{\beta}x_u(t)=y_0(t),\quad t\in(0,1],$$

$$\lim_{u\to+\infty}\sup_{t\to(t_i,t_{i+1}]}(t-t_i)^{2-\alpha}|x_u(t)-x_0(t)|=0,\quad i\in\mathbb{N}[0,m],$$

$$\lim_{u\to+\infty}\sup_{t\in(t_i,t_{i+1}]}(t-t_i)^{2+\beta-\alpha}|D_{0^+}^{\beta}x_u(t)-y_0(t)|=0,\quad i\in\mathbb{N}[0,m].$$

Now, denote $D_{0^+}^{\beta}x_u(t)=y_u(t)$ for $t\in(0,1]$. Then by Theorem 3.12 (with $\lambda=0$, $F(t)=y_n(t)$, $n=1$) there exist numbers $c_{u,\sigma}(\sigma\in\mathbb{N}[0,m])$ such that $x_u(t)=I_{0^+}^{\beta}y_u(t)+\sum_{\sigma=0}^{i}c_{u,\sigma}(t-t_\sigma)^{\beta-1}$ for $t\in(t_i,t_{i+1}]$, $i\in\mathbb{N}[0,m]$. So for $t\in(t_i,t_{i+1}]$, we have

$$\begin{aligned}
&|x_u(t)-\sum_{\sigma=0}^{i}c_{u,\sigma}(t-t_\sigma)^{\beta-1}-I_{0^+}^{\beta}(t-t_\sigma)^{\alpha-\beta-2}y_0(t)|\\
&=|I_{0^+}^{\beta}y_u(t)-I_{0^+}^{\beta}(t-t_\sigma)^{\alpha-\beta-2}y_0(t)|\\
&=|I_{0^+}^{\beta}D_{0^+}^{\beta}x_u(t)-I_{0^+}^{\beta}(t-t_\sigma)^{\alpha-\beta-2}y_0(t)|\\
&=|\int_0^t\frac{(t-s)^{\beta-1}}{\Gamma(\beta)}[D_{0^+}^{\beta}x_u(s)-(s-t_\sigma)^{\alpha-\beta-2}y_0(s)]ds|
\end{aligned}$$

$$
\begin{aligned}
&\le \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}|D_{0^+}^\beta x_u(s)-(s-t_\sigma)^{\alpha-\beta-2}y_0(s)|ds\\
&=\sum_{\sigma=0}^{i-1}\int_{t_\sigma}^{t_{\sigma+1}} \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}|D_{0^+}^\beta x_u(s)-y_0(s)|ds\\
&\quad+\int_{t_i}^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}|D_{0^+}^\beta x_u(s)-(s-t_\sigma)^{\alpha-\beta-2}y_0(s)|ds\\
&=\sum_{\sigma=0}^{i-1}\int_{t_\sigma}^{t_{\sigma+1}} \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}|D_{0^+}^\beta x_u(s)-(s-t_\sigma)^{\alpha-\beta-2}y_{0,\sigma}(s)|ds\\
&\quad+\int_{t_i}^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}|D_{0^+}^\beta x_u(s)-(s-t_i)^{\alpha-\beta-2}y_{0,i}(s)|ds\\
&=\sum_{\sigma=0}^{i-1}\int_{t_\sigma}^{t_{\sigma+1}} \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}(s-t_\sigma)^{\alpha-\beta-2}|(s-t_\sigma)^{2+\beta-\alpha}D_{0^+}^\beta x_u(s)-y_{0,\sigma}(s)|ds\\
&\quad+\int_{t_i}^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}(s-t_i)^{\alpha-\beta-2}|(s-t_\sigma)^{2+\beta-\alpha}D_{0^+}^\beta x_u(s)-y_{0,i}(s)|ds\\
&\le\sum_{\sigma=0}^{i-1}\int_{t_\sigma}^{t_{\sigma+1}} \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}(s-t_\sigma)^{\alpha-\beta-2}ds\sup_{t\in(t_\sigma,t_{\sigma+1}]}|(t-t_\sigma)^{2+\beta-\alpha}D_{0^+}^\beta x_u(t)-y_{0,\sigma}(t)|\\
&\quad+\int_{t_i}^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}(s-t_i)^{\alpha-\beta-2}ds\sup_{t\in(t_i,t_{i+1}]}|(t-t_\sigma)^{2+\beta-\alpha}D_{0^+}^\beta x_u(t)-y_{0,i}(t)|\\
&=\sum_{\sigma=0}^{i-1}(t-t_\sigma)^{\alpha-2}\int_0^{\frac{t_{\sigma+1}-t_\sigma}{t-t_\sigma}} \frac{(1-w)^{\beta-1}}{\Gamma(\beta)}w^{\alpha-\beta-2}dw\\
&\quad\times\sup_{t\in(t_\sigma,t_{\sigma+1}]}|(t-t_\sigma)^{2+\beta-\alpha}D_{0^+}^\beta x_u(t)-y_{0,\sigma}(t)|\\
&\quad+\int_0^1 \frac{(1-w)^{\beta-1}}{\Gamma(\beta)}w^{\alpha-\beta-2}dw\sup_{t\in(t_i,t_{i+1}]}|(t-t_\sigma)^{2+\beta-\alpha}D_{0^+}^\beta x_u(t)-y_{0,i}(t)|\\
&\le\sum_{\sigma=0}^{i-1}(t-t_\sigma)^{\alpha-2}\int_0^1 \frac{(1-w)^{\beta-1}}{\Gamma(\beta)}w^{\alpha-\beta-2}dw\\
&\quad\times\sup_{t\in(t_\sigma,t_{\sigma+1}]}|(t-t_\sigma)^{2+\beta-\alpha}D_{0^+}^\beta x_u(t)-y_{0,\sigma}(t)|\\
&\quad+\int_0^1 \frac{(1-w)^{\beta-1}}{\Gamma(\beta)}w^{\alpha-\beta-2}dw\sup_{t\in(t_i,t_{i+1}]}|(t-t_\sigma)^{2+\beta-\alpha}D_{0^+}^\beta x_u(t)-y_{0,i}(t)|\\
&\to 0\quad\text{as } u\to+\infty.
\end{aligned}
$$

It follows that

$$
\lim_{u\to+\infty}\big[x_u(t)-\sum_{\sigma=0}^{i}c_{u,\sigma}(t-t_\sigma)^{\beta-1}\big]=I_{0^+}^\beta(t-t_\sigma)^{\alpha-\beta-2}y_0(t).
$$

We have

$$
x_0(t)-\sum_{\sigma=0}^{i}c_{0,\sigma}(t-t_\sigma)^{\beta-1}=I_{0^+}^\beta(t-t_\sigma)^{\alpha-\beta-2}y_0(t),
$$

for $t \in (t_i, t_{i+1}]$, $i \in \mathbb{N}[0,m]$. Then for $t \in (t_i, t_{i+1}]$, we have

$$
\begin{aligned}
&(t-t_\sigma)^{\alpha-\beta-2}y_0(t)\\
&= D_{0^+}^{\beta} I_{0^+}^{\beta}(t-t_\sigma)^{\alpha-\beta-2}y_0(t)\\
&= \frac{1}{\Gamma(1-\beta)}\Big[\sum_{j=0}^{i-1}\int_{t_j}^{t_{j+1}}(t-s)^{-\beta}\Big(x_0(s)-\sum_{\sigma=0}^{j}c_{0,\sigma}(s-t_\sigma)^{\beta-1}\Big)ds\Big]'\\
&\quad + \frac{1}{\Gamma(1-\beta)}\Big[\int_{t_i}^{t}(t-s)^{-\beta}\Big(x_0(s)-\sum_{\sigma=0}^{i}c_{0,\sigma}(s-t_\sigma)^{\beta-1}\Big)ds\Big]'\\
&= \frac{1}{\Gamma(\beta)}\Big[\int_0^t (t-s)^{-\beta}x_0(s)ds\Big]'\\
&\quad - \frac{1}{\Gamma(1-\beta)}\Big[\sum_{j=0}^{i-1}\int_{t_j}^{t_{j+1}}(t-s)^{-\beta}\sum_{\sigma=0}^{j}c_{0,\sigma}(s-t_\sigma)^{\beta-1}ds\Big]'\\
&\quad - \frac{1}{\Gamma(1-\beta)}\Big[\int_{t_i}^{t}(t-s)^{-\beta}\sum_{\sigma=0}^{i}c_{0,\sigma}(s-t_\sigma)^{\beta-1}ds\Big]'\\
&= D_{0^+}^{\beta}x_0(t) - \frac{1}{\Gamma(1-\beta)}\Big[\sum_{j=0}^{i-1}\sum_{\sigma=0}^{j}c_{0,\sigma}\int_{\frac{t_j-t_\sigma}{t-t_\sigma}}^{\frac{t_{j+1}-t_\sigma}{t-t_\sigma}}(1-w)^{-\beta}w^{\beta-1}dw\Big]'\\
&\quad - \frac{1}{\Gamma(1-\beta)}\Big[\sum_{\sigma=0}^{i}c_{0,\sigma}\int_{\frac{t_i-t_\sigma}{t-t_\sigma}}^{1}(1-w)^{-\beta}w^{\beta-1}dw\Big]'\\
&= D_{0^+}^{\beta}x_0(t) - \frac{1}{\Gamma(1-\beta)}\Big[\sum_{\sigma=0}^{i-1}c_{0,\sigma}\sum_{j=\sigma}^{i-1}\int_{\frac{t_j-t_\sigma}{t-t_\sigma}}^{\frac{t_{j+1}-t_\sigma}{t-t_\sigma}}(1-w)^{-\beta}w^{\beta-1}dw\Big]'\\
&\quad - \frac{1}{\Gamma(1-\beta)}\Big[\sum_{\sigma=0}^{i}c_{0,\sigma}\int_{\frac{t_i-t_\sigma}{t-t_\sigma}}^{1}(1-w)^{-\beta}w^{\beta-1}dw\Big]'\\
&= D_{0^+}^{\beta}x_0(t) - \frac{1}{\Gamma(1-\beta)}\Big[\sum_{\sigma=0}^{i}c_{0,\sigma}\int_0^1(1-w)^{-\beta}w^{\beta-1}dw\Big]'\\
&= D_{0^+}^{\beta}x_0(t).
\end{aligned}
$$

Then $(t-t_\sigma)^{\alpha-\beta-2}y_0(t) = D_{0^+}^{\beta}x_0(t)$ for $t \in (t_i, t_{i+1}]$, $i \in \mathbb{N}[0,m]$. From above discussion, X is a Banach space. □

Lemma 5.8. *Let M be a subset of X. Then M is relatively compact if and only if the following conditions are satisfied:*

(i) *both $\{t \to (t-t_i)^{2-\alpha}x(t) : x \in M\}$ and $\{t \to (t-t_i)^{2+\beta-\alpha}D_{0^+}^{\beta}x(t) : x \in M\}$ are uniformly bounded,*

(ii) *both $\{t \to (t-t_i)^{2-\alpha}x(t) : x \in M\}$ and $\{t \to (t-t_i)^{2+\beta-\alpha}D_{0^+}^{\beta}x(t) : x \in M\}$ are equi-continuous in $(t_i, t_{i+1}](i \in N[0,m])$.*

Proof. " $\Leftarrow$ ". From lemma 5.7, we know X is a Banach space. To prove that the subset M is relatively compact in X, we only need to show M is totally bounded in X, that is for all $\epsilon > 0$, M has a finite ϵ-net.

For any given $\epsilon > 0$, by (i) and (ii), there exist a constants $A \geq 0, \delta > 0$ and a t_{s_0}, such that

$$\begin{gathered}
|(u_1 - t_i)^{2-\alpha} x(u_1) - (u_2 - t_i)^{2-\alpha} x(u_2)| \leq \frac{\epsilon}{3}, \\
\text{for } t_i < u_1, u_2 \leq t_{i+1},\ |u_1 - u_2| < \delta,\ x \in M, \\
|(u_1 - t_i)^{2+\beta-\alpha} D_{0^+}^{\beta} x(u_1) - (u_2 - t_i)^{2+\beta-\alpha} D_{0^+}^{\beta} x(u_2)| < \frac{\epsilon}{3}, \\
\text{for } t_i < u_1, u_2 \leq t_{i+1},\ |u_1 - u_2| < \delta,\ x \in M, \\
(t - t_i)^{2-\alpha} |x(t)|, (t - t_i)^{2+\beta-\alpha} |D_{0^+}^{\beta} x(t)| \leq A, \\
\text{for } t \in (t_i, t_{i+1}],\ i \in N[0, m],\ x \in M.
\end{gathered}$$

Define

$$\begin{aligned}
X|_{(t_i,t_{i+1}]} = \Big\{ & x : x\big|_{(t_i,t_{i+1}]}, D_{0^+}^{\beta} x \text{ is continuous on } (t_i, t_{i+1}] \text{ and} \\
& \lim_{t \to t_i^+} (t - t_i)^{2-\alpha} x(t),\ \lim_{t \to t_i^+} (t - t_i)^{2+\beta-\alpha} D_{0^+}^{\beta} x(t) \Big\}.
\end{aligned}$$

For $x \in X|_{(t_i,t_{i+1}]}$, define

$$\|x\|_i = \max \Big\{ \sup_{t \in (t_i,t_{i+1}]} (t - t_i)^{2-\alpha} |x(t)|,\ \sup_{t \in (t_i,t_{i+1}]} (t - t_i)^{2+\beta-\alpha} |D_{0^+}^{\beta} x(t)| \Big\}.$$

Similarly to Lemma 5.7, we can prove that $X_{(t_i,t_{i+1}]}$ is a Banach space. Let $M|_{(t_i,t_{i+1}]} = \{t \to x(t), t \in (t_i, t_{i+1}] : x \in M\}$. Then $M|_{(t_i,t_{i+1}]}$ is a subset of $X|_{(t_i,t_{i+1}]}$. By **(i)** and **(ii)**, and Ascoli-Arzela theorem, we can know that $M|_{(t_i,t_{i+1}]}$ is relatively compact. Thus, there exist $x_{i1}, x_{i2}, \dots, x_{ik_i} \in M|_{(t_i,t_{i+1}]}$ such that $\{U_{x_{i1}}, U_{x_{i2}}, \dots, U_{x_{ik_i}}\}$ is a finite ϵ-net of $M|_{(t_i,t_{i+1}]}$.

Denote $x_{l_0 l_1 l_2 \dots l_m}(t) = x_{il_i}(t)$, $t \in (t_i, t_{i+1}]$, $i \in \mathbb{N}[0, m]$, $l_i \in \mathbb{N}[1, k_{l_i}]$. For any $x \in M$, we have $x|_{(t_i,t_{i+1}]} \in M|_{(t_i,t_{i+1}]}$, so there exists $l_i \in \mathbb{N}[1, k_i]$ such that

$$\begin{aligned}
\|x|_{(t_i,t_{i+1}]} - x_{il_i}\|_i = \max \Big\{ & \sup_{t \in (t_i,t_{i+1}]} (t - t_i)^{2-\alpha} |x(t) - x_{il_i}(t)|, \\
& \sup_{t \in (t_i,t_{i+1}]} (t - t_i)^{2+\beta-\alpha} |D_{0^+}^{\beta} x(t) - D_{0^+}^{\beta} x_{il_i}(t)| \} \leq \epsilon.
\end{aligned}$$

Then, for $x \in M$,

$$\begin{aligned}
& \|x - x_{l_0 l_1 l_2 \dots l_m}\|_X \\
& = \max \Big\{ \sup_{t \in (t_i,t_{i+1}]} (t - t_i)^{2-\alpha} |x(t) - x_{l_0 l_1 l_2 \dots l_m}(t)|, \\
& \sup_{t \in (t_i,t_{i+1}]} (t - t_i)^{2+\beta-\alpha} |D_{0^+}^{\beta} x(t) - D_{0^+}^{\beta} x_{l_0 l_1 l_2 \dots l_m}(t)| : i \in \mathbb{N}[0, m] \Big\} \\
& \leq \max \Big\{ \sup_{t \in (t_i,t_{i+1}]} (t - t_i)^{2-\alpha} |x(t) - x_{il_i} t)|, \\
& \sup_{t \in (t_i,t_{i+1}]} (t - t_i)^{2+\beta-\alpha} |D_{0^+}^{\beta} x(t) - D_{0^+}^{\beta} x_{il_i}(t)| : i \in \mathbb{N}[0, m] \Big\} \\
& < \epsilon.
\end{aligned}$$

So, for any $\epsilon > 0$, M has a finite ϵ-net $\{U_{x_{l_0 l_1 l_2 \dots l_m}} : l_i \in N[1, k_i], i \in \mathbb{N}[0, m]\}$; that is, M is totally bounded in X. Hence, M is relatively compact in X.

$\Rightarrow$ Assume that M is relatively compact, then for any $\epsilon > 0$, there exists a finite ϵ-net of M. Let the finite ϵ-net be $\{U_{x_1}, U_{x_2}, \dots, U_{x_k}\}$ with $x_i \subset M$. Then for any $x \in M$, there exists U_{x_i} such that $x \in U_{x_i}$ and

$$\|x\| \le \|x - x_i\| + \|x_i\| \le \epsilon + \max\{\|x_i\| : i \in \mathbb{N}[1,k]\}.$$

It follows that M is uniformly bounded. Then (i) holds.

Furthermore, let $x \in M$, then there exists x_i such that $\|x - x_i\| < \epsilon$. Since $\lim_{t\to t_j^+}(t-t_i)^{2-\alpha}x_i(t)$ exists and x_i is continuous on $(t_j, t_{j+1}]$, then there exists $\delta > 0$ such that $u_1, u_2 \in (t_j, t_{j+1}]$ with $|u_1 - u_2| < \delta$ implies that $|(u_1-t_j)^{2-\alpha}x_i(u_1) - (u_2-t_j)^{2-\alpha}x_i(u_2)| < \epsilon$. Then we have for $u_1, u_2 \ge (t_j, t_{j+1}]$ with $|u_1 - u_2| < \delta$

$$\begin{aligned}
&|(u_1-t_j)^{2-\alpha}x(u_1) - (u_2-t_j)^{2-\alpha}x(u_2)| \\
&\le |(u_1-t_j)^{2-\alpha}x(u_1) - (u_1-t_j)^{2-\alpha}x_i(u_1)| + \big|(u_1-t_j)^{2-\alpha}x_i(u_1) \\
&\quad - (u_2-t_j)^{2-\alpha}x_i(u_2)\big| + |(u_2-t_j)^{2-\alpha}x_i(u_2) - (u_2-t_j)^{2-\alpha}x(u_2)| \\
&< 3\epsilon, \quad x \in M.
\end{aligned}$$

Similarly we have $t \to (t-t_j)^{2+\beta-\alpha}D_{0^+}^{\beta}x(t)$ is equi-continuous on $(t_j, t_{j+1}]$. Thus (iii) is valid. Similarly we can prove that (ii) holds. Consequently, the claim is proved. $\square$

5.1. **Impulsive multi-point boundary value problems.** In [137], the authors studied the impulsive boundary value problem

$$\begin{gathered}
{}^{RL}D_{0^+}^{\alpha}u(t) = f(t, v(t), {}^{RL}D_{0^+}^{p}v(t)), \\
{}^{RL}D_{0^+}^{\beta}v(t) = g(t, u(t), {}^{RL}D_{0^+}^{q}u(t)), \quad t \in (0,1), \\
\Delta u(t_i) = A_i(v(t_i), {}^{RL}D_{0^+}^{p}v(t_i))), \\
\Delta {}^{RL}D_{0^+}^{q}u(t_i)) = B_i(v(t_i), {}^{RL}D_{0^+}^{p}v(t_i))), \quad i \in \mathbb{N}[1,k], \\
\Delta v(t_i) = C_i(u(t_i), {}^{RL}D_{0^+}^{q}u(t_i))), \quad \Delta {}^{RL}D_{0^+}^{p}v(t_i)) = D_i(u(t_i), \\
{}^{RL}D_{0^+}^{q}u(t_i))), \quad i \in \mathbb{N}[1,k], \\
{}^{RL}D_{0^+}^{\alpha-1}u(0) = \sum_{i=1}^{m} a_i {}^{RL}D_{0^+}^{\alpha-1}u(\xi_i), \quad u(1) = \sum_{i=1}^{m} b_i \eta^{2-\alpha}u(\eta_i), \\
{}^{RL}D_{0^+}^{\beta-1}v(0) = \sum_{i=1}^{m} c_i {}^{RL}D_{0^+}^{\beta-1}v(\zeta_i), \quad v(1) = \sum_{i=1}^{m} d_i \eta^{2-\beta}u(\theta_i),
\end{gathered} \tag{5.12}$$

where

- (i) $\alpha, \beta \in (1,2)$, $p \in (0, \beta-1]$, $q \in (0, \alpha-1]$, $\{t_i : i \in \mathbb{N}[1,k]\}$, $\{\xi_i : i \in \mathbb{N}[1,m]\}$, $\{\eta_i : i \in \mathbb{N}[1,m]\}$, $\{\zeta_i : i \in \mathbb{N}[1,m]\}$, $\{\theta_i : i \in \mathbb{N}[1,m]\} \subset (0,1)$ are increasing sequences,
- (ii) $f, g : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ are Carathéodory functions,
- (iii) $A_i, B_i, C_i, D_i : R^2 \to \mathbb{R}$ are continuous functions, k, m are positive integers, a_i, b_i, c_i, d_i are fixed constants, $i \in \mathbb{N}[1,m]$.

Zhang et al. claimed that the following assumptions

$$\sum_{i=1}^{m} a_i = \sum_{i=1}^{m} b_i = \sum_{i=1}^{m} c_i = \sum_{i=1}^{m} d_i = 1, \quad \sum_{i=1}^{m} b_i\eta_i = \sum_{i=1}^{m} d_i\eta_i = 1 \tag{5.13}$$

make (5.12) be a resonant problem and tried to establish existence result for solutions of BVP (5.1) by using the coincidence degree theory due to Mawhin [79]. However, some mistakes occurred in [137, (2.10)–(2.14)]. There, it means that

$$ {}^{RL}D_{0^+}^{\alpha}u(t)=z_1(t),\quad \Delta u(t_i)=\delta_i,\quad \Delta {}^{RL}D_{0^+}^{q}u(t_i)=\omega_i,\quad i\in\mathbb{N}[1,m] \tag{5.14}$$

imply that

$$\begin{aligned} u(t)&=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}z_1(s)ds+\big(h_1+\frac{\Gamma(\alpha-q)}{\Gamma(\alpha)}\sum_{t_i<t}\omega_it_i^{q+1-\alpha}\big)t^{\alpha-1}\\ &\quad+\Big(h_2+\sum_{t_i<t}\delta_it_i^{2-\alpha}-\frac{\Gamma(\alpha-q)}{\Gamma(\alpha)}\sum_{t_i<t}\omega_it_i^{q+2-\alpha}\Big)t^{\alpha-2},\quad h_1,h_2\in\mathbb{R}. \end{aligned}$$

This claim is false because of the following two items:

(i) In fact, when $t\in(0,t_1]$, this claim is correct. When $t\in(t_i,t_{i+1}]$, $i\geq 1$, we have by Definition 2.2

$$\begin{aligned} &{}^{RL}D_{0^+}^{\alpha}u(t)\\ &=\frac{1}{\Gamma(2-\alpha)}\Big(\int_0^t(t-s)^{1-\alpha}u(s)ds\Big)''\\ &=\frac{1}{\Gamma(2-\alpha)}\Big(\sum_{j=0}^{i-1}\int_{t_j}^{t_{j+1}}(t-s)^{1-\alpha}u(s)ds\Big)''+\frac{1}{\Gamma(2-\alpha)}\Big(\int_{t_i}^{t}(t-s)^{1-\alpha}u(s)ds\Big)''\\ &=\frac{1}{\Gamma(2-\alpha)}\Big[\sum_{j=0}^{i-1}\int_{t_j}^{t_{j+1}}(t-s)^{1-\alpha}\Big(\frac{1}{\Gamma(\alpha)}\int_0^s(s-u)^{\alpha-1}z_1(u)du\\ &\quad+(h_1+\frac{\Gamma(\alpha-q)}{\Gamma(\alpha)}\sum_{\tau=1}^{j}\omega_\tau t_\tau^{q+1-\alpha})s^{\alpha-1}\\ &\quad+(h_2+\sum_{\tau=1}^{j}\delta_\tau t_\tau^{2-\alpha}-\frac{\Gamma(\alpha-q)}{\Gamma(\alpha)}\sum_{\tau=1}^{j}\omega_\tau t_\tau^{q+2-\alpha})s^{\alpha-2}\Big)ds\Big]''\\ &\quad+\frac{1}{\Gamma(2-\alpha)}\Big[\int_{t_i}^{t}(t-s)^{1-\alpha}\Big(\frac{1}{\Gamma(\alpha)}\int_0^s(s-u)^{\alpha-1}z_1(u)du\\ &\quad+\Big(h_1+\frac{\Gamma(\alpha-q)}{\Gamma(\alpha)}\sum_{\tau=1}^{i}\omega_\tau t_\tau^{q+1-\alpha}\Big)s^{\alpha-1}\\ &\quad+\Big(h_2+\sum_{\tau=1}^{j}\delta_\tau t_\tau^{2-\alpha}-\frac{\Gamma(\alpha-q)}{\Gamma(\alpha)}\sum_{\tau=1}^{i}\omega_\tau t_\tau^{q+2-\alpha}\Big)s^{\alpha-2}\Big)ds\Big]''\\ &=\frac{1}{\Gamma(2-\alpha)}\Big[\sum_{j=0}^{i-1}\int_{t_j}^{t_{j+1}}(t-s)^{1-\alpha}\Big(\frac{1}{\Gamma(\alpha)}\int_0^s(s-u)^{\alpha-1}z_1(u)du\\ &\quad+\Big(h_1+\frac{\Gamma(\alpha-q)}{\Gamma(\alpha)}\sum_{\tau=1}^{j}\omega_\tau t_\tau^{q+1-\alpha}\Big)s^{\alpha-1}\Big)ds\Big]''\\ &\quad+\frac{1}{\Gamma(2-\alpha)}\Big[\int_{t_i}^{t}(t-s)^{1-\alpha}\Big(\frac{1}{\Gamma(\alpha)}\int_0^s(s-u)^{\alpha-1}z_1(u)du \end{aligned}$$

$$
\begin{aligned}
&+\Big(h_1+\frac{\Gamma(\alpha-q)}{\Gamma(\alpha)}\sum_{\tau=1}^{i}\omega_\tau t_\tau^{q+1-\alpha}\Big)s^{\alpha-1}\Big)ds\Big]''\\
&+\frac{1}{\Gamma(2-\alpha)}\Big[\int_0^t(t-s)^{1-\alpha}\Big(h_2+\sum_{\tau=1}^{j}\delta_\tau t_\tau^{2-\alpha}-\frac{\Gamma(\alpha-q)}{\Gamma(\alpha)}\sum_{\tau=1}^{i}\omega_\tau t_\tau^{q+2-\alpha}\Big)s^{\alpha-2}ds\Big]''\\
&\neq z_1(t),\quad \text{by direct computation.}
\end{aligned}
$$

(ii) Problem (5.14) is unsuitable proposed. By Corollary 5.3, one sees the piecewise continuous solutions of ${}^{RL}D_{0^+}^{\alpha}u(t)=z_1(t)$ are given by

$$
u(t)=\sum_{\sigma=0}^{i}\Big(\frac{c_{\sigma 1}}{\Gamma(\alpha)}(t-t_\sigma)^{\alpha-1}+\frac{c_{\sigma 2}}{\Gamma(\alpha-1)}(t-t_\sigma)^{\alpha-2}\Big)+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}z_1(s)ds,
$$

for $t\in(t_i,t_{i+1}]$, $i\in\mathbb{N}[0,m]$. It is easy to see that u may be discontinuous at $t=t_i$ if $c_{i2}\neq 0$. So $\Delta u(t_i)=\delta_i$ is unsuitable. Since the expression of u on $(t_i,t_{i+1}], i\in N[0,m]$ are different from each other, then the resonant conditions (5.13) may be false.

To show the readers a correct result, now, we consider the boundary value problem (for ease expression, we consider the one of a impulsive fractional differential equation, not a fractional differential system):

$$
\begin{gathered}
{}^{RL}D_{0^+}^{\alpha}u(t)=p(t)f(t,u(t),{}^{RL}D_{0^+}^{\beta}u(t)),\quad t\in(t_i,t_{i+1}],\ i\in\mathbb{N}[0,m],\\
\lim_{t\to t_i^+}(t-t_i)^{2-\alpha}u(t)=I(t_i,u(t_i),{}^{RL}D_{0^+}^{\beta}u(t_i)),\quad i\in\mathbb{N}[1,m],\\
\Delta\,{}^{RL}D_{0^+}^{\alpha-1}u(t_i))=J(t_i,u(t_i),{}^{RL}D_{0^+}^{\beta}u(t_i)),\quad i\in\mathbb{N}[1,m],\\
{}^{RL}D_{0^+}^{\alpha-1}u(0)=\sum_{i=1}^{m}a_i\,{}^{RL}D_{0^+}^{\alpha-1}u(\xi_i),\quad u(1)=\sum_{i=1}^{m}b_iu(\eta_i),
\end{gathered}
\tag{5.15}
$$

where:

(5.A1) $\alpha\in(1,2)$, $\beta\in(0,\alpha-1]$, $\{t_i:i\in\mathbb{N}[1,m]\}$, $\{\xi_i:i\in\mathbb{N}[1,m]\}$, $\{\eta_i:i\in\mathbb{N}[1,m]\}$ are increasing sequences with $\xi_i,\eta_i\in(t_i,t_{i+1}]$, $i\in\mathbb{N}[0,m]$, $a_i,b_i\in\mathbb{R}$ are fixed constants, m is a positive integer,

(5.A2) $f:(0,1)\times\mathbb{R}^2\to\mathbb{R}$ satisfies the following items: $t\to f(t,(t-t_i)^{\alpha-2}u,(t-t_i)^{\alpha-\beta-2}v)$ is measurable on $(0,1)$ for each $(u,v)\in\mathbb{R}^2$, $(u,v)\to f(t,(t-t_i)^{\alpha-2}u,(t-t_i)^{\alpha-\beta-2}v)$ is continuous on $\mathbb{R}^2$ for all $t\in(t_i,t_{i+1}]$, $i\in\mathbb{N}[0,m]$, for each $r>0$ there exists a constant $M_r\geq 0$ such that $|f(t,(t-t_i)^{\alpha-2}u,(t-t_i)^{\alpha-\beta-2}v)|\leq M_r$ holds for all $t\in(t_i,t_{i+1}]$, $i\in\mathbb{N}[0,m]$ and $|u|,|v|\leq r$;

(5.A3) $I,J:\{t_i:i\in\mathbb{N}[1,m]\}\times\mathbb{R}^2\to\mathbb{R}$ satisfies the following items: $(u,v)\to I$ or $J(t_i,(t_i-t_{i-1})^{\alpha-2}u,(t_i-t_{i-1})^{\alpha-\beta-2}v)$ is continuous on $\mathbb{R}^2$ for all $i\in\mathbb{N}[1,m]$, for each $r>0$ there exists a constant $M_r\geq 0$ such that $|I$ or $J(t_i,(t_i-t_{i-1})^{\alpha-2}u,(t_i-t_{i-1})^{\alpha-\beta-2}v)|\leq M_r$ holds for all $i\in\mathbb{N}[1,m]$ and $|u|,|v|\leq r$;

(5.A4) $p:(0,1)\to\mathbb{R}$ satisfies that there exist number $k>-1$, $l\in(\max\{-\alpha,-2-k\},0]$ and $\beta-\alpha<l\leq 0$ such that $|p(t)|\leq t^k(1-t)^l$ for all $t\in(0,1)$.

The homogeneous problem of (5.15) is as follows:

$$\begin{gathered}
{}^{RL}D_{0^+}^{\alpha}u(t)=0,\quad t\in(t_i,t_{i+1}],\ i\in\mathbb{N}[0,m],\\
\lim_{t\to t_i^+}(t-t_i)^{2-\alpha}u(t)=\Delta^{RL}D_{0^+}^{\alpha-1}u(t_i))=0,\quad i\in\mathbb{N}[1,m],\\
{}^{RL}D_{0^+}^{\alpha-1}u(0)=\sum_{i=1}^m a_i{}^{RL}D_{0^+}^{\alpha-1}u(\xi_i),\quad u(1)=\sum_{i=1}^m b_iu(\eta_i),
\end{gathered}\tag{5.16}$$

By Corollary 5.3, ${}^{RL}D_{0^+}^{\alpha}u(t)=0$ with $t\in(t_i,t_{i+1}]$, $i\in\mathbb{N}[0,m]$ implies that there exist constants $c_{\sigma1},c_{\sigma2}\in\mathbb{R}$ such that

$$u(t)=\sum_{\sigma=0}^{i}(c_{\sigma1}(t-t_\sigma)^{\alpha-1}+c_{\sigma2}(t-t_\sigma)^{\alpha-2}),\quad t\in(t_i,t_{i+1}],\ i\in\mathbb{N}[0,m].\tag{5.17}$$

So Definition 2.2 implies for $t\in(t_j,t_{j+1}]$, $i\in\mathbb{N}[0,m]$ that

$$\begin{aligned}
&{}^{RL}D_{0^+}^{\alpha-1}u(t)\\
&=\frac{1}{\Gamma(2-\alpha)}\Big[\int_0^t(t-s)^{1-\alpha}u(s)ds\Big]'\\
&=\frac{1}{\Gamma(2-\alpha)}\Big[\sum_{i=0}^{j-1}\int_{t_i}^{t_{i+1}}(t-s)^{1-\alpha}\sum_{\sigma=0}^{i}(c_{\sigma1}(s-t_\sigma)^{\alpha-1}+c_{\sigma2}(s-t_\sigma)^{\alpha-2})ds\Big]'\\
&\quad+\frac{1}{\Gamma(2-\alpha)}\Big[\int_{t_j}^{t}(t-s)^{1-\alpha}\sum_{\sigma=0}^{j}(c_{\sigma1}(s-t_\sigma)^{\alpha-1}+c_{\sigma2}(s-t_\sigma)^{\alpha-2})ds\Big]'\\
&=\frac{1}{\Gamma(2-\alpha)}\Big[\sum_{i=0}^{j-1}\sum_{\sigma=0}^{i}(c_{\sigma1}\int_{t_i}^{t_{i+1}}(t-s)^{1-\alpha}(s-t_\sigma)^{\alpha-1}ds\\
&\quad+c_{\sigma2}\int_{t_i}^{t_{i+1}}(t-s)^{1-\alpha}(s-t_\sigma)^{\alpha-2}ds)\Big]'\\
&\quad+\frac{1}{\Gamma(2-\alpha)}\Big[\sum_{\sigma=0}^{j}(c_{\sigma1}\int_{t_j}^{t}(t-s)^{1-\alpha}(s-t_\sigma)^{\alpha-1}ds\\
&\quad+c_{\sigma2}\int_{t_j}^{t}(t-s)^{1-\alpha}(s-t_\sigma)^{\alpha-2}ds)\Big]'\\
&=\frac{1}{\Gamma(2-\alpha)}\Big[\sum_{i=0}^{j-1}\sum_{\sigma=0}^{i}(c_{\sigma1}(t-t_\sigma)\int_{\frac{t_i-t_\sigma}{t-t_\sigma}}^{\frac{t_{i+1}-t_\sigma}{t-t_\sigma}}(1-w)^{1-\alpha}w^{\alpha-1}dw\\
&\quad+c_{\sigma2}\int_{\frac{t_i-t_\sigma}{t-t_\sigma}}^{\frac{t_{i+1}-t_\sigma}{t-t_\sigma}}(1-w)^{1-\alpha}w^{\alpha-2}dw)\Big]'\\
&\quad+\frac{1}{\Gamma(2-\alpha)}\Big[\sum_{\sigma=0}^{j}(c_{\sigma1}(t-t_\sigma)\int_{\frac{t_j-t_\sigma}{t-t_\sigma}}^{1}(1-w)^{1-\alpha}w^{\alpha-1}dw\\
&\quad+c_{\sigma2}\int_{\frac{t_j-t_\sigma}{t-t_\sigma}}^{1}(1-w)^{1-\alpha}w^{\alpha-2}dw)\Big]'
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\Gamma(2-\alpha)}\Big[\sum_{\sigma=0}^{j-1}\sum_{i=\sigma}^{j-1}(c_{\sigma 1}(t-t_\sigma)\int_{\frac{t_i-t_\sigma}{t-t_\sigma}}^{\frac{t_{i+1}-t_\sigma}{t-t_\sigma}}(1-w)^{1-\alpha}w^{\alpha-1}dw\\
&\quad + c_{\sigma 2}\int_{\frac{t_i-t_\sigma}{t-t_\sigma}}^{\frac{t_{i+1}-t_\sigma}{t-t_\sigma}}(1-w)^{1-\alpha}w^{\alpha-2}dw)\Big]'\\
&\quad + \frac{1}{\Gamma(2-\alpha)}\Big[\sum_{\sigma=0}^{j}\Big(c_{\sigma 1}(t-t_\sigma)\int_{\frac{t_j-t_\sigma}{t-t_\sigma}}^{1}(1-w)^{1-\alpha}w^{\alpha-1}dw\\
&\quad + c_{\sigma 2}\int_{\frac{t_j-t_\sigma}{t-t_\sigma}}^{1}(1-w)^{1-\alpha}w^{\alpha-2}dw\Big)\Big]'\\
&= \frac{1}{\Gamma(2-\alpha)}\Big[\sum_{\sigma=0}^{j}\Big(c_{\sigma 1}(t-t_\sigma)\int_0^1(1-w)^{1-\alpha}w^{\alpha-1}dw\\
&\quad + c_{\sigma 2}\int_0^1(1-w)^{1-\alpha}w^{\alpha-2}dw\Big)\Big]'\\
&= \Gamma(\alpha)\sum_{\sigma=0}^{j}c_{\sigma 1}.
\end{aligned}
$$

It follows that

$$
{}^{RL}D_{0^+}^{\alpha-1}u(t) = \Gamma(\alpha)\sum_{\sigma=0}^{i}c_{\sigma 1}, \quad t\in(t_i,t_{i+1}],\ i\in\mathbb{N}[0,m]. \tag{5.18}
$$

Then the other equations in (5.16) with (5.17) and (5.18) imply

$$
\begin{gathered}
c_{i2}=0,\quad c_{i1}=0, i\in\mathbb{N}[1,m],\\
\Gamma(\alpha)c_{01}=\sum_{i=1}^{m}a_i\sum_{\sigma=0}^{i}\Gamma(\alpha)c_{\sigma 1},\\
\sum_{\sigma=0}^{m}(c_{\sigma 1}(1-t_\sigma)^{\alpha-1}+c_{\sigma 2}(1-t_\sigma)^{\alpha-2})\\
=\sum_{i=1}^{m}b_i\sum_{\sigma=0}^{i}(c_{\sigma 1}(\eta_i-t_\sigma)^{\alpha-1}+c_{\sigma 2}(\eta_i-t_\sigma)^{\alpha-2}).
\end{gathered}
\tag{5.19}
$$

Thus (5.19) implies that $c_{\sigma 1}=0$ and $c_{\sigma 2}=0$ for all $\sigma\in\mathbb{N}[1,m]$. So by (5.19) we have

$$
c_{01}=c_{01}\sum_{i=1}^{m}a_i,\quad c_{01}(1-\sum_{i=1}^{m}b_i\eta_i^{\alpha-1})=c_{02}\Big[\sum_{i=1}^{m}b_i\eta_i^{\alpha-2}-1\Big]=0. \tag{5.20}
$$

From (5.20) it is easy to see the following:
Case (i): (5.16) has a unique trivial solution $u(t)=0$ if and only if

$$
\sum_{i=1}^{m}a_i\neq 1,\quad \sum_{i=1}^{m}b_i\eta_i^{\alpha-2}\neq 1. \tag{5.21}
$$

Case (ii): (5.16) has a group of nontrivial solutions $u(t) = c_{02}t^{\alpha-2}, c_{02} \in \mathbb{R}$ if and only if

$$\sum_{i=1}^{m} a_i \neq 1, \quad \sum_{i=1}^{m} b_i\eta_i^{\alpha-2} = 1. \tag{5.22}$$

Case (iii): (5.16) has a group of nontrivial solutions

$$u(t) = c_{01}t^{\alpha-1} + c_{01}\frac{1-\sum_{i=1}^{m} b_i\eta_i^{\alpha-1}}{\sum_{i=1}^{m} b_i\eta_i^{\alpha-2}-1}t^{\alpha-2}, \quad c_{01}, c_{02} \in \mathbb{R}$$

if and only if

$$\sum_{i=1}^{m} a_i = 1, \quad \sum_{i=1}^{m} b_i\eta_i^{\alpha-1} \neq 1, \quad \sum_{i=1}^{m} b_i\eta_i^{\alpha-2} \neq 1. \tag{5.23}$$

Case (iv): (5.16) has a group of nontrivial solutions $u(t) = c_{02}t^{\alpha-2}$, $c_{02} \in \mathbb{R}$ if and only if

$$\sum_{i=1}^{m} a_i = 1, \quad \sum_{i=1}^{m} b_i\eta_i^{\alpha-1} \neq 1, \quad \sum_{i=1}^{m} b_i\eta_i^{\alpha-2} = 1. \tag{5.24}$$

Case (v): (5.16) has a group of nontrivial solutions $u(t) = c_{01}t^{\alpha-1}$, $c_{01} \in \mathbb{R}$ if and only if

$$\sum_{i=1}^{m} a_i = 1, \quad \sum_{i=1}^{m} b_i\eta_i^{\alpha-1} = 1, \quad \sum_{i=1}^{m} b_i\eta_i^{\alpha-2} = 1. \tag{5.25}$$

If (5.21) holds, then BVP (5.15) is a un-resonant problem. While (5.22) or (5.23) or (5.24) or (5.25) implies that BVP(5.15) is a resonant problem. Concerning Case (i), we establish an existence result for solutions of (5.15). Similar results can be established for other cases. The readers should try it.

Lemma 5.9. *Suppose that (5.21) holds, denote* $\Delta_1 = 1 - \sum_{i=1}^{m} a_i$ *and* $\Delta_2 = 1 - \sum_{i=1}^{m} b_i\eta_i^{\alpha-2}$, $\Delta_3 = \sum_{i=1}^{m} b_i\eta_i^{\alpha-1} - 1$, σ *is continuous on* $(0,1)$ *and there exist* $k > -1$ *and* $l \in (\max\{-\alpha, -2-k\}, 0]$ *such that* $|\sigma(t)| \le t^k(1-t)^l$ *for all* $t \in (0,1)$. *Then* x *is a solution of the problem*

$$\begin{gathered}
{}^{RL}D_{0^+}^{\alpha}x(t) = \sigma(t), \quad t \in (t_i, t_{i+1}],\ i \in \mathbb{N}[0,m], \\
\lim_{t\to t_i^+}(t-t_i)^{2-\alpha}x(t) = I_i, \quad i \in \mathbb{N}[1,m], \\
\Delta\, {}^{RL}D_{0^+}^{\alpha-1}x(t_i)) = J_i, \quad i \in \mathbb{N}[1,m], \\
{}^{RL}D_{0^+}^{\alpha-1}x(0) = \sum_{i=1}^{m} a_i\, {}^{RL}D_{0^+}^{\alpha-1}x(\xi_i), \quad x(1) = \sum_{i=1}^{m} b_i x(\eta_i)
\end{gathered} \tag{5.26}$$

if and only if

$$\begin{aligned} & x(t)\\ &= \frac{1}{\Gamma(\alpha)}\Big[\frac{t^{\alpha-1}}{\Delta_1}\sum_{\sigma=1}^{m}\sum_{i=\sigma}^{m}a_i + \frac{\Delta_3}{\Delta_1}\frac{t^{\alpha-2}}{\Delta_2}\sum_{\sigma=1}^{m}\sum_{i=\sigma}^{m}a_i + \frac{t^{\alpha-2}}{\Delta_2}\sum_{\sigma=1}^{m}\sum_{i=\sigma}^{m}b_i(\eta_i-t_\sigma)^{\alpha-1}\Big]J_\sigma\\ &\quad + \frac{t^{\alpha-2}}{\Delta_2}\sum_{\sigma=1}^{m}\sum_{i=\sigma}^{m}b_i(\eta_i-t_\sigma)^{\alpha-2}I_\sigma + \frac{1}{\Gamma(\alpha)}\sum_{\sigma=1}^{i}(t-t_\sigma)^{\alpha-1}J_\sigma + \sum_{\sigma=1}^{i}(t-t_\sigma)^{\alpha-2}I_\sigma\\ &\quad + \frac{1}{\Gamma(\alpha)}\frac{t^{\alpha-2}}{\Delta_2}\frac{\Delta_3}{\Delta_1}\sum_{i=1}^{m}a_i\int_0^{\xi_i}\sigma(s)ds + \frac{t^{\alpha-2}}{\Delta_2}\sum_{i=1}^{m}b_i\int_0^{\eta_i}\frac{(\eta_i-s)^{\alpha-1}}{\Gamma(\alpha)}\sigma(s)ds\\ &\quad - \frac{t^{\alpha-2}}{\Delta_2}\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\sigma(s)ds + \frac{1}{\Gamma(\alpha)}\frac{t^{\alpha-1}}{\Delta_1}\sum_{i=1}^{m}a_i\int_0^{\xi_i}\sigma(s)ds\\ &\quad + \int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\sigma(s)ds, \quad t\in(t_i,t_{i+1}],\ i\in\mathbb{N}[0,m]. \end{aligned} \tag{5.27}$$

Proof. By Theorem 3.12, we have ${}^{RL}D_{0^+}^{\alpha}x(t)=\sigma(t)$, $t\in(t_i,t_{i+1}]$, $i\in\mathbb{N}[0,m]$ if and only if there exit numbers $c_{\sigma1},c_{\sigma2}$, $i\in\mathbb{N}[0,m]$ such that

$$x(t)=\sum_{\sigma=0}^{i}[c_{\sigma1}(t-t_\sigma)^{\alpha-1}+c_{\sigma2}(t-t_\sigma)^{\alpha-2}]+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\sigma(s)ds, \tag{5.28}$$

for $t\in(t_i,t_{i+1}]$, $i\in\mathbb{N}[0,m]$. Then

$${}^{RL}D_{0^+}^{\alpha-1}x(t)=\Gamma(\alpha)\sum_{\sigma=0}^{i}c_{\sigma1}+\int_0^t\sigma(s)ds, \quad t\in(t_i,t_{i+1}],\ i\in\mathbb{N}[0,m]. \tag{5.29}$$

Furthermore, by direct computations we have

$$\begin{aligned} & {}^{RL}D_{0^+}^{\beta}x(t)\\ &= \frac{1}{\Gamma(1-\beta)}\Big[\int_0^t(t-s)^{-\beta}x(s)ds\Big]'\\ &= \frac{1}{\Gamma(1-\beta)}\Big[\sum_{j=0}^{i-1}\int_{t_j}^{t_{j+1}}(t-s)^{-\beta}x(s)ds\Big]' + \frac{1}{\Gamma(1-\beta)}\Big[\int_{t_i}^{t}(t-s)^{-\beta}x(s)ds\Big]'\\ &= \sum_{\sigma=0}^{i}\Big(c_{\sigma1}\frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta+1)}(t-t_\sigma)^{\alpha-\beta-1}+c_{\sigma2}\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-\beta-1)}(t-t_\sigma)^{\alpha-\beta-2}\Big)\\ &\quad + \int_0^t\frac{(t-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\sigma(s)ds, \quad t\in(t_i,t_{i+1}],\ i\in N[0,m]. \end{aligned} \tag{5.30}$$

From $\lim_{t\to t_i^+}(t-t_i)^{2-\alpha}x(t)=I_i$, we have $c_{\sigma2}=I_\sigma$ for all $\sigma\in\mathbb{N}[1,m]$. From $\Delta\,{}^{RL}D_{0^+}^{\alpha-1}x(t_i))=J_i$, $i\in\mathbb{N}[1,m]$, one has $c_{\sigma1}=\frac{J_\sigma}{\Gamma(\alpha)}$ for all $\sigma\in\mathbb{N}[1,m]$. From the boundary conditions in (5.25), we obtain

$$\Gamma(\alpha)c_{01}=\sum_{i=1}^{m}a_i\Big[\Gamma(\alpha)\sum_{\sigma=0}^{i}c_{\sigma1}+\int_0^{\xi_i}\sigma(s)ds\Big],$$

$$\sum_{\sigma=0}^{m}[c_{\sigma 1}(1-t_\sigma)^{\alpha-1}+c_{\sigma 2}(1-t_\sigma)^{\alpha-2}]+\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\sigma(s)ds$$
$$=\sum_{i=1}^{m}b_i\Big[\sum_{\sigma=0}^{i}[c_{\sigma 1}(\eta_i-t_\sigma)^{\alpha-1}+c_{\sigma 2}(\eta_i-t_\sigma)^{\alpha-2}]+\int_0^{\eta_i}\frac{(\eta_i-s)^{\alpha-1}}{\Gamma(\alpha)}\sigma(s)ds\Big].$$

It follows that

$$\begin{aligned} c_{01}&=\frac{1}{\Gamma(\alpha)}\frac{1}{\Delta_1}\Big[\sum_{i=1}^{m}a_i\sum_{\sigma=1}^{i}J_\sigma+\sum_{i=1}^{m}a_i\int_0^{\xi_i}\sigma(s)ds\Big]\\ &=\frac{1}{\Gamma(\alpha)}\frac{1}{\Delta_1}\sum_{\sigma=1}^{m}\sum_{i=\sigma}^{m}a_iJ_\sigma+\frac{1}{\Gamma(\alpha)}\frac{1}{\Delta_1}\sum_{i=1}^{m}a_i\int_0^{\xi_i}\sigma(s)ds,\end{aligned}$$

$$\begin{aligned} c_{02}&=\frac{1}{\Delta_2}\Big[\frac{1}{\Gamma(\alpha)}\frac{\Delta_3}{\Delta_1}\sum_{\sigma=1}^{m}\sum_{i=\sigma}^{m}a_iJ_\sigma+\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{m}b_i\sum_{\sigma=1}^{i}(\eta_i-t_\sigma)^{\alpha-1}J_\sigma\\ &\quad+\sum_{i=1}^{m}b_i\sum_{\sigma=1}^{i}(\eta_i-t_\sigma)^{\alpha-2}I_\sigma+\frac{1}{\Gamma(\alpha)}\frac{\Delta_3}{\Delta_1}\sum_{i=1}^{m}a_i\int_0^{\xi_i}\sigma(s)ds\\ &\quad+\sum_{i=1}^{m}b_i\int_0^{\eta_i}\frac{(\eta_i-s)^{\alpha-1}}{\Gamma(\alpha)}\sigma(s)ds-\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\sigma(s)ds\Big]\\ &=\frac{1}{\Delta_2}\Big[\frac{1}{\Gamma(\alpha)}\frac{\Delta_3}{\Delta_1}\sum_{\sigma=1}^{m}\sum_{i=\sigma}^{m}a_iJ_\sigma+\frac{1}{\Gamma(\alpha)}\sum_{\sigma=1}^{m}\sum_{i=\sigma}^{m}b_i(\eta_i-t_\sigma)^{\alpha-1}J_\sigma\\ &\quad+\sum_{\sigma=1}^{m}\sum_{i=\sigma}^{m}b_i(\eta_i-t_\sigma)^{\alpha-2}I_\sigma+\frac{1}{\Gamma(\alpha)}\frac{\Delta_3}{\Delta_1}\sum_{i=1}^{m}a_i\int_0^{\xi_i}\sigma(s)ds\\ &\quad+\sum_{i=1}^{m}b_i\int_0^{\eta_i}\frac{(\eta_i-s)^{\alpha-1}}{\Gamma(\alpha)}\sigma(s)ds-\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\sigma(s)ds\Big].\end{aligned}$$

Substituting $c_{\sigma 1}, c_{\sigma 2}$ into (5.28), we obtain (5.27) by changing the order of the terms. It is easy to show from (5.28) and (5.30) that $x\in X$. The proof is compete. □

To simplify notation, let $H_x(t)=H(t,x(t),{}^{RL}D_{0^+}^{\beta}x(t))$ for functions $H:(0,1)\times\mathbb{R}^2\to\mathbb{R}$ and $x:(0,1]\to\mathbb{R}$. Define the operator $(Tx)(t)$ for $x\in X$ by

$$\begin{aligned} &(Tx)(t)\\ &=\frac{1}{\Gamma(\alpha)}\Big[\frac{t^{\alpha-1}}{\Delta_1}\sum_{\sigma=1}^{m}\sum_{i=\sigma}^{m}a_i+\frac{\Delta_3}{\Delta_1}\frac{t^{\alpha-2}}{\Delta_2}\sum_{\sigma=1}^{m}\sum_{i=\sigma}^{m}a_i+\frac{t^{\alpha-2}}{\Delta_2}\sum_{\sigma=1}^{m}\sum_{i=\sigma}^{m}b_i(\eta_i-t_\sigma)^{\alpha-1}\Big]J_x(t_\sigma)\\ &\quad+\frac{t^{\alpha-2}}{\Delta_2}\sum_{\sigma=1}^{m}\sum_{i=\sigma}^{m}b_i(\eta_i-t_\sigma)^{\alpha-2}I_x(t_\sigma)+\frac{1}{\Gamma(\alpha)}\sum_{\sigma=1}^{i}(t-t_\sigma)^{\alpha-1}J_x(t_\sigma)\\ &\quad+\sum_{\sigma=1}^{i}(t-t_\sigma)^{\alpha-2}I_x(t_\sigma)+\frac{1}{\Gamma(\alpha)}\frac{t^{\alpha-2}}{\Delta_2}\frac{\Delta_3}{\Delta_1}\sum_{i=1}^{m}a_i\int_0^{\xi_i}p(s)f_x(s)ds\\ &\quad+\frac{t^{\alpha-2}}{\Delta_2}\sum_{i=1}^{m}b_i\int_0^{\eta_i}\frac{(\eta_i-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)f_x(s)ds-\frac{t^{\alpha-2}}{\Delta_2}\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)f_x(s)ds\end{aligned}$$

$$+\frac{1}{\Gamma(\alpha)}\frac{t^{\alpha-1}}{\Delta_1}\sum_{i=1}^{m}a_i\int_0^{\xi_i}p(s)f_x(s)ds+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)f_x(s)ds,$$

for $t\in(t_i,t_{i+1}]$ and $i\in\mathbb{N}[0,m]$.

Theorem 5.10. *Suppose that* (5.21), (5.A1)–(5.A4) *hold and*

(H1) *there exist non-decreasing functions* $M_f,M_I,M_J:[0,+\infty)\times[0,+\infty)\to[0,+\infty)$ *such that*

$$|f(t,(t-t_i)^{\alpha-2}u,(t-t_i)^{\alpha-\beta-2}v)|\le M_f(|u|,|v|),$$
$$\text{for } t\in(t_i,t_{i+1}],\ i\in\mathbb{N}[0,m],u,v\in\mathbb{R},$$
$$|I(t_i,(t_i-t_{i-1})^{\alpha-2}u,(t_i-t_{i-1})^{\alpha-\beta-2}v)|\le M_I(|u|,|v|),\quad i\in\mathbb{N}[1,m],u,v\in\mathbb{R},$$
$$|J(t_i,(t_i-t_{i-1})^{\alpha-2}u,(t_i-t_{i-1})^{\alpha-\beta-2}v)|\le M_J(|u|,|v|),\quad i\in\mathbb{N}[1,m],u,v\in\mathbb{R}$$

Then (5.15) *has at least one solution if there is* $r_0>0$ *such that* $A_1M_J(r_0,r_0)+A_2M_I(r_0,r_0)+A_3M_f(r_0,r_0)\le r_0$, *where*

$$\begin{aligned}A_1=\max\Big\{&\frac{1}{\Gamma(\alpha)}\Big[\Big(\frac{1}{|\Delta_1|}+\frac{|\Delta_3|}{|\Delta_1|}\frac{1}{|\Delta_2|}\Big)\sum_{i=1}^{m}i|a_i|+\frac{1}{|\Delta_2|}\sum_{i=1}^{m}i|b_i|+m\Big],\\
&\Big(\frac{1}{\Gamma(\alpha-\beta+1)|\Delta_1|}+\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-\beta-1)}\frac{1}{\Gamma(\alpha)|\Delta_2|}\frac{|\Delta_3|}{|\Delta_1|}\Big)\sum_{i=1}^{m}i|a_i|\\
&+\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-\beta-1)}\frac{1}{\Gamma(\alpha)|\Delta_2|}\sum_{i=1}^{m}i|b_i|+\frac{m}{\Gamma(\alpha-\beta+1)}\Big\};\end{aligned}$$

$$\begin{aligned}A_2=\max\Big\{&\frac{1}{|\Delta_2|}\sum_{\sigma=1}^{m}\sum_{i=\sigma}^{m}|b_i|(\eta_i-t_\sigma)^{\alpha-2}+m,\\
&\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-\beta-1)}\frac{1}{|\Delta_2|}\sum_{\sigma=1}^{m}\sum_{i=\sigma}^{m}|b_i|(\eta_i-t_\sigma)^{\alpha-2}+\frac{m\Gamma(\alpha-1)}{\Gamma(\alpha-\beta-1)}\Big\};\end{aligned}$$

$$\begin{aligned}A_3=\max\Big\{&\Big(\frac{\mathbf{B}(l+1,k+1)}{\Gamma(\alpha)|\Delta_2|}\frac{|\Delta_3|}{|\Delta_1|}+\frac{\mathbf{B}(l+1,k+1)}{\Gamma(\alpha)|\Delta_1|}\Big)\sum_{i=1}^{m}|a_i|\\
&+\frac{\mathbf{B}(\alpha+l,k+1)}{\Gamma(\alpha)|\Delta_2|}\sum_{i=1}^{m}|b_i|+\frac{\mathbf{B}(\alpha+l,k+1)}{\Gamma(\alpha)|\Delta_2|}+\frac{\mathbf{B}(l+1,k+1)}{\Gamma(\alpha)},\\
&\frac{\mathbf{B}(l+1,k+1)}{\Gamma(\alpha)}\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-\beta-1)}\frac{1}{|\Delta_2|}\frac{|\Delta_3|}{|\Delta_1|}\sum_{i=1}^{m}|a_i|\\
&+\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-\beta-1)}\frac{1}{|\Delta_2|}\sum_{i=1}^{m}|b_i|\frac{\mathbf{B}(\alpha+l,k+1)}{\Gamma(\alpha)}+\frac{\mathbf{B}(\alpha+l,k+1)}{\Gamma(\alpha)}\\
&+\frac{\mathbf{B}(l+1,k+1)}{\Gamma(\alpha-\beta+1)}\frac{1}{\Delta_1}+\frac{\mathbf{B}(\alpha-\beta+l,k+1)}{\Gamma(\alpha-\beta)}\Big\}.\end{aligned}$$

Proof. Let X be defined above. By Lemma 5.9, we know that x is a solution of (5.15) if and only if x is a fixed point of T. By a standard proof, we can see that $T:X\to X$ is a completely continuous operator. From (H1), for $x\in X$ we have

$$|f_x(t)|=|f(t,x(t),{}^{RL}D_{0^+}^{\beta}x(t))|\le M_f(\|x\|,\|x\|),\quad t\in(t_i,t_{i+1}],\ i\in\mathbb{N}[0,m],$$
$$|I(t_i,x(t_i),{}^{RL}D_{0^+}^{\beta}x(t_i))|\le M_I(\|x\|,\|x\|),\quad i\in\mathbb{N}[1,m],\ x\in\mathbb{R},$$

$$|J(t_i,x(t_i),{}^{RL}D_{0^+}^{\beta}x(t_i))|\le M_J(\|x\|,\|x\|),\quad i\in\mathbb{N}[1,m],\ x\in\mathbb{R}.$$

We consider the set $\Omega=\{x\in X: x=\lambda(Tx),\ \text{for some }\lambda\in[0,1)\}$. For $x\in\Omega$, by the definition of T for $t\in(t_i,t_{i+1}]$ we have

$$\begin{aligned}
&{}^{RL}D_{0^+}^{\beta}(Tx)(t)\\
&=\frac{1}{\Gamma(\alpha-\beta+1)}\frac{t^{\alpha-\beta-1}}{\Delta_1}\sum_{\sigma=1}^{m}\sum_{i=\sigma}^{m}a_iJ_\sigma+\frac{1}{\Gamma(\alpha)}\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-\beta-1)}\frac{t^{\alpha-\beta-2}}{\Delta_2}\frac{\Delta_3}{\Delta_1}\sum_{\sigma=1}^{m}\sum_{i=\sigma}^{m}a_iJ_\sigma\\
&\quad+\frac{1}{\Gamma(\alpha)}\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-\beta-1)}\frac{t^{\alpha-\beta-2}}{\Delta_2}\sum_{\sigma=1}^{m}\sum_{i=\sigma}^{m}b_i(\eta_i-t_\sigma)^{\alpha-1}J_\sigma\\
&\quad+\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-\beta-1)}\frac{t^{\alpha-\beta-2}}{\Delta_2}\sum_{\sigma=1}^{m}\sum_{i=\sigma}^{m}b_i(\eta_i-t_\sigma)^{\alpha-2}I_\sigma\\
&\quad+\sum_{\sigma=1}^{i}\Big(\frac{1}{\Gamma(\alpha-\beta+1)}(t-t_\sigma)^{\alpha-\beta-1}J_\sigma+\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-\beta-1)}(t-t_\sigma)^{\alpha-\beta-2}I_\sigma\Big)\\
&\quad+\frac{1}{\Gamma(\alpha)}\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-\beta-1)}\frac{t^{\alpha-\beta-2}}{\Delta_2}\frac{\Delta_3}{\Delta_1}\sum_{i=1}^{m}a_i\int_0^{\xi_i}\sigma(s)ds\\
&\quad+\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-\beta-1)}\frac{t^{\alpha-\beta-2}}{\Delta_2}\sum_{i=1}^{m}b_i\int_0^{\eta_i}\frac{(\eta_i-s)^{\alpha-1}}{\Gamma(\alpha)}\sigma(s)ds\\
&\quad-\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\sigma(s)ds+\frac{1}{\Gamma(\alpha-\beta+1)}\frac{t^{\alpha-\beta-1}}{\Delta_1}\sum_{i=1}^{m}a_i\int_0^{\xi_i}\sigma(s)ds\\
&\quad+\int_0^t\frac{(t-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\sigma(s)ds,\quad t\in(t_i,t_{i+1}],\ i\in\mathbb{N}[0,m].
\end{aligned}$$

So

$$\begin{aligned}
&(t-t_i)^{2-\alpha}|(Tx)(t)|\\
&\le\frac{1}{\Gamma(\alpha)}\Big[\frac{1}{|\Delta_1|}\sum_{\sigma=1}^{m}\sum_{i=\sigma}^{m}|a_i|+\frac{|\Delta_3|}{|\Delta_1|}\frac{1}{|\Delta_2|}\sum_{\sigma=1}^{m}\sum_{i=\sigma}^{m}|a_i|\\
&\quad+\frac{1}{|\Delta_2|}\sum_{\sigma=1}^{m}\sum_{i=\sigma}^{m}|b_i|(\eta_i-t_\sigma)^{\alpha-1}\Big]M_J(\|x\|,\|x\|)\\
&\quad+\frac{1}{|\Delta_2|}\sum_{\sigma=1}^{m}\sum_{i=\sigma}^{m}|b_i|(\eta_i-t_\sigma)^{\alpha-2}M_I(\|x\|,\|x\|)\\
&\quad+\frac{1}{\Gamma(\alpha)}\sum_{\sigma=1}^{i}M_J(\|x\|,\|x\|)+\sum_{\sigma=1}^{i}M_I(\|x\|,\|x\|)\\
&\quad+\frac{1}{\Gamma(\alpha)}\frac{1}{|\Delta_2|}\frac{|\Delta_3|}{|\Delta_1|}\sum_{i=1}^{m}|a_i|\int_0^{\xi_i}s^k(1-s)^l dsM_f(\|x\|,\|x\|)\\
&\quad+\frac{1}{|\Delta_2|}\sum_{i=1}^{m}|b_i|\int_0^{\eta_i}\frac{(\eta_i-s)^{\alpha-1}}{\Gamma(\alpha)}s^k(1-s)^l dsM_f(\|x\|,\|x\|)\\
&\quad+\frac{1}{|\Delta_2|}\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}s^k(1-s)^l dsM_f(\|x\|,\|x\|)
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{1}{\Gamma(\alpha)}\frac{1}{|\Delta_1|}\sum_{i=1}^{m}|a_i|\int_0^{\xi_i} s^k(1-s)^l ds M_f(\|x\|,\|x\|)\\
&+ (t-t_i)^{2-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} s^k(1-s)^l ds M_f(\|x\|,\|x\|)\\
\le{}& \frac{1}{\Gamma(\alpha)}\Big[\Big(\frac{1}{|\Delta_1|}+\frac{|\Delta_3|}{|\Delta_1|}\frac{1}{|\Delta_2|}\Big)\sum_{i=1}^{m} i|a_i| + \frac{1}{|\Delta_2|}\sum_{i=1}^{m} i|b_i| + m\Big]M_J(\|x\|,\|x\|)\\
&+ \Big[\frac{1}{|\Delta_2|}\sum_{\sigma=1}^{m}\sum_{i=\sigma}^{m}|b_i|(\eta_i-t_\sigma)^{\alpha-2}+m\Big]M_I(\|x\|,\|x\|)\\
&+ \Big[\Big(\frac{\mathbf{B}(l+1,k+1)}{\Gamma(\alpha)|\Delta_2|}\frac{|\Delta_3|}{|\Delta_1|}+\frac{\mathbf{B}(l+1,k+1)}{\Gamma(\alpha)|\Delta_1|}\Big)\sum_{i=1}^{m}|a_i| + \frac{\mathbf{B}(\alpha+l,k+1)}{\Gamma(\alpha)|\Delta_2|}\sum_{i=1}^{m}|b_i|\\
&+ \frac{\mathbf{B}(\alpha+l,k+1)}{\Gamma(\alpha)|\Delta_2|}+\frac{\mathbf{B}(l+1,k+1)}{\Gamma(\alpha)}\Big]M_f(\|x\|,\|x\|).
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
&(t-t_\sigma)^{2+\beta-\alpha}|^{RL}D_{0^+}^{\beta}(Tx)(t)|\\
&\le \frac{1}{\Gamma(\alpha-\beta+1)}\frac{1}{|\Delta_1|}\sum_{\sigma=1}^{m}\sum_{i=\sigma}^{m}|a_i|M_J(\|x\|,\|x\|)\\
&\quad + \frac{1}{\Gamma(\alpha)}\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-\beta-1)}\frac{1}{|\Delta_2|}\frac{|\Delta_3|}{|\Delta_1|}\sum_{\sigma=1}^{m}\sum_{i=\sigma}^{m}|a_i|M_J(\|x\|,\|x\|)\\
&\quad + \frac{1}{\Gamma(\alpha)}\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-\beta-1)}\frac{1}{|\Delta_2|}\sum_{\sigma=1}^{m}\sum_{i=\sigma}^{m}|b_i|M_J(\|x\|,\|x\|)\\
&\quad + \frac{\Gamma(\alpha-1)}{\Gamma(\alpha-\beta-1)}\frac{1}{|\Delta_2|}\sum_{\sigma=1}^{m}\sum_{i=\sigma}^{m}|b_i|(\eta_i-t_\sigma)^{\alpha-2}M_I(\|x\|,\|x\|)\\
&\quad + \sum_{\sigma=1}^{i}\Big(\frac{1}{\Gamma(\alpha-\beta+1)}M_J(\|x\|,\|x\|)+\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-\beta-1)}M_I(\|x\|,\|x\|)\Big)\\
&\quad + \frac{1}{\Gamma(\alpha)}\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-\beta-1)}\frac{1}{|\Delta_2|}\frac{|\Delta_3|}{|\Delta_1|}\sum_{i=1}^{m}|a_i|\int_0^{\xi_i} s^k(1-s)^l ds M_f(\|x\|,\|x\|)\\
&\quad + \frac{\Gamma(\alpha-1)}{\Gamma(\alpha-\beta-1)}\frac{1}{|\Delta_2|}\sum_{i=1}^{m}|b_i|\int_0^{\eta_i}\frac{(\eta_i-s)^{\alpha-1}}{\Gamma(\alpha)} s^k(1-s)^l ds M_f(\|x\|,\|x\|)\\
&\quad + \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} s^k(1-s)^l ds M_f(\|x\|,\|x\|)\\
&\quad + \frac{1}{\Gamma(\alpha-\beta+1)}\frac{1}{\Delta_1}\sum_{i=1}^{m}a_i\int_0^{\xi_i} s^k(1-s)^l ds M_f(\|x\|,\|x\|)\\
&\quad + (t-t_i)^{2+\beta-\alpha}\int_0^t \frac{(t-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} s^k(1-s)^l ds M_f(\|x\|,\|x\|)\\
&\le \Big[\Big(\frac{1}{\Gamma(\alpha-\beta+1)|\Delta_1|}+\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-\beta-1)}\frac{1}{\Gamma(\alpha)|\Delta_2|}\frac{|\Delta_3|}{|\Delta_1|}\Big)\sum_{i=1}^{m} i|a_i|
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{\Gamma(\alpha-1)}{\Gamma(\alpha-\beta-1)} \frac{1}{\Gamma(\alpha)|\Delta_2|} \sum_{i=1}^{m} i|b_i| + \frac{m}{\Gamma(\alpha-\beta+1)}\Big] M_J(\|x\|,\|x\|) \\
&+ \Big[\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-\beta-1)} \frac{1}{|\Delta_2|} \sum_{\sigma=1}^{m}\sum_{i=\sigma}^{m} |b_i|(\eta_i-t_\sigma)^{\alpha-2} + \frac{m\Gamma(\alpha-1)}{\Gamma(\alpha-\beta-1)}\Big] M_I(\|x\|,\|x\|) \\
&+ \Big[\frac{\mathbf{B}(l+1,k+1)}{\Gamma(\alpha)} \frac{\Gamma(\alpha-1)}{\Gamma(\alpha-\beta-1)} \frac{1}{|\Delta_2|} \frac{|\Delta_3|}{|\Delta_1|} \sum_{i=1}^{m} |a_i| \\
&+ \frac{\Gamma(\alpha-1)}{\Gamma(\alpha-\beta-1)} \frac{1}{|\Delta_2|} \sum_{i=1}^{m} |b_i| \frac{\mathbf{B}(\alpha+l,k+1)}{\Gamma(\alpha)} + \frac{\mathbf{B}(\alpha+l,k+1)}{\Gamma(\alpha)} \\
&+ \frac{\mathbf{B}(l+1,k+1)}{\Gamma(\alpha-\beta+1)} \frac{1}{\Delta_1} + \frac{\mathbf{B}(\alpha-\beta+l,k+1)}{\Gamma(\alpha-\beta)}\Big] M_f(\|x\|,\|x\|).
\end{aligned}
$$

It follows that

$$
\|Tx\| \le A_1 M_J(\|x\|,\|x\|) + A_2 M_I(\|x\|,\|x\|) + A_3 M_f(\|x\|,\|x\|).
$$

From the assumption, we choose $\Omega = \{x \in X : \|x\| \le r_0\}$. For $x \in \partial\Omega$, we obtain $x \ne \lambda(Tx)$ for any $\lambda \in [0,1]$. In fact, if there exists $x \in \partial\Omega$ such that $x = \lambda(Tx)$ for some $\lambda \in [0,1]$. Then

$$
r_0 = \|x\| = \lambda\|Tx\| < \|Tx\| \le A_1 M_J(r_0,r_0) + A_2 M_I(r_0,r_0) + A_3 M_f(r_0,r_0) \le r_0,
$$

which is a contradiction.

As a consequence of Schaefer's fixed point theorem, we deduce that T has a fixed point which is a solution of problem (5.15). The proof is complete. □

Theorem 5.11. *Suppose that* (5.21), (5.A1)-(5.A4), (H1) *hold and*

$$
\lim_{r\to 0^+} \frac{M_f(r,r)}{r} = \lim_{r\to 0^+} \frac{M_I(r,r)}{r} = \lim_{r\to 0^+} \frac{M_J(r,r)}{r} = 0
$$

or

$$
\lim_{r\to +\infty} \frac{M_f(r,r)}{r} = \lim_{r\to +\infty} \frac{M_I(r,r)}{r} = \lim_{r\to +\infty} \frac{M_J(r,r)}{r} = 0.
$$

Then (5.15) *has at least one solution.*

Proof. Let X be defined above. By Lemma 5.9, we know that x is a solution of (5.15) if and only if x is a fixed point of T. By a standard proof, we can see that $T : X \to X$ is a completely continuous operator.

From (H1), as in the proof of Theorem 5.10, we have

$$
\|Tx\| \le A_1 M_J(\|x\|,\|x\|) + A_2 M_I(\|x\|,\|x\|) + A_3 M_f(\|x\|,\|x\|).
$$

Choose $\delta_0 > 0$ such that $(A_1 + A_2 + A_3)\delta_0 \le 1$. By the assumption, we know that there exist a constant $M > 0$ such that

$$
M_f(r,r) \le \delta_0 r, \quad M_I(r,r) \le \delta_0 r, \quad M_J(r,r) \le \delta_0 r, \quad r \in [0,M] \text{ or } r \in [M,+\infty).
$$

We choose $\Omega = \{x \in X : \|x\| < M\}$. Then Ω is an open bounded subset of X and $0 \in \Omega$. For $x \in \partial\Omega$, we have $\|x\| = M$. Thus

$$
\begin{aligned}
\|Tx\| &\le A_1 M_J(\|x\|,\|x\|) + A_2 M_I(\|x\|,\|x\|) + A_3 M_f(\|x\|,\|x\|) \\
&\le A_1\delta_0 M + A_2\delta_0 M + A_3\delta_0 M \le M = \|x\|.
\end{aligned}
$$

As a consequence of Theorem 3.10, we deduce that T has a fixed point which is a solution of (5.15). The proof is complete. □

5.2. **Impulsive Sturm-Liouville boundary value problems.** Zhang and Feng [130] studied the Sturm-Liouville boundary value problem of impulsive fractional differential equation

$$\begin{gathered} {}^C D_{0^+}^q x(t) = \omega(t) f(t, x(t), x'(t)), \quad t \in (t_i, t_{i+1}],\ i \in \mathbb{N}[0, m], \\ \Delta x(t_i) = I_i(x(t_i)), \quad i \in \mathbb{N}[1, m], \\ \Delta x'(t_i) = J_i(x(t_i)), \quad i \in \mathbb{N}[1, m], \\ \alpha_1 x(0) - \beta_1 x'(0) = \alpha_2 x(1) + \beta_2 x'(1) = 0, \end{gathered} \tag{5.31}$$

where $q \in (1, 2]$, ${}^C D_{0^+}^q u$ is the Caputo fractional derivative, $\omega : [0, 1] \to [0, +\infty)$, $f : [0, 1] \times \mathbb{R}^2 \to \mathbb{R}$, $I_i, J_i : \mathbb{R} \to \mathbb{R}$ are continuous functions, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = 1$, $\alpha_1\alpha_2 + \alpha_1\beta_2 + \alpha_2\beta_1 \neq 0$. The existence and uniqueness of solutions of BVP(5.31) were established under the assumptions that f, I_i, J_i are bounded functions. The following theorem was proved in [130].

Theorem 5.12 ([130])**.** *Suppose that $\sigma \in C[0, 1]$, I_i, J_i are continuous, and $\eta = \alpha_1\alpha_2 + \alpha_1\beta_2 + \alpha_1\beta_1 \neq 0$. The solution of the problem*

$$\begin{gathered} {}^C D_{0^+}^q x(t) = \sigma(t), \quad t \in (t_i, t_{i+1}],\ i \in \mathbb{N}[0, m], \\ \Delta x(t_i) = I_i(x(t_i)), \quad i \in \mathbb{N}[1, m], \\ \Delta x'(t_i) = J_i(x(t_i)), \quad i \in \mathbb{N}[1, m], \\ \alpha_1 x(0) - \beta_1 x'(0) = \alpha_2 x(1) + \beta_2 x'(1) = 0, \end{gathered} \tag{5.32}$$

can be expressed as

$$\begin{aligned} x(t) &= \int_{t_k}^t \frac{(t-s)^{q-1}}{\Gamma(q)} \sigma(s) ds + \sum_{i=1}^{m+1} G'_{1s}(t, t_i) \int_{t_{i-1}}^{t_i} \frac{(t_i - s)^{q-1}}{\Gamma(q)} \sigma(s) ds \\ &\quad - \sum_{i=1}^{n} G_1(t, t_i) \int_{t_{i-1}}^{t_i} \frac{(t_i - s)^{q-2}}{\Gamma(q-1)} \sigma(s) ds + \sum_{i=1}^{n} G'_{1s}(t, t_i) I_i(x(t_i)) \\ &\quad - \sum_{i=1}^{n} G_1(t, t_i) J_i(x(t_i)), \quad t \in (t_k, t_{k+1}],\ k \in \mathbb{N}[0, m], \end{aligned} \tag{5.33}$$

where

$$G_1(t, s) = -\frac{1}{\eta} \begin{cases} (\beta_1 + \alpha_1 t)(\alpha_2(1-s) + \beta_2), & t \le s, \\ (\beta_1 + \alpha_1 s)(\alpha_2(1-t) + \beta_2), & s \le t. \end{cases}$$

This result is wrong. In fact, suppose that u is a solution of (5.32). By Theorem 3.11 (with $\lambda = 0$), we know from ${}^C D_{0^+}^q x(t) = \sigma(t)$, $t \in (t_i, t_{i+1}]$, $i \in \mathbb{N}[0, m]$ that there exist constants $c_{\sigma v} \in \mathbb{R}$ such that

$$x(t) = \sum_{\sigma=0}^{i} c_\sigma + \sum_{\sigma=0}^{i} d_\sigma (t - t_\sigma) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \sigma(s) ds, \quad t \in (t_i, t_{i+1}],\ i \in N[0, m].$$

Then

$$x'(t) = \sum_{\sigma=0}^{i} d_\sigma + \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} \sigma(s) ds, \quad t \in (t_i, t_{i+1}],\ i \in N[0, m].$$

From $\Delta x(t_i) = I_i(x(t_i)), \Delta x'(t_i) = J_i(x(t_i)), i \in \mathbb{N}[1, m]$, we obtain $c_i = I_i(x(t_i))$ and $d_i = J_i(x(t_i)), i \in \mathbb{N}[1, m]$. By $\alpha_1 x(0) - \beta_1 x'(0) = \alpha_2 x(1) + \beta_2 x'(1) = 0$, we have

$$\begin{gathered}\alpha_1c_0-\beta_1d_0=0,\\ \alpha_2\Big[\sum_{\sigma=0}^{m}c_\sigma+\sum_{\sigma=0}^{m}d_\sigma(1-t_\sigma)+\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\sigma(s)ds\Big]\\ +\beta_2\Big[\sum_{\sigma=0}^{m}d_\sigma+\int_0^1\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\sigma(s)ds\Big]=0.\end{gathered}$$

It follows that

$$\begin{gathered}\alpha_1c_0-\beta_1d_0=0,\\ \alpha_2c_0+(\alpha_2+\beta_2)d_0=-\alpha_2\sum_{\sigma=1}^{m}I_\sigma(x(t_\sigma))-\sum_{\sigma=1}^{m}[\alpha_2(1-t_\sigma)+\beta_2]J_\sigma(x(t_\sigma))\\ -\int_0^1[\frac{\alpha_2(1-s)^{\alpha-1}}{\Gamma(\alpha)}+\frac{\beta_2(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}]\sigma(s)ds.\end{gathered}$$

Then

$$\begin{aligned}c_0=\beta_1\Big(&-\alpha_2\sum_{\sigma=1}^{m}I_\sigma(x(t_\sigma))-\sum_{\sigma=1}^{m}[\alpha_2(1-t_\sigma)+\beta_2]J_\sigma(x(t_\sigma))\\ &-\int_0^1[\frac{\alpha_2(1-s)^{\alpha-1}}{\Gamma(\alpha)}-\frac{\beta_2(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}]\sigma(s)ds\Big)/\eta,\\ d_0=\alpha_1\Big(&-\alpha_2\sum_{\sigma=1}^{m}I_\sigma(x(t_\sigma))-\sum_{\sigma=1}^{m}[\alpha_2(1-t_\sigma)+\beta_2]J_\sigma(x(t_\sigma))\\ &-\int_0^1[\frac{\alpha_2(1-s)^{\alpha-1}}{\Gamma(\alpha)}+\frac{\beta_2(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}]\sigma(s)ds\Big)/\eta.\end{aligned}$$

Hence

$$\begin{aligned}x(t)=\beta_1\Big(&-\alpha_2\sum_{\sigma=1}^{m}I_\sigma(x(t_\sigma))-\sum_{\sigma=1}^{m}[\alpha_2(1-t_\sigma)+\beta_2]J_\sigma(x(t_\sigma))\\ &-\int_0^1[\frac{\alpha_2(1-s)^{\alpha-1}}{\Gamma(\alpha)}-\frac{\beta_2(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}]\sigma(s)ds\Big)/\eta\\ &+\alpha_1\Big(-\alpha_2\sum_{\sigma=1}^{m}I_\sigma(x(t_\sigma))-\sum_{\sigma=1}^{m}[\alpha_2(1-t_\sigma)+\beta_2]J_\sigma(x(t_\sigma))\\ &-\int_0^1[\frac{\alpha_2(1-s)^{\alpha-1}}{\Gamma(\alpha)}+\frac{\beta_2(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}]\sigma(s)ds\Big)t/\eta\\ &\times\sum_{\sigma=1}^{i}I_\sigma(x(t_\sigma))+\sum_{\sigma=1}^{i}J_\sigma(x(t_\sigma))(t-t_\sigma)+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\sigma(s)ds,\end{aligned}$$

for $t\in(t_i,t_{i+1}]$ and $i\in N[0,m]$. This is the correct expression of solutions of (5.32). This result shows that [130, Theorem ZF] is wrong.

Theorem 5.13. *Suppose that $\sigma\in C[0,1]$, I_i,J_i are continuous, and*

$$\zeta=\alpha_1\Big[\alpha_2\sum_{\tau=0}^{m-1}(t_{\tau+1}-t_\tau)+[\alpha_2+\beta_2](1-t_m)\Big]+\alpha_2\beta_1\neq0. \tag{5.34}$$

Then the solution of BVP for fractional differential equation with multiple starting points t_i,

$$\begin{gathered}
{}^C D^q_{t_i^+} x(t) = \sigma(t), \quad t \in (t_i, t_{i+1}],\ i \in \mathbb{N}[0,m], \\
\Delta x(t_i) = I_i(x(t_i)), \quad i \in \mathbb{N}[1,m], \\
\Delta x'(t_i) = J_i(x(t_i)), \quad i \in \mathbb{N}[1,m], \\
\alpha_1 x(0) - \beta_1 x'(0) = \alpha_2 x(1) + \beta_2 x'(1) = 0,
\end{gathered} \tag{5.35}$$

can be expressed as

$$x(t) = \begin{cases}
c_0 + d_0 t + \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}\sigma(s)ds, \quad t \in (0,t_1], \\
c_0 + \sum_{\tau=0}^{i-1}(t_{\tau+1}-t_\tau)d_0 + \sum_{j=1}^{i-1}\sum_{\tau=j}^{i-1}(t_{\tau+1}-t_\tau)J_j(x(t_j)) \\
+ \sum_{\tau=1}^{i} I_\tau(x(t_\tau)) + \sum_{\tau=0}^{i-1}\int_{t_\tau}^{t_{\tau+1}} \frac{(t_{\tau+1}-s)^{\alpha-1}}{\Gamma(\alpha)}\sigma(s)ds \\
+ \sum_{j=0}^{i-2}\sum_{\tau=j+1}^{i-1}(t_{\tau+1}-t_\tau)\int_{t_j}^{t_{j+1}} \frac{(t_{j+1}-s)^{\alpha-2}}{\Gamma(\alpha-1)}\sigma(s)ds \\
+ \big[d_0 + \sum_{j=0}^{i-1}\int_{t_j}^{t_{j+1}} \frac{(t_{j+1}-s)^{\alpha-2}}{\Gamma(\alpha-1)}\sigma(s)ds + \sum_{j=1}^{i} J_j(x(t_j))\big](t-t_i) \\
+ \int_{t_i}^{t} \frac{(t-s)^{q-1}}{\Gamma(q)}\sigma(s)ds, \quad t \in (t_i, t_{i+1}],\ i \in \mathbb{N}[1,m].
\end{cases} \tag{5.36}$$

Here c_0, d_0 are defined by

$$\begin{aligned}
d_0 = -\frac{\alpha_1}{\zeta}\Big[&\alpha_2 \sum_{j=1}^{m-1}\sum_{\tau=j}^{m-1}(t_{\tau+1}-t_\tau)J_j(x(t_j)) + \sum_{j=1}^{m} J_j(x(t_j)) + \alpha_2 \sum_{\tau=1}^{m} I_\tau(x(t_\tau)) \\
&+ \alpha_2 \sum_{\tau=0}^{m-1}\int_{t_\tau}^{t_{\tau+1}} \frac{(t_{\tau+1}-s)^{\alpha-1}}{\Gamma(\alpha)}\sigma(s)ds \\
&+ \alpha_2 \sum_{j=0}^{m-2}\sum_{\tau=j+1}^{m-1}(t_{\tau+1}-t_\tau)\int_{t_j}^{t_{j+1}} \frac{(t_{j+1}-s)^{\alpha-2}}{\Gamma(\alpha-1)}\sigma(s)ds \\
&+ [\alpha_2+\beta_2](1-t_m)\sum_{j=0}^{m-1}\int_{t_j}^{t_{j+1}} \frac{(t_{j+1}-s)^{\alpha-2}}{\Gamma(\alpha-1)}\sigma(s)ds \\
&+ \alpha_2 \int_{t_m}^{1} \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\sigma(s)ds + \beta_2 \int_{t_m}^{1} \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\sigma(s)ds\Big],
\end{aligned}$$

$$\begin{aligned}
c_0 = -\frac{\beta_1}{\zeta}\Big[&\alpha_2 \sum_{j=1}^{m-1}\sum_{\tau=j}^{m-1}(t_{\tau+1}-t_\tau)J_j(x(t_j)) + \sum_{j=1}^{m} J_j(x(t_j)) + \alpha_2 \sum_{\tau=1}^{m} I_\tau(x(t_\tau)) \\
&+ \alpha_2 \sum_{\tau=0}^{m-1}\int_{t_\tau}^{t_{\tau+1}} \frac{(t_{\tau+1}-s)^{\alpha-1}}{\Gamma(\alpha)}\sigma(s)ds \\
&+ \alpha_2 \sum_{j=0}^{m-2}\sum_{\tau=j+1}^{m-1}(t_{\tau+1}-t_\tau)\int_{t_j}^{t_{j+1}} \frac{(t_{j+1}-s)^{\alpha-2}}{\Gamma(\alpha-1)}\sigma(s)ds \\
&+ [\alpha_2+\beta_2](1-t_m)\sum_{j=0}^{m-1}\int_{t_j}^{t_{j+1}} \frac{(t_{j+1}-s)^{\alpha-2}}{\Gamma(\alpha-1)}\sigma(s)ds \\
&+ \alpha_2 \int_{t_m}^{1} \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\sigma(s)ds + \beta_2 \int_{t_m}^{1} \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\sigma(s)ds\Big].
\end{aligned}$$

Proof. Suppose that x is a solution of (5.35). From Theorem 3.11 and ${}^CD^q_{t_i^+}x(t) = \sigma(t), t \in (t_i, t_{i+1}]$, there exist constants $c_i, d_i \in \mathbb{R}$ such that

$$x(t) = c_i + d_i(t - t_i) + \int_{t_i}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\sigma(s)ds, \quad t \in (t_i, t_{i+1}],\ i \in N[0, m].$$

Then

$$x'(t) = d_i + \int_{t_i}^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}\sigma(s)ds, \quad t \in (t_i, t_{i+1}],\ i \in N[0, m].$$

By $\Delta x(t_i) = I_i(x(t_i)), \Delta x'(t_i) = J_i(x(t_i)), i \in \mathbb{N}[1, m]$, we obtain

$$c_i - \Big[c_{i-1} + d_{i-1}(t_i - t_{i-1}) + \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)}\sigma(s)ds\Big] = I_i(x(t_i)),$$
$$d_i - \Big[d_{i-1} + \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}\sigma(s)ds\Big] = J_i(x(t_i)), \quad i \in \mathbb{N}[1, m].$$

It follows that

$$d_i = d_0 + \sum_{j=0}^{i-1}\int_{t_j}^{t_{j+1}} \frac{(t_{j+1}-s)^{\alpha-2}}{\Gamma(\alpha-1)}\sigma(s)ds + \sum_{j=1}^{i} J_j(x(t_j)), \quad i \in \mathbb{N}[1, m].$$

Then

$$\begin{aligned}
c_i &= c_{i-1} + (t_i - t_{i-1})d_0 + (t_i - t_{i-1})\sum_{j=1}^{i-1} J_j(x(t_j)) + I_i(x(t_i))\\
&\quad + \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)}\sigma(s)ds + (t_i - t_{i-1})\sum_{j=0}^{i-2}\int_{t_j}^{t_{j+1}} \frac{(t_{j+1}-s)^{\alpha-2}}{\Gamma(\alpha-1)}\sigma(s)ds\\
&= c_0 + \sum_{\tau=0}^{i-1}(t_{\tau+1} - t_\tau)d_0 + \sum_{\tau=0}^{i-1}(t_{\tau+1} - t_\tau)\sum_{j=1}^{\tau} J_j(x(t_j)) + \sum_{\tau=1}^{i} I_\tau(x(t_\tau))\\
&\quad + \sum_{\tau=0}^{i-1}\int_{t_\tau}^{t_{\tau+1}} \frac{(t_{\tau+1}-s)^{\alpha-1}}{\Gamma(\alpha)}\sigma(s)ds\\
&\quad + \sum_{\tau=0}^{i-1}(t_{\tau+1} - t_\tau)\sum_{j=0}^{\tau-1}\int_{t_j}^{t_{j+1}} \frac{(t_{j+1}-s)^{\alpha-2}}{\Gamma(\alpha-1)}\sigma(s)ds\\
&= c_0 + \sum_{\tau=0}^{i-1}(t_{\tau+1} - t_\tau)d_0 + \sum_{j=1}^{i-1}\sum_{\tau=j}^{i-1}(t_{\tau+1} - t_\tau)J_j(x(t_j)) + \sum_{\tau=1}^{i} I_\tau(x(t_\tau))\\
&\quad + \sum_{\tau=0}^{i-1}\int_{t_\tau}^{t_{\tau+1}} \frac{(t_{\tau+1}-s)^{\alpha-1}}{\Gamma(\alpha)}\sigma(s)ds\\
&\quad + \sum_{j=0}^{i-2}\sum_{\tau=j+1}^{i-1}(t_{\tau+1} - t_\tau)\int_{t_j}^{t_{j+1}} \frac{(t_{j+1}-s)^{\alpha-2}}{\Gamma(\alpha-1)}\sigma(s)ds, \quad i \in \mathbb{N}[1, m].
\end{aligned}$$

By $\alpha_1 x(0) - \beta_1 x'(0) = \alpha_2 x(1) + \beta_2 x'(1) = 0$, we have

$$\alpha_1 c_0 - \beta_1 d_0 = 0,$$

$$\begin{aligned}&\alpha_2\Big[c_m+d_m(1-t_m)+\int_{t_m}^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\sigma(s)ds\Big]\\&+\beta_2\Big[d_m+\int_{t_m}^1\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\sigma(s)ds\Big]=0.\end{aligned}$$

It follows that

$$\alpha_1c_0-\beta_1d_0=0,$$

$$\begin{aligned}&\alpha_2c_0+\Big[\alpha_2\sum_{\tau=0}^{m-1}(t_{\tau+1}-t_\tau)+[\alpha_2+\beta_2](1-t_m)\Big]d_0\\&+\alpha_2\sum_{j=1}^{m-1}\sum_{\tau=j}^{m-1}(t_{\tau+1}-t_\tau)J_j(x(t_j))+\sum_{j=1}^{m}J_j(x(t_j))+\alpha_2\sum_{\tau=1}^{m}I_\tau(x(t_\tau))\\&+\alpha_2\sum_{\tau=0}^{m-1}\int_{t_\tau}^{t_{\tau+1}}\frac{(t_{\tau+1}-s)^{\alpha-1}}{\Gamma(\alpha)}\sigma(s)ds\\&+\alpha_2\sum_{j=0}^{m-2}\sum_{\tau=j+1}^{m-1}(t_{\tau+1}-t_\tau)\int_{t_j}^{t_{j+1}}\frac{(t_{j+1}-s)^{\alpha-2}}{\Gamma(\alpha-1)}\sigma(s)ds\\&+[\alpha_2+\beta_2](1-t_m)\sum_{j=0}^{m-1}\int_{t_j}^{t_{j+1}}\frac{(t_{j+1}-s)^{\alpha-2}}{\Gamma(\alpha-1)}\sigma(s)ds\\&+\alpha_2\int_{t_m}^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\sigma(s)ds+\beta_2\int_{t_m}^1\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\sigma(s)ds=0.\end{aligned}$$

Hence

$$\begin{aligned}d_0=&-\frac{\alpha_1}{\zeta}\Big[\alpha_2\sum_{j=1}^{m-1}\sum_{\tau=j}^{m-1}(t_{\tau+1}-t_\tau)J_j(x(t_j))+\sum_{j=1}^{m}J_j(x(t_j))+\alpha_2\sum_{\tau=1}^{m}I_\tau(x(t_\tau))\\&+\alpha_2\sum_{\tau=0}^{m-1}\int_{t_\tau}^{t_{\tau+1}}\frac{(t_{\tau+1}-s)^{\alpha-1}}{\Gamma(\alpha)}\sigma(s)ds\\&+\alpha_2\sum_{j=0}^{m-2}\sum_{\tau=j+1}^{m-1}(t_{\tau+1}-t_\tau)\int_{t_j}^{t_{j+1}}\frac{(t_{j+1}-s)^{\alpha-2}}{\Gamma(\alpha-1)}\sigma(s)ds\\&+[\alpha_2+\beta_2](1-t_m)\sum_{j=0}^{m-1}\int_{t_j}^{t_{j+1}}\frac{(t_{j+1}-s)^{\alpha-2}}{\Gamma(\alpha-1)}\sigma(s)ds\\&+\alpha_2\int_{t_m}^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\sigma(s)ds+\beta_2\int_{t_m}^1\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\sigma(s)ds\Big],\end{aligned}$$

$$\begin{aligned}c_0=&-\frac{\beta_1}{\zeta}\Big[\alpha_2\sum_{j=1}^{m-1}\sum_{\tau=j}^{m-1}(t_{\tau+1}-t_\tau)J_j(x(t_j))+\sum_{j=1}^{m}J_j(x(t_j))+\alpha_2\sum_{\tau=1}^{m}I_\tau(x(t_\tau))\\&+\alpha_2\sum_{\tau=0}^{m-1}\int_{t_\tau}^{t_{\tau+1}}\frac{(t_{\tau+1}-s)^{\alpha-1}}{\Gamma(\alpha)}\sigma(s)ds\end{aligned}$$

$$
\begin{aligned}
&+\alpha_2\sum_{j=0}^{m-2}\sum_{\tau=j+1}^{m-1}(t_{\tau+1}-t_\tau)\int_{t_j}^{t_{j+1}}\frac{(t_{j+1}-s)^{\alpha-2}}{\Gamma(\alpha-1)}\sigma(s)ds\\
&+[\alpha_2+\beta_2](1-t_m)\sum_{j=0}^{m-1}\int_{t_j}^{t_{j+1}}\frac{(t_{j+1}-s)^{\alpha-2}}{\Gamma(\alpha-1)}\sigma(s)ds\\
&+\alpha_2\int_{t_m}^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\sigma(s)ds+\beta_2\int_{t_m}^1\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\sigma(s)ds\Big].
\end{aligned}
$$

It follows that

$$
x(t)=\begin{cases}
c_0+d_0t+\int_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}\sigma(s)ds, \quad t\in(0,t_1],\\
c_0+\sum_{\tau=0}^{i-1}(t_{\tau+1}-t_\tau)d_0+\sum_{j=1}^{i-1}\sum_{\tau=j}^{i-1}(t_{\tau+1}-t_\tau)J_j(x(t_j))\\
+\sum_{\tau=1}^{i}I_\tau(x(t_\tau))+\sum_{\tau=0}^{i-1}\int_{t_\tau}^{t_{\tau+1}}\frac{(t_{\tau+1}-s)^{\alpha-1}}{\Gamma(\alpha)}\sigma(s)ds\\
+\sum_{j=0}^{i-2}\sum_{\tau=j+1}^{i-1}(t_{\tau+1}-t_\tau)\int_{t_j}^{t_{j+1}}\frac{(t_{j+1}-s)^{\alpha-2}}{\Gamma(\alpha-1)}\sigma(s)ds\\
+\Big[d_0+\sum_{j=0}^{i-1}\int_{t_j}^{t_{j+1}}\frac{(t_{j+1}-s)^{\alpha-2}}{\Gamma(\alpha-1)}\sigma(s)ds+\sum_{j=1}^{i}J_j(x(t_j))\Big](t-t_i)\\
+\int_{t_i}^t\frac{(t-s)^{q-1}}{\Gamma(q)}\sigma(s)ds, \quad t\in(t_i,t_{i+1}],\ i\in\mathbb{N}[1,m].
\end{cases}
$$

This is (5.36). From Theorem 5.13, Result 6.26 is wrong even $D_{0^+}^q$ with a single starting point in (5.32) is replaced by $D_{t_i^+}^q$ with multiple starting points $\{t_i\}$. □

Karaca and Tokmak [54] studied the kind of impulsive Sturm-Liouville boundary value problem

$$
\begin{gathered}
{}'=f(t,x(t)), \quad t\in(t_i,t_{i+1}],\ i\in\mathbb{N}[0,m],\\
\Delta x(t_i)=I_i(x(t_i)), \quad i\in\mathbb{N}[1,m],\\
\Delta x'(t_i)=J_i(x(t_i)), \quad i\in\mathbb{N}[1,m],\\
\alpha_1x(0)-\beta_1x'(0)=\alpha_2x(1)+\beta_2x'(1)=0,
\end{gathered}
\tag{5.37}
$$

where $\beta\in(0,1]$, ${}^CD_{0^+}^\beta u$ is the Caputo fractional derivative, $f:[0,1]\times\mathbb{R}\to\mathbb{R}$, $I_i,J_i:\mathbb{R}\to\mathbb{R}$ are continuous functions, $0=t_0<t_1<\dots<t_m<t_{m+1}=1$, $\alpha_1\alpha_2+\alpha_1\beta_2+\alpha_2\beta_1\neq0$. The existence and uniqueness of solutions of (5.37) were established under the assumptions that f, I_i,J_i are bounded functions.

In [139, 140], the authors studied the existence and uniqueness of solution for the boundary value problems for the semilinear impulsive fractional integro-differential equations:

$$
\begin{gathered}
{}^CD^qx(t)=\lambda x(t)+f(t,x(t),(Kx)(t),(Hx)(t)), \quad t\in(t_i,t_{i+1}],\ i\in\mathbb{N}[0,m],\\
\Delta x(t_i)=I_i(x(t_i)), \quad i\in\mathbb{N}[1,m],\\
\Delta x'(t_i)=J_i(x(t_i)), \quad i\in\mathbb{N}[1,m],\\
\alpha_1x(0)-\beta_1x'(0)=x_0, \quad \alpha_2x(1)+\beta_2x'(1)=x_1,
\end{gathered}
$$

where $q\in(1,2]$, $\lambda\geq0$, $\alpha_1\geq0$, $\beta_1>0$, $\alpha_2\geq0$, $\beta_2>0$, $\Gamma=\alpha_1\alpha_2+\alpha_1\beta_2+\alpha_2\beta_1\neq0$, and $x_0,x_1\in\mathbb{R}$, $f:[0,1]\times\mathbb{R}^3\to\mathbb{R}$, $I_i,J_i:\mathbb{R}\to\mathbb{R}$ are continuous functions, $0=t_0<t_1<\dots<t_m<t_{m+1}=1$, and H,K are integral operators with integral kernels

$$
(Kx)(t)=\int_0^tk(t,s)x(s)ds, \quad (Hx)(t)=\int_0^1h(t,s)x(s)ds, \quad t\in[0,1],
$$

where $k : D_0 \to \mathbb{R}$ and $h : D \to \mathbb{R}$ satisfies $\sup_{t\in[0,1]} \int_0^t k(t,s)|ds < +\infty$ and $\sup_{t\in[0,1]} \int_0^1 h(t,s)|ds < +\infty$. The existence and uniqueness of solutions were established under the assumptions that f, I_i, J_i are bounded functions and satisfies the Lipschitz conditions. Some examples were presented in [139, 140]. These examples are unsuitable. Since [139, Lemma 2.3]) and [140, Lemma 2.3]) are from [99] in which the derivative is the Caputo derivative with multiple start point $t_i (i \in N_0)$, see [99, Lemma 2.2] and [100]. So ${}^C D_{0^+}^q$ in examples in [139, 140] should be replaced by the one with multiple start point $t_i (i \in N_0)$. But The derivative in Examples in [139, 140] is the Riemann-Liouville derivative with single start point $t = 0$.

There has been no papers concerning with the solvability of Sturm-Liouville boundary value problems of impulsive fractional differential equations involving the other fractional derivatives such as the Riemman-Liouville fractional derivatives. Now we consider the problem

$$\begin{gathered}
{}^{RL}D_{0^+}^{\alpha} x(t) = p(t) f(t, x(t), {}^{RL}D_{0^+}^{\beta} x(t)), \quad t \in (t_i, t_{i+1}],\ i \in \mathbb{N}[0, m], \\
\lim_{t\to t_i^+} (t - t_i)^{2-\alpha} x(t) = I(t_i, x(t_i), {}^{RL}D_{0^+}^{\beta} x(t_i)), \quad i \in \mathbb{N}[1, m], \\
\Delta {}^{RL}D_{0^+}^{\alpha-1} x(t_i) = J(t_i, x(t_i), {}^{RL}D_{0^+}^{\beta} x(t_i)), \quad i \in \mathbb{N}[1, m], \\
\alpha_1 \lim_{t\to 0^+} t^{2-\alpha} x(t) - \beta_1 {}^{RL}D_{0^+}^{\alpha-1} x(0) = \alpha_2 x(1) + \beta_2 {}^{RL}D_{0^+}^{\alpha-1} x(1) = 0,
\end{gathered} \tag{5.38}$$

where $\alpha \in (1, 2]$, $\beta \in (0, \alpha - 1]$, ${}^{RL}D_{0^+}^{*} u$ is the Riemann-Liouville fractional derivative of order $*$, $p : (0, 1) \to \mathbb{R}$ satisfies (5.A4) in (5.15), $f : (0, 1) \times \mathbb{R}^2 \to \mathbb{R}$ satisfies (5.A2) in (5.15), and $I, J : \{t_i : i \in \mathbb{N}[1, m] \times \mathbb{R}^2 \to \mathbb{R}$ in (5.15) satisfies (5.A3) , $0 = t_0 < t_1 < \dots < t_m < t_{m+1} = 1$, $\alpha_1\alpha_2 + \alpha_1\beta_2 + \alpha_2\beta_1 \neq 0$.

Lemma 5.14. *Suppose that $\Theta = \alpha_1\alpha_2 + \Gamma(\alpha)\alpha_1\beta_2 + \Gamma(\alpha)\alpha_2\beta_1 \neq 0$, σ is continuous on $(0, 1)$ and there exist $k > -1$ and $l \in (\max\{-\alpha, -2 - k\}, 0]$ such that $|\sigma(t)| \le t^k (1 - t)^l$ for all $t \in (0, 1)$. Then x is a solution of the problem*

$$\begin{gathered}
{}^{RL}D_{0^+}^{\alpha} x(t) = \sigma(t), \quad t \in (t_i, t_{i+1}],\ i \in \mathbb{N}[0, m], \\
\lim_{t\to t_i^+} (t - t_i)^{2-\alpha} x(t) = I_i, \quad i \in \mathbb{N}[1, m], \\
\Delta {}^{RL}D_{0^+}^{\alpha-1} x(t_i) = J_i, \quad i \in \mathbb{N}[1, m], \\
\alpha_1 \lim_{t\to 0^+} t^{2-\alpha} x(t) - \beta_1 {}^{RL}D_{0^+}^{\alpha-1} x(0) = \alpha_2 x(1) + \beta_2 {}^{RL}D_{0^+}^{\alpha-1} x(1) = 0,
\end{gathered} \tag{5.39}$$

if and only if

$$\begin{aligned}
&x(t) \\
&= \frac{\alpha_1}{\Theta} \Big[\sum_{\sigma=1}^{m} \frac{(1 - t_\sigma)^{\alpha-1}}{\Gamma(\alpha)} J_\sigma + \sum_{\sigma=1}^{m} (1 - t_\sigma)^{\alpha-2} I_\sigma + \beta_2 \int_0^1 \sigma(s) ds \\
&\quad + \alpha_2 \int_0^1 \frac{(1 - s)^{\alpha-1}}{\Gamma(\alpha)} \sigma(s) ds \Big] t^{\alpha-1} - \frac{\Gamma(\alpha)\beta_1}{\Theta} \Big[\sum_{\sigma=1}^{m} \frac{(1 - t_\sigma)^{\alpha-1}}{\Gamma(\alpha)} J_\sigma \\
&\quad + \sum_{\sigma=1}^{m} (1 - t_\sigma)^{\alpha-2} I_\sigma + \beta_2 \int_0^1 \sigma(s) ds + \alpha_2 \int_0^1 \frac{(1 - s)^{\alpha-1}}{\Gamma(\alpha)} \sigma(s) ds \Big] t^{\alpha-2} \\
&\quad + \sum_{\sigma=1}^{i} \frac{(1 - t_\sigma)^{\alpha-1}}{\Gamma(\alpha)} J_\sigma + \sum_{\sigma=1}^{i} (1 - t_\sigma)^{\alpha-2} I_\sigma + \int_0^t \frac{(t - s)^{\alpha-1}}{\Gamma(\alpha)} \sigma(s) ds,
\end{aligned} \tag{5.40}$$

for $t\in(t_i,t_{i+1}]$ and $i\in N[0,m]$.

Proof. By Theorem 3.12, we have ${}^{RL}D_{0^+}^{\alpha}x(t)=\sigma(t)$, $t\in(t_i,t_{i+1}]$, $i\in\mathbb{N}[0,m]$ if and only if there exit numbers $c_{\sigma 1},c_{\sigma 2},i\in\mathbb{N}[0,m]$ such that

$$x(t)=\sum_{\sigma=0}^{i}[c_{\sigma 1}(t-t_\sigma)^{\alpha-1}+c_{\sigma 2}(t-t_\sigma)^{\alpha-2}]+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\sigma(s)ds, \tag{5.41}$$

for $t\in(t_i,t_{i+1}]$ and $i\in\mathbb{N}[0,m]$. Then

$${}^{RL}D_{0^+}^{\alpha-1}x(t)=\Gamma(\alpha)\sum_{\sigma=0}^{i}c_{\sigma 1}+\int_0^t\sigma(s)ds,\quad t\in(t_i,t_{i+1}],\ i\in\mathbb{N}[0,m]. \tag{5.42}$$

Furthermore, by direct computations we have

$$\begin{aligned}
&{}^{RL}D_{0^+}^{\beta}x(t)\\
&=\frac{1}{\Gamma(1-\beta)}\Big[\int_0^t(t-s)^{-\beta}x(s)ds\Big]'\\
&=\frac{1}{\Gamma(1-\beta)}\Big[\sum_{j=0}^{i-1}\int_{t_j}^{t_{j+1}}(t-s)^{-\beta}x(s)ds\Big]'+\frac{1}{\Gamma(1-\beta)}\Big[\int_{t_i}^t(t-s)^{-\beta}x(s)ds\Big]'\\
&=\sum_{\sigma=0}^{i}\Big(c_{\sigma 1}\frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta+1)}(t-t_\sigma)^{\alpha-\beta-1}+c_{\sigma 2}\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-\beta-1)}(t-t_\sigma)^{\alpha-\beta-2}\Big)\\
&\quad+\int_0^t\frac{(t-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\sigma(s)ds,\quad t\in(t_i,t_{i+1}],\ i\in\mathbb{N}[0,m].
\end{aligned} \tag{5.43}$$

From $\lim_{t\to t_i^+}(t-t_i)^{2-\alpha}x(t)=I_i$, we have $c_{\sigma 2}=I_\sigma$ for all $\sigma\in\mathbb{N}[1,m]$. From $\Delta^{RL}D_{0^+}^{\alpha-1}x(t_i))=J_i$, $i\in\mathbb{N}[1,m]$, one has $c_{\sigma 1}=\frac{J_\sigma}{\Gamma(\alpha)}$ for all $\sigma\in\mathbb{N}[1,m]$. From the boundary conditions in (5.39), we obtain

$$\begin{gathered}
\alpha_1c_{02}-\Gamma(\alpha)\beta_1c_{01}=0,\\
\alpha_2\Big[\sum_{\sigma=0}^{m}(c_{\sigma 1}(1-t_\sigma)^{\alpha-1}+c_{\sigma 2}(1-t_\sigma)^{\alpha-2})+\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\sigma(s)ds\Big]\\
+\beta_2\Big[\Gamma(\alpha)\sum_{\sigma=0}^{m}c_{\sigma 1}+\int_0^1\sigma(s)ds\Big]=0.
\end{gathered}$$

It follows that

$$\begin{gathered}
c_{01}=\frac{\alpha_1}{\Theta}\Big[\sum_{\sigma=1}^{m}\frac{(1-t_\sigma)^{\alpha-1}}{\Gamma(\alpha)}J_\sigma+\sum_{\sigma=1}^{m}(1-t_\sigma)^{\alpha-2}I_\sigma\\
+\beta_2\int_0^1\sigma(s)ds+\alpha_2\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\sigma(s)ds\Big],\\
c_{02}=-\frac{\Gamma(\alpha)\beta_1}{\Theta}\Big[\sum_{\sigma=1}^{m}\frac{(1-t_\sigma)^{\alpha-1}}{\Gamma(\alpha)}J_\sigma+\sum_{\sigma=1}^{m}(1-t_\sigma)^{\alpha-2}I_\sigma\\
+\beta_2\int_0^1\sigma(s)ds+\alpha_2\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\sigma(s)ds\Big].
\end{gathered}$$

Substituting $c_{\sigma 1}, c_{\sigma 2}$ into (5.41), we obtain (5.40) by changing the order of the terms. It is easy to show from (5.40) and (5.43) that $x \in X$. The proof is compete. □

To abbreviate notation, let $H_x(t) = H(t, x(t), {}^{RL}D_{0+}^{\beta} x(t))$ for functions $H : (0,1) \times \mathbb{R}^2 \to \mathbb{R}$ and $x : (0,] \to \mathbb{R}$. Define the operator $(Tx)(t)$ for $x \in X$ by

$$\begin{aligned}
(Tx)(t) = \frac{\alpha_1}{\Theta} \Big[&\sum_{\sigma=1}^{m} \frac{(1-t_\sigma)^{\alpha-1}}{\Gamma(\alpha)} J_x(t_\sigma) + \sum_{\sigma=1}^{m} (1-t_\sigma)^{\alpha-2} I_x(t_\sigma) \\
&+ \beta_2 \int_0^1 p(s) f_x(s) ds + \alpha_2 \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} p(s) f_x(s) ds \Big] t^{\alpha-1} \\
&- \frac{\Gamma(\alpha)\beta_1}{\Theta} \Big[\sum_{\sigma=1}^{m} \frac{(1-t_\sigma)^{\alpha-1}}{\Gamma(\alpha)} J_x(t_\sigma) + \sum_{\sigma=1}^{m} (1-t_\sigma)^{\alpha-2} I_x(t_\sigma) \\
&+ \beta_2 \int_0^1 p(s) f_x(s) ds + \alpha_2 \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} p(s) f_x(s) ds \Big] t^{\alpha-2} \\
&+ \sum_{\sigma=1}^{i} \frac{(1-t_\sigma)^{\alpha-1}}{\Gamma(\alpha)} J_x(t_\sigma)(t-t_\sigma)^{\alpha-1} + \sum_{\sigma=1}^{i} (1-t_\sigma)^{\alpha-2} I_x(t_\sigma)(t-t_\sigma)^{\alpha-2} \\
&+ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} p(s) f_x(s) ds, \quad t \in (t_i, t_{i+1}],\ i \in N[0,m].
\end{aligned}$$

Theorem 5.15. *Suppose that $\Theta \neq 0$ and (H1) in Theorem 5.10 holds. Then (5.38) has at least one solution if there exists $r_0 > 0$ such that*

$$A_1 M_J(r_0, r_0) + A_2 M_I(r_0, r_0) + A_3 M_f(r_0, r_0) \le r_0,$$

where

$$\begin{aligned}
A_1 = \max\Big\{ &\frac{|\alpha_1|}{|\Theta|} \sum_{\sigma=1}^{m} \frac{(1-t_\sigma)^{\alpha-1}}{\Gamma(\alpha)} + \sum_{\sigma=1}^{m} \frac{(1-t_\sigma)^{\alpha-1}}{\Gamma(\alpha)} + \frac{\Gamma(\alpha)|\beta_1|}{|\Theta|} \sum_{\sigma=1}^{m} \frac{(1-t_\sigma)^{\alpha-1}}{\Gamma(\alpha)}, \\
&\frac{|\alpha_1|}{|\Theta|} \frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)} \sum_{\sigma=1}^{m} \frac{(1-t_\sigma)^{\alpha-1}}{\Gamma(\alpha)} + \sum_{\sigma=1}^{m} \frac{(1-t_\sigma)^{\alpha-1}}{\Gamma(\alpha)} \frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)} \\
&+ \frac{\Gamma(\alpha)|\beta_1|}{|\Theta|} \frac{\Gamma(\alpha-1)}{\Gamma(\alpha-\beta-1)} \sum_{\sigma=1}^{m} \frac{(1-t_\sigma)^{\alpha-1}}{\Gamma(\alpha)} \Big\},
\end{aligned}$$

and

$$\begin{aligned}
A_2 = \max\Big\{ &\frac{\Gamma(\alpha)|\beta_1|}{|\Theta|} \sum_{\sigma=1}^{m} (1-t_\sigma)^{\alpha-2} + \sum_{\sigma=1}^{m} (1-t_\sigma)^{\alpha-2} + \frac{|\alpha_1|}{|\Theta|} \sum_{\sigma=1}^{m} (1-t_\sigma)^{\alpha-2}, \\
&\frac{\Gamma(\alpha)|\beta_1|}{|\Theta|} \frac{\Gamma(\alpha-1)}{\Gamma(\alpha-\beta-1)} \sum_{\sigma=1}^{m} (1-t_\sigma)^{\alpha-2} + \sum_{\sigma=1}^{m} (1-t_\sigma)^{\alpha-2} \frac{\Gamma(\alpha-1)}{\Gamma(\alpha-\beta-1)} \\
&+ \frac{|\alpha_1|}{|\Theta|} \frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)} \sum_{\sigma=1}^{m} (1-t_\sigma)^{\alpha-2} \Big\},
\end{aligned}$$

$$A_3 = \max\Big\{ \frac{|\alpha_1| |\beta_2| \mathbf{B}(l+1, k+1)}{|\Theta|} + \frac{|\alpha_1| |\alpha_2|}{|\Theta|} \frac{\mathbf{B}(\alpha+l, k+1)}{\Gamma(\alpha)}$$

$$
\begin{aligned}
&+\frac{\Gamma(\alpha)|\beta_1||\beta_2|\mathbf{B}(l+1,k+1)}{|\Theta|}+\frac{\Gamma(\alpha)|\beta_1||\alpha_2|}{|\Theta|}\frac{\mathbf{B}(\alpha+l,k+1)}{\Gamma(\alpha)}\\
&+\frac{\mathbf{B}(\alpha+l,k+1)}{\Gamma(\alpha)},\ \frac{|\alpha_1||\beta_2|\mathbf{B}(l+1,k+1)}{|\Theta|}\frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)}\\
&+\frac{|\alpha_1||\alpha_2|}{|\Theta|}\frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)}\frac{\mathbf{B}(\alpha+l,k+1)}{\Gamma(\alpha)}\\
&+\frac{\Gamma(\alpha)|\beta_1||\beta_2|\mathbf{B}(l+1,k+1)}{|\Theta|}\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-\beta-1)}\\
&+\frac{\Gamma(\alpha)|\beta_1||\alpha_2|}{|\Theta|}\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-\beta-1)}\frac{\mathbf{B}(\alpha+l,k+1)}{\Gamma(\alpha)}+\frac{\mathbf{B}(\alpha-\beta+l,k+1)}{\Gamma(\alpha-\beta)}\Big\}.
\end{aligned}
$$

Proof. Let X be defined above. By Lemma 5.14, we know that x is a solution of (5.38) if and only if x is a fixed point of T. By a standard proof, we can see that $T : X \to X$ is a completely continuous operator.

From (H1), for $x \in X$ we have

$$
\begin{gathered}
|f_x(t)|=|f(t,x(t),{}^{RL}D_{0^+}^{\beta}x(t))|\le M_f(\|x\|,\|x\|),\quad t\in(t_i,t_{i+1}],\ i\in\mathbb{N}[0,m],\\
|I(t_i,x(t_i),{}^{RL}D_{0^+}^{\beta}x(t_i))|\le M_I(\|x\|,\|x\|),\quad i\in\mathbb{N}[1,m],\ x\in\mathbb{R},\\
|J(t_i,x(t_i),{}^{RL}D_{0^+}^{\beta}x(t_i))|\le M_J(\|x\|,\|x\|),\quad i\in\mathbb{N}[1,m],\ x\in\mathbb{R}.
\end{gathered}
$$

We consider the set $\Omega = \{x \in X : x = \lambda(Tx),\ \text{for some } \lambda \in [0, 1)\}$. For $x \in \Omega$, we have by definition of T for $t \in (t_i, t_{i+1}]$ that

$$
\begin{aligned}
&{}^{RL}D_{0^+}^{\beta}(Tx)(t)\\
&=\frac{\alpha_1}{\Theta}\Big[\sum_{\sigma=1}^{m}\frac{(1-t_\sigma)^{\alpha-1}}{\Gamma(\alpha)}J_x(t_\sigma)+\sum_{\sigma=1}^{m}(1-t_\sigma)^{\alpha-2}I_x(t_\sigma)\\
&\quad+\beta_2\int_0^1p(s)f_x(s)ds+\alpha_2\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)f_x(s)ds\Big]\frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)}t^{\alpha-\beta-1}\\
&\quad-\frac{\Gamma(\alpha)\beta_1}{\Theta}\Big[\sum_{\sigma=1}^{m}\frac{(1-t_\sigma)^{\alpha-1}}{\Gamma(\alpha)}J_x(t_\sigma)+\sum_{\sigma=1}^{m}(1-t_\sigma)^{\alpha-2}I_x(t_\sigma)\\
&\quad+\beta_2\int_0^1p(s)f_x(s)ds+\alpha_2\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)f_x(s)ds\Big]\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-\beta-1)}t^{\alpha-\beta-2}\\
&\quad+\sum_{\sigma=1}^{i}\frac{(1-t_\sigma)^{\alpha-1}}{\Gamma(\alpha)}J_x(t_\sigma)\frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)}(t-t_\sigma)^{\alpha-1}\\
&\quad+\sum_{\sigma=1}^{i}(1-t_\sigma)^{\alpha-2}I_x(t_\sigma)\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-\beta-1)}(t-t_\sigma)^{\alpha-2}\\
&\quad+\int_0^t\frac{(t-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}p(s)f_x(s)ds,\quad t\in(t_i,t_{i+1}],\ i\in\mathbb{N}[0,m].
\end{aligned}
$$

So

$$
\begin{aligned}
&(t-t_i)^{2-\alpha}|(Tx)(t)|\\
&\le\frac{|\alpha_1|}{|\Theta|}\Big[\sum_{\sigma=1}^{m}\frac{(1-t_\sigma)^{\alpha-1}}{\Gamma(\alpha)}M_J(\|x\|,\|x\|)+\sum_{\sigma=1}^{m}(1-t_\sigma)^{\alpha-2}M_I(\|x\|,\|x\|)
\end{aligned}
$$

$$
\begin{aligned}
&+|\beta_2|\int_0^1 s^k(1-s)^l ds M_f(\|x\|,\|x\|)\\
&+|\alpha_2|\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}s^k(1-s)^l ds M_f(\|x\|,\|x\|)\Big]\\
&+\frac{\Gamma(\alpha)|\beta_1|}{|\Theta|}\Big[\sum_{\sigma=1}^m \frac{(1-t_\sigma)^{\alpha-1}}{\Gamma(\alpha)}M_J(\|x\|,\|x\|)+\sum_{\sigma=1}^m(1-t_\sigma)^{\alpha-2}M_I(\|x\|,\|x\|)\\
&+|\beta_2|\int_0^1 s^k(1-s)^l ds M_f(\|x\|,\|x\|)\\
&+|\alpha_2|\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}s^k(1-s)^l ds M_f(\|x\|,\|x\|)\Big]\\
&+\sum_{\sigma=1}^m \frac{(1-t_\sigma)^{\alpha-1}}{\Gamma(\alpha)}M_J(\|x\|,\|x\|)+\sum_{\sigma=1}^m(1-t_\sigma)^{\alpha-2}M_I(\|x\|,\|x\|)\\
&+(t-t_i)^{2-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}s^k(1-s)^l ds M_f(\|x\|,\|x\|)\\
\le{}&\Big[\frac{|\alpha_1|}{|\Theta|}\sum_{\sigma=1}^m \frac{(1-t_\sigma)^{\alpha-1}}{\Gamma(\alpha)}+\sum_{\sigma=1}^m \frac{(1-t_\sigma)^{\alpha-1}}{\Gamma(\alpha)}\\
&+\frac{\Gamma(\alpha)|\beta_1|}{|\Theta|}\sum_{\sigma=1}^m \frac{(1-t_\sigma)^{\alpha-1}}{\Gamma(\alpha)}\Big]M_J(\|x\|,\|x\|)+\Big[\frac{\Gamma(\alpha)|\beta_1|}{|\Theta|}\sum_{\sigma=1}^m(1-t_\sigma)^{\alpha-2}\\
&+\sum_{\sigma=1}^m(1-t_\sigma)^{\alpha-2}+\frac{|\alpha_1|}{|\Theta|}\sum_{\sigma=1}^m(1-t_\sigma)^{\alpha-2}\Big]M_I(\|x\|,\|x\|)\\
&+\Big[\frac{|\alpha_1||\beta_2|\mathbf{B}(l+1,k+1)}{|\Theta|}+\frac{|\alpha_1||\alpha_2|}{|\Theta|}\frac{\mathbf{B}(\alpha+l,k+1)}{\Gamma(\alpha)}\\
&+\frac{\Gamma(\alpha)|\beta_1||\beta_2|\mathbf{B}(l+1,k+1)}{|\Theta|}+\frac{\Gamma(\alpha)|\beta_1||\alpha_2|}{|\Theta|}\frac{\mathbf{B}(\alpha+l,k+1)}{\Gamma(\alpha)}\\
&+\frac{\mathbf{B}(\alpha+l,k+1)}{\Gamma(\alpha)}\Big]M_f(\|x\|,\|x\|).
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
&(t-t_\sigma)^{2+\beta-\alpha}|{}^{RL}D_{0^+}^{\beta}(Tx)(t)|\\
&\le\Big[\frac{|\alpha_1|}{|\Theta|}\frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)}\sum_{\sigma=1}^m \frac{(1-t_\sigma)^{\alpha-1}}{\Gamma(\alpha)}+\sum_{\sigma=1}^m \frac{(1-t_\sigma)^{\alpha-1}}{\Gamma(\alpha)}\frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)}\\
&\quad+\frac{\Gamma(\alpha)|\beta_1|}{|\Theta|}\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-\beta-1)}\sum_{\sigma=1}^m \frac{(1-t_\sigma)^{\alpha-1}}{\Gamma(\alpha)}\Big]M_J(\|x\|,\|x\|)\\
&\quad+\Big[\frac{\Gamma(\alpha)|\beta_1|}{|\Theta|}\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-\beta-1)}\sum_{\sigma=1}^m(1-t_\sigma)^{\alpha-2}+\sum_{\sigma=1}^m(1-t_\sigma)^{\alpha-2}\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-\beta-1)}\\
&\quad+\frac{|\alpha_1|}{|\Theta|}\frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)}\sum_{\sigma=1}^m(1-t_\sigma)^{\alpha-2}\Big]M_I(\|x\|,\|x\|)\\
&\quad+\Big[\frac{|\alpha_1||\beta_2|\mathbf{B}(l+1,k+1)}{|\Theta|}\frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)}+\frac{|\alpha_1||\alpha_2|}{|\Theta|}\frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)}\frac{\mathbf{B}(\alpha+l,k+1)}{\Gamma(\alpha)}
\end{aligned}
$$

$$+\frac{\Gamma(\alpha)|\beta_1||\beta_2|\mathbf{B}(l+1,k+1)}{|\Theta|}\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-\beta-1)}$$
$$+\frac{\Gamma(\alpha)|\beta_1||\alpha_2|}{|\Theta|}\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-\beta-1)}\frac{\mathbf{B}(\alpha+l,k+1)}{\Gamma(\alpha)}+\frac{\mathbf{B}(\alpha-\beta+l,k+1)}{\Gamma(\alpha-\beta)}\Big]M_f(\|x\|,\|x\|).$$

It follows that

$$\|Tx\|\leq A_1M_J(\|x\|,\|x\|)+A_2M_I(\|x\|,\|x\|)+A_3M_f(\|x\|,\|x\|).$$

From the assumption, we choose $\Omega=\{x\in X:\|x\|\leq r_0\}$. For $x\in\partial\Omega$, we obtain $x\neq\lambda(Tx)$ for any $\lambda\in[0,1]$. In fact, if there exists $x\in\partial\Omega$ such that $x=\lambda(Tx)$ for some $\lambda\in[0,1]$. Then

$$r_0=\|x\|=\lambda\|Tx\|<\|Tx\|\leq A_1M_J(r_0,r_0)+A_2M_I(r_0,r_0)+A_3M_f(r_0,r_0)\leq r_0,$$

which is a contradiction.

As a consequence of Schaefer's fixed point theorem, we deduce that T has a fixed point which is a solution of (5.38). The proof is complete. □

Theorem 5.16. *Suppose that $\Theta\neq 0$, (1.A1)–(1.A4) and (H1) in Theorem 5.10 hold and*

$$\lim_{r\to0^+}\frac{M_f(r,r)}{r}=\lim_{r\to0^+}\frac{M_I(r,r)}{r}=\lim_{r\to0^+}\frac{M_J(r,r)}{r}=0$$

or

$$\lim_{r\to+\infty}\frac{M_f(r,r)}{r}=\lim_{r\to+\infty}\frac{M_I(r,r)}{r}=\lim_{r\to+\infty}\frac{M_J(r,r)}{r}=0.$$

Then BVP (5.38) has at least one solution.

The proof of the above theorem is similar to that of Theorem 5.11 and is omitted.

Example 5.17. Consider the problem

$$\begin{gathered}
{}^{RL}D_{0^+}^{3/2}x(t)=t^{-\frac18}(1-t)^{-\frac18}\Big[\overline{A}_1+\overline{B}_1((t-t_i)^{1/2}x(t))^\sigma \\
+\overline{C}_1((t-t_i)^{5/8\,RL}D_{0^+}^{1/4}x(t))^\sigma\Big],\quad t\in(t_i,t_{i+1}],\ i\in\mathbb{N}[0,m],\\
\lim_{t\to t_i^+}(t-t_i)^{1/2}x(t)=\overline{A}_2+\overline{B}_2((t_i-t_{i-1})^{1/2}x(t_i))^\sigma \\
+\overline{C}_2((t_i-t_{i-1})^{5/8\,RL}D_{0^+}^{1/4}x(t_i))^\sigma,\\
\Delta^{RL}D_{0^+}^{1/2}x(t_i)=\overline{A}_3+\overline{B}_3((t_i-t_{i-1})^{1/2}x(t_i))^\sigma \\
+\overline{C}_3((t_i-t_{i-1})^{5/8\,RL}D_{0^+}^{1/4}x(t_i))^\sigma,\\
\lim_{t\to0^+}t^{1/2}x(t)-{}^{RL}D_{0^+}^{1/2}x(0)=x(1)+{}^{RL}D_{0^+}^{1/2}x(1)=0,
\end{gathered}\tag{5.44}$$

where $A_1,B_i,C_i\in\mathbb{R}(i\in\mathbb{N}[1,3],\sigma\geq0$ are constants, $0=t_0<t_1<\dots<t_m<t_{m+1}=1$. Then (5.44) has at least one solution if one of the following items holds:

- (i) $\sigma\in[0,1]$ or
- (ii) $\sigma=1$ with $23.5780m|\overline{B}_1|+23.5780m|\overline{C}_1|+14.2468m|\overline{B}_2|+14.2468m|\overline{C}_2|+34.6784m|\overline{B}_3|+34.6784m|\overline{C}_3|<1$ or
- (iii) $\sigma>1$ with $[23.5780m|\overline{A}_1|+14.2468m|\overline{A}_2|+34.6784m|\overline{A}_3|]^{\sigma-1}[23.5780m|\overline{B}_1|+23.5780m|\overline{C}_1|+14.2468m|\overline{B}_2|+14.2468m|\overline{C}_2|+A_3|\overline{B}_3|+34.6784m|\overline{C}_3|]\leq\frac{(\sigma-1)^{\sigma-1}}{\sigma^\sigma}$.

Proof. Corresponding to (5.38), we have $\alpha = \frac{3}{2}$, $\beta = \frac{1}{4}$, $\alpha_1 = \beta_1 = \alpha_2 = \beta_1 = 1$, $0 = t_0 < t_1 < \dots < t_m < t_{m+1} = 1$, $p(t) = t^{-\frac{1}{8}}(1-t)^{-\frac{1}{8}}$ with $k = l = -1/8$, and

$$f(t,u,v) = \overline{A}_1 + \overline{B}_1((t-t_i)^{1/2}u)^\sigma + \overline{C}_1((t-t_i)^{5/8}v)^\sigma, \quad t \in (t_i, t_{i+1}],\ i \in \mathbb{N}_0^m,$$
$$I(t_i,u,v) = \overline{A}_2 + \overline{B}_2((t_i-t_{i-1})^{1/2}u)^\sigma + \overline{C}_2((t_i-t_{i-1})^{5/8}v)^\sigma, \quad i \in \mathbb{N}_1^m,$$
$$J(t_i,u,v) = \overline{A}_3 + \overline{B}_3((t_i-t_{i-1})^{1/2}u)^\sigma + \overline{C}_3((t_i-t_{i-1})^{5/8}v)^\sigma, \quad i \in \mathbb{N}_1^m.$$

We see that

$$|f(t,(t-t_i)^{\alpha-2}u,(t-t_i)^{\alpha-\beta-2}v)| \le M_f(|u|,|v|) = |\overline{A}_1| + |\overline{B}_1|u^\sigma + |\overline{C}_1|v^\sigma,$$
$$t \in (t_i,t_{i+1}],\ i \in \mathbb{N}[0,m], u,v \in \mathbb{R},$$
$$|I(t_i,(t_i-t_{i-1})^{\alpha-2}u,(t_i-t_{i-1})^{\alpha-\beta-2}v)| \le M_I(|u|,|v|) = |\overline{A}_2| + |\overline{B}_2|u^\sigma + |\overline{C}_2|v^\sigma,$$
$$i \in \mathbb{N}[1,m], u,v \in \mathbb{R},$$
$$|J(t_i,(t_i-t_{i-1})^{\alpha-2}u,(t_i-t_{i-1})^{\alpha-\beta-2}v)| \le M_J(|u|,|v|) = |\overline{A}_3| + |\overline{B}_3|u^\sigma + |\overline{C}_3|v^\sigma,$$
$$i \in \mathbb{N}[1,m],\ u,v \in \mathbb{R}.$$

By direct computations, we obtain $\Theta = 1 + 2\Gamma(3/2)$, and by using Mathlab 7.0 that

$$\begin{aligned}
A_1 &= \max\Big\{\frac{1}{1+2\Gamma(3/2)}\sum_{\sigma=1}^m \frac{(1-t_\sigma)^{1/2}}{\Gamma(3/2)} + \sum_{\sigma=1}^m \frac{(1-t_\sigma)^{1/2}}{\Gamma(3/2)} \\
&\quad + \frac{\Gamma(3/2)}{1+2\Gamma(3/2)}\sum_{\sigma=1}^m \frac{(1-t_\sigma)^{1/2}}{\Gamma(3/2)}, \frac{1}{1+2\Gamma(3/2)}\frac{\Gamma(3/2)}{\Gamma(11/8)}\sum_{\sigma=1}^m \frac{(1-t_\sigma)^{1/2}}{\Gamma(3/2)} \\
&\quad + \sum_{\sigma=1}^i \frac{(1-t_\sigma)^{1/2}}{\Gamma(3/2)}\frac{\Gamma(3/2)}{\Gamma(11/8)} + \frac{\Gamma(3/2)}{1+2\Gamma(3/2)}\frac{\Gamma(1/2)}{\Gamma(3/8)}\sum_{\sigma=1}^m \frac{(1-t_\sigma)^{1/2}}{\Gamma(3/2)}\Big\} \\
&\le m\max\{\frac{2+3\Gamma(3/2)}{(1+2\Gamma(3/2))\Gamma(3/2)}, \frac{(2+2\Gamma(3/2))\Gamma(3/8)+\Gamma(11/8)\Gamma(1/2)}{(1+2\Gamma(3/2))\Gamma(11/8)\Gamma(3/8)}\Big\} \\
&< 23.5780m,
\end{aligned}$$

$$\begin{aligned}
A_2 &= \max\Big\{\frac{\Gamma(3/2)}{1+2\Gamma(3/2)}\sum_{\sigma=1}^m (1-t_\sigma)^{-1/2} + \sum_{\sigma=1}^m (1-t_\sigma)^{-1/2} \\
&\quad + \frac{1}{1+2\Gamma(3/2)}\sum_{\sigma=1}^m (1-t_\sigma)^{-1/2}, \frac{\Gamma(3/2)}{1+2\Gamma(3/2)}\frac{\Gamma(1/2)}{\Gamma(3/8)}\sum_{\sigma=1}^m (1-t_\sigma)^{-1/2} \\
&\quad + \sum_{\sigma=1}^i (1-t_\sigma)^{-1/2}\frac{\Gamma(1/2)}{\Gamma(3/8)} + \frac{1}{1+2\Gamma(3/2)}\frac{\Gamma(3/2)}{\Gamma(11/8)}\sum_{\sigma=1}^m (1-t_\sigma)^{-1/2}\Big\} \\
&\le \frac{m}{\sqrt{1-t_m}}\max\Big\{\frac{2+3\Gamma(3/2)}{1+2\Gamma(3/2)}, \frac{\Gamma(3/2)}{1+2\Gamma(3/2)}\frac{\Gamma(1/2)}{\Gamma(3/8)} + \frac{\Gamma(1/2)}{\Gamma(3/8)} \\
&\quad + \frac{1}{1+2\Gamma(3/2)}\frac{\Gamma(3/2)}{\Gamma(11/8)}\Big\} < 14.2468m,
\end{aligned}$$

and

$$\begin{aligned}
A_3 = \max\Big\{&\frac{(1+\Gamma(3/2))\mathbf{B}(7/8,7/8)}{1+2\Gamma(3/2)} + \frac{2+3\Gamma(3/2)}{1+2\Gamma(3/2)}\frac{\mathbf{B}(11/8,7/8)}{\Gamma(3/2)}, \\
&\frac{\mathbf{B}(7/8,7/8)}{1+2\Gamma(3/2)}\frac{\Gamma(3/2)}{\Gamma(11/8)} + \frac{1}{1+2\Gamma(3/2)}\frac{\Gamma(3/2)}{\Gamma(11/8)}\frac{\mathbf{B}(11/8,7/8)}{\Gamma(3/2)}
\end{aligned}$$

$$+ \frac{\Gamma(3/2)\mathbf{B}(7/8,7/8)}{1+2\Gamma(3/2)}\frac{\Gamma(1/2)}{\Gamma(3/8)} + \frac{\Gamma(3/2)}{1+2\Gamma(3/2)}\frac{\Gamma(1/2)}{\Gamma(3/8)}\frac{\mathbf{B}(11/8,7/8)}{\Gamma(3/2)}$$
$$+ \frac{\mathbf{B}(5/4,7/8)}{\Gamma(11/8)}\Big\} < 34.6784m.$$

By Theorem 5.15, BVP (5.44) has at least one solution if

$$\begin{aligned} &A_1|\overline{A}_1| + A_2|\overline{A}_2| + A_3|\overline{A}_3| + [A_1|\overline{B}_1| + A_1|\overline{C}_1| \\ &+ A_2|\overline{B}_2| + A_2|\overline{C}_2| + A_3|\overline{B}_3| + A_3|\overline{C}_3|]r^\sigma \le r \end{aligned} \tag{5.45}$$

has a positive solution r_0.

It is easy to see that $\sigma \in [0,1]$ or

$$\sigma = 1 \text{ with } A_1|\overline{B}_1| + A_1|\overline{C}_1| + A_2|\overline{B}_2| + A_2|\overline{C}_2| + A_3|\overline{B}_3| + A_3|\overline{C}_3| < 1$$

or $\sigma > 1$ with

$$\begin{aligned} &[A_1|\overline{A}_1| + A_2|\overline{A}_2| + A_3|\overline{A}_3|]^{\sigma-1}[A_1|\overline{B}_1| + A_1|\overline{C}_1| + A_2|\overline{B}_2| \\ &+ A_2|\overline{C}_2| + A_3|\overline{B}_3| + A_3|\overline{C}_3|] \le \frac{(\sigma-1)^{\sigma-1}}{\sigma^\sigma} \end{aligned}$$

implies that (5.45) holds. Hence (5.44) has at least one solution if (i) or (ii) or (iii) holds. The proof is complete. □

5.3. **Impulsive anti-periodic boundary value problems.** The solvability of anti-periodic boundary value problems of impulsive fractional differential equations involving the Caputo fractional derivatives with multiple start points were studied by many authors, see [6, 71, 100, 99] and the references therein. In [91], authors presented a new method to converting the impulsive fractional differential equation (with the Caputo fractional derivative) to an equivalent integral equation and established existence and uniqueness results for some boundary value problems of impulsive fractional differential equations involving the Caputo fractional derivatives. There has been no paper concerning the solvability of anti-periodic boundary value problems of impulsive fractional differential equations involving other fractional derivatives with single start point.

Now we consider the problem

$$\begin{gathered} {}^{RL}D_{0^+}^{\alpha}x(t) = p(t)f(t,x(t),{}^{RL}D_{0^+}^{\beta}x(t)), \quad t \in (t_i,t_{i+1}],\ i \in \mathbb{N}[0,m], \\ \Delta x(t_i) = I(t_i,x(t_i),{}^{RL}D_{0^+}^{\beta}x(t_i)), \quad i \in \mathbb{N}[1,m], \\ \Delta {}^{RL}D_{0^+}^{\alpha-1}x(t_i) = J(t_i,x(t_i),{}^{RL}D_{0^+}^{\beta}x(t_i)), \quad i \in \mathbb{N}[1,m], \\ \lim_{t\to 0^+} t^{2-\alpha}x(t) + x(1) = \lim_{t\to 0^+} t^{2+\beta-\alpha}{}^{RL}D_{0^+}^{\alpha-1}x(t) + {}^{RL}D_{0^+}^{\alpha-1}x(1) = 0, \end{gathered} \tag{5.46}$$

where $\alpha \in (1,2]$, $\beta \in (0,\alpha-1]$, ${}^{RL}D_{0^+}^{*}u$ is the Riemann-Liouville fractional derivative of order $*$, $p:(0,1) \to \mathbb{R}$ in (5.15) satisfies (5.A4), $f:(0,1)\times\mathbb{R}^2 \to \mathbb{R}$ in (5.15) satisfies (5.A2), and $I,J:\{t_i : i \in \mathbb{N}[1,m]\}\times\mathbb{R}^2 \to \mathbb{R}$ in (5.15) satisfy (5.A3), $0 = t_0 < t_1 < \dots < t_m < t_{m+1} = 1$.

Lemma 5.18. *Suppose that σ is continuous on $(0,1)$ and there exist $k > -1$ and $l \in (\max\{-\alpha,-2-k\},0]$ such that $|\sigma(t)| \le t^k(1-t)^l$ for all $t \in (0,1)$. Then x is* *a solution of the problem*

$$\begin{gathered}
{}^{RL}D_{0^+}^{\alpha}x(t)=\sigma(t),\quad t\in(t_i,t_{i+1}],\ i\in\mathbb{N}[0,m],\\
\Delta x(t_i)=I_i,\quad \Delta^{RL}D_{0^+}^{\alpha-1}x(t_i)=J_i,\quad i\in\mathbb{N}[1,m],\\
\lim_{t\to 0^+}t^{2-\alpha}x(t)+x(1)=\lim_{t\to 0^+}t^{2+\beta-\alpha RL}D_{0^+}^{\alpha-1}x(t)+{}^{RL}D_{0^+}^{\alpha-1}x(1)=0,
\end{gathered}\tag{5.47}$$

if and only if

$$\begin{aligned}
x(t)&=-\frac{1}{2\Gamma(\alpha)}\Big[\sum_{\sigma=1}^{m}J_\sigma+\int_0^1\sigma(s)ds\Big]t^{\alpha-1}-\frac{1}{2}\Big[\sum_{\sigma=1}^{m}\frac{(1-t_\sigma)^{\alpha-1}}{2\Gamma(\alpha)}J_\sigma\\
&\quad+\sum_{\sigma=1}^{m}(1-t_\sigma)^{\alpha-2}I_\sigma-\frac{1}{2\Gamma(\alpha)}\int_0^1\sigma(s)ds\\
&\quad+\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\sigma(s)ds\Big]t^{\alpha-2}+\sum_{\sigma=1}^{i}\frac{(1-t_\sigma)^{\alpha-1}}{\Gamma(\alpha)}J_\sigma(t-t_\sigma)^{\alpha-1}\\
&\quad+\sum_{\sigma=1}^{i}(1-t_\sigma)^{\alpha-2}I_\sigma(t-t_\sigma)^{\alpha-2}+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\sigma(s)ds,
\end{aligned}\tag{5.48}$$

for $t\in(t_i,t_{i+1}]$, $i\in\mathbb{N}[0,m]$.

Proof. By Theorem 3.12, we have ${}^{RL}D_{0^+}^{\alpha}x(t)=\sigma(t)$, $t\in(t_i,t_{i+1}]$, $i\in\mathbb{N}[0,m]$ if and only if there exit numbers $c_{\sigma1},c_{\sigma2},i\in\mathbb{N}[0,m]$ such that

$$x(t)=\sum_{\sigma=0}^{i}[c_{\sigma1}(t-t_\sigma)^{\alpha-1}+c_{\sigma2}(t-t_\sigma)^{\alpha-2}]+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\sigma(s)ds,\tag{5.49}$$

for $t\in(t_i,t_{i+1}]$, $i\in N[0,m]$. Then

$${}^{RL}D_{0^+}^{\alpha-1}x(t)=\Gamma(\alpha)\sum_{\sigma=0}^{i}c_{\sigma1}+\int_0^t\sigma(s)ds,\quad t\in(t_i,t_{i+1}],\ i\in\mathbb{N}[0,m].\tag{5.50}$$

Furthermore, by direct computations we have

$$\begin{aligned}
&{}^{RL}D_{0^+}^{\beta}x(t)\\
&=\frac{1}{\Gamma(1-\beta)}\Big[\int_0^t(t-s)^{-\beta}x(s)ds\Big]'\\
&=\frac{1}{\Gamma(1-\beta)}\Big[\sum_{j=0}^{i-1}\int_{t_j}^{t_{j+1}}(t-s)^{-\beta}x(s)ds\Big]'+\frac{1}{\Gamma(1-\beta)}\Big[\int_{t_i}^{t}(t-s)^{-\beta}x(s)ds\Big]'\\
&=\sum_{\sigma=0}^{i}\Big(c_{\sigma1}\frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta+1)}(t-t_\sigma)^{\alpha-\beta-1}+c_{\sigma2}\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-\beta-1)}(t-t_\sigma)^{\alpha-\beta-2}\Big)\\
&\quad+\int_0^t\frac{(t-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\sigma(s)ds,\quad t\in(t_i,t_{i+1}],\ i\in\mathbb{N}[0,m].
\end{aligned}\tag{5.51}$$

From $\lim_{t\to t_i^+}(t-t_i)^{2-\alpha}x(t)=I_i$, we have $c_{\sigma2}=I_\sigma$ for all $\sigma\in\mathbb{N}[1,m]$. From $\Delta^{RL}D_{0^+}^{\alpha-1}x(t_i))=J_i,\ i\in\mathbb{N}[1,m]$, one has $c_{\sigma1}=\frac{J_\sigma}{\Gamma(\alpha)}$ for all $\sigma\in\mathbb{N}[1,m]$. From the boundary conditions in (5.39), we obtain

$$c_{02}+\sum_{\sigma=0}^{m}[c_{\sigma1}(1-t_\sigma)^{\alpha-1}+c_{\sigma2}(1-t_\sigma)^{\alpha-2}]+\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\sigma(s)ds=0,$$

$$\Gamma(\alpha)c_{01}+\Gamma(\alpha)\sum_{\sigma=0}^{m}c_{\sigma1}+\int_0^1\sigma(s)ds=0.$$

It follows that

$$c_{01}=-\frac{1}{2\Gamma(\alpha)}\Big[\sum_{\sigma=1}^{m}J_\sigma+\int_0^1\sigma(s)ds\Big],$$

$$\begin{aligned}c_{02}=-\frac{1}{2}\Big[&\sum_{\sigma=1}^{m}\frac{(1-t_\sigma)^{\alpha-1}}{2\Gamma(\alpha)}J_\sigma+\sum_{\sigma=1}^{m}(1-t_\sigma)^{\alpha-2}I_\sigma\\&-\frac{1}{2\Gamma(\alpha)}\int_0^1\sigma(s)ds+\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\sigma(s)ds\Big].\end{aligned}$$

Substituting $c_{\sigma1},c_{\sigma2}$ into (5.49), we obtain (5.48) by changing the order of the terms. It is easy to show from (5.49) and (5.51) that $x\in X$. The proof is compete. □

To abbreviate notation let $H_x(t)=H(t,x(t),{}^{RL}D_{0^+}^{\beta}x(t))$ for functions $H:(0,1)\times\mathbb{R}^2\to\mathbb{R}$ and $x:(0,]\to\mathbb{R}$. Define the operator $(Tx)(t)$ for $x\in X$ by

$$\begin{aligned}(Tx)(t)=&-\frac{1}{2\Gamma(\alpha)}\Big[\sum_{\sigma=1}^{m}J_x(t_\sigma)+\int_0^1p(s)f_x(s)ds\Big]t^{\alpha-1}-\frac{1}{2}\Big[\sum_{\sigma=1}^{m}\frac{(1-t_\sigma)^{\alpha-1}}{2\Gamma(\alpha)}J_x(t_\sigma)\\&+\sum_{\sigma=1}^{m}(1-t_\sigma)^{\alpha-2}I_x(t_\sigma)-\frac{1}{2\Gamma(\alpha)}\int_0^1p(s)f_x(s)ds\\&+\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)f_x(s)ds\Big]t^{\alpha-2}+\sum_{\sigma=1}^{i}\frac{(1-t_\sigma)^{\alpha-1}}{\Gamma(\alpha)}J_x(t_\sigma)(t-t_\sigma)^{\alpha-1}\\&+\sum_{\sigma=1}^{i}(1-t_\sigma)^{\alpha-2}I_x(t_\sigma)(t-t_\sigma)^{\alpha-2}+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)f_x(s)ds,\end{aligned}$$

for $t\in(t_i,t_{i+1}]$ and $i\in N[0,m]$.

Theorem 5.19. *Suppose that* (H1) *in Theorem 5.10 holds. Then* (5.46) *has at least one solution if there is* $r_0>0$ *such that* $A_1M_J(r_0,r_0)+A_2M_I(r_0,r_0)+A_3M_f(r_0,r_0)\le r_0$, *where*

$$\begin{aligned}A_1=\max\Big\{&\frac{m}{2\Gamma(\alpha)}+\sum_{\sigma=1}^{m}\frac{(1-t_\sigma)^{\alpha-1}}{\Gamma(\alpha)}+\sum_{\sigma=1}^{m}\frac{(1-t_\sigma)^{\alpha-1}}{4\Gamma(\alpha)},\frac{m}{2\Gamma(\alpha-\beta)}\\&+\sum_{\sigma=1}^{m}\frac{(1-t_\sigma)^{\alpha-1}}{\Gamma(\alpha)}+\frac{\Gamma(\alpha-1)}{2\Gamma(\alpha-\beta-1)}\sum_{\sigma=1}^{m}\frac{(1-t_\sigma)^{\alpha-1}}{2\Gamma(\alpha)}\Big\},\end{aligned}$$

$$\begin{aligned}A_2=\max\Big\{&\sum_{\sigma=1}^{m}\frac{(1-t_\sigma)^{\alpha-2}}{2}+\sum_{\sigma=1}^{m}(1-t_\sigma)^{\alpha-2},\ \frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)}\\&+\sum_{\sigma=1}^{m}(1-t_\sigma)^{\alpha-2}\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-\beta-1)}+\frac{\Gamma(\alpha-1)}{2\Gamma(\alpha-\beta-1)}\sum_{\sigma=1}^{m}(1-t_\sigma)^{\alpha-2}\Big\},\end{aligned}$$

$$
\begin{aligned}
A_3 = \max\Big\{ &\frac{\mathbf{B}(l+1,k+1)}{4\Gamma(\alpha)} + \frac{\mathbf{B}(\alpha+l,k+1)}{2\Gamma(\alpha)} + \frac{\mathbf{B}(l+1,k+1)}{2\Gamma(\alpha)} \\
&+ \frac{\mathbf{B}(\alpha+l,k+1)}{\Gamma(\alpha)},\ \frac{\mathbf{B}(l+1,k+1)}{2\Gamma(\alpha-\beta)} + \frac{\Gamma(\alpha-1)}{2\Gamma(\alpha-\beta-1)}\frac{\mathbf{B}(l+1,k+1)}{2\Gamma(\alpha)} \\
&+ \frac{\Gamma(\alpha-1)}{2\Gamma(\alpha-\beta-1)}\frac{\mathbf{B}(\alpha+l,k+1)}{\Gamma(\alpha)} + \frac{\mathbf{B}(\alpha-\beta+l,k+1)}{\Gamma(\alpha-\beta)}\Big\}.
\end{aligned}
$$

Proof. Let X be defined above. By Lemma 5.18, we know that x is a solution of (5.46) if and only if x is a fixed point of T. By a standard proof, we can see that $T : X \to X$ is a completely continuous operator.

From (H1), for $x \in X$ we have

$$
\begin{gathered}
|f_x(t)| = |f(t, x(t), {}^{RL}D_{0^+}^{\beta}x(t))| \le M_f(\|x\|,\|x\|), \quad t \in (t_i, t_{i+1}],\ i \in \mathbb{N}[0,m], \\
|I(t_i, x(t_i), {}^{RL}D_{0^+}^{\beta}x(t_i))| \le M_I(\|x\|,\|x\|), \quad i \in \mathbb{N}[1,m],\ x \in \mathbb{R}, \\
|J(t_i, x(t_i), {}^{RL}D_{0^+}^{\beta}x(t_i))| \le M_J(\|x\|,\|x\|), \quad i \in \mathbb{N}[1,m],\ x \in \mathbb{R}.
\end{gathered}
$$

We consider the set $\Omega = \{x \in X : x = \lambda(Tx),\ \text{for some } \lambda \in [0,1)\}$. For $x \in \Omega$, we have by definition of T for $t \in (t_i, t_{i+1}]$ that

$$
\begin{aligned}
&{}^{RL}D_{0^+}^{\beta}(Tx)(t) \\
&= -\frac{1}{2\Gamma(\alpha)}\Big[\sum_{\sigma=1}^{m} J_x(t_\sigma) + \int_0^1 p(s)f_x(s)ds\Big]\frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)}t^{\alpha-\beta-1} \\
&\quad - \frac{1}{2}\Big[\sum_{\sigma=1}^{m}\frac{(1-t_\sigma)^{\alpha-1}}{2\Gamma(\alpha)}J_x(t_\sigma) + \sum_{\sigma=1}^{m}(1-t_\sigma)^{\alpha-2}I_x(t_\sigma) - \frac{1}{2\Gamma(\alpha)}\int_0^1 p(s)f_x(s)ds \\
&\quad + \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)f_x(s)ds\Big]\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-\beta-1)}t^{\alpha-\beta-2} \\
&\quad + \sum_{\sigma=1}^{i}\frac{(1-t_\sigma)^{\alpha-1}}{\Gamma(\alpha)}J_x(t_\sigma)\frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)}(t-t_\sigma)^{\alpha-\beta-1} \\
&\quad + \sum_{\sigma=1}^{i}(1-t_\sigma)^{\alpha-2}I_x(t_\sigma)\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-\beta-1)}(t-t_\sigma)^{\alpha-\beta-2} \\
&\quad + \int_0^t \frac{(t-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}p(s)f_x(s)ds, \quad t \in (t_i, t_{i+1}],\ i \in \mathbb{N}[0,m].
\end{aligned}
$$

So

$$
\begin{aligned}
&(t-t_i)^{2-\alpha}|(Tx)(t)| \\
&\le \frac{1}{2\Gamma(\alpha)}\big[mM_J(\|x\|,\|x\|) + \int_0^1 s^k(1-s)^l dsM_f(\|x\|,\|x\|)\big] \\
&\quad + \frac{1}{2}\Big[\sum_{\sigma=1}^{m}\frac{(1-t_\sigma)^{\alpha-1}}{2\Gamma(\alpha)}M_J(\|x\|,\|x\|) + \sum_{\sigma=1}^{m}(1-t_\sigma)^{\alpha-2}M_I(\|x\|,\|x\|) \\
&\quad + \frac{1}{2\Gamma(\alpha)}\int_0^1 s^k(1-s)^l dsM_f(\|x\|,\|x\|) \\
&\quad + \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}s^k(1-s)^l dsM_f(\|x\|,\|x\|)\Big]
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{\sigma=1}^{i}\frac{(1-t_\sigma)^{\alpha-1}}{\Gamma(\alpha)}M_J(\|x\|,\|x\|)+\sum_{\sigma=1}^{i}(1-t_\sigma)^{\alpha-2}M_I(\|x\|,\|x\|)\\
&+(t-t_i)^{2-\alpha}\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}s^k(1-s)^l dsM_f(\|x\|,\|x\|)\\
\le&\Big[\frac{m}{2\Gamma(\alpha)}+\sum_{\sigma=1}^{m}\frac{(1-t_\sigma)^{\alpha-1}}{\Gamma(\alpha)}+\sum_{\sigma=1}^{m}\frac{(1-t_\sigma)^{\alpha-1}}{4\Gamma(\alpha)}\Big]M_J(\|x\|,\|x\|)\\
&+\big[\sum_{\sigma=1}^{m}\frac{(1-t_\sigma)^{\alpha-2}}{2}+\sum_{\sigma=1}^{m}(1-t_\sigma)^{\alpha-2}\big]M_I(\|x\|,\|x\|)\\
&+\big[\frac{\mathbf{B}(l+1,k+1)}{4\Gamma(\alpha)}+\frac{\mathbf{B}(\alpha+l,k+1)}{2\Gamma(\alpha)}+\frac{\mathbf{B}(l+1,k+1)}{2\Gamma(\alpha)}\\
&+\frac{\mathbf{B}(\alpha+l,k+1)}{\Gamma(\alpha)}\big]M_f(\|x\|,\|x\|).
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
&(t-t_\sigma)^{2+\beta-\alpha}|^{RL}D_{0^+}^{\beta}(Tx)(t)|\\
&\le\Big[\frac{m}{2\Gamma(\alpha-\beta)}+\sum_{\sigma=1}^{m}\frac{(1-t_\sigma)^{\alpha-1}}{\Gamma(\alpha)}+\frac{\Gamma(\alpha-1)}{2\Gamma(\alpha-\beta-1)}\sum_{\sigma=1}^{m}\frac{(1-t_\sigma)^{\alpha-1}}{2\Gamma(\alpha)}\Big]M_J(\|x\|,\|x\|)\\
&\quad+\Big[\frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)}+\sum_{\sigma=1}^{m}(1-t_\sigma)^{\alpha-2}\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-\beta-1)}+\frac{\Gamma(\alpha-1)}{2\Gamma(\alpha-\beta-1)}\sum_{\sigma=1}^{m}(1-t_\sigma)^{\alpha-2}\Big]\\
&\quad\times M_I(\|x\|,\|x\|)+\Big[\frac{\mathbf{B}(l+1,k+1)}{2\Gamma(\alpha-\beta)}+\frac{\Gamma(\alpha-1)}{2\Gamma(\alpha-\beta-1)}\frac{\mathbf{B}(l+1,k+1)}{2\Gamma(\alpha)}\\
&\quad+\frac{\Gamma(\alpha-1)}{2\Gamma(\alpha-\beta-1)}\frac{\mathbf{B}(\alpha+l,k+1)}{\Gamma(\alpha)}+\frac{\mathbf{B}(\alpha-\beta+l,k+1)}{\Gamma(\alpha-\beta)}\Big]M_f(\|x\|,\|x\|).
\end{aligned}
$$

It follows that

$$
\|Tx\|\le A_1M_J(\|x\|,\|x\|)+A_2M_I(\|x\|,\|x\|)+A_3M_f(\|x\|,\|x\|).
$$

From the assumption, we choose $\Omega=\{x\in X:\|x\|\le r_0\}$. For $x\in\partial\Omega$, we obtain $x\ne\lambda(Tx)$ for any $\lambda\in[0,1]$. In fact, if there exists $x\in\partial\Omega$ such that $x=\lambda(Tx)$ for some $\lambda\in[0,1]$. Then

$$
r_0=\|x\|=\lambda\|Tx\|\le\|Tx\|\le A_1M_J(r_0,r_0)+A_2M_I(r_0,r_0)+A_3M_f(r_0,r_0)<r_0,
$$

which is a contradiction.

As a consequence of Schaefer's fixed point theorem, we deduce that T has a fixed point which is a solution of (5.46). The proof is complete. □

Theorem 5.20. *Suppose that* (H1) *in Theorem 5.10 holds and*

$$
\lim_{r\to0^+}\frac{M_f(r,r)}{r}=\lim_{r\to0^+}\frac{M_I(r,r)}{r}=\lim_{r\to0^+}\frac{M_J(r,r)}{r}=0
$$

or

$$
\lim_{r\to+\infty}\frac{M_f(r,r)}{r}=\lim_{r\to+\infty}\frac{M_I(r,r)}{r}=\lim_{r\to+\infty}\frac{M_J(r,r)}{r}=0.
$$

Then (5.46) *has at least one solution.*

6. Comments on some published articles

In some recently published articles, the existence and uniqueness of solutions of initial or boundary value problems for impulsive fractional differential equations have been studied, see [109, 112, 113, 126, 127, 131, 133, 67, 136]. However, we find that some results in such papers are wrong from a mathematical point of view. To avoiding misleading readers, in this section we make some comments on these papers.

6.1. **Corrected results from** [136]**.** In [136], the authors studied the solvability of the initial value problems for the impulsive fractional differential equation

$$\begin{gathered}{}_0D_t^q x(t)=f(t,x(t)),\quad t\in[0,T],\ t\neq t_1,t_2,\dots,t_m,\\ \Delta x(t_k)=I_k(x(t_k^-)),\quad k=1,2,\dots,m,\ x(0)=x_0,\end{gathered}\tag{6.1}$$

where $q\in(0,1)$, $x_0\in\mathbb{R}$, ${}_0D_t^q$ is the is the Caputo fractional derivative in interval $[0,t]$, $f:[0,T]\times\times\mathbb{R}\mapsto\mathbb{R}$ is an appropriate continuous function, $I_k:\mathbb{R}\mapsto\mathbb{R}(k=1,2,\dots,m)$ are continuous functions, $0=t_0<t_1<\dots<t_m<t_{m+1}=T$, $\Delta x(t_k)=x(t_k^+)-x(t_k^-)=\lim_{\epsilon\to0^+}x(t_k+\epsilon)-\lim_{\epsilon\to0^+}x(t_k-\epsilon)$. We find that the main result ([136, Theorem 2.1] with its long proof) is as follows:

Result 6.1. System (6.1) is equivalent to the integral equation

$$x(t)=\begin{cases}x_0+\frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}f(s,x(s))ds,\quad t\in(t_0,t_1],\\ x_0+\sum_{k=1}^n I_k(x(t_k^-))+\frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}f(s,x(s))ds\\ +h\sum_{k=1}^n\Big[I_k(x(t_k^-))\Big(\int_0^{t_k}(t_k-s)^{q-1}f(s,x(s))ds\\ +\int_{t_k}^t(t-s)^{q-1}f(s,x(s))ds-\int_0^t(t-s)^{q-1}f(s,x(s))ds\Big)\Big]/\Gamma(q),\\ \quad t\in(t_n,t_{n+1}],\ n=1,2,\dots,m\end{cases}\tag{6.2}$$

provided that the integral in (6.2) exists, where $q\in(0,1),h\in\mathbb{R}$ are constants.

However, this result is in-correct. In fact, the following example was given [136, Example 1]:

$${}_0D_t^{1/4}x(t)=t,\quad t\in[0,2]\setminus\{1\},\quad x(1)-x(1^-)=I(x(1^-)),\quad x(0)=0.\tag{6.3}$$

It was claimed that the general solution of (6.3) is

$$x(t)=\begin{cases}\frac{16}{5\Gamma(1/4)}t^{5/4}, & t\in(0,1],\\ I_1(x(1^-))+\frac{16}{5\Gamma(1/4)}t^{5/4}\\ +\frac{4hI_1(x(1^-))}{5\Gamma(1/4)}[4+(t-1)^{1/4}(4t+1)-4t^{5/4}], & t\in(1,2],\end{cases}\tag{6.4}$$

where h is a constant. Let x be defined by (6.4). One notes (by [136, Definition 2.2]) for $t\in(1,2]$ that

$$\begin{aligned}{}_0D_t^{1/4}x(t)&=\frac{1}{\Gamma(1-1/4)}\int_0^t(t-s)^{-1/4}x'(s)ds\\ &=\frac{1}{\Gamma(3/4)}\int_0^1(t-s)^{-1/4}x'(s)ds+\frac{1}{\Gamma(3/4)}\int_1^t(t-s)^{-1/4}x'(s)ds\\ &=\frac{1}{\Gamma(3/4)}\int_0^1(t-s)^{-1/4}\frac{16}{5\Gamma(1/4)}[s^{5/4}]'ds\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(3/4)}\int_1^t (t-s)^{-1/4}\Big[I_1(x(1^-))+\frac{16}{5\Gamma(1/4)}s^{5/4}\\
&+\frac{4hI_1(x(1^-))}{5\Gamma(1/4)}(4+(s-1)^{1/4}(4s+1)-4s^{5/4})\Big]'ds.
\end{aligned}
$$

When $w=s/t$, we obtain

$$
\begin{aligned}
{}_0D_t^{1/4}x(t) &= \frac{1}{\Gamma(3/4)}\frac{4}{\Gamma(1/4)}\int_0^t (t-s)^{-1/4}s^{1/4}ds\\
&\quad+\frac{1}{\Gamma(3/4)}\int_1^t (t-s)^{-1/4}\Big[\frac{4hI_1(x(1^-))}{5\Gamma(1/4)}\Big(4(s-1)^{1/4}+s(s-1)^{-3/4}\\
&\quad-\frac{1}{4}(s-1)^{-3/4}-5s^{1/4}\Big)\Big]ds\\
&= \frac{1}{\Gamma(3/4)}\frac{4}{\Gamma(1/4)}t\int_0^1 (1-w)^{-1/4}w^{1/4}dw\\
&\quad+\frac{1}{\Gamma(3/4)}\int_1^t (t-s)^{-1/4}\Big[\frac{4hI_1(x(1^-))}{5\Gamma(1/4)}(4(s-1)^{1/4}+s(s-1)^{-3/4}\\
&\quad-\frac{1}{4}(s-1)^{-3/4}-5s^{1/4})\Big]ds \neq t.
\end{aligned}
$$

This shows us that x does not satisfy ${}_0D_t^{1/4}x(t)=t$ on $(1,2]$. In fact, we have

$$
\begin{aligned}
&\frac{1}{\Gamma(1/4)}\int_1^t (t-s)^{\frac14-1}s\,ds\\
&=\frac{1}{\Gamma(1/4)}\int_1^t (t-s)^{\frac14-1}(s-1)ds+\frac{1}{\Gamma(1/4)}\int_1^t (t-s)^{\frac14-1}ds\\
&=\frac{1}{\Gamma(1/4)}(t-1)^{5/4}\int_0^1 (1-w)^{\frac14-1}w\,dw+\frac{4}{\Gamma(1/4)}(t-1)^{1/4}\\
&\qquad\qquad (\text{using } \frac{s-1}{t-1}=w),\\
&\qquad \mathbf{B}(1/4,2)=\frac{\Gamma(1/4)\Gamma(2)}{\Gamma(9/4)}\\
&\qquad\qquad =\frac{1}{\Gamma(9/4)}(t-1)^{5/4}+\frac{4}{\Gamma(1/4)}(t-1)^{1/4}\\
&\qquad\qquad =\frac{4}{5\Gamma(1/4)}(t-1)^{1/4}(4t+1).
\end{aligned}
$$

This mistake comes from the following incorrect equality used in [136]:

$$
\frac{1}{\Gamma(1/4)}\int_1^t (t-s)^{\frac14-1}s\,ds=\frac{4}{5\Gamma(1/4)}(t-1)^{1/4}(4t+1).
$$

The correct formula of solution of (6.3) is

$$
x(t)=\begin{cases}\frac{16}{5\Gamma(1/4)}t^{5/4}, & t\in(0,1],\\ I_1(x(1^-))+\frac{16}{5\Gamma(1/4)}t^{5/4}, & t\in(1,2].\end{cases}
$$

We consider the more general problem

$$\begin{gathered} {}^cD_{0+}^{\alpha}x(t) - \lambda x(t) = f(t,x(t)), \quad t \in (t_i,t_{i+1}], i \in \mathbb{N}_0^m, \\ \Delta x(t_i) = I(t_i, x(t_i^-)), \quad x(0) = x_0, \end{gathered} \tag{6.5}$$

where $\alpha \in (0,1)$, $\lambda \in \mathbb{R}$, ${}^cD_{0+}^{*}$ is the Caputo fractional derivative of order $*$, $f : [0,1] \times \mathbb{R} \mapsto \mathbb{R}$, $I : \{t_i : i \in \mathbb{N}_1^m\} \times \mathbb{R} \mapsto \mathbb{R}$ are continuous functions, $0 = t_0 < t_1 < \dots < t_m < t_{m+1} = 1$, $x_0, \overline{y}_0 \in \mathbb{R}$.

Theorem 6.2. *x is a solution of* (6.5) *if and only if*

$$\begin{aligned} x(t) &= x_0\mathbf{E}_{\alpha-\beta,1}(\lambda t^{\alpha-\beta}) - \lambda x_0 t^{\alpha-\beta}\mathbf{E}_{\alpha-\beta,\alpha-\beta+1}(\lambda t^{\alpha-\beta}) \\ &\quad + \sum_{j=1}^{i} I(t_j, x(t_j))\mathbf{E}_{\alpha-\beta,1}(\lambda(t-t_j)^{\alpha-\beta}) \\ &\quad - \lambda\sum_{j=1}^{i} I(t_j, x(t_j))(t-t_j)^{\alpha-\beta}\mathbf{E}_{\alpha-\beta,\alpha-\beta+1}(\lambda(t-t_j)^{\alpha-\beta}) \\ &\quad + \int_0^t (t-s)^{\alpha-1}\mathbf{E}_{\alpha-\beta,\alpha}(\lambda(t-s)^{\alpha-\beta})f(s,x(s))ds, \end{aligned} \tag{6.6}$$

for $t \in (t_i, t_{i+1}]$ and $i \in \mathbb{N}_0^m$.

Proof. By Theorem 3.11 (with $\alpha = 1$), we know that x is a solution of (6.5) if and only if there exist constants $d_j \in \mathbb{R} (j \in \mathbb{N}_0^m)$ such that

$$\begin{aligned} x(t) &= \sum_{j=0}^{i} d_j\mathbf{E}_{\alpha-\beta,1}(\lambda(t-t_j)^{\alpha-\beta}) - \lambda\sum_{j=0}^{i} d_j(t-t_j)^{\alpha-\beta} \\ &\quad \times \mathbf{E}_{\alpha-\beta,\alpha-\beta+1}(\lambda(t-t_j)^{\alpha-\beta}) \\ &\quad + \int_0^t (t-s)^{\alpha-1}\mathbf{E}_{\alpha-\beta,\alpha}(\lambda(t-s)^{\alpha-\beta})f(s,x(s))ds, \end{aligned} \tag{6.7}$$

$t \in (t_i, t_{i+1}]$ and $i \in \mathbb{N}_0^m$. From $x(0) = x_0$, we obtain $d_0 = x_0$. By $\Delta x(t_i) = I(t_i, x(t_i^-))$, we have $d_i = I(t_i, x(t_i^-))$ $(i \in \mathbb{N}_1^m)$. Substituting d_i into (6.7), we obtain (6.6). The proof is complete. □

Remark 6.3. Let $\lambda = 0$. Then Theorem 6.2 implies that

$$\begin{gathered} {}^cD_{0+}^{\alpha}x(t) = f(t,x(t)), \quad t \in (t_i,t_{i+1}],\ i \in \mathbb{N}_0^m, \\ \Delta x(t_i) = I(t_i, x(t_i^-)), x(0) = x_0, \end{gathered}$$

is equivalent to

$$x(t) = x_0 + \sum_{j=1}^{i} I(t_j, x(t_j)) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,x(s))ds, \quad t \in (t_i,t_{i+1}],\ i \in \mathbb{N}_0^m.$$

It is easy to see that Result 6.1 (the equivalent integral equation (6.1)) is wrong. Theorem 6.2 (with $\lambda = 0$) is a corrected version of the result.

Remark 6.4. Similar to Theorem 6.2, we can establish equivalent integral equations for the following problems:

$$\begin{gathered} {}^cD_{0+}^{\alpha}x(t) - \lambda x(t) = f(t,x(t)), \quad t \in (0,1] \setminus \{t_1,\dots,t_p\}, \\ \Delta x(t_i) = I_i(x(t_i)), \quad \Delta x'(t_i) = J_i(x(t_i)), \quad i \in \mathbb{N}_1^p, \end{gathered}$$

$$x(0) = x_0, \quad x'(0) = x_1;$$

$$\begin{gathered} {}^cD^{\alpha}_{0^+}x(t) - \lambda x(t) = f(t, x(t)), \quad t \in (0, 1] \setminus \{t_1, \dots, t_p\}, \\ \Delta x(t_i) = I_i(x(t_i)), \quad \Delta x'(t_i) = J_i(x(t_i)), \quad i \in \mathbb{N}_1^p, \\ x(0) + \phi(x) = x_0, \quad x'(0) = x_1; \end{gathered}$$

$$\begin{gathered} {}^cD^{\beta}_{0^+}x(t) - \lambda x(t) = f(t, x(t)), \quad t \in (0, 1] \setminus \{t_1, \dots, t_p\}, \\ \Delta x(t_i) = I_i(x(t_i)), \quad i \in \mathbb{N}_1^p, \\ ax(0) + bx(1) = 0; \end{gathered}$$

$$\begin{gathered} {}^cD^{\alpha}_{0^+}x(t) - \lambda x(t) = f(t, x(t)), \quad t \in (0, 1] \setminus \{t_1, \dots, t_p\}, \\ \Delta x(t_i) = I_i(x(t_i)), \quad \Delta x'(t_i) = J_i(x(t_i)), \ i \in \mathbb{N}_1^p, \\ ax(0) - bx'(0) = x_0, \quad cx(1) + dx'(1) = x_1; \end{gathered}$$

and

$$\begin{gathered} {}^cD^{\alpha}_{0^+}x(t) - \lambda x(t) = f(t, x(t)), \quad t \in (0, 1] \setminus \{t_1, \dots, t_p\}, \\ \Delta x(t_i) = I_i(x(t_i)), \quad \Delta x'(t_i) = J_i(x(t_i)), \quad i \in \mathbb{N}_1^p, \\ x(0) - ax(\xi) = x(1) - bx(\eta) = 0. \end{gathered}$$

These problems are generalized forms of the problems discussed in [91].

6.2. **Corrected results from** [126]**.** In a recent article, Zhang [126] studied the solvability of the boundary value problems for the impulsive differential equations with fractional derivative

$$\begin{gathered} {}_0D^q_t y(t) = f(t, y(t)), \quad t \in [0, T], \ t \neq t_k, \ t \neq \bar{t}_l, \ k = 1, 2, \dots, m, \ l = 1, 2, \dots, p, \\ \Delta y(t_k) = I_k(y(t_k^-)), \quad k = 1, 2, \dots, m, \quad \Delta y'(\bar{t}_l) = \overline{I}_l(y(\bar{t}_l^-)), \quad l = 1, 2, \dots, p, \\ y(0) = y_0, \quad y'(0) = \overline{y}_0, \end{gathered} \tag{6.8}$$

and its special case

$$\begin{gathered} {}_0D^q_t y(t) = f(t, y(t)), \quad t \in [0, T], \ t \neq t_k, \ k = 1, 2, \dots, m, \\ \Delta y(t_k) = I_k(y(t_k^-)), \quad \Delta y'(t_k) = \overline{I}_k(y(t_k^-)), \ k = 1, 2, \dots, m, \\ y(0) = y_0, \quad y'(0) = \overline{y}_0, \end{gathered} \tag{6.9}$$

where $q \in (1, 2)$, $y_0, \overline{y}_0 \in \mathbb{R}$, ${}_0D^q_t$ is the Caputo fractional derivative in interval $[0, t]$, $f : [0, T] \times \times \mathbb{R} \mapsto \mathbb{R}$ is an appropriate continuous function, $I_k, \overline{I}_l : \mathbb{R} \mapsto \mathbb{R}(k = 1, 2, \dots, m, l = 1, 2, \dots, p)$ are continuous functions, $0 = t_0 < t_1 < \dots < t_m < t_{m+1} = T$, $0 = \bar{t}_0 < \bar{t}_1 < \dots < \bar{t}_p < \bar{t}_{p+1} = T$, $\Delta y(t_k) = y(t_k^+) - y(t_k^-) = \lim_{\epsilon \to 0^+} y(t_k + \epsilon) - \lim_{\epsilon \to 0^+} y(t_k - \epsilon)$, $\Delta y'(t_k) = y'(t_k^+) - y'(t_k^-) = \lim_{\epsilon \to 0^+} y'(t_k + \epsilon) - \lim_{\epsilon \to 0^+} y'(t_k - \epsilon)$. The main theorem ([126, Theorem 2.1]) claims that

Result 6.5. System (6.8) is equivalent to the integral equation

$$y(t)=\begin{cases} y_0+\overline{y}_0t+\int_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}f(s,y(s))ds, \quad t\in J_0',\\ y_0+\overline{y}_0t+\sum_{1\le k\le n_1}I_k(y(t_k^-))+\sum_{1\le l\le n_2}(t-\overline{t}_l)\overline{I}_l(y(\overline{t}_l^-))\\ +\int_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}f(s,y(s))ds+\xi\sum_{1\le k\le n_1}\Big[I_k(y(t_k^-))\\ \times\Big(\int_0^{t_k}(t_k-s)^{q-1}f(s,y(s))ds+\int_{t_k}^t(t-s)^{q-1}f(s,y(s))ds\\ -\int_0^t(t-s)^{q-1}f(s,y(s))ds\Big)\Big]/\Gamma(q)\\ +\zeta\sum_{1\le l\le n_2}\Big[\overline{I}_l(y(\overline{t}_l^-))\Big(\int_0^{\overline{t}_l}(\overline{t}_l-s)^{q-1}f(s,y(s))ds\\ +\int_{\overline{t}_l}^t(t-s)^{q-1}f(s,y(s))ds-\int_0^t(t-s)^{q-1}f(s,y(s))ds\Big)\Big]/\Gamma(q)\\ +\frac{\xi}{\Gamma(q-1)}\sum_{1\le k\le n_1}\Big[(t-t_k)I_k(y(t_k^-))\int_0^{t_k}(t_k-s)^{q-2}f(s,y(s))ds\Big]\\ +\frac{\zeta}{\Gamma(q-1)}\sum_{1\le l\le n_2}\Big[(t-\overline{t}_l)\overline{I}_l(y(\overline{t}_l^-))\int_0^{\overline{t}_l}(\overline{t}_l-s)^{q-2}f(s,y(s))ds\Big],\\ \quad t\in J_n',\ n=1,2,\dots,\Delta, \end{cases}$$

provided that the integral exists, where $q\in(1,2)$, $\xi,\zeta\in\mathbb{R}$ are two constants,

$$\{t_1,t_2,\dots,t_m,\overline{t}_1,\overline{t}_2,\dots,\overline{t}_p\}=\{t_1',t_2'\dots,t_\Delta\}$$

with $0=t_0'<t_1'<t_2'<\dots<t_\Delta<t_{\Delta+1}=T$, $J_k'=(t_k',t_{k+1}']$ $(k=0,1,2,\dots,\Delta)$.

The following statement was also claimed [126, Corollary 2.4].

Result 6.6. System (6.9) is equivalent to the integral equation

$$y(t)=\begin{cases} y_0+\overline{y}_0t+\int_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}f(s,y(s))ds, \quad t\in[0,t_1],\\ y_0+\overline{y}_0t+\sum_{1\le i\le k}I_k(y(t_k^-))+\sum_{1\le i\le k}(t-\overline{t}_i)\overline{I}_i(y(\overline{t}_i^-))\\ +\int_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}f(s,y(s))ds+\sum_{1\le i\le k}\Big[(\xi I_i(y(t_i^-))\\ +\zeta\overline{I}_i(y(t_i^-)))\Big(\int_0^{t_i}(t_i-s)^{q-1}f(s,y(s))ds\\ +\int_{t_i}^t(t-s)^{q-1}f(s,y(s))ds-\int_0^t(t-s)^{q-1}f(s,y(s))ds\Big)\Big]/\Gamma(q)\\ +\frac{\sum_{1\le i\le k}\Big[(\xi I_i(y(t_i^-))+\zeta\overline{I}_i(y(t_i^-)))(t-t_i)\int_0^{t_i}(t_i-s)^{q-2}f(s,y(s))ds\Big]/\Gamma(q-1)}{t\in(t_k,t_{k+1}],\ k=1,2,\dots,m.}, \end{cases}$$

We find that Results 6.5 and 6.6 are also incorrect.

It is easy to see that (6.8) and (6.9) can be generalized by the IVP

$$\begin{gathered} {}^cD_{0^+}^\alpha y(t)-\lambda y(t)=f(t,y(t)),\quad t\in(t_i,t_{i+1}],\ i\in\mathbb{N}_0^m,\\ \Delta y(t_i)=I(t_i,y(t_i^-)),\Delta y'(t_i)=\overline{I}(t_i,y(t_i^-)),\quad i\in\mathbb{N}_1^m,\\ y(0)=y_0,\quad y'(0)=\overline{y}_0, \end{gathered}\tag{6.10}$$

where $\alpha\in(1,2)$, ${}^cD_{0^+}^*$ is the Caputo fractional derivative of order $*$, $f:[0,1]\times\mathbb{R}\mapsto\mathbb{R}$, $I,J:\{t_i:i\in\mathbb{N}_1^m\}\times\mathbb{R}\mapsto\mathbb{R}$ are continuous functions, $\lambda\in\mathbb{R}$, $0=t_0<t_1<\dots<t_m<t_{m+1}=1$, $y_0,\overline{y}_0\in\mathbb{R}$. One sees that BVP(6.10) generalizes BVP(6.8) and (6.9). We now establish existence results for (6.10).

Theorem 6.7. *A function y is a solution of* (6.10) *if and only if*

$$
\begin{aligned}
y(t) &= y_0\mathbf{E}_{\alpha,1}(\lambda t^\alpha) + \overline{y}_0 t\mathbf{E}_{\alpha,2}(\lambda t^\alpha) + \sum_{j=1}^{i} I_j(t_j, y(t_j))\mathbf{E}_{\alpha,1}(\lambda(t-t_j)^\alpha) \\
&\quad + \sum_{j=1}^{i} \overline{I}_j(t_j, y(t_j))(t-t_j)\mathbf{E}_{\alpha,2}(\lambda(t-t_j)^\alpha) \\
&\quad + \int_0^t (t-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t-s)^\alpha) f(s, y(s))ds, \quad t \in (t_i, t_{i+1}],\ i \in \mathbb{N}_0^m.
\end{aligned}
\tag{6.11}
$$

Proof. From Theorem 3.11 (with $n = 2$), y is a solution of (6.10) if and only if there exist constants $c_j, d_j \in \mathbb{R}(j \in \mathbb{N}_0^m)$ such that

$$
\begin{aligned}
y(t) &= \sum_{j=0}^{i} c_j\mathbf{E}_{\alpha,1}(\lambda(t-t_j)^\alpha) + \sum_{j=0}^{i} d_j(t-t_j)\mathbf{E}_{\alpha,2}(\lambda(t-t_j)^\alpha) \\
&\quad + \int_0^t (t-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t-s)^\alpha) f(s, y(s))ds,
\end{aligned}
\tag{6.12}
$$

for $t \in (t_i, t_{i+1}]$ and $i \in \mathbb{N}_0^m$. By direct computations we obtain

$$
\begin{aligned}
y'(t) &= \lambda\sum_{j=0}^{i} c_j(t-t_j)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t-t_j)^\alpha) + \sum_{j=0}^{i} d_j\mathbf{E}_{\alpha-1,1}(\lambda(t-t_j)^{\alpha-1}) \\
&\quad + \int_0^t (t-s)^{\alpha-2}\mathbf{E}_{\alpha,\alpha-1}(\lambda(t-s)^\alpha) f(s, y(s))ds,
\end{aligned}
\tag{6.13}
$$

for $t \in (t_i, t_{i+1}]$ and $i \in \mathbb{N}_0^m$. From $y(0) = y_0, y'(0) = \overline{y}_0$, we obtain $c_0 = y_0$ and $d_0 = \overline{y}_0$. From $\Delta y(t_i) = I(t_i, y(t_i^-)), \Delta y'(t_i) = \overline{I}(t_i, y(t_i^-))$, we obtain $c_i = I(t_i, y(t_i^-))$ and $d_i = \overline{I}(t_i, y(t_i^-))$ for all $i \in \mathbb{N}_1^m$. Substituting c_i, d_i into (6.12), we obtain (6.11). The proof is complete. □

Remark 6.8. Let $\lambda = 0$. By (6.11), we obtain that the IVP

$$
\begin{gathered}
{}^cD_{0^+}^\alpha y(t) = f(t, y(t)), \quad t \in (t_i, t_{i+1}],\ i \in \mathbb{N}_0^m, \\
\Delta y(t_i) = I(t_i, y(t_i^-)), \quad \Delta y'(t_i) = \overline{I}(t_i, y(t_i^-)),\ i \in \mathbb{N}_1^m, \\
y(0) = y_0, \quad y'(0) = \overline{y}_0,
\end{gathered}
$$

is equivalent to the integral equation

$$
\begin{aligned}
y(t) &= y_0 + \overline{y}_0 t + \sum_{j=1}^{i} I_j(t_j, y(t_j)) + \sum_{j=1}^{i} \overline{I}_j(t_j, y(t_j))(t-t_j) \\
&\quad + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, y(s))ds, \quad t \in (t_i, t_{i+1}],\ i \in \mathbb{N}_0^m.
\end{aligned}
$$

It is easy to see that Results 6.5 and 6.6 (obtained in [126]) are wrong.

6.3. **Corrected results from** [127]**.** Zhang [127] considered the initial value problem for fractional differential equation with Caputo-Hadamard fractional derivative and impulsive effect,

$$
\begin{gathered}
{}^{ch}D_{a^+}^q x(t) = f(t, x(t)), \quad t \in (a, T],\ t \neq t_k,\ k = 1, 2, \dots, m, \\
\Delta x(t_k) = \Delta_k(x(t_k^-)), \quad k = 1, 2, \dots, m,\ x(a) = x_a,
\end{gathered}
\tag{6.14}
$$

where $q \in \mathbb{C}$ and $\mathbb{R}(q) \in (0,1)$, ${}^{ch}D^q_{a^+}$ denotes the left-sided Caputo-Hadamard fractional derivative of order q with the low limit $a(>0)$, $a = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$, $f : (a,T] \times \mathbb{C} \mapsto \mathbb{C}$ is an appropriate continuous function, $x(t_k^+) = \lim_{\epsilon\to 0^+} x(t_k+\epsilon)$ and $x(t_k^-) = \lim_{\epsilon\to 0^-} x(t_k+\epsilon)$ represent the right and left limits of $x(t)$ at $t = t_k$, respectively. The following result was claimed [127, Theorem 3.2].

Result 6.9. System (6.14) is equivalent to the fractional integral equation

$$x(t) = \begin{cases} x_a + \int_a^t \frac{(\ln t-\ln s)^{q-1}}{\Gamma(q)} f(s,x(s))\frac{ds}{s}, \quad t \in (a,t_1], \\ x_a + \sum_{i=1}^k \Delta_i(x(t_i^-)) + \int_a^t \frac{(\ln t-\ln s)^{q-1}}{\Gamma(q)} f(s,x(s))\frac{ds}{s} \\ + \sum_{i=1}^k \frac{h\Delta_i(x(t_i^-))}{\Gamma(q)}\Big[\int_a^{t_i} \frac{(\ln t_i-\ln s)^{q-1}}{\Gamma(q)} f(s,x(s))\frac{ds}{s} \\ + \int_{t_i}^t \frac{(\ln t-\ln s)^{q-1}}{\Gamma(q)} f(s,x(s))\frac{ds}{s} - \int_a^t \frac{(\ln t-\ln s)^{q-1}}{\Gamma(q)} f(s,x(s))\frac{ds}{s}\Big], \\ \quad t \in (t_i,t_{i+1}],\ i = 1,2,\dots,m, \end{cases}$$

where h is a constant. We find that this result is wrong. We consider the more general problem

$$\begin{gathered} {}^{ch}D^q_{0^+}x(t) - \lambda x(t) = f(t,x(t)), \quad t \in (t_i,t_{i+1}],\ i \in \mathbb{N}_0^m, \\ \Delta x(t_i) = x(t_i^+) - x(t_i^-) = I(t_i,x(t_i)), \quad i \in \mathbb{N}_1^m,\ x(1) = x_0, \end{gathered} \tag{6.15}$$

where $q \in (0,1)$, $p \in (0,q)$, ${}^{ch}D^*_{0^+}$ is the Caputo-Hadamard type fractional derivative of order $*$, $\lambda \in \mathbb{R}$, $f : [1,e] \times \mathbb{R} \mapsto \mathbb{R}$, $I : \{t_i : i \in \mathbb{N}_1^m\} \mapsto \mathbb{R}$ are continuous functions, $1 = t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = e$, $x_0 \in \mathbb{R}$. One sees that (6.15) is a generalization of (6.14) ($a = 1$ and $\lambda = 0$).

Theorem 6.10. *BVP (6.15) is equivalent to the integral equation*

$$\begin{aligned} x(t) &= x_0\mathbf{E}_{q,1}(\lambda(\ln t)^q) + \sum_{j=1}^i I(t_j,x(t_j))\mathbf{E}_{q,1}(\lambda(\ln t-\ln t_j)^q) \\ &\quad + \int_0^t (\ln t-\ln s)^{q-1}\mathbf{E}_{q,q}(\lambda(\ln t-\ln s)^q) f(s,x(s))\frac{ds}{s}, \end{aligned} \tag{6.16}$$

for $t \in (t_i,t_{i+1}]$ *and* $i \in \mathbb{N}_0^m$.

Proof. By Theorem 3.14, x is a solution of BVP(6.15) if and only if there exist constants $d_j \in \mathbb{R}(j \in \mathbb{N}_0^m)$ such that

$$\begin{aligned} x(t) &= \sum_{j=0}^i d_j\mathbf{E}_{q,1}(\lambda(\ln t-\ln t_j)^q) \\ &\quad + \int_0^t (\ln t-\ln s)^{q-1}\mathbf{E}_{q,q}(\lambda(\ln t-\ln s)^q) f(s,x(s))\frac{ds}{s}, \end{aligned} \tag{6.17}$$

for $t \in (t_i,t_{i+1}]$ and $i \in \mathbb{N}_0^m$. From $\Delta x(t_i) = I(t_i,x(t_i))$, we obtain $d_i = I(t_i,x(t_i))$ ($i \in \mathbb{N}_1^m$). From $x(1) = x_0$, we obtain $d_0 = x_0$. Substituting d_j into (6.17), we obtain (6.16). The proof is complete. □

Remark 6.11. From (6.16), letting $\lambda = 0$, we obtain that the system

$$\begin{gathered} {}^{ch}D^q_{0^+}x(t) = f(t,x(t)), \quad t \in (t_i,t_{i+1}],\ i \in \mathbb{N}_0^m, \\ \Delta x(t_i) = I(t_i,x(t_i)), \quad i \in \mathbb{N}_1^m,\ x(1) = x_0 \end{gathered}$$

is equivalent to

$$x(t)=x_0+\sum_{j=1}^{i}I(t_j,x(t_j))+\int_0^t\frac{(\ln t-\ln s)^{q-1}}{\Gamma(q)}f(s,x(s))\frac{ds}{s},\quad t\in(t_i,t_{i+1}],\ i\in\mathbb{N}_0^m.$$

Then we know that Result 6.9 obtained in [127] is wrong.

6.4. **Corrected results from** [112, 113]**.** In [112, 113], the authors studied the existence of solutions of the anti-periodic boundary value problems for impulsive fractional differential equations,

$$\begin{gathered}D_{0^+}^q x(t)+\lambda(t)x(t)=f(t,x(t)),\quad t\in[0,1]\setminus\{t_1,t_2,\dots,t_m\},\\ I_{0^+}^\alpha x(t_i^+)-I_{0^+}^\alpha(t_i^-)=J_i(x(t_i)),\quad i=1,2,\dots,m,\ t^{1-q}x(t)|_{t=0}+t^{1-q}x(t)|_{t=1}=0,\end{gathered}$$

and

$$\begin{gathered}{}^cD_{0^+}^q x(t)+\lambda((t)x(t)=f(t,x(t)),\quad t\in[0,1]\setminus\{t_1,t_2,\dots,t_m\},\\ I_{0^+}^\alpha x(t_i^+)-I_{0^+}^\alpha(t_i^-)=J_i(x(t_i)),\quad i=1,2,\dots,m,\ x(0)+x(1)=0,\end{gathered}$$

where $q,\alpha\in(0,1)$, ${}^cD_{0^+}^q$ is the Caputo fractional derivative, $D_{0^+}^q$ is the Riemann-Liouville fractional derivative, $I_{0^+}^\alpha$ is the Riemann-Liouville fractional integral, $0=t_0<t_1<\dots<t_m<t_{m+1}=1$, $\lambda\in C^0([0,1],R)$ satisfies $\lambda_0=:\max_{t\in[0,1]}\lambda(t)>0$, $J_k:\mathbb{R}\mapsto\mathbb{R}$ is continuous, f is a given piecewise continuous function. The main results in [113] are based upon the following two results:

Result 6.12 ([113, Lemma2.7])**.** Suppose that $q,\alpha\in(0,1)$. Then x is a solution of

$$\begin{gathered}D_{0^+}^q x(t)+\lambda_0x(t)=f(t,x(t)),\quad t\in[0,1]\setminus\{t_1,t_2,\dots,t_m\},\\ I_{0^+}^\alpha x(t_i^+)-I_{0^+}^\alpha(t_i^-)=J_i(x(t_i)),\quad i=1,2,\dots,m,\ t^{1-q}x(t)|_{t=0}+t^{1-q}x(t)|_{t=1}=0\end{gathered}\tag{6.18}$$

if and only if x is a fixed point of the operator $T_q:PC_q(0,1]\mapsto PC_q(0,1]$, where T_q is defined by

$$\begin{aligned}(T_qx)(t)=\frac{\Gamma(q)t^{q-1}\mathbf{E}_{q,q}(-\lambda_0t^q)}{1+\Gamma(q)\mathbf{E}_{q,q}(-\lambda_0)}\Big[&\sum_{i=1}^m\frac{J_i(x(t_i))}{\Gamma(q)t_i^{\alpha+q-1}\mathbf{E}_{q,q+\alpha}(-\lambda_0t_i^q)}\\ &-\int_0^1(1-s)^{q-1}\mathbf{E}_{q,q}(-\lambda_0(1-s)^q)f(s,x(s))ds\Big]\\ &-t^{q-1}\mathbf{E}_{q,q}(-\lambda_0t^q)\sum_{t\le t_i<1}\frac{J_i(x(t_i))}{t_i^{\alpha+q-1}\mathbf{E}_{q,q+\alpha}(-\lambda_0t_i^q)}\\ &+\int_0^t(t-s)^{q-1}\mathbf{E}_{q,q}(-\lambda_0(t-s)^q)f(s,x(s))ds.\end{aligned}$$

Result 6.13 (cite[Lemma 2.8]wl3)**.** Suppose that $q,\alpha\in(0,1)$. Then x is a solution of

$$\begin{gathered}{}^cD_{0^+}^q x(t)+\lambda_0x(t)=f(t,x(t)),\quad t\in[0,1]\setminus\{t_1,t_2,\dots,t_m\},\\ I_{0^+}^\alpha x(t_i^+)-I_{0^+}^\alpha(t_i^-)=J_i(x(t_i)),\quad i=1,2,\dots,m,\ x(0)+x(1)=0\end{gathered}\tag{6.19}$$

if and only if x is a fixed point of the operator $T : PC(0,1] \mapsto PC(0,1]$, where T is defined by

$$\begin{aligned}(Tx)(t) = \frac{\mathbf{E}_{q,1}(-\lambda_0 t^q)}{1+\mathbf{E}_{q,1}(-\lambda_0)}\Big[\sum_{i=1}^{m} \frac{J_i(x(t_i))}{t_i^{\alpha}\mathbf{E}_{q,1+\alpha}(-\lambda_0 t_i^q)} \\ - \int_0^1 (1-s)^{q-1}\mathbf{E}_{q,q}(-\lambda_0(1-s)^q)f(s,x(s))ds\Big] \\ - \mathbf{E}_{q,1}(-\lambda_0 t^q)\sum_{t\le t_i<1} \frac{J_i(x(t_i))}{t_i^{\alpha}\mathbf{E}_{q,1+\alpha}(-\lambda_0 t_i^q)} \\ + \int_0^t (t-s)^{q-1}\mathbf{E}_{q,q}(-\lambda_0(t-s)^q)f(s,x(s))ds.\end{aligned}$$

We consider the problem

$$\begin{gathered} {}^cD_{0^+}^{\alpha}x(t) + \lambda x(t) = f(t,x(t)), \quad t\in(t_i,t_{i+1}],\ i\in\mathbb{N}_0^m, \\ \Delta x(t_i) = x(t_i^+) - x(t_i^-) = I(t_i, x(t)i)), \quad i\in\mathbb{N}_1^m,\ x(0)+x(1)=0, \end{gathered} \tag{6.20}$$

where $\lambda\in\mathbb{R}$, $f:[0,1]\times\mathbb{R}\mapsto\mathbb{R}$, $I:\{t_i : i\in\mathbb{N}_1^m\}\times\mathbb{R}\mapsto\mathbb{R}$ are continuous functions, ${}^cD_{0^+}^{\alpha}$ is the Caputo fractional derivative with the order $\alpha\in(0,1)$, $0=t_0<t_1<t_2<\dots<t_m<t_{m+1}=1$.

By Theorem 3.11, we know x is a solution of (6.20) if and only if there exists constants $c_i (i\in\mathbb{N}_0^m)$ such that

$$x(t) = \sum_{v=0}^{j} c_v\mathbf{E}_{\alpha,1}(-\lambda(t-t_v)^{\alpha}) + \int_0^t (t-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(-\lambda(t-s)^{\alpha})f(s,x(s))ds, \tag{6.21}$$

for $t\in(t_j,t_{j+1}]$, $j\in\mathbb{N}_0^m$. From $\Delta x(t_i) = I(t_i,x(t)i))$, we have $c_i = I(t_i,x(t)i))$ $(i\in\mathbb{N}_1^m)$. By $x(0)+x(1)=0$, we have

$$c_0 + \sum_{v=0}^{m} c_v\mathbf{E}_{\alpha,1}(-\lambda(1-t_v)^{\alpha}) + \int_0^1 (1-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(-\lambda(t-s)^{\alpha})f(s,x(s))ds = 0.$$

It follows that

$$\begin{aligned} c_0 = -\frac{1}{1+\mathbf{E}_{\alpha,1}(-\lambda)}\Big[\sum_{v=1}^{m} I(t_v,x(t_v))\mathbf{E}_{\alpha,1}(-\lambda(1-t_v)^{\alpha}) \\ + \int_0^1 (1-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(-\lambda(t-s)^{\alpha})f(s,x(s))ds\Big]. \end{aligned}$$

Substituting c_v into (6.21), we obtain

$$\begin{aligned} x(t) = -\frac{\mathbf{E}_{\alpha,1}(\lambda t^{\alpha})}{1+\mathbf{E}_{\alpha,1}(-\lambda)}\Big[\sum_{v=1}^{m} I(t_v,x(t_v))\mathbf{E}_{\alpha,1}(-\lambda(1-t_v)^{\alpha}) \\ + \int_0^1 (1-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(-\lambda(t-s)^{\alpha})f(s,x(s))ds\Big] \\ + \sum_{v=1}^{j} I(t_v,x(t_v))\mathbf{E}_{\alpha,1}(-\lambda(t-t_v)^{\alpha}) \\ + \int_0^t (t-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(-\lambda(t-s)^{\alpha})f(s,x(s))ds, \end{aligned} \tag{6.22}$$

for $t \in (t_j, t_{j+1}]$ and $j \in \mathbb{N}_0^m$. From (6.22), Result 6.13 is wrong.

We consider problem (6.18). By Theorem 3.12, we know x is a solution of (6.18) if and only if there exists constants $c_i (i \in \mathbb{N}_0^m)$ such that

$$\begin{aligned} x(t) &= \sum_{v=0}^{j} c_v (t-t_v)^{q-1} \mathbf{E}_{q,q}(-\lambda_0 (t-t_v)^q) \\ &\quad + \int_0^t (t-s)^{q-1} \mathbf{E}_{q,q}(-\lambda_0 (t-s)^q) f(s, x(s)) ds, \end{aligned} \tag{6.23}$$

for $t \in (t_j, t_{j+1}]$ and $j \in \mathbb{N}_0^m$. By Definition 2.1 and direct computations, we have

$$\begin{aligned} I_{0^+}^{\alpha} x(t) &= \sum_{v=0}^{j} c_v (t-t_v)^{q+\alpha-1} \mathbf{E}_{q,q+\alpha}(-\lambda_0 (t-t_v)^q) \\ &\quad + \int_0^t (t-s)^{q+\alpha-1} \mathbf{E}_{q,q+\alpha}(-\lambda_0 (t-s)^q) f(s, x(s)) ds, \end{aligned} \tag{6.24}$$

for $t \in (t_j, t_{j+1}]$ and $j \in \mathbb{N}_0^m$. Using $t^{1-q} x(t)|_{t=0} + t^{1-q} x(t)|_{t=1} = 0$, we obtain

$$\begin{aligned} &\frac{c_0}{\Gamma(q)} + \sum_{v=0}^{m} c_v (1-t_v)^{q-1} \mathbf{E}_{q,q}(-\lambda_0 (1-t_v)^q) \\ &+ \int_0^1 (1-s)^{q-1} \mathbf{E}_{q,q}(-\lambda_0 (1-s)^q) f(s, x(s)) ds = 0. \end{aligned}$$

Case 1: $\alpha + q < 1$. From (6.23) we know that the existence of $I_{0^+}^{\alpha} x(t_i^+)$ implies $c_i = 0 (i \in \mathbb{N}[1, m])$. So

$$I_{0^+}^{\alpha} x(t) = c_0 t^{q+\alpha-1} \mathbf{E}_{q,q+\alpha}(-\lambda_0 t^q) + \int_0^t (t-s)^{q+\alpha-1} \mathbf{E}_{q,q+\alpha}(-\lambda_0 (t-s)^q) f(s, x(s)) ds,$$

for $t \in (t_j, t_{j+1}]$, $j \in \mathbb{N}_0^m$. Then $I_{0^+}^{\alpha} x(t_i^+) - I_{0^+}^{\alpha}(t_i^-) = J_i(x(t_i))$ implies that $J_i(x(t_i)) = 0 (i \in \mathbb{N}[1, m]$ So this impulse model is unsuitable.

Case 2: $\alpha + q = 1$. From $I_{0^+}^{\alpha} x(t_i^+) - I_{0^+}^{\alpha}(t_i^-) = J_i(x(t_i))$ and (6.23), we obtain

$$c_i - \sum_{v=0}^{i-1} c_v \mathbf{E}_{q,q+\alpha}(-\lambda_0 (t_i - t_v)^q) = J_i(x(t_i)), \quad i \in \mathbb{N}_1^m.$$

Case 3: $\alpha + q > 1$. From $I_{0^+}^{\alpha} x(t_i^+) - I_{0^+}^{\alpha}(t_i^-) = J_i(x(t_i))$ and (6.23), we obtain

$$-\sum_{v=0}^{i-1} (t_i - t_v)^{\alpha+q-1} c_v \mathbf{E}_{q,q+\alpha}(-\lambda_0 (t_i - t_v)^q) = J_i(x(t_i)), \quad i \in \mathbb{N}_1^m.$$

We know that Result 6.12 is wrong.

6.5. **Corrected results from** [115]**.** In [115], authors established the existence of solutions for a class of nonlinear impulsive Hadamard fractional differential equations with initial condition of the form

$$\begin{gathered} {}^{rh}D_{1^+}^{\alpha} x(t) = f(t, x(t)), \quad t \in (1, e] \setminus \{t_1, t_2, \dots, t_m\}, \\ \Delta^* x(t_i) = {}^{h}\mathfrak{J}_{1^+}^{1-\alpha} x(t_i^+) - {}^{h}\mathfrak{J}_{1^+}^{1-\alpha} x(t_i^-) = p_i, \quad i = 1, 2, \dots, m, \\ {}^{h}\mathfrak{J}_{1^+}^{1-\alpha} u(1) = u_0, \end{gathered} \tag{6.25}$$

where ${}^{rh}D^{\alpha}_{1^+}$ is the left-side Riemann-Liouville type Hadamard fderivative of order $\alpha\in(0,1)$ with the starting point 1 and ${}^{h}\mathfrak{J}^{1-\alpha}_{1^+}$ denotes left-side Hadamard fractional integral of order $1-\alpha$, $1=t_0<t_1<\dots<t_m<t_{m+1}=e$, $u_0,p_i\in\mathbb{R}(i=1,2,\dots,m)$, $f:[1,e]\times\mathbb{R}\mapsto\mathbb{R}$ is a continuous function. It was claimed the following result [115, Lemma 2.9, p. 87]:

Result 6.14. Let $f:[1,e]\times\mathbb{R}\mapsto\mathbb{R}$ and $t\mapsto(\ln t)^{1-\alpha}f(t,u)$ are continuous functions. Then x is a solution of the fractional integral equation

$$x(t)=\begin{cases}\frac{u_0}{\Gamma(\alpha)}(\ln t)^{\alpha-1}+\int_1^t\frac{(\ln t-\ln s)^{\alpha-1}}{\Gamma(\alpha)}f(s,x(s))\frac{ds}{s}, \quad t\in(1,t_1],\\ \frac{u_0}{\Gamma(\alpha)}(\ln t)^{\alpha-1}+\sum_{j=1}^i\frac{p_j}{\Gamma(\alpha)}(\ln t)^{\alpha-1}+\int_1^t\frac{(\ln t-\ln s)^{\alpha-1}}{\Gamma(\alpha)}f(s,x(s))\frac{ds}{s},\\ \quad t\in(t_i,t_{i+1}],i=1,2,\dots,m\end{cases}$$

if and only if x is a solution of IVP (6.25).

We note that Result 6.14 is also wrong. Now we consider the initial value problem for impulsive fractional differential equation involving the Riemann-Liouville type Hadamard fractional derivatives

$$\begin{gathered}{}^{rh}D^{\alpha}_{1^+}x(t)-\lambda{}^{rh}D^{\beta}_{1^+}x(t)=f(t,x(t)),\quad t\in(t_i,t_{i+1}],\ i\in\mathbb{N}_0^m,\\ \Delta^*x(t_i)=I(t_i,{}^{h}\mathfrak{J}^{1-\alpha}_{1^+}x(t_i)),\quad i\in\mathbb{N}_1^m,\\ {}^{h}\mathfrak{J}^{1-\alpha}_{1^+}u(1)=u_0,\end{gathered}\tag{6.26}$$

where $\alpha\in(0,1)$ and $\beta\in(0,\beta)$, ${}^{rh}D^{*}_{1^+}$ is the left-side Riemann-Liouville type Hadamard fderivative of order $*\in(0,1)$ with the starting point 1 and ${}^{h}\mathfrak{J}^{1-\alpha}_{1^+}$ denotes left-side Hadamard fractional integral of order $1-\alpha$, $u_0\in\mathbb{R}$, $1=t_0<t_1<\dots<t_m<t_{m+1}=e$, $f:[1,e]\times\mathbb{R}\mapsto\mathbb{R}$ is a V-Carathéodory function, $I:\{t_i:i\in\mathbb{N}_1^m\}\times\mathbb{R}\mapsto\mathbb{R}$ is a discrete V-Carathéodory function,

$$\Delta^*x(t_i)={}^{h}\mathfrak{J}^{1-\alpha}_{1^+}x(t_i^+)-{}^{h}\mathfrak{J}^{1-\alpha}_{1^+}x(t_i^-).$$

It is easy to see that (6.26) generalizes (6.25) ($\lambda=0$ and $I(t_i,u)=p_i$).

Theorem 6.15. *Suppose that $\alpha\in(0,1)$. Then x is a solution of (6.26) if and only x is the solution of the integral equation*

$$\begin{aligned}x(t)&=u_0(\ln t)^{\alpha-1}\mathbf{E}_{\alpha-\beta,\alpha}(\lambda(\ln t)^{\alpha-\beta})\\&\quad+\sum_{j=1}^iI(t_j,x(t_j))(\ln t-\ln t_j)^{\alpha-1}\mathbf{E}_{\alpha-\beta,\alpha}(\lambda(\ln t-\ln t_j)^{\alpha-\beta})\\&\quad+\int_1^t(\ln t-\ln s)^{\alpha-1}\mathbf{E}_{\alpha-\beta,\alpha}(\lambda(\ln t-\ln s)^{\alpha-\beta})f(s,x(s))\frac{ds}{s},\end{aligned}\tag{6.27}$$

for $t\in(t_i,t_{i+1}]$, $i\in\mathbb{N}_0^m$.

Proof. We can prove that x is a piecewise continuous solution of

$${}^{rh}D^{\alpha}_{1^+}x(t)-\lambda{}^{rh}D^{\beta}_{1^+}x(t)=f(t,x(t)),\quad t\in(t_i,t_{i+1}],\ i\in\mathbb{N}_0^m.$$

if and only if there exist constants $d_j \in \mathbb{R}(j \in \mathbb{N}_0^m)$ such that

$$\begin{aligned} x(t) &= \sum_{j=0}^{i} d_j(\ln t - \ln t_j)^{\alpha-1}\mathbf{E}_{\alpha-\beta,\alpha}(\lambda(\ln t - \ln t_j)^{\alpha-\beta}) \\ &\quad + \int_1^t (\ln t - \ln s)^{\alpha-1}\mathbf{E}_{\alpha-\beta,\alpha}(\lambda(\ln t - \ln s)^{\alpha-\beta})f(s, x(s))\frac{ds}{s}, \end{aligned} \tag{6.28}$$

for $t \in (t_i, t_{i+1}]$ and $i \in \mathbb{N}_0^m$. For $t \in (t_i, t_{i+1}]$, using Definition 2.1-(ii) we have

$$\begin{aligned} &{}^{h}\mathfrak{I}_{1^+}^{1-\alpha}x(t) \\ &= \int_1^t \frac{(\ln t - \ln s)^{-\alpha}}{\Gamma(1-\alpha)}x(s)\frac{ds}{s} \\ &= \Big(\sum_{\tau=0}^{i-1}\int_{t_\tau}^{t_{\tau+1}} (\ln t - \ln s)^{-\alpha}x(s)\frac{ds}{s} + \int_{t_i}^t (\ln t - \ln s)^{-\alpha}x(s)\frac{ds}{s}\Big)\Gamma(1-\alpha) \\ &= \Big(\sum_{\tau=0}^{i-1}\int_{t_\tau}^{t_{\tau+1}} (\ln t - \ln s)^{-\alpha}\Big[\sum_{j=0}^{\tau} d_j(\ln s - \ln t_j)^{\alpha-1}\mathbf{E}_{\alpha-\beta,\alpha}\big(\lambda(\ln s - \ln t_j)^{\alpha-\beta}\big) \\ &\quad + \int_1^s (\ln s - \ln u)^{\alpha-1}\mathbf{E}_{\alpha-\beta,\alpha}(\lambda(\ln s - \ln u)^{\alpha-\beta})f(u, x(u))\frac{du}{u}\Big]\frac{ds}{s}\Big)/\Gamma(1-\alpha) \\ &\quad + \Big(\int_{t_i}^t (\ln t - \ln s)^{-\alpha}\Big[\sum_{j=0}^{i} d_j(\ln s - \ln t_j)^{\alpha-1}\mathbf{E}_{\alpha-\beta,\alpha}\big(\lambda(\ln s - \ln t_j)^{\alpha-\beta}\big) \\ &\quad + \int_1^s (\ln s - \ln u)^{\alpha-1}\mathbf{E}_{\alpha-\beta,\alpha}(\lambda(\ln s - \ln u)^{\alpha-\beta})f(u, x(u))\frac{du}{u}\Big]\frac{ds}{s}\Big)/\Gamma(1-\alpha) \\ &= \Big(\sum_{\tau=0}^{i-1}\int_{t_\tau}^{t_{\tau+1}} (\ln t - \ln s)^{-\alpha}\Big[\sum_{j=0}^{\tau} d_j(\ln s - \ln t_j)^{\alpha-1} \\ &\quad \times \mathbf{E}_{\alpha-\beta,\alpha}\big(\lambda(\ln s - \ln t_j)^{\alpha-\beta}\big)\Big]\frac{ds}{s}\Big)/\Gamma(1-\alpha) \\ &\quad + \Big(\int_{t_i}^t (\ln t - \ln s)^{-\alpha}\Big[\sum_{j=0}^{i} d_j(\ln s - \ln t_j)^{\alpha-1} \\ &\quad \times \mathbf{E}_{\alpha-\beta,\alpha}\big(\lambda(\ln s - \ln t_j)^{\alpha-\beta}\big)\Big]\frac{ds}{s}\Big)/\Gamma(1-\alpha) \\ &\quad + \Big(\int_1^t (\ln t - \ln s)^{-\alpha}\Big[\int_1^s (\ln s - \ln u)^{\alpha-1}\mathbf{E}_{\alpha-\beta,\alpha}(\lambda(\ln s - \ln u)^{\alpha-\beta}) \\ &\quad \times f(u, x(u))\frac{du}{u}\Big]\frac{ds}{s}\Big)/\Gamma(1-\alpha) \\ &\quad \text{(by changing the order of the sum and the integral)} \\ &= \Big(\sum_{j=0}^{i-1}\sum_{\tau=j}^{i-1} d_j\int_{t_\tau}^{t_{\tau+1}} (\ln t - \ln s)^{-\alpha}(\ln s - \ln t_j)^{\alpha-1} \\ &\quad \times \mathbf{E}_{\alpha-\beta,\alpha}\big(\lambda(\ln s - \ln t_j)^{\alpha-\beta}\big)\frac{ds}{s}\Big)/\Gamma(1-\alpha) \end{aligned}$$

$$
\begin{aligned}
&+\Big(\sum_{j=0}^{i} d_j \int_{t_i}^{t} (\ln t-\ln s)^{-\alpha}(\ln s-\ln t_j)^{\alpha-1}\\
&\times \mathbf{E}_{\alpha-\beta,\alpha}\big(\lambda(\ln s-\ln t_j)^{\alpha-\beta}\big)\frac{ds}{s}\Big)/\Gamma(1-\alpha)\\
&+\Big(\int_1^t\int_u^t (\ln t-\ln s)^{-\alpha}(\ln s-\ln u)^{\alpha-1}\\
&\times \mathbf{E}_{\alpha-\beta,\alpha}(\lambda(\ln s-\ln u)^{\alpha-\beta})\frac{ds}{s} f(u,x(u))\frac{du}{u}\Big)/\Gamma(1-\alpha)\\
=&\Big(\sum_{j=0}^{i} d_j \int_{t_j}^{t} (\ln t-\ln s)^{-\alpha}(\ln s-\ln t_j)^{\alpha-1}\\
&\times \mathbf{E}_{\alpha-\beta,\alpha}\big(\lambda(\ln s-\ln t_j)^{\alpha-\beta}\big)\frac{ds}{s}\Big)/\Gamma(1-\alpha)\\
&+\Big(\int_1^t\int_u^t (\ln t-\ln s)^{-\alpha}(\ln s-\ln u)^{\alpha-1}\\
&\times \mathbf{E}_{\alpha-\beta,\alpha}(\lambda(\ln s-\ln u)^{\alpha-\beta})\frac{ds}{s} f(u,x(u))\frac{du}{u}\Big)/\Gamma(1-\alpha)\\
=&\Big(\sum_{j=0}^{i} d_j \int_{t_j}^{t} (\ln t-\ln s)^{-\alpha}(\ln s-\ln t_j)^{\alpha-1}\\
&\times \sum_{\chi=0}^{\infty}\frac{\lambda^{\chi}}{\Gamma(\chi(\alpha-\beta)+\alpha)}(\ln s-\ln t_j)^{\chi(\alpha-\beta)}\frac{ds}{s}\Big)/\Gamma(1-\alpha)\\
&\text{(using } \frac{\ln s-\ln t_j}{\ln t-\ln t_j}=w)\\
&+\Big(\int_1^t\int_u^t (\ln t-\ln s)^{-\alpha}(\ln s-\ln u)^{\alpha-1}\\
&\times \sum_{\chi=0}^{\infty}\frac{\lambda^{\chi}}{\Gamma(\chi(\alpha-\beta)+\alpha)}(\ln s-\ln u)^{\chi(\alpha-\beta)}\frac{ds}{s} f(u,x(u))\frac{du}{u}\Big)/\Gamma(1-\alpha)\\
&\text{(using } \frac{\ln s-\ln u}{\ln t-\ln s}=w)\\
=&\Big(\sum_{j=0}^{i} d_j \sum_{\chi=0}^{\infty}\frac{\lambda^{\chi}}{\Gamma(\chi(\alpha-\beta)+\alpha)}(\ln t-\ln t_j)^{\chi(\alpha-\beta)}\\
&\times \int_0^1 (1-w)^{-\alpha} w^{\alpha-1+\chi(\alpha-\beta)}dw\Big)/\Gamma(1-\alpha)\\
&+\Big(\sum_{\chi=0}^{\infty}\frac{\lambda^{\chi}}{\Gamma(\chi(\alpha-\beta)+\alpha)}\int_1^t (\ln t-\ln u)^{\chi(\alpha-\beta)}\\
&\times \int_0^1 (1-w)^{-\alpha} w^{\alpha-1+\chi(\alpha-\beta)}dw f(u,x(u))\frac{du}{u}\Big)/\Gamma(1-\alpha)\\
=&\sum_{j=0}^{i} d_j \sum_{\chi=0}^{\infty}\frac{\lambda^{\chi}}{\Gamma(\chi(\alpha-\beta)+1)}(\ln t-\ln t_j)^{\chi(\alpha-\beta)}
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{\chi=0}^{\infty}\frac{\lambda^{\chi}}{\Gamma(\chi(\alpha-\beta)+1)}\int_1^t(\ln t-\ln u)^{\chi(\alpha-\beta)}f(u,x(u))\frac{du}{u}\\
&=\sum_{j=0}^{i}d_j\mathbf{E}_{\alpha-\beta,1}(\lambda(\ln t-\ln t_j)^{\alpha-\beta})+\int_1^t\mathbf{E}_{\alpha-\beta,1}(\lambda(\ln t-\ln s)^{\alpha-\beta})f(u,x(u))\frac{du}{u}.
\end{aligned}
$$

Then

$$
\begin{aligned}
{}^h\mathfrak{J}_{1^+}^{1-\alpha}x(t)&=\sum_{j=0}^{i}d_j\mathbf{E}_{\alpha-\beta,1}(\lambda(\ln t-\ln t_j)^{\alpha-\beta})\\
&\quad+\int_1^t\mathbf{E}_{\alpha-\beta,1}(\lambda(\ln t-\ln s)^{\alpha-\beta})f(u,x(u))\frac{du}{u},
\end{aligned}
\tag{6.29}
$$

for $t\in(t_i,t_{i+1}]$ and $i\in\mathbb{N}_0^m$. Using the above inequality, $\Delta^*x(t_i)=I(t_i,{}^h\mathfrak{J}_{1^+}^{1-\alpha}x(t_i))$, $i\in\mathbb{N}_1^m$, and ${}^h\mathfrak{J}_{1^+}^{1-\alpha}u(1)=u_0$, we obtain $d_i=I(t_i,{}^h\mathfrak{J}_{1^+}^{1-\alpha}x(t_i))$, $i\in\mathbb{N}_1^m$ and $d_0=u_0$. Substituting d_i into (6.28), we obtain (6.28). □

Now we show that Result 6.14 is wrong. Note that $\mathbf{E}_{\alpha-\beta,\alpha}(0)=\frac{1}{\Gamma(\alpha)}$. Let $A=0$ and $I(t_i,x)=p_i$. We know from Theorem 6.15 that x is a solution of (6.25) if and only if

$$
x(t)=\frac{u_0}{\Gamma(\alpha)}(\ln t)^{\alpha-1}+\sum_{j=1}^{i}\frac{p_j}{\Gamma(\alpha)}(\ln t-\ln t_j)^{\alpha-1}+\int_1^t\frac{(\ln t-\ln s)^{\alpha-1}}{\Gamma(\alpha)}f(s,x(s))\frac{ds}{s},
$$

for $t\in(t_i,t_{i+1}]$ and $i\in\mathbb{N}_0^m$. This shows that Result 6.14 is wrong.

6.6. **Corrected results from** [67, 131, 133]**.** The following BVP was studied in [133]:

$$
\begin{gathered}
{}^cD_{0^+}^{\gamma}x(t)+ax(t)=f(t,x(t),y(t)),\quad t\in[0,1]\setminus\{t_1,\dots,t_m\},\\
{}^cD_{0^+}^{\gamma}y(t)+by(t)=g(t,x(t),y(t)),\quad t\in[0,1]\setminus\{t_1,\dots,t_m\},\\
x(0)=-\sum_{i=1}^{m}\alpha_ix(\tau_i),\quad y(0)=-\sum_{i=1}^{m}\beta_iy(\tau_i),\\
\Delta x(t_i)=I_i(x(t_i)),\quad \Delta y(t_i)=J_i(y(t_i)),\quad i=1,2,\dots,m,
\end{gathered}
\tag{6.30}
$$

where $\gamma\in(0,1)$, $a,b>0$, $0=t_0<t_1<\dots<t_m<t_{m+1}=1$, $\tau_i\in(t_i,t_{i+1})$, $\alpha_i,\beta_i\in\mathbb{R}$ with $1+\sum_{i=1}^m\alpha_i\neq0$ and $1+\sum_{i=1}^m\beta_i\neq0$, $\Delta x(t_i)=\lim_{t\to t_i^+}x(t)-\lim_{t\to t_i^-}x(t)$ and $\Delta y(t_i)=\lim_{t\to t_i^+}y(t)-\lim_{t\to t_i^-}y(t)$, ${}^cD_{0^+}^{\gamma}$ is the Caputo type fractional derivative of order γ with starting point 0, $I_i,J_i:\mathbb{R}\mapsto\mathbb{R}$ are continuous, f,g are jointly continuous functions. The following claim was made:

Result 6.16 ([133]). *BVP* (6.30) *is equivalent to the integral system*

$$
\begin{aligned}
x(t) &= -\alpha\sum_{i=1}^{m}\alpha_i\Big[\sum_{0<t_i<\tau_i}\mathbf{E}_{\gamma,1}(-at_i^{\gamma})I_i(x(t_i))\\
&\quad+\int_0^{\tau_i}(\tau_i-s)^{\gamma-1}\mathbf{E}_{\gamma,\gamma}(-a(\tau_i-s)^{\gamma})f(s,x(s),y(s))ds\Big]\\
&\quad+\sum_{0<t_i<t}\mathbf{E}_{\gamma,1}(-at^{\gamma})I_i(x(t_i))\\
&\quad+\int_0^{t}(t-s)^{\gamma-1}\mathbf{E}_{\gamma,\gamma}(-a(t-s)^{\gamma})f(s,x(s),y(s))ds,\\
y(t) &= -\beta\sum_{i=1}^{m}\beta_i\Big[\sum_{0<t_i<\tau_i}\mathbf{E}_{\gamma,1}(-bt_i^{\gamma})J_i(y(t_i))\\
&\quad+\int_0^{\tau_i}(\tau_i-s)^{\gamma-1}\mathbf{E}_{\gamma,\gamma}(-b(\tau_i-s)^{\gamma})g(s,x(s),y(s))ds\Big]\\
&\quad+\sum_{0<t_i<t}\mathbf{E}_{\gamma,1}(-bt^{\gamma})J_i(y(t_i))\\
&\quad+\int_0^{t}(t-s)^{\gamma-1}\mathbf{E}_{\gamma,\gamma}(-b(t-s)^{\gamma})g(s,x(s),y(s))ds,
\end{aligned}
$$

where $\alpha=[1+\sum_{i=1}^{m}\alpha_i\mathbf{E}_{\gamma,1}(-a\tau_i^{\gamma})]^{-1}$ *and* $\beta=[1+\sum_{i=1}^{m}\beta_i\mathbf{E}_{\gamma,1}(-b\tau_i^{\gamma})]^{-1}$.

Theorem 6.17. *BVP* (6.30) *is equivalent to the integral system*

$$
\begin{aligned}
x(t) &= -\alpha\sum_{i=1}^{m}\alpha_i\Big[\sum_{0<t_i<\tau_i}\mathbf{E}_{\gamma,1}(-at_i^{\gamma})I_i(x(t_i))\\
&\quad+\int_0^{\tau_i}(\tau_i-s)^{\gamma-1}\mathbf{E}_{\gamma,\gamma}(-a(\tau_i-s)^{\gamma})f(s,x(s),y(s))ds\Big]\\
&\quad+\sum_{0<t_i<t}\mathbf{E}_{\gamma,1}(-at^{\gamma})I_i(x(t_i))\\
&\quad+\int_0^{t}(t-s)^{\gamma-1}\mathbf{E}_{\gamma,\gamma}(-a(t-s)^{\gamma})f(s,x(s),y(s))ds,\\
y(t) &= -\beta\sum_{i=1}^{m}\beta_i\Big[\sum_{0<t_i<\tau_i}\mathbf{E}_{\gamma,1}(-bt_i^{\gamma})J_i(y(t_i))\\
&\quad+\int_0^{\tau_i}(\tau_i-s)^{\gamma-1}\mathbf{E}_{\gamma,\gamma}(-b(\tau_i-s)^{\gamma})g(s,x(s),y(s))ds\Big]\\
&\quad+\sum_{0<t_i<t}\mathbf{E}_{\gamma,1}(-bt^{\gamma})J_i(y(t_i))\\
&\quad+\int_0^{t}(t-s)^{\gamma-1}\mathbf{E}_{\gamma,\gamma}(-b(t-s)^{\gamma})g(s,x(s),y(s))ds,
\end{aligned}
\tag{6.31}
$$

Proof. Suppose that (x,y) is a solution of (6.30). By Theorem 3.14 (choose $\alpha=\gamma\in(0,1)$ in (3.26)), we know that there exist constants $c_i,d_i\in\mathbb{R}(i\in\mathbb{N}[0,m])$ such that

$$x(t)=\sum_{j=0}^{i}c_j\mathbf{E}_{\gamma,1}(-a(\log t-\log t_j)^{\gamma})+\int_1^t(\log t-\log s)^{\gamma-1}f(s,x(s),y(s))\frac{ds}{s},\quad t\in(t_i,t_{i+1}],\ i\in\mathbb{N}[0,m],$$

$$y(t)=\sum_{j=0}^{i}d_j\mathbf{E}_{\gamma,1}(-b(\log t-\log t_j)^{\gamma})+\int_1^t(\log t-\log s)^{\gamma-1}g(s,x(s),y(s))\frac{ds}{s},\quad t\in(t_i,t_{i+1}],\ i\in\mathbb{N}[0,m].$$

By $\Delta x(t_i)=I_i(x(t_i))$, $\Delta y(t_i)=J_i(y(t_i))$, $i\in\mathbb{N}[1,m]$, we obtain $c_i=I_i(x(t_i)),\ d_i=J_i(y(t_i))$, $i\in\mathbb{N}[1,m]$. By $x(0)=-\sum_{i=1}^m\alpha_ix(\tau_i)$, we obtain

$$c_0+\sum_{i=1}^m\alpha_i\Big[\sum_{j=0}^{i}c_j\mathbf{E}_{\gamma,1}(-a(\log\tau_i-\log t_j)^{\gamma})+\int_1^{\tau_i}(\log\tau_i-\log s)^{\gamma-1}f(s,x(s),y(s))\frac{ds}{s}\Big]=0.$$

It follows that

$$c_0=-\Big(\sum_{i=1}^m\alpha_i\Big[\sum_{j=1}^{i}I_j(x(t_j))\mathbf{E}_{\gamma,1}(-a(\log\tau_i-\log t_j)^{\gamma})+\int_1^{\tau_i}(\log\tau_i-\log s)^{\gamma-1}f(s,x(s),y(s))\frac{ds}{s}\Big]\Big)/\Big(1+\sum_{i=1}^m\alpha_i\mathbf{E}_{\gamma,1}(-a(\log\tau_i))\Big).$$

Hence

$$\begin{aligned}x(t)&=-\Big(\sum_{i=1}^m\alpha_i\Big[\sum_{j=1}^{i}I_j(x(t_j))\mathbf{E}_{\gamma,1}(-a(\log\tau_i-\log t_j)^{\gamma})\\&\quad+\int_1^{\tau_i}(\log\tau_i-\log s)^{\gamma-1}f(s,x(s),y(s))\frac{ds}{s}\Big]\Big)\mathbf{E}_{\gamma,1}(-a(\log t)^{\gamma})\\&\quad\div\Big(1+\sum_{i=1}^m\alpha_i\mathbf{E}_{\gamma,1}(-a(\log\tau_i))\Big)\\&\quad+\sum_{j=1}^{i}I_j(x(t_j))\mathbf{E}_{\gamma,1}(-a(\log t-\log t_j)^{\gamma})\\&\quad+\int_1^t(\log t-\log s)^{\gamma-1}f(s,x(s),y(s))\frac{ds}{s},\quad t\in(t_i,t_{i+1}],\ i\in\mathbb{N}[0,m].\end{aligned}$$

Similarly we obtain

$$\begin{aligned}y(t)&=-\Big(\sum_{i=1}^m\beta_i\Big[\sum_{j=1}^{i}J_j(y(t_j))\mathbf{E}_{\gamma,1}(-b(\log\tau_i-\log t_j)^{\gamma})\\&\quad+\int_1^{\tau_i}(\log\tau_i-\log s)^{\gamma-1}g(s,x(s),y(s))\frac{ds}{s}\Big]\Big)\mathbf{E}_{\gamma,1}(-b(\log t)^{\gamma})\\&\quad\div\Big(1+\sum_{i=1}^m\beta_i\mathbf{E}_{\gamma,1}(-b(\log\tau_i))\Big)\end{aligned}$$

$$+\sum_{j=1}^{i} J_j(y(t_j))\mathbf{E}_{\gamma,1}(-b(\log t-\log t_j)^\gamma)$$
$$+\int_1^t (\log t-\log s)^{\gamma-1} g(s,x(s),y(s))\frac{ds}{s},\quad t\in(t_i,t_{i+1}],\ i\in\mathbb{N}[0,m].$$

On the other hand, if (x,y) satisfies (6.31), we can prove that (x,y) is a solution of (6.30). The proof is complete. ☐

From Theorem 6.17, we know the Result 6.16 claimed in [133] is wrong.

In [131], authors studied the fractional impulsive boundary value problem on infinite intervals

$$\begin{gathered}
D_{0^+}^{\alpha}u(t)+f(t,u(t))=0,\quad t\in(0,+\infty),\ t\neq t_k, k=1,2,\dots,m,\\
u(t_k^+)-u(t_k^-)=-I_k(u(t_k)),\quad k=1,2,\dots,m,\\
u(0)=0,\quad D_{0^+}^{\alpha-1}u(+\infty)=0,
\end{gathered}\tag{6.32}$$

where $\alpha\in(1,2]$, $D_{0^+}^{*}$ is the Riemann-Liouville fractional derivatives of orders $*>0$, $t_0=0$, $1<t_1<\dots<t_m<+\infty$, $u(t_k^+)=\lim_{t\to t_k^+}u(t)$ and $u(t_k^-)=\lim_{t\to t_k^-}u(t)$, $D_{0^+}^{\alpha-1}u(+\infty)=\lim_{t\to+\infty}D_{0^+}^{\alpha-1}u(t)$, $(t,u)\to f(t,(1+t^\alpha)u)$ is nonnegative, continuous on $[0,+\infty)\times[0,+\infty)$ and $u\to I_k(u)$ is nonnegative, continuous and bounded. Existence, uniqueness and computational method of unbounded positive solutions of (6.32) were established. [131, Lemma 3.1] claimed the following result:

Result 6.18. Let $y\in C^0[0,\infty)$ with $\int_0^\infty y(t)dt$ convergent, $\alpha\in(1,2)$. If u is a solution of

$$u(t)=\int_0^\infty G(t,s)y(s)ds+\sum_{i=1}^{m}W_i(t,u(t_i)),\tag{6.33}$$

where

$$G(t,s)=\begin{cases}\frac{t^{\alpha-1}-(t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0\le s\le t<\infty,\\ \frac{t^{\alpha-1}}{\Gamma(\alpha)}, & 0\le t\le s<\infty,\end{cases}$$
$$W_i(t,u(t_i))=\begin{cases}\frac{I_i(u(t_i))t^{\alpha-1}}{t_i^{\alpha-1}-t_i^{\alpha-2}}, & 0\le t\le t_i,\\ \frac{I_i(u(t_i))t^{\alpha-2}}{t_i^{\alpha-1}-t_i^{\alpha-2}}, & t_i<t<\infty\end{cases}$$

then u is a solution of

$$\begin{gathered}
D_{0^+}^{\alpha}u(t)+y(t)=0,\quad t\in(0,+\infty),\ t\neq t_k,\ k=1,2,\dots,m,\\
u(t_k^+)-u(t_k^-)=-I_k(u(t_k)),\quad k=1,2,\dots,m,\\
u(0)=0,\quad D_{0^+}^{\alpha-1}u(+\infty)=0.
\end{gathered}\tag{6.34}$$

We find that this result is wrong. In fact, (6.33) can be re-written as

$$u(t)=-\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)ds+\frac{t^{\alpha-1}}{\Gamma(\alpha)}\int_0^\infty y(s)ds$$
$$+\sum_{i=k+1}^{m}\frac{I_i(u(t_i))t^{\alpha-1}}{t_i^{\alpha-1}-t_i^{\alpha-2}}+\sum_{i=1}^{k}\frac{I_i(u(t_i))t^{\alpha-2}}{t_i^{\alpha-1}-t_i^{\alpha-2}},\quad t\in(t_k,t_{k+1}],\ k=0,1,2,\dots.$$

Hence for $t\in(t_k,t_{k+1}](k=1,2,\dots,m)$ we have

$$D_{0^+}^{\alpha}u(t)$$

$$
\begin{aligned}
&= \frac{1}{\Gamma(2-\alpha)}\Big[\int_0^t (t-s)^{1-\alpha}u(s)ds\Big]''\\
&= \Big[\sum_{\nu=0}^{k-1}\int_{t_\nu}^{t_{\nu+1}}(t-s)^{1-\alpha}u(s)ds+\int_{t_k}^t(t-s)^{1-\alpha}u(s)ds\Big]''/\Gamma(2-\alpha)\\
&= \Big[\sum_{\nu=0}^{k-1}\int_{t_\nu}^{t_{\nu+1}}(t-s)^{1-\alpha}(-\int_0^s\frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}y(u)du+\frac{s^{\alpha-1}}{\Gamma(\alpha)}\int_0^\infty y(u)du\\
&\quad +\sum_{i=\nu+1}^m\frac{I_i(u(t_i))s^{\alpha-1}}{t_i^{\alpha-1}-t_i^{\alpha-2}}+\sum_{i=1}^{\nu}\frac{I_i(u(t_i))s^{\alpha-2}}{t_i^{\alpha-1}-t_i^{\alpha-2}})ds\Big]''/\Gamma(2-\alpha)\\
&\quad +\Big[\int_{t_k}^t(t-s)^{1-\alpha}(-\int_0^s\frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}y(u)du+\frac{s^{\alpha-1}}{\Gamma(\alpha)}\int_0^\infty y(u)du\\
&\quad +\sum_{i=k+1}^m\frac{I_i(u(t_i))s^{\alpha-1}}{t_i^{\alpha-1}-t_i^{\alpha-2}}+\sum_{i=1}^{k}\frac{I_i(u(t_i))s^{\alpha-2}}{t_i^{\alpha-1}-t_i^{\alpha-2}})ds\Big]''/\Gamma(2-\alpha)\\
&= \Big[\sum_{\nu=0}^{k-1}\sum_{i=\nu+1}^m\frac{I_i(u(t_i))}{t_i^{\alpha-1}-t_i^{\alpha-2}}\int_{t_\nu}^{t_{\nu+1}}(t-s)^{1-\alpha}s^{\alpha-1}ds\\
&\quad +\sum_{\nu=0}^{k-1}\sum_{i=1}^{\nu}\frac{I_i(u(t_i))}{t_i^{\alpha-1}-t_i^{\alpha-2}}\int_{t_\nu}^{t_{\nu+1}}(t-s)^{1-\alpha}s^{\alpha-2}ds\Big]''/\Gamma(2-\alpha)\\
&\quad +\Big[\sum_{i=k+1}^m\frac{I_i(u(t_i))}{t_i^{\alpha-1}-t_i^{\alpha-2}}\int_{t_k}^t(t-s)^{1-\alpha}s^{\alpha-1}ds\\
&\quad +\sum_{i=1}^{k}\frac{I_i(u(t_i))}{t_i^{\alpha-1}-t_i^{\alpha-2}}\int_{t_k}^t(t-s)^{1-\alpha}s^{\alpha-2}ds\Big]''/\Gamma(2-\alpha)\\
&\quad +\Big[-\int_0^t(t-s)^{1-\alpha}\int_0^s\frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}y(u)\,du\,ds\\
&\quad +\int_0^t(t-s)^{1-\alpha}\frac{s^{\alpha-1}}{\Gamma(\alpha)}\int_0^\infty y(u)\,du\,ds\Big]''/\Gamma(2-\alpha)\\
&\quad \text{(by changing the order of the sum and the integral, we obtain)}\\
&= \Big[\sum_{i=1}^{k}\sum_{\nu=0}^{i-1}\frac{I_i(u(t_i))}{t_i^{\alpha-1}-t_i^{\alpha-2}}\int_{t_\nu}^{t_{\nu+1}}(t-s)^{1-\alpha}s^{\alpha-1}ds\Big]''/\Gamma(2-\alpha)\\
&\quad +\Big[\sum_{i=k+1}^m\sum_{\nu=0}^{k-1}\frac{I_i(u(t_i))}{t_i^{\alpha-1}-t_i^{\alpha-2}}\int_{t_\nu}^{t_{\nu+1}}(t-s)^{1-\alpha}s^{\alpha-1}ds\\
&\quad +\sum_{i=1}^{k-1}\sum_{\nu=i}^{k-1}\frac{I_i(u(t_i))}{t_i^{\alpha-1}-t_i^{\alpha-2}}\int_{t_\nu}^{t_{\nu+1}}(t-s)^{1-\alpha}s^{\alpha-2}ds\Big]''/\Gamma(2-\alpha)\\
&\quad \Big[\sum_{i=k+1}^m\frac{I_i(u(t_i))}{t_i^{\alpha-1}-t_i^{\alpha-2}}\int_{t_k}^t(t-s)^{1-\alpha}s^{\alpha-1}ds\\
&\quad +\sum_{i=1}^{k}\frac{I_i(u(t_i))}{t_i^{\alpha-1}-t_i^{\alpha-2}}\int_{t_k}^t(t-s)^{1-\alpha}s^{\alpha-2}ds\Big]''/\Gamma(2-\alpha)
\end{aligned}
$$

$$
\begin{aligned}
&\quad + \Big[-\int_0^t \int_u^t (t-s)^{1-\alpha} \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)} ds y(u) du \\
&\quad + \int_0^\infty \int_0^t (t-s)^{1-\alpha} \frac{s^{\alpha-1}}{\Gamma(\alpha)} ds y(u) du \Big]'' / \Gamma(2-\alpha) \\
&\quad (\text{using that } \frac{s-u}{t-u} = w, \frac{s}{t} = w) \\
&= \Big[\sum_{i=1}^k \frac{I_i(u(t_i))}{t_i^{\alpha-1} - t_i^{\alpha-2}} \int_0^{t_i} (t-s)^{1-\alpha} s^{\alpha-1} ds \Big]'' / \Gamma(2-\alpha) \\
&\quad + \Big[\sum_{i=k+1}^m \frac{I_i(u(t_i))}{t_i^{\alpha-1} - t_i^{\alpha-2}} \int_0^t (t-s)^{1-\alpha} s^{\alpha-1} ds \\
&\quad + \sum_{i=1}^k \frac{I_i(u(t_i))}{t_i^{\alpha-1} - t_i^{\alpha-2}} \int_{t_i}^t (t-s)^{1-\alpha} s^{\alpha-2} ds \Big]'' / \Gamma(2-\alpha) \\
&\quad + \Big[-\int_0^t (t-u) \int_0^1 (1-w)^{1-\alpha} \frac{w^{\alpha-1}}{\Gamma(\alpha)} dw y(u) du \\
&\quad + \int_0^\infty t \int_0^1 (1-w)^{1-\alpha} \frac{w^{\alpha-1}}{\Gamma(\alpha)} dw y(u) du \Big]'' / \Gamma(2-\alpha) \\
&\quad (\text{using that } \mathbf{B}(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1} dx = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}) \\
&= y(t) + \Big[\sum_{i=1}^k \frac{I_i(u(t_i))}{t_i^{\alpha-1} - t_i^{\alpha-2}} t \int_0^{\frac{t_i}{t}} (1-w)^{1-\alpha} w^{\alpha-1} dw \Big]'' / \Gamma(2-\alpha) \\
&\quad + \Big[\sum_{i=k+1}^m \frac{I_i(u(t_i))}{t_i^{\alpha-1} - t_i^{\alpha-2}} t \int_0^1 (1-w)^{1-\alpha} w^{\alpha-1} dw \\
&\quad + \sum_{i=1}^k \frac{I_i(u(t_i))}{t_i^{\alpha-1} - t_i^{\alpha-2}} t \int_{\frac{t_i}{t}}^1 (1-w)^{1-\alpha} w^{\alpha-2} dw \Big]'' / \Gamma(2-\alpha) \\
&= y(t) + \Big[\sum_{i=1}^k \frac{I_i(u(t_i))}{t_i^{\alpha-1} - t_i^{\alpha-2}} t \int_0^{\frac{t_i}{t}} (1-w)^{1-\alpha} w^{\alpha-1} dw \Big]'' / \Gamma(2-\alpha) \\
&\quad - \Big[\sum_{i=1}^k \frac{I_i(u(t_i))}{t_i^{\alpha-1} - t_i^{\alpha-2}} t \int_0^{\frac{t_i}{t}} (1-w)^{1-\alpha} w^{\alpha-2} dw \Big]'' / \Gamma(2-\alpha) \\
&= y(t) + \Big[\sum_{i=1}^k \frac{I_i(u(t_i))}{t_i^{\alpha-1} - t_i^{\alpha-2}} t \int_0^{\frac{t_i}{t}} (1-w)^{1-\alpha} [w^{\alpha-1} - w^{\alpha-2}] dw \Big]'' / \Gamma(2-\alpha) \\
&\neq y(t), \quad t \in (t_k, t_{k+1}], \ k = 1, 2, \dots, m.
\end{aligned}
$$

Thus u given by (6.33) is not a solution of (6.32). Hence Result 6.18 is wrong.

To correct the results in [131], we consider the BVP

$$
\begin{gathered}
D_{0^+}^{\alpha} u(t) + f(t, u(t)) = 0, \quad t \in (0, +\infty), \ t \neq t_k, \ k \in \mathbb{N}[0, \infty), \\
\Delta D_{0^+}^{2-\alpha} u(t_k) = J(t_k, u(t_k)), \quad \Delta I_{0^+}^{2-\alpha} u(t_k) = I(t_k, u(t_k)), \quad k \in \mathbb{N}[1, \infty), \\
I_{0^+}^{2-\alpha} u(0) = 0, \quad D_{0^+}^{\alpha-1} u(+\infty) = 0,
\end{gathered}
\tag{6.35}
$$

where $\alpha \in (1,2]$, $D_{0^+}^{*}$ is the Riemann-Liouville fractional derivatives of orders $* > 0$, $I_{0^+}^{*}$ is the Riemann-Liouville fractional integral of order $* > 0$, $t_0 = 0$, $1 < t_1 < \dots < t_m < \dots < +\infty$,

$$\begin{gathered}
\Delta D_{0^+}^{\alpha-1}u(t_k) = \lim_{\epsilon \to 0^+} D_{0^+}^{\alpha-1}u(t_k+\epsilon) - \lim_{\epsilon \to 0^-} D_{0^+}^{\alpha-1}u(t_k+\epsilon), \\
\Delta I_{0^+}^{2-\alpha}u(t_k) = \lim_{\epsilon \to 0^+} I_{0^+}^{2-\alpha}u(t_k+\epsilon) - \lim_{\epsilon \to 0^-} I_{0^+}^{2-\alpha}u(t_k+\epsilon), \\
D_{0^+}^{\alpha-1}u(+\infty) = \lim_{t \to +\infty} D_{0^+}^{\alpha-1}u(t),
\end{gathered}$$

and f, I, J satisfy some suitable assumptions.

Theorem 6.19. *A function u is a solution of* (6.35) *if and only if* (6.35).

Proof. Suppose that u is a solution of (6.35). By Theorem 3.12 (with $\lambda = 0$, $\alpha \in (1,2)$, there exist constants $c_i, d_i \in \mathbb{R}$ such that

$$\begin{aligned}
u(t) &= \sum_{j=0}^{i} \frac{c_j}{\Gamma(\alpha)}(t-t_j)^{\alpha-1} + \sum_{j=0}^{i} \frac{d_j}{\Gamma(\alpha-1)}(t-t_j)^{\alpha-2} \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_1^t (t-s)^{\alpha-1} f(s,u(s))ds, \quad t \in (t_i, t_{i+1}],\ i \in \mathbb{N}[0,\infty).
\end{aligned}$$

By direct computations, we obtain

$$\begin{gathered}
I_{0^+}^{2-\alpha}u(t) = \sum_{j=0}^{i} c_j(t-t_j) + \sum_{j=0}^{i} d_j + \int_1^t (t-s) f(s,u(s))ds, \\
t \in (t_i, t_{i+1}],\ i \in \mathbb{N}[0,\infty), \\
D_{0^+}^{\alpha-1}u(t) = \sum_{j=0}^{i} c_j + \int_1^t f(s,u(s))ds, \quad t \in (t_i, t_{i+1}],\ i \in \mathbb{N}[0,\infty).
\end{gathered}$$

From $\Delta D_{0^+}^{2-\alpha}u(t_k) = J(t_k, u(t_k))$, $\Delta I_{0^+}^{2-\alpha}u(t_k) = I(t_k, u(t_k))$, $k \in \mathbb{N}[1,\infty)$, we know that $c_k = J(t_k, u(t_k))$ and $d_k = I(t_k, u(t_k))$, $k \in \mathbb{N}[1,\infty)$. By $I_{0^+}^{2-\alpha}u(0) = 0$, $D_{0^+}^{\alpha-1}u(+\infty) = 0$, we know that $d_0 = u_0$ and $c_0 = u_2$. Hence we obtain

$$\begin{aligned}
u(t) &= u_2 t^{\alpha-1} + u_1 t^{\alpha-2} + \sum_{j=1}^{i} \frac{J(t_j, u(t_j))}{\Gamma(\alpha)}(t-t_j)^{\alpha-1} + \sum_{j=1}^{i} \frac{I(t_j, u(t_j))}{\Gamma(\alpha-1)}(t-t_j)^{\alpha-2} \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_1^t (t-s)^{\alpha-1} f(s,u(s))ds, \quad t \in (t_i, t_{i+1}], ; i \in \mathbb{N}[0,\infty).
\end{aligned}$$

On the other hand, if u satisfies above integral equation, we can prove that u satisfies (6.35). The proof is complete. $\square$

Liu [67] studied the existence of positive solutions of the boundary value problem of fractional impulsive differential equation

$$\begin{gathered}
D_{0^+}^{\alpha}x(t) + f(t,x(t)), \quad t \in (0,1), \quad t \ne t_i,\ i \in \mathbb{N}[1,m], \\
x(0) = x(1) = 0, \quad x(t_i^+) - x(t_i^-) = c_i x(t_i^-), \quad i \in \mathbb{N}[1,m],
\end{gathered} \tag{6.36}$$

where $\alpha \in (1,2]$, $0 = t_0 < t_1 < \dots < t_m < t_{m+1} = 1$, $D_{0^+}^{\alpha}$ is the standard Riemann-Liouville fractional derivative, $c_i \in (0, \frac{1}{2})$, $f : [0,1] \times \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function. [67, Lemma 3.1] claimed that:

Result 6.20. If $u \in PC([0,1])$ is a fixed point of the operator A defined by

$$Ax(t) = \int_0^1 G(t,s)f(s,x(s))ds + t^{\alpha-1}\sum_{t<t_k<1}\frac{c_k}{1-c_k}t_k^{1-\alpha}x(t_k), \tag{6.37}$$

for all $x \in PC([0,1])$, then u is a solution of(6.36) (x is continuous at each point $t \neq t_i$, right continuous at t_i, the left limit $\lim_{t\to t_i^-} x(t)$ is finite and satisfies (6.36)), where

$$G(t,s) = \frac{1}{\Gamma(\alpha)}\begin{cases}[t(1-s)]^{\alpha-1} - (t-s)^{\alpha-1}, & 0 \le s \le t \le 1,\\ [t(1-s)]^{\alpha-1}, & 0 \le t \le s \le 1.\end{cases}$$

We find that Result 6.20 is wrong. In fact, if u is a fixed point of A, the we obtain

$$x(t) = \int_0^1 G(t,s)f(s,x(s))ds + t^{\alpha-1}\sum_{t<t_k<1}\frac{c_k}{1-c_k}t_k^{1-\alpha}x(t_k).$$

This is re-written as

$$\begin{aligned} x(t) = -\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,x(s))ds + t^{\alpha-1}\Big[\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,x(s))ds \\ + \sum_{i=1}^m \frac{c_i}{1-c_i}t_i^{1-\alpha}x(t_i)\Big] - t^{\alpha-1}\sum_{i=1}^{k-1}\frac{c_i}{1-c_i}t_i^{1-\alpha}x(t_i), \end{aligned}$$

for $t \in [t_{k-1}, t_k)$ and $k = 1, \dots, m+1$. One can easily verify that $D_{0^+}^\alpha x(t) \neq f(t,x(t))$, $t \in (t_i, t_{i+1}]$, $i = 1, \dots, m$ similarly to above discussion. We can improve (6.36) and correct Result 6.20. We omit the details.

6.7. **Corrected results from** [128, 129, 135]**.** In [135], the authors studied the impulsive system with Hadamard fractional derivative:

$$\begin{gathered} {}_HD_{a^+}^q u(t) = f(t,u(t)), \quad t \in (a,T],\ t \neq t_i, \bar{t}_j,\ i \in \mathbb{N}[1,m],\ j \in \mathbb{N}[1,n],\\ \Delta_H I_{a^+}^{2-q}u(t_i) = {}_HI_{a^+}^{2-q}u(t_i^+) - {}_HI_{a^+}^{2-q}u(t_i^-) = \Delta_i(u(t_i^-)), \quad i \in \mathbb{N}[1,m],\\ \Delta_H D_{a^+}^{q-1}u(\bar{t}_j) = {}_HD_{a^+}^{q-1}u(\bar{t}_j^+) - {}_HD_{a^+}^{q-1}u(\bar{t}_j^-) = \overline{\Delta}_j(u(\bar{t}_j^-)), \quad j \in \mathbb{N}[1,n],\\ {}_HI_{a^+}^{2-q}u(a) = u_1, \quad {}_HD_{a^+}^{q-1}u(a) = u_2, \end{gathered} \tag{6.38}$$

and its special case

$$\begin{gathered} {}_HD_{a^+}^q u(t) = f(t,u(t)), \quad t \in (a,T],\ t \neq t_i,\ i \in \mathbb{N}[1,m],\\ \Delta_H I_{a^+}^{2-q}u(t_i) = {}_HI_{a^+}^{2-q}u(t_i^+) - {}_HI_{a^+}^{2-q}u(t_i^-) = \Delta_i(u(t_i^-)), \quad i \in \mathbb{N}[1,m],\\ \Delta_H D_{a^+}^{q-1}u(t_j) = {}_HD_{a^+}^{q-1}u(t_j^+) - {}_HD_{a^+}^{q-1}u(t_j^-) = \overline{\Delta}_j(u(t_j^-)), \quad j \in \mathbb{N}[1,n],\\ {}_HI_{a^+}^{2-q}u(a) = u_1, \quad {}_HD_{a^+}^{q-1}u(a) = u_2, \end{gathered} \tag{6.39}$$

where $a > 0$, ${}_HD_{a^+}^q$ denotes left-sided Riemann-Liouville type Hadamard fractional derivative of order $q \in (1,2)$ with the starting point a, $f : [a,T] \times \mathbb{R} \to \mathbb{R}$ is an appropriate continuous function, $a = t_0 < t_1 < \dots < t_m < t_{m+1} = T$, and $a = \bar{t}_0 < \bar{t}_1 < \dots < \bar{t}_m < \bar{t}_{n+1} = T$, ${}_HI_{a^+}^*$ denotes the left-sided Hadamard fractional integral of order $* > 0$, and

$${}_HI_{a^+}^{2-q}u(t_i^+) = \lim_{\epsilon\to 0^+} {}_HI_{a^+}^{2-q}u(t_i+\epsilon), \quad {}_HI_{a^+}^{2-q}u(t_i^-) = \lim_{\epsilon\to 0^+} {}_HI_{a^+}^{2-q}u(t_i-\epsilon)$$

represent the right and left limits of ${}_H I_{a^+}^{2-q}u(t)$ at $t=t_i$, respectively. The derivatives ${}_H D_{a^+}^{q-1}u(t_i^+)$ and ${}_H D_{a^+}^{q-1}u(t_i^-)$ have a similar meaning for ${}_H D_{a^+}^{q-1}u(t)$. [135, Theorem 3.4 and Corollary 3.5] are as follows:

Theorem 6.21 ([135]). *Let $q\in(1,2)$, $\overline{\lambda},\overline{h}$ be constants. Then system (6.38) is equivalent to the fractional integral equation*

$$
\begin{aligned}
u(t)&=\frac{u_1}{\Gamma(q)}\Big(\int_a^t\frac{ds}{s}\Big)^{q-1}+\frac{u_2}{\Gamma(q-1)}\Big(\int_a^t\frac{ds}{s}\Big)^{q-2}\\
&\quad+\frac{1}{\Gamma(q)}\int_a^t(\log\frac{t}{s})^{q-1}f(s,u(s))\frac{ds}{s},\quad t\in(1,t_1];\\
u(t)&=\frac{u_1}{\Gamma(q)}\Big(\int_a^t\frac{ds}{s}\Big)^{q-1}+\frac{u_2}{\Gamma(q-1)}\Big(\int_a^t\frac{ds}{s}\Big)^{q-2}+\frac{1}{\Gamma(q)}\int_a^t(\log\frac{t}{s})^{q-1}f(s,u(s))\frac{ds}{s}\\
&\quad+\sum_{i=1}^{k_0}\frac{\Delta_i(u(t_i^-))}{\Gamma(q-1)}\Big(\int_{t_i}^t\frac{ds}{s}\Big)^{q-2}+\sum_{j=1}^{k_1}\frac{\overline{\Delta}_j(u(\overline{t}_j^-))}{\Gamma(q)}\Big(\int_{\overline{t}_j}^t\frac{ds}{s}\Big)^{q-1}\\
&\quad-\overline{\lambda}\sum_{i=1}^{k_0}\Delta_i(u(t_i^-))\Big[\frac{u_1}{\Gamma(q)}\Big(\int_a^t\frac{ds}{s}\Big)^{q-1}+\frac{u_2}{\Gamma(q-1)}\Big(\int_a^t\frac{ds}{s}\Big)^{q-2}\\
&\quad+\frac{1}{\Gamma(q)}\int_a^t(\log\frac{t}{s})^{q-1}f(s,u(s))\frac{ds}{s}\\
&\quad-\frac{u_1+\int_a^{t_i}f(s,u(s))\frac{ds}{s}}{\Gamma(q)}\Big(\int_{t_i}^t\frac{ds}{s}\Big)^{q-1}\\
&\quad-\frac{u_1\log\frac{t_i}{a}+u_2+\int_a^{t_i}\log\frac{t_i}{s}f(s,u(s))\frac{ds}{s}}{\Gamma(q-1)}\Big(\int_{t_i}^t\frac{ds}{s}\Big)^{q-2}\\
&\quad-\frac{1}{\Gamma(q)}\int_{t_i}^t(\log\frac{t}{s})^{q-1}f(s,u(s))\frac{ds}{s}\Big]\\
&\quad-\overline{h}\sum_{j=1}^{k_1}\overline{\Delta}_j(u(t_j^-))\Big[\frac{u_1}{\Gamma(q)}(\int_a^t\frac{ds}{s})^{q-1}+\frac{u_2}{\Gamma(q-1)}(\int_a^t\frac{ds}{s})^{q-2}\\
&\quad+\frac{1}{\Gamma(q)}\int_a^t(\log\frac{t}{s})^{q-1}f(s,u(s))\frac{ds}{s}\\
&\quad-\frac{u_1+\int_a^{\overline{t}_j}f(s,u(s))\frac{ds}{s}}{\Gamma(q)}\Big(\int_{\overline{t}_j}^t\frac{ds}{s}\Big)^{q-1}\\
&\quad-\frac{u_1\log\frac{\overline{t}_j}{a}+u_2+\int_a^{\overline{t}_j}\log\frac{\overline{t}_j}{s}f(s,u(s))\frac{ds}{s}}{\Gamma(q-1)}\Big(\int_{\overline{t}_j}^t\frac{ds}{s}\Big)^{q-2}\\
&\quad-\frac{1}{\Gamma(q)}\int_{\overline{t}_j}^t(\log\frac{t}{s})^{q-1}f(s,u(s))\frac{ds}{s}\Big],
\end{aligned}
$$

for $t\in(t_k',t_{k+1}']$ and $k=1,2,\dots,\Omega$, where $a,t_1,\dots,t_m,\overline{t}_1,\dots,\overline{t}_n,T$ are queued to $a=t_0'<t_1'<t_2'<\dots<t_\Omega'<t_\Omega'=T$ so that $\{t_1,t_2,\dots,t_m,\overline{t_1},\dots,\overline{t}_n\}=\{t_1',t_2',\dots,t_\Omega'\}$.

Corollary 6.22 ([135]). *Let $q \in (1,2)$, $\overline{\lambda}, \overline{h}$ be constants. Then system (6.39) is equivalent to the fractional integral equation*

$$u(t) = \begin{cases} \frac{u_1}{\Gamma(q)}\Big(\int_a^t \frac{ds}{s}\Big)^{q-1} + \frac{u_2}{\Gamma(q-1)}\Big(\int_a^t \frac{ds}{s}\Big)^{q-2} + \frac{1}{\Gamma(q)}\int_a^t (\log\frac{t}{s})^{q-1} f(s,u(s))\frac{ds}{s}, \\ \quad t\in(1,t_1]; \\ \frac{u_1}{\Gamma(q)}\Big(\int_a^t \frac{ds}{s}\Big)^{q-1} + \frac{u_2}{\Gamma(q-1)}\Big(\int_a^t \frac{ds}{s}\Big)^{q-2} + \frac{1}{\Gamma(q)}\int_a^t (\log\frac{t}{s})^{q-1} f(s,u(s))\frac{ds}{s} \\ +\sum_{i=1}^k \Big[\frac{\Delta_i(u(t_i^-))}{\Gamma(q-1)}\Big(\int_{t_i}^t \frac{ds}{s}\Big)^{q-2} + \frac{\overline{\Delta}_i(u(t_i^-))}{\Gamma(q)}\Big(\int_{t_i}^t \frac{ds}{s}\Big)^{q-1}\Big] \\ -\sum_{i=1}^{k_0}[\overline{\lambda}\Delta_i(u(t_i^-)) + \overline{h\Delta}_j(u(t_j^-))]\Big[\frac{u_1}{\Gamma(q)}(\int_a^t \frac{ds}{s})^{q-1} + \frac{u_2}{\Gamma(q-1)}(\int_a^t \frac{ds}{s})^{q-2} \\ +\frac{1}{\Gamma(q)}\int_a^t (\log\frac{t}{s})^{q-1} f(s,u(s))\frac{ds}{s} - \frac{u_1+\int_a^{t_i} f(s,u(s))\frac{ds}{s}}{\Gamma(q)}\Big(\int_{t_i}^t \frac{ds}{s}\Big)^{q-1} \\ -\frac{u_1\log\frac{t_i}{a} + u_2 + \int_a^{t_i}\log\frac{t_i}{s} f(s,u(s))\frac{ds}{s}}{\Gamma(q-1)}\Big(\int_{t_i}^t \frac{ds}{s}\Big)^{q-2} \\ -\frac{1}{\Gamma(q)}\int_{t_i}^t (\log\frac{t}{s})^{q-1} f(s,u(s))\frac{ds}{s}\Big], \\ \quad t\in(t_k,t_{k+1}],\ k=1,2,\dots,m. \end{cases} \tag{6.40}$$

Theorem 6.23. *A function x is a solution of (6.39) if and only if x satisfies the integral equation*

$$\begin{aligned} x(t) &= \frac{u_2}{\Gamma(q)}(\log t)^{q-1} + \frac{u_1}{\Gamma(q-1)}(\log t)^{q-2} + \sum_{j=1}^i \frac{\overline{\Delta}_j(u(t_j^-))}{\Gamma(q)}(\log\frac{t}{t_j})^{q-1} \\ &\quad + \sum_{j=1}^i \frac{\Delta_j(u(t_j^-))}{\Gamma(q-1)}(\log\frac{t}{t_j})^{q-2} + \frac{1}{\Gamma(q)}\int_1^t (\log\frac{t}{s})^{q-1} f(s,x(s))\frac{ds}{s}, \end{aligned}$$

for $t \in (t_i, t_{i+1}]$, $i \in \mathbb{N}[0,m]$.

Proof. Suppose that x is a solution of BVP(6.39) with $a = 1$ and $T = e$. By Theorem 3.13 (with $\lambda = 0$), we know from ${}_HD_{a^+}^q x(t) = f(t,x(t))$ and $q \in (1,2)$ that there exist constants $c_i, d_i \in \mathbb{R}(i \in \mathbb{N}[0,m]$ such that

$$\begin{aligned} x(t) &= \sum_{j=0}^i \frac{c_j}{\Gamma(q)}(\log\frac{t}{t_j})^{q-1} + \sum_{j=0}^i \frac{d_j}{\Gamma(q-1)}(\log\frac{t}{t_j})^{q-2} \\ &\quad + \frac{1}{\Gamma(q)}\int_1^t (\log\frac{t}{s})^{q-1} f(s,x(s))\frac{ds}{s}, \quad t\in(t_i,t_{i+1}],\ i\in\mathbb{N}[0,m]. \end{aligned} \tag{6.41}$$

Then by Definitions 2.4 and 2.5, for $t \in (t_i, t_{i+1}]$, we have

$$\begin{aligned} &{}_HI_{1^+}^{2-q}x(t) \\ &= \frac{1}{\Gamma(2-q)}\int_1^t (\log\frac{t}{s})^{1-q} x(s)\frac{ds}{s} \\ &= \Big(\sum_{\tau=0}^{i-1}\int_{t_\tau}^{t_{\tau+1}} (\log\frac{t}{s})^{1-q} x(s)\frac{ds}{s} + \int_{t_i}^t (\log\frac{t}{s})^{1-q} x(s)\frac{ds}{s}\Big)/\Gamma(2-q) \\ &= \Big(\sum_{\tau=0}^{i-1}\int_{t_\tau}^{t_{\tau+1}} (\log\frac{t}{s})^{1-q}\Big[\sum_{j=0}^{\tau}\frac{c_j}{\Gamma(q)}(\log\frac{s}{t_j})^{q-1} \end{aligned}$$

$$
\begin{aligned}
&+\sum_{j=0}^{\tau}\frac{d_j}{\Gamma(q-1)}(\log\frac{s}{t_j})^{q-2}+\frac{1}{\Gamma(q)}\int_1^s(\log\frac{s}{u})^{q-1}f(u,x(u))\frac{du}{u}\Big]\frac{ds}{s}\Big)/\Gamma(2-q)\\
&+\Big(\int_{t_i}^t(\log\frac{t}{s})^{1-q}\Big[\sum_{j=0}^{i}\frac{c_j}{\Gamma(q)}(\log\frac{s}{t_j})^{q-1}+\sum_{j=0}^{i}\frac{d_j}{\Gamma(q-1)}(\log\frac{s}{t_j})^{q-2}\\
&+\frac{1}{\Gamma(q)}\int_1^s(\log\frac{s}{u})^{q-1}f(u,x(u))\frac{du}{u}\Big]\frac{ds}{s}\Big)/\Gamma(2-q)\\
=&\Big(\sum_{\tau=0}^{i-1}\sum_{j=0}^{\tau}\frac{c_j}{\Gamma(q)}\int_{t_\tau}^{t_{\tau+1}}(\log\frac{t}{s})^{1-q}(\log\frac{s}{t_j})^{q-1}\frac{ds}{s}\\
&+\sum_{\tau=0}^{i-1}\sum_{j=0}^{\tau}\frac{d_j}{\Gamma(q-1)}\int_{t_\tau}^{t_{\tau+1}}(\log\frac{t}{s})^{1-q}(\log\frac{s}{t_j})^{q-2}\frac{ds}{s}\Big)/\Gamma(2-q)\\
&+\Big(\int_{t_i}^t(\log\frac{t}{s})^{1-q}\sum_{j=0}^{i}\frac{c_j}{\Gamma(q)}(\log\frac{s}{t_j})^{q-1}\frac{ds}{s}\\
&+\int_{t_i}^t(\log\frac{t}{s})^{1-q}\sum_{j=0}^{i}\frac{d_j}{\Gamma(q-1)}(\log\frac{s}{t_j})^{q-2}\frac{ds}{s}\Big)/\Gamma(2-q)\\
&+\Big(\frac{1}{\Gamma(q)}\int_1^t(\log\frac{t}{s})^{1-q}\int_1^s(\log\frac{s}{u})^{q-1}f(u,x(u))\frac{du}{u}\frac{ds}{s}\Big)/\Gamma(2-q)
\end{aligned}
$$

(changing the order of the sums and integral)

$$
\begin{aligned}
=&\Big(\sum_{j=0}^{i-1}\sum_{\tau=j}^{i-1}\frac{c_j}{\Gamma(q)}\int_{t_\tau}^{t_{\tau+1}}(\log\frac{t}{s})^{1-q}(\log\frac{s}{t_j})^{q-1}\frac{ds}{s}\\
&+\sum_{j=0}^{i-1}\sum_{\tau=j}^{i-1}\frac{d_j}{\Gamma(q-1)}\int_{t_\tau}^{t_{\tau+1}}(\log\frac{t}{s})^{1-q}(\log\frac{s}{t_j})^{q-2}\frac{ds}{s}\Big)/\Gamma(2-q)\\
&+\Big(\sum_{j=0}^{i}\frac{c_j}{\Gamma(q)}\int_{t_i}^t(\log\frac{t}{s})^{1-q}(\log\frac{s}{t_j})^{q-1}\frac{ds}{s}\\
&+\sum_{j=0}^{i}\frac{d_j}{\Gamma(q-1)}\int_{t_i}^t(\log\frac{t}{s})^{1-q}(\log\frac{s}{t_j})^{q-2}\frac{ds}{s}\Big)/\Gamma(2-q)\\
&+\Big(\frac{1}{\Gamma(q)}\int_1^t\int_u^t(\log\frac{t}{s})^{1-q}(\log\frac{s}{u})^{q-1}\frac{ds}{s}f(u,x(u))\frac{du}{u}\Big)/\Gamma(2-q)\\
=&\Big(\sum_{j=0}^{i}\frac{c_j}{\Gamma(q)}\int_{t_j}^t(\log\frac{t}{s})^{1-q}(\log\frac{s}{t_j})^{q-1}\frac{ds}{s}\\
&+\sum_{j=0}^{i}\frac{d_j}{\Gamma(q-1)}\int_{t_j}^t(\log\frac{t}{s})^{1-q}(\log\frac{s}{t_j})^{q-2}\frac{ds}{s}\Big)/\Gamma(2-q)\\
&+\Big(\frac{1}{\Gamma(q)}\int_1^t\int_u^t(\log\frac{t}{s})^{1-q}(\log\frac{s}{u})^{q-1}\frac{ds}{s}f(u,x(u))\frac{du}{u}\Big)/\Gamma(2-q)
\end{aligned}
$$

$$
\begin{aligned}
&\quad \text{using } \frac{\log s-\log t_j}{\log t-\log t_j}=w, \frac{\log s-\log u}{\log t-\log u}=w\\
&=\Big(\sum_{j=0}^{i}\frac{c_j}{\Gamma(q)}(\log\frac{t}{t_j})\int_0^1(1-w)^{1-q}w^{q-1}dw\\
&\quad +\sum_{j=0}^{i}\frac{d_j}{\Gamma(q-1)}\int_0^1(1-w)^{1-q}w^{q-2}dw\Big)/\Gamma(2-q)\\
&\quad +\Big(\frac{1}{\Gamma(q)}\int_1^t(\log\frac{t}{u})\int_0^1(1-w)^{1-q}w^{q-1}dwf(u,x(u))\frac{du}{u}\Big)/\Gamma(2-q)\\
&=\sum_{j=0}^{i}c_j(\log\frac{t}{t_j})+\sum_{j=0}^{i}d_j+\int_1^t(\log\frac{t}{u})f(u,x(u))\frac{du}{u}.
\end{aligned}
$$

and similarly

$$
\begin{aligned}
&{}_HD_{1^+}^{q-1}x(t)\\
&=\Big(\sum_{\tau=0}^{i-1}(t\frac{d}{dt})\int_1^t(\log\frac{t}{s})^{1-q}x(s)\frac{ds}{s}+(t\frac{d}{dt})\int_{t_i}^t(\log\frac{t}{s})^{1-q}x(s)\frac{ds}{s}\Big)/\Gamma(2-q)\\
&=\Big(\sum_{\tau=0}^{i-1}(t\frac{d}{dt})\int_1^t(\log\frac{t}{s})^{1-q}\Big[\sum_{j=0}^{\tau}\frac{c_j}{\Gamma(q)}(\log\frac{s}{t_j})^{q-1}\\
&\quad +\sum_{j=0}^{\tau}\frac{d_j}{\Gamma(q-1)}(\log\frac{s}{t_j})^{q-2}+\frac{1}{\Gamma(q)}\int_1^s(\log\frac{s}{u})^{q-1}f(u,x(u))\frac{du}{u}\Big]\frac{ds}{s}\Big)/\Gamma(2-q)\\
&\quad +\Big((t\frac{d}{dt})\int_{t_i}^t(\log\frac{t}{s})^{1-q}\Big[\sum_{j=0}^{i}\frac{c_j}{\Gamma(q)}(\log\frac{s}{t_j})^{q-1}\\
&\quad +\sum_{j=0}^{i}\frac{d_j}{\Gamma(q-1)}(\log\frac{s}{t_j})^{q-2}+\frac{1}{\Gamma(q)}\int_1^s(\log\frac{s}{u})^{q-1}f(u,x(u))\frac{du}{u}\Big]\frac{ds}{s}\Big)/\Gamma(2-q)\\
&=\Big(\sum_{\tau=0}^{i-1}(t\frac{d}{dt})\int_1^t(\log\frac{t}{s})^{1-q}\sum_{j=0}^{\tau}\frac{c_j}{\Gamma(q)}(\log\frac{s}{t_j})^{q-1}\frac{ds}{s}\\
&\quad +\sum_{\tau=0}^{i-1}(t\frac{d}{dt})\int_1^t(\log\frac{t}{s})^{1-q}\sum_{j=0}^{\tau}\frac{d_j}{\Gamma(q-1)}(\log\frac{s}{t_j})^{q-2}\frac{ds}{s}\Big)/\Gamma(2-q)\\
&\quad +\Big((t\frac{d}{dt})\int_{t_i}^t(\log\frac{t}{s})^{1-q}\sum_{j=0}^{i}\frac{c_j}{\Gamma(q)}(\log\frac{s}{t_j})^{q-1}\frac{ds}{s}\\
&\quad +(t\frac{d}{dt})\int_{t_i}^t(\log\frac{t}{s})^{1-q}\sum_{j=0}^{i}\frac{d_j}{\Gamma(q-1)}(\log\frac{s}{t_j})^{q-2}\frac{ds}{s}\Big)/\Gamma(2-q)\\
&\quad +\Big(\frac{1}{\Gamma(q)}(t\frac{d}{dt})\int_1^t(\log\frac{t}{s})^{1-q}\int_1^s(\log\frac{s}{u})^{q-1}f(u,x(u))\frac{du}{u}\frac{ds}{s}\Big)/\Gamma(2-q)\\
&=\Big(\sum_{j=0}^{i}\frac{c_j}{\Gamma(q)}(t\frac{d}{dt})\int_{t_j}^t(\log\frac{t}{s})^{1-q}(\log\frac{s}{t_j})^{q-1}\frac{ds}{s}
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{j=0}^{i}\frac{d_j}{\Gamma(q-1)}(t\frac{d}{dt})\int_{t_j}^{t}(\log\frac{t}{s})^{1-q}(\log\frac{s}{t_j})^{q-2}\frac{ds}{s}\Big)/\Gamma(2-q)\\
&+\Big(\frac{1}{\Gamma(q)}(t\frac{d}{dt})\int_1^t\int_u^t(\log\frac{t}{s})^{1-q}(\log\frac{s}{u})^{q-1}\frac{ds}{s}f(u,x(u))\frac{du}{u}\Big)/\Gamma(2-q)\\
=&\Big(\sum_{j=0}^{i}\frac{c_j}{\Gamma(q)}(t\frac{d}{dt})(\log\frac{t}{t_j})\int_0^1(1-w)^{1-q}w^{q-1}dw\\
&+\sum_{j=0}^{i}\frac{d_j}{\Gamma(q-1)}(t\frac{d}{dt})\int_0^1(1-w)^{1-q}w^{q-2}dw\Big)/\Gamma(2-q)\\
&+\Big(\frac{1}{\Gamma(q)}(t\frac{d}{dt})\int_1^t(\log\frac{t}{u})\int_0^1(1-w)^{1-q}w^{q-1}dwf(u,x(u))\frac{du}{u}\Big)/\Gamma(2-q)\\
=&\sum_{j=0}^{i}c_j+\int_1^t f(s,x(s))\frac{ds}{s}.
\end{aligned}
$$

It follows that

$$
\begin{gathered}
{}_HI_{1^+}^{2-\alpha}x(t)=\sum_{j=0}^{i}c_j(\log\frac{t}{t_j})+\sum_{j=0}^{i}d_j+\int_1^t(\log\frac{t}{u})f(u,x(u))\frac{du}{u},\\
t\in(t_i,t_{i+1}],\ i\in\mathbb{N}[0,m],\\
{}_HD_{1^+}^{\alpha-1}x(t)=\sum_{j=0}^{i}c_j+\int_1^t f(s,x(s))\frac{ds}{s},\quad t\in(t_i,t_{i+1}],\ i\in\mathbb{N}[0,m].
\end{gathered}
\tag{6.42}
$$

From $\Delta_H I_{1^+}^{2-q}u(t_i)=\Delta_i(u(t_i^-))$, $i\in\mathbb{N}[1,m]$ and (6.42), we obtain $d_i=\Delta_i(u(t_i^-))$, $i\in\mathbb{N}[1,m]$. From $\Delta_H D_{1^+}^{q-1}u(t_j)=\overline{\Delta}_j(u(t_j^-))$, $j\in\mathbb{N}[1,m]$ and (6.42), we obtain $c_j=\overline{\Delta}_j(u(t_j^-))$, $j\in\mathbb{N}[1,m]$. From ${}_HI_{1^+}^{2-q}u(1)=u_1$, ${}_HD_{1^+}^{q-1}u(1)=u_2$ and (6.42), we obtain $d_0=u_1, c_0=u_2$. Substituting c_i,d_i into (6.41), we obtain

$$
\begin{aligned}
x(t)=&\frac{u_2}{\Gamma(q)}(\log t)^{q-1}+\frac{u_1}{\Gamma(q-1)}(\log t)^{q-2}+\sum_{j=1}^{i}\frac{\overline{\Delta}_j(u(t_j^-))}{\Gamma(q)}(\log\frac{t}{t_j})^{q-1}\\
&+\sum_{j=1}^{i}\frac{\Delta_j(u(t_j^-))}{\Gamma(q-1)}(\log\frac{t}{t_j})^{q-2}+\frac{1}{\Gamma(q)}\int_1^t(\log\frac{t}{s})^{q-1}f(s,x(s))\frac{ds}{s},
\end{aligned}
\tag{6.43}
$$

for $t\in(t_i,t_{i+1}]$, $i\in\mathbb{N}[0,m]$. On the other hand, if x satisfies (6.43), we can prove that x is a solution of (6.39) by direct computation. The proof is complete. □

Example 6.24 ([135])**.** Consider the problem

$$
\begin{gathered}
{}_HD_{1^+}^{3/2}x(t)=\ln t,\quad t\in(1,3],\ t\neq2,\\
\Delta_H I_{1^+}^{1/2}u(2)={}_HI_{1^+}^{1/2}u(2^+)-{}_HI_{1^+}^{1/2}u(2^-)=\delta,\\
\Delta_H D_{1^+}^{1/2}u(2)={}_HD_{1^+}^{1/2}u(2^+)-{}_HD_{1^+}^{1/2}u(2^-)=\overline{\delta},\\
{}_HI_{1^+}^{1/2}u(1)=u_1,\quad {}_HD_{1^+}^{1/2}u(1)=u_2.
\end{gathered}
\tag{6.44}
$$

By [135, Theorem 3.4], its solution is

$$x(t)=\begin{cases}\frac{u_1}{\Gamma(3/2)}\Big(\int_1^t\frac{ds}{s}\Big)^{3/2-1}+\frac{u_2}{\Gamma(3/2-1)}\Big(\int_1^t\frac{ds}{s}\Big)^{3/2-2}\\ +\frac{1}{\Gamma(3/2)}\int_1^t(\log\frac{t}{s})^{3/2-1}\log s\frac{ds}{s},\\ \quad t\in(1,2];\\ \frac{u_1}{\Gamma(3/2)}\Big(\int_1^t\frac{ds}{s}\Big)^{3/2-1}+\frac{u_2}{\Gamma(3/2-1)}\Big(\int_1^t\frac{ds}{s}\Big)^{3/2-2}\\ +\frac{1}{\Gamma(3/2)}\int_1^t(\log\frac{t}{s})^{3/2-1}\log s\frac{ds}{s}+\frac{\delta}{\Gamma(3/2-1)}\Big(\int_2^t\frac{ds}{s}\Big)^{3/2-2}\\ +\frac{\overline{\delta}}{\Gamma(3/2)}\Big(\int_2^t\frac{ds}{s}\Big)^{3/2-1}-[\overline{\lambda}\delta+\overline{h\delta}]\Big[\frac{u_1}{\Gamma(3/2)}\Big(\int_1^t\frac{ds}{s}\Big)^{3/2-1}\\ +\frac{u_2}{\Gamma(3/2-1)}\Big(\int_1^t\frac{ds}{s}\Big)^{3/2-2}+\frac{1}{\Gamma(3/2)}\int_1^t(\log\frac{t}{s})^{3/2-1}\log s\frac{ds}{s}\\ -\frac{u_1+\int_1^2\log s\frac{ds}{s}}{\Gamma(3/2)}(\int_2^t\frac{ds}{s})^{3/2-1}-\frac{u_1\log 2+u_2+\int_1^2\log\frac{2}{s}\log s\frac{ds}{s}}{\Gamma(3/2-1)}\Big(\int_2^t\frac{ds}{s}\Big)^{3/2-2}\\ -\frac{1}{\Gamma(3/2)}\int_2^t(\log\frac{t}{s})^{3/2-1}\log s\frac{ds}{s}\Big],\\ \quad t\in(2,3],\end{cases}$$

where $\overline{\lambda},\overline{h}$ are constants. It is easy to see that

$$\int_1^t(\log\frac{t}{s})^{3/2-1}\log s\frac{ds}{s}=(\log t)^{5/2}\int_0^1(1-w)^{3/2-1}wdw=(\log t)^{5/2}\mathbf{B}(3/2,2).$$

Then

$$x(t)=\begin{cases}\frac{u_1}{\Gamma(3/2)}(\log t)^{1/2}+\frac{u_2}{\Gamma(1/2)}(\log t)^{-1/2}+\frac{1}{\Gamma(7/2)}(\log t)^{5/2},\\ \quad t\in(1,2];\\ \frac{u_1}{\Gamma(3/2)}(\log t)^{1/2}+\frac{u_2}{\Gamma(1/2)}(\log t)^{-1/2}+\frac{1}{\Gamma(7/2)}(\log t)^{5/2}+\frac{\delta}{\Gamma(1/2)}(\log\frac{t}{2})^{-1/2}\\ +\frac{\overline{\delta}}{\Gamma(3/2)}(\log\frac{t}{2})^{1/2}-[\overline{\lambda}\delta+\overline{h\delta}]\Big[\frac{u_1}{\Gamma(3/2)}(\log t)^{1/2}+\frac{u_2}{\Gamma(1/2)}(\log t)^{-1/2}\\ +\frac{1}{\Gamma(7/2)}(\log t)^{5/2}-\frac{u_1+\frac{1}{2}(\log 2)^2}{\Gamma(3/2)}(\log\frac{t}{2})^{1/2}\\ -\frac{u_1\log 2+u_2+\frac{1}{6}(\log 2)^3}{\Gamma(1/2)}(\log\frac{t}{2})^{-1/2}-\frac{1}{\Gamma(7/2)}(\log t)^{5/2}\Big],\\ \quad t\in(2,3]\end{cases}$$

$$=\begin{cases}\frac{u_1}{\Gamma(3/2)}(\log t)^{1/2}+\frac{u_2}{\Gamma(1/2)}(\log t)^{-1/2}+\frac{1}{\Gamma(7/2)}(\log t)^{5/2},\quad t\in(1,2];\\ \frac{u_1+\overline{\delta}+[\overline{\lambda}\delta+\overline{h\delta}]u_1}{\Gamma(3/2)}(\log t)^{1/2}+\frac{u_2+\delta+[\overline{\lambda}\delta+\overline{h\delta}]u_2}{\Gamma(1/2)}(\log t)^{-1/2}+\frac{1}{\Gamma(7/2)}(\log t)^{5/2}\\ -\frac{[\overline{\lambda}\delta+\overline{h\delta}](u_1+\frac{1}{2}(\log 2)^2)}{\Gamma(3/2)}(\log\frac{t}{2})^{1/2}-\frac{[\overline{\lambda}\delta+\overline{h\delta}](u_1\log 2+u_2+\frac{1}{6}(\log 2)^3)}{\Gamma(1/2)}(\log\frac{t}{2})^{-1/2},\\ \quad t\in(2,3].\end{cases}$$

First, this example shows us that (6.44) has infinitely many solutions since $\overline{\lambda},\overline{h}\in\mathbb{R}$ are two variables. Second, we can obtain

$${}_HI_{1^+}^{1/2}x(t)=\begin{cases}u_1(\log t)+u_2+\frac{1}{\Gamma(4)}(\log t)^3, & t\in(1,2];\\ (u_1+\overline{\delta}+[\overline{\lambda}\delta+\overline{h\delta}]u_1)(\log t)+(u_2+\delta+[\overline{\lambda}\delta+\overline{h\delta}]u_2)\\ +\frac{1}{\Gamma(4)}(\log t)^3-([\overline{\lambda}\delta+\overline{h\delta}](u_1+\frac{1}{2}(\log 2)^2))(\log\frac{t}{2})\\ -([\overline{\lambda}\delta+\overline{h\delta}](u_1\log 2+u_2+\frac{1}{6}(\log 2)^3)), & t\in(2,3].\end{cases}$$

It is easy to see that

$$\Delta_H I_{1^+}^{1/2}u(2)={}_HI_{1^+}^{\frac{1}{2}}u(2^+)-{}_HI_{1^+}^{1/2}u(2^-)\neq\delta.$$

Hence Corollary 6.22 is wrong. In fact, the correct expression for the solution of (6.44) is

$$\begin{aligned} x(t) &= \frac{u_2}{\Gamma(3/2)}(\log t)^{3/2-1} + \frac{u_1}{\Gamma(3/2-1)}(\log t)^{3/2-2} + \sum_{j=1}^{i} \frac{\overline{\delta}}{\Gamma(3/2)}(\log \frac{t}{t_j})^{3/2-1} \\ &\quad + \sum_{j=1}^{i} \frac{\delta}{\Gamma(3/2-1)}(\log \frac{t}{t_j})^{3/2-2} + \frac{1}{\Gamma(3/2)} \int_1^t (\log \frac{t}{s})^{3/2-1} f(s,x(s)) \frac{ds}{s}, \end{aligned}$$

for $t \in (t_i, t_{i+1}]$ and $i \in \mathbb{N}[0,1]$. Our result is easy to understand.

Remark 6.25. A similar initial value problem for impulsive fractional differential equation involving the Riemann-Liouville fractional derivatives were studied in [129]. Similarly we remark that [129, Theorem 3.5, page 920]) and [129, Corollary 3.6, page 927]) are wrong. We omit the details.

Zhang [128] studied a class of higher-order nonlinear Riemann-Liouville fractional differential equations with Riemann-Stieltjes integral boundary value conditions and impulses as follows:

$$\begin{gathered} -D_{0^+}^{\alpha} u(t) = \lambda a(t) f(t,u(t)), \quad t \in (0,1) \setminus \{t_1, t_2, \dots, t_m\}, \\ \Delta u(t_k) = I_k(u(t_k)), \quad k = 1, 2, \dots, m, \\ u(0) = u'(0) = \dots = u^{(n-2)}(0) = 0, \quad u'(1) = \int_0^1 u(s) dH(s), \end{gathered} \tag{6.45}$$

where $D_{0^+}^{\alpha}$ is the standard Riemann-Liouville fractional derivative of order $n-1 < \alpha \leq n$, $n \geq 3$. The number n is the smallest integer greater than or equal to α. The impulsive point sequence $\{t_k\}_{k=1}^m$ satisfies $0 = t_0 < t_1 < \dots < t_m < t_{m+1} = 1$, $\Delta u(t_k) = \lim_{\epsilon \to 0^+} u(t_k + \epsilon) - \lim_{\epsilon \to 0^-} u(t_k + \epsilon)$, $\lambda > 0$ is a parameter, $f \in C([0,1] \times [0,+\infty), [0,+\infty))$, $a \in C((0,1), [0,+\infty))$, $I_k \in C([0,+\infty), [0,+\infty))$, the integral $\int_0^1 u(s) dH(s)$ is the Riemann-Stieltjes integral with $H : [0,1] \to \mathbb{R}$. [128, Lemma 2.4] claimed the following result:

Result 6.26. Suppose that $H : [0,1] \to \mathbb{R}$ is a function of bounded variation $\delta = \int_0^1 s^{\alpha-1} dH(s) \neq \alpha - 1$, $h \in C[0,1]$. Then the unique solution of

$$\begin{gathered} -D_{0^+}^{\alpha} u(t) = h(t), \quad t \in (0,1) \setminus \{t_1, t_2, \dots, t_m\}, \\ \Delta u(t_k) = I_k(u(t_k)), \quad k \in \mathbb{N}[1,m], \\ u(0) = u'(0) = \dots = u^{(n-2)}(0) = 0, \quad u'(1) = \int_0^1 u(s) dH(s), \end{gathered} \tag{6.46}$$

is

$$u(t) = \int_0^1 G(t,s) h(s) ds + t^{\alpha-1} \sum_{t \leq t_k < 1} t_k^{1-\alpha} I_k(u(t_k)), \quad t \in (0,1], \tag{6.47}$$

where $G(t,s) = G_1(t,s) + G_2(t,s)$ with

$$G_1(t,s) = \begin{cases} \frac{t^{\alpha-1}(1-s)^{\alpha-2} - (t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leq s \leq t \leq 1, \\ \frac{t^{\alpha-1}(1-s)^{\alpha-2}}{\Gamma(\alpha)}, & 0 \leq t \leq s \leq 1, \end{cases}$$

$$G_2(t,s) = \frac{t^{\alpha-1}}{\alpha - 1 - \delta} \int_0^1 G_1(\tau, s) dH(\tau).$$

This result is wrong. In fact, (6.47) can be re-written as

$$\begin{aligned}
u(t) &= -\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds + t^{\alpha-1}\int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha)} h(s)ds \\
&\quad + \frac{t^{\alpha-1}}{\alpha-1-\delta}\int_0^1\int_0^1 G_1(\tau,s)dH(\tau)h(s)ds \\
&\quad + t^{\alpha-1}\Big[\sum_{k=1}^{m} t_k^{1-\alpha} I_k(u(t_k)) - \sum_{j=1}^{k-1} t_j^{1-\alpha} I_j(u(t_j))\Big] \\
&=: -\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds + A_k t^{\alpha-1}, \quad t\in(t_{k-1},t_k],\ k\in\mathbb{N}[1,m].
\end{aligned}$$

By Definition 2.2, for $\alpha\in(n-1,n)$, and $t\in(t_i,t_{i+1}]$ we have

$$\begin{aligned}
&D_{0^+}^{\alpha} x(t) \\
&= \Big[\int_0^t (t-s)^{n-\alpha-1}x(s)ds\Big]^{(n)}/\Gamma(n-\alpha) \\
&= \Big[\sum_{\tau=0}^{i-1}\int_{t_\tau}^{t_{\tau+1}} (t-s)^{n-\alpha-1}x(s)ds + \int_{t_i}^{t}(t-s)^{n-\alpha-1}x(s)ds\Big]^{(n)}/\Gamma(n-\alpha) \\
&= \Big[\sum_{\tau=0}^{i-1}\int_{t_\tau}^{t_{\tau+1}} (t-s)^{n-\alpha-1}\Big(-\int_0^s \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}h(u)du + A_{\tau+1}s^{\alpha-1}\Big)ds\Big]^{(n)}/\Gamma(n-\alpha) \\
&\quad + \Big[\int_{t_i}^{t}(t-s)^{n-\alpha-1}\Big(-\int_0^s \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}h(u)du + A_{i+1}s^{\alpha-1}\Big)ds\Big]^{(n)}/\Gamma(n-\alpha) \\
&= \Big[\sum_{\tau=0}^{i-1} A_{\tau+1}\int_{t_\tau}^{t_{\tau+1}} (t-s)^{n-\alpha-1}\sum_{\tau=0}^{i-1}\int_{t_\tau}^{t_{\tau+1}}(t-s)^{n-\alpha-1}s^{\alpha-1}ds\Big]^{(n)}/\Gamma(n-\alpha) \\
&\quad + \Big[-\int_0^t (t-s)^{n-\alpha-1}\int_0^s \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}h(u)\,du\,ds \\
&\quad + A_{i+1}\int_{t_i}^t (t-s)^{n-\alpha-1}s^{\alpha-1}ds\Big]^{(n)}/\Gamma(n-\alpha) \\
&= \Big[\sum_{\tau=0}^{i-1} A_{\tau+1}\int_{t_\tau}^{t_{\tau+1}} (t-s)^{n-\alpha-1}s^{\alpha-1}ds\Big]^{(n)}\Gamma(n-\alpha) \\
&\quad + \Big[-\int_0^t\int_u^t (t-s)^{n-\alpha-1}\frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}dsh(u)du \\
&\quad + A_{i+1}\int_{t_i}^t (t-s)^{n-\alpha-1}s^{\alpha-1}ds\Big]^{(n)}/\Gamma(n-\alpha) \\
&= \Big[\sum_{\tau=0}^{i-1} A_{\tau+1}t^{n-1}\int_{\frac{t_\tau}{t}}^{\frac{t_{\tau+1}}{t}} (1-w)^{n-\alpha-1}w^{\alpha-1}dw\Big]^{(n)}/\Gamma(n-\alpha) \\
&\quad + \Big[-\int_0^t (t-u)^{n-1}\int_0^1 (1-w)^{n-\alpha-1}\frac{w^{\alpha-1}}{\Gamma(\alpha)}dwh(u)du \\
&\quad + A_{i+1}t^{n-1}\int_{\frac{t_i}{t}}^1 (1-w)^{n-\alpha-1}w^{\alpha-1}dw\Big]^{(n)}/\Gamma(n-\alpha) = h(t)
\end{aligned}$$

if and only if $A_1 = A_2 = \cdots = A_{i+1}$ if and only if

$$I_1(u(t_1)) = I_2(u(t_2)) = I_3(u(t_3)) = \cdots = I_m(u(t_m)) = 0.$$

Hence the impulse functions are not suitable. Then Result 6.26 is wrong.

We consider the improved problem

$$\begin{gathered}
-D_{0^+}^{\alpha}u(t) = h(t), \quad t \in (t_i, t_{i+1}],\ i \in \mathbb{N}[0,m], \\
\Delta I_{0^+}^{n-\alpha}u(t_k) = I_{nk}, \quad \Delta D_{0^+}^{\alpha-i}u(t_k) = I_{ik},\ k \in \mathbb{N}[1,m],\ i \in \mathbb{N}[1,n-1] \\
I_{0^+}^{n-\alpha}u(0) = D_{0^+}^{\alpha-n-1}u(0) = \cdots = D_{0^+}^{n-2}u(0) = 0, \\
D_{0^+}^{\alpha-(n-1)}u(1) = \int_0^1 I_{0^+}^{n-\alpha}u(s)dH(s),
\end{gathered} \tag{6.48}$$

Theorem 6.27. *Suppose that*

$$\delta = \frac{1}{\Gamma(n-1)} - \sum_{\tau=0}^{m} \frac{t_{\tau+1}^n - t_\tau^n}{\Gamma(n+1)} \neq 0.$$

Then u is a solution of (6.48) *if and only if u satisfies the integral equation*

$$\begin{aligned}
u(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}\frac{1}{\delta}\Big[&\int_0^1 \frac{(t-s)^{n-2}}{\Gamma(i)}h(s)ds - \sum_{\sigma=1}^{m}\sum_{v=1}^{n-1}\frac{I_{v\sigma}}{\Gamma(n-v)}(1-t_\sigma)^{n-1-v} \\
&+ \int_0^1 \frac{(t-s)^{n-2}}{\Gamma(n-1)}h(s)ds \\
&+ \sum_{\sigma=1}^{m}\sum_{\tau=\sigma}^{m}\sum_{v=1}^{n}\frac{I_{v\sigma}[(t_{\tau+1}-t_\sigma)^{n-v+1} - (t_\tau - t_\sigma)^{n-v+1}]}{\Gamma(n-v+2)} \\
&- \int_0^1 (\int_u^1 \frac{(s-u)^{n-1}}{\Gamma(n)}dH(s))h(u)du\Big] + \sum_{\sigma=1}^{i}\sum_{v=1}^{n}\frac{I_{v\sigma}}{\Gamma(\alpha-v+1)}(t-t_\sigma)^{\alpha-v} \\
&- \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds, \quad t \in (t_i, t_{i+1}],\ i \in \mathbb{N}[0,m].
\end{aligned} \tag{6.49}$$

Proof. Suppose that u is a solution of (6.48). By Theorem 3.12 (with $\lambda = 0$), there exist constants $c_{\sigma v} \in \mathbb{R}(\sigma \in \mathbb{N}[0,m],\ v \in \mathbb{N}[1,n])$ such that

$$u(t) = \sum_{\sigma=0}^{i}\sum_{v=1}^{n}\frac{c_{\sigma v}}{\Gamma(\alpha-v+1)}(t-t_\sigma)^{\alpha-v} - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds, \tag{6.50}$$

for $t \in (t_i, t_{i+1}]$ and $i \in \mathbb{N}[0,m]$. Then we obtain

$$I_{0^+}^{n-\alpha}u(t) = \sum_{\sigma=0}^{i}\sum_{v=1}^{n}\frac{c_{\sigma v}}{\Gamma(n-v+1)}(t-t_\sigma)^{n-v} - \int_0^t \frac{(t-s)^{n-1}}{\Gamma(n)}h(s)ds, \tag{6.51}$$

for $t \in (t_i, t_{i+1}]$, $i \in \mathbb{N}[0,m]$, and

$$D_{0^+}^{\alpha-i}u(t) = \sum_{\sigma=0}^{\tau}\sum_{v=1}^{i}\frac{c_{\sigma v}}{\Gamma(i-v+1)}(t-t_\sigma)^{i-v} - \int_0^t \frac{(t-s)^{i-1}}{\Gamma(i)}h(s)ds, \tag{6.52}$$

for $t \in (t_\tau, t_{\tau+1}]$, $\tau \in \mathbb{N}[0,m]$, $i \in \mathbb{N}[1,n-1]$.

From $\Delta I_{0^+}^{n-\alpha}u(t_k) = I_{nk}$ and (6.51), we obtain $c_{kn} = I_{nk}(k \in \mathbb{N}[1,m])$. From $\Delta D_{0^+}^{\alpha-i}u(t_k) = I_{ik}$ and (6.52) and (6.51), we obtain $c_{ki} = I_{ik}(k \in \mathbb{N}[1,m], i \in$

$\mathbb{N}[1, n-1]$). From $I_{0^+}^{n-\alpha}u(0) = 0$ and (6.51), we obtain $c_{0n} = 0$. From $D_{0^+}^{\alpha-n-1}u(0) = \dots = D_{0^+}^{n-2}u(0) = 0$ and (6.52), we obtain $c_{0i} = 0 (i \in \mathbb{N}[2, n-1])$. From $D_{0^+}^{\alpha-(n-1)}u(1) = \int_0^1 I_{0^+}^{n-\alpha}u(s)dH(s)$ and (6.52), we obtain

$$\begin{aligned}
&\sum_{\sigma=0}^{m}\sum_{v=1}^{n-1}\frac{c_{\sigma v}}{\Gamma(n-v)}(1-t_\sigma)^{n-1-v} - \int_0^1 \frac{(t-s)^{n-2}}{\Gamma(n-1)}h(s)ds \\
&= \sum_{\tau=0}^{m}\int_{t_\tau}^{t_{\tau+1}}(\sum_{\sigma=0}^{\tau}\sum_{v=1}^{n}\frac{c_{\sigma v}}{\Gamma(n-v+1)}(s-t_\sigma)^{n-v} - \int_0^s \frac{(s-u)^{n-1}}{\Gamma(n)}h(u)du)dH(s).
\end{aligned}$$

It follows that

$$\begin{aligned}
c_{01} = &\frac{1}{\frac{1}{\Gamma(n-1)} - \sum_{\tau=0}^{m}\frac{t_{\tau+1}^n - t_\tau^n}{\Gamma(n+1)}}\Big[\int_0^1 \frac{(t-s)^{n-2}}{\Gamma(i)}h(s)ds \\
&- \sum_{\sigma=1}^{m}\sum_{v=1}^{n-1}\frac{I_{v\sigma}}{\Gamma(n-v)}(1-t_\sigma)^{n-1-v} + \int_0^1 \frac{(t-s)^{n-2}}{\Gamma(n-1)}h(s)ds \\
&+ \sum_{\sigma=1}^{m}\sum_{\tau=\sigma}^{m}\sum_{v=1}^{n}\frac{I_{v\sigma}\Big[(t_{\tau+1}-t_\sigma)^{n-v+1} - (t_\tau - t_\sigma)^{n-v+1}\Big]}{\Gamma(n-v+2)} \\
&- \int_0^1 (\int_u^1 \frac{(s-u)^{n-1}}{\Gamma(n)}dH(s))h(u)du\Big].
\end{aligned}$$

Substituting $c_{\sigma v}$ into (6.50), we obtain

$$\begin{aligned}
u(t) = &\frac{t^{\alpha-1}}{\Gamma(\alpha)}\frac{1}{\delta}\Big[\int_0^1 \frac{(t-s)^{n-2}}{\Gamma(i)}h(s)ds - \sum_{\sigma=1}^{m}\sum_{v=1}^{n-1}\frac{I_{v\sigma}}{\Gamma(n-v)}(1-t_\sigma)^{n-1-v} \\
&+ \int_0^1 \frac{(t-s)^{n-2}}{\Gamma(n-1)}h(s)ds \\
&+ \sum_{\sigma=1}^{m}\sum_{\tau=\sigma}^{m}\sum_{v=1}^{n}\frac{I_{v\sigma}[(t_{\tau+1}-t_\sigma)^{n-v+1} - (t_\tau - t_\sigma)^{n-v+1}]}{\Gamma(n-v+2)} \\
&- \int_0^1 (\int_u^1 \frac{(s-u)^{n-1}}{\Gamma(n)}dH(s))h(u)du\Big] + \sum_{\sigma=1}^{i}\sum_{v=1}^{n}\frac{I_{v\sigma}}{\Gamma(\alpha-v+1)}(t-t_\sigma)^{\alpha-v} \\
&- \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds, \quad t \in (t_i, t_{i+1}],\ i \in \mathbb{N}[0, m].
\end{aligned}$$

This is (6.49). On the other hand, if u satisfies (6.49), we can prove that u is a solution of (6.48) by direct computation similar to the one in the proof of Theorem 3.12. □

7. Applications of impulsive fractional differential equations

It is generally known that integer-order derivatives and integrals have clear physical and geometric interpretations. In [88], it is shown that geometric interpretation of fractional integration is "Shadows on the walls" and its Physical interpretation is "Shadows of the past". Geometric and physical interpretation of fractional integration and fractional differentiation were introduced in [89]. The physical meaning of initial value problems for fractional differential equations was expressed in [45, 51].

It was shown that some fractional operators describe in a better way some complex physical phenomena, especially when dealing with memory processes or viscoelastic and viscoplastic materials [17]. Well known references about the application of fractional operators in rheology modeling are [22, 23]. One of the most important advantage of fractional order models in comparison with integer order ones is that fractional integrals and derivatives are a powerful tool for the description of memory and hereditary properties of some materials. Notice that integer order derivatives are local operators, but the fractional order derivative of a function in a point depends on the past values of such function. This features motivated the successful use of fractional calculus in population dynamics, control theory, physics, biology, medicine and so forth see [1, 14, 36, 90] and [88, Chap. 10].

A fractional differential equations (FDEs)-based theory involving 1- and 2-term equations was developed to predict the nonlinear survival and growth curves of food-borne pathogens. It is interesting to note that the solution of 1-term FDE leads to the Weibull model. Two-term FDE was more successful in describing the complex shapes of microbial survival and growth curves as compared to the linear and Weibull models [56]. The Schrödinger equation control the dynamical behaviour of quantum particles. In [2], F. B. Adda and J. Cresson, considered to study of α-differential equations and discussed a fundamental problem concerning the Schrödinger equation in the framework of Nottale's scale relativity theory.

Impulsive fractional differential equations represent a real framework for mathematical modeling to real world problems. Significant progress has been made in the theory of impulsive fractional differential equations [4, 30]. Xu et al. in their paper [117] have described an impulsive delay fishing model. In [77], the authors introduced the fractional impulsive logistic model. Fractional impulsive neural networks, fractional impulsive biological models, Lasota-Wazewska models, Lotka-Volterra models. Kolmogorov-type models and fractional impulsive models in economics were introduce in recent book [96].

It is well known that $x'(t) = t$ has solutions $x(t) = c + \frac{1}{2}t^2$ which is continuous on $\mathbb{R}$, where $c \in \mathbb{R}$. Let $0 = t_0 < t_1 < t_2 < \dots < t_m < t_{m+1} = 1$ be fixed points on $\mathbb{R}$. It also has piecewise continuous solutions $x(t) = \sum_{j=0}^{i} c_i + \frac{1}{2}t^2$, where $c_i \in \mathbb{R}(i \in \mathbb{N}[0, m]$.

Example 7.1. We consider the fractional differential equation ${}^C D_{0^+}^{\alpha} x(t) = t$ with $\alpha \in (0, 1)$. It has continuous solutions

$$\begin{aligned}
x(t) &= c + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} s ds \\
&= c + \frac{1}{\Gamma(\alpha)} t^{\alpha+1} \int_0^1 (1-w)^{\alpha-1} w dw \\
&= c + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)}, \quad t \geq 0, \ c \in \mathbb{R}.
\end{aligned}$$

By Theorem 3.11, it also has piecewise continuous solutions

$$x(t) = \sum_{j=0}^{i} c_j + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} s ds = \sum_{j=0}^{i} c_j + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)},$$

for $t \in (t_i, t_{i+1}]$, $c_i \in \mathbb{R}$ and $i \in \mathbb{N}_0^m$.

Example 7.2. We consider the fractional differential equation ${}^{RL}D^{\alpha}_{0^+}x(t) = t$ with $\alpha \in (0, 1)$. It has continuous solutions

$$x(t) = ct^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} s ds = ct^{\alpha-1} + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)}, \quad t > 0,\ c \in \mathbb{R}.$$

By Theorem 3.12, it also has piecewise continuous solutions

$$x(t) = \sum_{j=0}^{i} c_j (t-t_j)^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} s ds = \sum_{j=0}^{i} c_j (t-t_j)^{\alpha-1} + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)},$$

for $t \in (t_i, t_{i+1}]$ and $i \in \mathbb{N}[0, m]$. Here the $c_i \in \mathbb{R}$ are constants.

Example 7.3. We consider the fractional differential equation ${}^{CH}D^{\alpha}_{1^+}x(t) = \log t$ with $\alpha \in (0, 1)$. It has continuous solutions

$$x(t) = c + \frac{1}{\Gamma(\alpha)} \int_1^t (\log t - \log s)^{\alpha-1} \log s \frac{ds}{s} = c + \frac{(\log t)^{\alpha+1}}{\Gamma(\alpha+2)}, \quad t \geq 1, c \in \mathbb{R}.$$

Let $1 = s_0 < s_1 < \cdots < s_m < s_{m+1} = e$ be fixed. By Theorem 3.14, it also has piecewise continuous solutions

$$x(t) = \sum_{j=0}^{i} c_j + \frac{1}{\Gamma(\alpha)} \int_1^t (\log t - \log s)^{\alpha-1} \log s \frac{ds}{s} = \sum_{j=0}^{i} c_j + \frac{(\log t)^{\alpha+1}}{\Gamma(\alpha+2)},$$

for $t \in (s_i, s_{i+1}]$, $c_i \in \mathbb{R}$ and $i \in \mathbb{N}[0, m]$.

Example 7.4. We consider the fractional differential equation ${}^{RLH}D^{\alpha}_{1^+}x(t) = \log t$ with $\alpha \in (0, 1)$. It has continuous solutions

$$x(t) = c(\log t)^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_1^t (\log t - \log s)^{\alpha-1} \log s \frac{ds}{s} = c(\log t)^{\alpha-1} + \frac{(\log t)^{\alpha+1}}{\Gamma(\alpha+2)},$$

for $t > 1$ and $c \in \mathbb{R}$. Let $1 = s_0 < s_1 < \cdots < s_m < s_{m+1} = e$ be fixed. By Theorem 3.13, it also has piecewise continuous solutions

$$\begin{aligned} x(t) &= \sum_{j=0}^{i} c_j (\log \frac{t}{t_j})^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_1^t (\log t - \log s)^{\alpha-1} \log s \frac{ds}{s} \\ &= \sum_{j=0}^{i} c_j (\log \frac{t}{t_j})^{\alpha-1} + \frac{(\log t)^{\alpha+1}}{\Gamma(\alpha+2)}, \quad t \in (s_i, s_{i+1}],\ c_i \in \mathbb{R},\ i \in \mathbb{N}[0, m]. \end{aligned}$$

A typical application of the Logistic equation

$$u'(t) = \rho u(t)(1 - u(t)), \tag{7.1}$$

is a common model of population growth. Let $u(t)$ represents the population size and t represents the time where the constant $\rho > 0$ defines the growth rate. Another application of Logistic curve is in medicine, where the Logistic differential equation is used to model the growth of tumors. This application can be considered an extension of the above mentioned use in the framework of ecology. Denoting with $u(t)$ the size of the tumor at time t.

The fractional order Logistic model

$$D^{\alpha}_{*+}u(t) = \rho u(t)(1 - u(t)), \quad t \geq 0,\ \alpha \in (0, 1), \tag{7.2}$$

can be obtained by applying the fractional derivative operator on the Logistic equation. The model is initially published by Pierre Verhulst in 1838 [37, 76].

In (7.2), $D_{0^+}^\alpha$ denotes the fractional derivative. One sees that the exact solutions of (7.1) is $u(t)=\frac{u_0}{(1-u_0)e^{-\rho t}+u_0}$. While it is difficult to solve (7.2). When $D_{*^+}u(t)={}^CD_{0^+}^\alpha u(t)$, by using the Picard iterative method, we can get its iterative solutions:

$$\begin{gathered}
\phi_0(t)=u_0,\\
\phi_1(t)=u_0+\rho\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}u_0(1-u_0)ds=u_0+\frac{\rho u_0(1-u_0)}{\Gamma(\alpha+1)}t^\alpha,\\
\begin{aligned}
\phi_2(t)&=u_0+\rho\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\phi_1(s)(1-\phi_1(s))ds\\
&=u_0+\frac{\rho u_0}{\Gamma(\alpha+1)}t^\alpha+\frac{\rho^2u_0(1-u_0)}{\Gamma(2\alpha+1)}t^{2\alpha},
\end{aligned}\\
\phi_i(t)=u_0+\rho\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\phi_{i-1}(s)(1-\phi_{i-1}(s))ds,\quad \dots.
\end{gathered}$$

Remark 7.5. We see that (1.7) with $\lambda=0$ can be re-written as

$$\begin{gathered}
{}^{RL}D_{0^+}^\beta x(t)=p(t)f(t,x(t)),\quad t\in(t_i,t_{i+1}],\ i\in\mathbb{N}[0,m],\\
\lim_{t\to0^+}t^{2-\beta}x(t)=\int_0^1\phi(s)G(s,x(s))ds,\quad x(1)=\int_0^1\psi(s)H(s,x(s))ds,\\
\lim_{t\to t_i^+}(t-t_i)^{2-\beta}[x(t)-x(t_i)]=I(t_i,x(t_i)),\quad \Delta{}^{RL}D_{0^+}^{\beta-1}x(t_i)=J(t_i,x(t_i)),
\end{gathered}\tag{7.3}$$

for $i\in\mathbb{N}[1,m]$. When $\lambda=0$, BVP (1.8) becomes

$$\begin{gathered}
{}^CD_{0^+}^\beta x(t)=p(t)f(t,x(t)),\quad t\in(t_i,t_{i+1}],\ i\in\mathbb{N}[0,m],\\
\lim_{t\to0^+}x(t)=\int_0^1\phi(s)G(s,x(s))ds,\quad x'(1)=\int_0^1\psi(s)H(s,x(s))ds,\\
\Delta x(t_i)=I(t_i,x(t_i)),\ \ \Delta x'(t_i)=J(t_i,x(t_i)),\quad i\in\mathbb{N}[1,m].
\end{gathered}\tag{7.4}$$

If $\beta\to2$, we obtain both that (7.3) and (7.4) become the following Dirichlet type BVP for second order impulsive differential equation

$$\begin{gathered}
x''(t)=p(t)f(t,x(t)),\quad t\in(t_i,t_{i+1}],\ i\in\mathbb{N}[0,m],\\
x(0)=\int_0^1\phi(s)G(s,x(s))ds,\quad ;x(1)=\int_0^1\psi(s)H(s,x(s))ds,\\
\Delta x(t_i)=I(t_i,x(t_i)),\quad \Delta{}^{RL}D_{0^+}^{\beta-1}x(t_i)=J(t_i,x(t_i)),\quad i\in\mathbb{N}[1,m].
\end{gathered}\tag{7.5}$$

From the equivalent integral equation (3.33) (Lemma 3.15) of BVP (1.7) with $\lambda=0$, we know that the equivalent integral equation of (7.3) become the integral equation

$$\begin{aligned}
x(t)&=t\Big[\int_0^1\psi(s)H(s,x(s))ds-\int_0^1\phi(s)G(s,x(s))ds-\int_0^1(1-s)\sigma(s)ds\\
&\quad-\sum_{\sigma=1}^m((1-t_\sigma)J(t_\sigma,x(t_\sigma))+I(t_\sigma,x(t_\sigma)))\Big]+\int_0^1\phi(s)G(s,x(s))ds\\
&\quad+\sum_{\sigma=1}^i[(t-t_\sigma)J(t_\sigma,x(t_\sigma))+I(t_\sigma,x(t_\sigma))]+\int_0^t(t-s)\sigma(s)ds,
\end{aligned}\tag{7.6}$$

for $t \in (t_i, t_{i+1}]$ and $i \in \mathbb{N}[0, m]$. From the equivalent integral equation (3.41) (Lemma 3.17) of BVP(1.8) with $\lambda = 0$, we know that the equivalent integral equation of (7.4) becomes the integral equation (7.6) when $\beta \to 2$.

When $G(t, x) = H(t, x) = 0$, (7.6) is the so called Dirichlat boundary value problem for second order impulsive differential equation. Its equivalent integral equation becomes

$$\begin{aligned} x(t) = t\Big[-\int_0^1 (1-s)\sigma(s)ds - \sum_{\sigma=1}^{m}((1-t_\sigma)J(t_\sigma, x(t_\sigma)) + I(t_\sigma, x(t_\sigma)))\Big] \\ + \sum_{\sigma=1}^{i}[(t-t_\sigma)J(t_\sigma, x(t_\sigma)) + I(t_\sigma, x(t_\sigma))] + \int_0^t (t-s)\sigma(s)ds, \end{aligned}$$

for $t \in (t_i, t_{i+1}]$ and $i \in \mathbb{N}[0, m]$. This result was established in [142] when $I(t, x) = 0$. The solvability of Dirichlet boundary value problems (7.5) or its special cases were studied in [65, 70, 87, 141].

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8. Addendum posted February 13, 2017

In response to comments from readers, the author wants to correct some typos and other mistakes in the original article. More precise proofs and extension of the current results will be presented in a future article.

Page 6, (1.A4): $l \in \max\{-\beta, -2-k, 0]$ should be replaced by $l \in (\max\{-\beta+1, -2-k\}, 0]$.

Page 6, (1.A7): $k > 1-\beta$ should be replaced by $k > -1$;
$l \in \max\{-\beta, -\beta-k, 0]$ should be replaced by $l \in (\max\{-\beta+1, -\beta-k+1\}, 0]$.

Page 7, (1.A10): $l \le 0$, $2+k+l > 0$ should be replaced by $l \in (\max\{-\beta+1, -2-k\}, 0]$.

Page 8, (1.A13): $l \le 0$, $\beta+k+l > 0$ should be replaced by $l \in (\max\{-\beta+1, -\beta-k+1\}, 0]$.

Page 9, Definitions 2.1: this definition should be replaced by

Definition 2.1 ([58, page 69]) Let $-\infty < a < b < +\infty$. The Riemann-Liouville fractional integrals $I^\alpha_{a^+}g$ and $I^\alpha_{b^-}g$ of order $\alpha \in \mathbb{C}$ with $(\mathrm{Re}(\alpha) > 0)$ are defined by

$$I^\alpha_{a^+}g(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}g(s)ds, t > a,$$
$$I^\alpha_{b^-}g(t) = \frac{1}{\Gamma(\alpha)}\int_t^b (s-t)^{\alpha-1}g(s)ds, t < b$$

respectively. These integrals are called he left side and the right side fractional integrals.

Page 9, Definition 2.2: this definition should be replaced by

Definition 2.2 ([58, page 70]) Let $-\infty < a < b < +\infty$. The Riemann-Liouville fractional derivatives $D^\alpha_{a^+}g$ and $D^\alpha_{b^-}g$ of order $\alpha \in \mathbb{C}$ with with $\mathrm{Re}(\alpha) \ge 0$ are defined by

$$ {}^{RL}D_{a+}^{\alpha}g(t) = (\frac{d}{dt})^n I_{a+}^{n-\alpha}g(t) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_a^t \frac{g(s)}{(t-s)^{\alpha-n+1}}ds, t>a, $$

$$ {}^{RL}D_{b-}^{\alpha}g(t) == (-\frac{d}{dt})^n I_{b-}^{n-\alpha}g(t) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_t^b \frac{g(s)}{(s-t)^{\alpha-n+1}}ds, t<b $$

where $n=[\mathrm{Re}(\alpha)]+1$. In particular, when $\alpha=n\in\mathbb{N}$, then $D_{a+}^0 g(t)=D_{b-}^0 g(t)=g(t)$ and $D_{a+}^n g(t)=g^{(n)}(t), D_{b-}^n g(t)=(-1)^n g^{(n)}(t)$, where $g^{(n)}(t)$ is the usual derivative of $g(t)$ of order n.

Page 9, Definition 2.3: this definition should be replaced by

Definition 2.3 [58, page 91] Let $-\infty<a<b<+\infty$. The Caputo fractional derivatives ${}^C D_{a+}^{\alpha}g$ and ${}^C D_{b-}^{\alpha}g$ of order $\alpha\in\mathbb{C}$ with with $\mathrm{Re}(\alpha)\geq 0$ are defined via the fractional integrals by

$$ {}^C D_{a+}^{\alpha}g(t) = \int_a^t \frac{(t-s)^{n-\alpha-1}}{\Gamma(n-\alpha)}g^{(n)}(s)ds, \quad t>a, $$

$$ {}^C D_{b-}^{\alpha}g(t) = \int_t^b \frac{(s-t)^{n-\alpha-1}}{\Gamma(n-\alpha)}g^{(n)}(s)ds, \quad t<b $$

respectively, where $n=[\mathrm{Re}(\alpha)]+1$ for $\alpha\notin\mathbb{N}$ and $n=\alpha$ for $\alpha\in\mathbb{N}$. These derivatives are called left side and right side Caputo fractional derivatives of order α.

Page 10, Definition 2.4: this definition should be replaced by

Definition 2.4 [58, page 110] Let $0<a<b<+\infty$. The left side and the right side Hadamard fractional integrals ${}^H I_{a+}^{\alpha}g$ and ${}^H I_{b-}^{\alpha}g$ of order $\alpha\in\mathbb{C}(\mathrm{Re}(\alpha)>0)$ are defined by

$$ {}^H I_{a+}^{\alpha}g(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (\log\frac{t}{s})^{\alpha-1}g(s)\frac{ds}{s}, \quad t>a, $$

$$ {}^H I_{b-}^{\alpha}g(t) = \frac{1}{\Gamma(\alpha)}\int_t^b (\log\frac{s}{t})^{\alpha-1}g(s)\frac{ds}{s}, \quad t<b $$

respectively.

Page 10, Definition 2.5: this definition should be replaced by

Definition 2.5 [58, page 111] Let $0<a<b<+\infty$. The left side and the right side Hadamard fractional derivatives ${}^{RLH}D_{a+}^{\alpha}g$ and ${}^{RRH}D_{b-}^{\alpha}g$ of order $\alpha\in\mathbb{C}$ with $\mathrm{Re}(\alpha)\geq 0$ are defined by

$$ {}^{RLH}D_{a+}^{\alpha}g(t) = \frac{1}{\Gamma(n-\alpha)}(t\frac{d}{dt})^n\int_a^t (\log\frac{t}{s})^{n-\alpha-1}g(s)\frac{ds}{s}, \quad t>a, $$

$$ {}^{RRH}D_{b-}^{\alpha}g(t) = \frac{1}{\Gamma(n-\alpha)}(-t\frac{d}{dt})^n\int_t^b (\log\frac{s}{t})^{n-\alpha-1}g(s)\frac{ds}{s}, \quad t<b $$

respectively, where $n=[\mathrm{Re}(\alpha)]+1$.

Page 10, Definition 2.6: this definition should be replaced by

Definition 2.6 [53] Let $0<a<b<+\infty$. The left side and right side Caputo type Hadamard fractional derivatives ${}^{CH}D_{a+}^{\alpha}g$ and ${}^{CH}D_{b-}^{\alpha}g$ of order $\alpha\in\mathbb{C}$ with $\mathrm{Re}(\alpha)\geq 0$ are defined by

$$ {}^{CH}D_{a+}^{\alpha}g(t) = \int_a^t \frac{(\log t-\log s)^{n-\alpha-1}}{\Gamma(n-\alpha)}(s\frac{d}{ds})^n g(s)\frac{ds}{s}, \quad t>a, $$

$$ {}^{CH}D_{b^-}^{\alpha}g(t)=\int_t^b\frac{(\log s-\log t)^{n-\alpha-1}}{\Gamma(n-\alpha)}(s\frac{d}{ds})^n g(s)\frac{ds}{s},\quad t<b $$

respectively, where $n=[\mathrm{Re}(\alpha)]+1$ for $\alpha\notin\mathbb{N}$ and $n=\alpha$ for $\alpha\in\mathbb{N}$.

Page 12, line 10: $j\in N[0,n-1]$ should be replaced by $j\in\mathbb{N}[0,n-1]$.

Page 12, equation (3.3) should be replaced by

$$ \begin{gathered} {}^{RLH}D_{1^+}^{\alpha}x(t)=B(t)x(t)+G(t),\quad \text{a.e. } t\in(1,e),\\ \lim_{t\to1^+}(\log t)^{n-\alpha}x(t)=\frac{\eta_n}{\Gamma(\alpha-n+1)},\\ \lim_{t\to1^+}{}^{RLH}D_{1^+}^{\alpha-j}x(t)=\eta_j,\quad j\in\mathbb{N}[1,n-1], \end{gathered} \tag{3.3} $$

Page 12, equation (3.4) should be replaced by

$$ \begin{gathered} {}^{CH}D_{1^+}^{\alpha}x(t)=B(t)x(t)+G(t),\quad \text{a.e. } t\in(1,e),\\ \lim_{t\to1^+}(t\frac{d}{dt})^j x(t)=\eta_j,\quad j\in\mathbb{N}[0,n-1]. \end{gathered} \tag{3.4} $$

Page 12, (3.A1): the assumptions should be replaced by

(3.A1) there exist constants $k_i>-\alpha+n-1$, $l_i\le 0$ with $l_i>\max\{-\alpha+n-1,-\alpha-k_i+n-1\}(i=1,2)$, $M_A\ge0$ and $M_F\ge0$ such that $|A(t)|\le M_At^{k_1}(1-t)^{l_1}$ and $|F(t)|\le M_Ft^{k_2}(1-t)^{l_2}$ for all $t\in(0,1)$

Page 16, line 3: $\frac{\eta_j}{j!}$" should be replaced by $\frac{\eta_j}{j!}t^j$.

Page 20 line -5: From Cases 1, 2 and 3 should be replaced by From Claims 1, 2 and 3.

Page 20, line -4: $j\in N[0,n-1]$ should be replaced by $j\in\mathbb{N}[0,n-1]$.

Page 52, equation (3.25) should be replaced by

$$ {}^{RLH}D_{1^+}^{\alpha}x(t)=\lambda x(t)+G(t),\quad \text{a.e. } t\in(s_i,s_{i+1}],\ i\in\mathbb{N}[0,m], \tag{3.25} $$

Page 52, equation (3.26) should be replaced by

$$ {}^{CH}D_{1^+}^{\alpha}x(t)=\lambda x(t)+G(t),\quad \text{a.e. } t\in(s_i,s_{i+1}],\ i\in\mathbb{N}[0,m], \tag{3.26} $$

Page 52, line 8: where $n-1<\alpha<n$, dots (3.25) and (3.26). should be replaced by where $0=t_0<t_1<\dots<t_m<t_{m+1}=1$ in (3.23) and (3.24), $1=s_0<s_1<\dots<s_m<s_{m+1}=e$ in (3.25) and (3.26).

Page 74, line -1: $P_1C_{1-\alpha}(0,1]$ should be replaced by $P_mC_{2-\beta}(0,1]$.

Page 75, line 3: for $(x,y)\in\overline{\Omega}$ should be replaced by $x\in\overline{\Omega}$.

Page 75, line 12: Using (3.33) should be replaced by Using the definition of T.

Page 76 line -12: Ascoli-CArzela should be replaced by Ascoli-Arzela

Page 101: equations (5.1)–(5.4) should be replaced by

$$ {}^{C}D_{0^+}^{\alpha}x(t)=G(t),\quad \text{a.e. } t\in(t_i,t_{i+1}],\ i\in\mathbb{N}[0,m], \tag{5.1} $$

$$ {}^{RL}D_{0^+}^{\alpha}x(t)=G(t),\quad \text{a.e. } t\in(t_i,t_{i+1}],\ i\in\mathbb{N}[0,m], \tag{5.2} $$

$$ {}^{RLH}D_{1^+}^{\alpha}x(t)=G(t),\quad \text{a.e. } t\in(s_i,s_{i+1}],\ i\in\mathbb{N}[0,m], \tag{5.3} $$

$$ {}^{CH}D_{1^+}^{\alpha}x(t)=G(t),\quad \text{a.e. } t\in(s_i,s_{i+1}],\ i\in\mathbb{N}[0,m]. \tag{5.4} $$

Page 151: equation (6.31) should be replaced by

$$ x(t)=\sum_{j=1}^{i}I_j(x(t_j))\mathbf{E}_{\gamma,1}(-a(\log t-\log t_j)^{\gamma}) $$

$$
\begin{aligned}
&- \Big[\Big[\sum_{i=1}^{m}\alpha_i\Big[\sum_{j=1}^{i} I_j(x(t_j))\mathbf{E}_{\gamma,1}(-a(\log\tau_i-\log t_j)^\gamma) \\
&+ \int_1^{\tau_i}(\log\tau_i-\log s)^{\gamma-1}f(s,x(s),y(s))\frac{ds}{s}\Big]\Big] \\
&\div \Big[1+\sum_{i=1}^{m}\alpha_i\mathbf{E}_{\gamma,1}(-a(\log\tau_i))\Big]\Big]\mathbf{E}_{\gamma,1}(-a(\log t)^\gamma) \\
&+ \int_1^{t}(\log t-\log s)^{\gamma-1}f(s,x(s),y(s))\frac{ds}{s}, \quad t\in(t_i,t_{i+1}],\ i\in\mathbb{N}[0,m].
\end{aligned}
$$

$$
\begin{aligned}
y(t) &= \sum_{j=1}^{i} J_j(y(t_j))\mathbf{E}_{\gamma,1}(-b(\log t-\log t_j)^\gamma) \\
&- \Big[\Big[\sum_{i=1}^{m}\beta_i\Big[\sum_{j=1}^{i} J_j(y(t_j))\mathbf{E}_{\gamma,1}(-b(\log\tau_i-\log t_j)^\gamma) \\
&+ \int_1^{\tau_i}(\log\tau_i-\log s)^{\gamma-1}g(s,x(s),y(s))\frac{ds}{s}\Big]\Big] \\
&\div \Big[1+\sum_{i=1}^{m}\beta_i\mathbf{E}_{\gamma,1}(-b(\log\tau_i))\Big]\Big]\mathbf{E}_{\gamma,1}(-b(\log t)^\gamma) \\
&+ \int_1^{t}(\log t-\log s)^{\gamma-1}g(s,x(s),y(s))\frac{ds}{s}, \quad t\in(t_i,t_{i+1}],\ i\in\mathbb{N}[0,m].
\end{aligned} \tag{6.31}
$$

Page 170, line 1: $D_{0^+}^{\alpha}$ should be replaced by $D_{*^+}^{\alpha}$.
Page 170, equation (7.5): ";" should be deleted.
Page 171, line 3: label (7.6) should be replaced by label (7.5).
End of addendum

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