Electronic Journal of Differential Equations, Vol. 2016 (2016), No. 301, pp. 1–16.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

MULTIPLE POSITIVE SOLUTIONS FOR KIRCHHOFF TYPE PROBLEMS INVOLVING CONCAVE AND CONVEX NONLINEARITIES IN $\mathbb{R}^3$

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Abstract. In this article, we consider the multiplicity of positive solutions for a class of Kirchhoff type problems with concave and convex nonlinearities. Under appropriate assumptions, we prove that the problem has at least two positive solutions, moreover, one of which is a positive ground state solution. Our approach is mainly based on the Nehari manifold, Ekeland variational principle and the theory of Lagrange multipliers.

1. Introduction and statement of main results

In this article, we consider the multiplicity of positive solutions for the Kirchhoff type problem

$$\begin{gathered}-(a+b\int_{\mathbb{R}^3}|\nabla u|^2\,dx)\Delta u+V(x)u=f(x)|u|^{q-2}u+g(x)|u|^{p-2}u,\\ u\in H^1(\mathbb{R}^3),\end{gathered}\tag{1.1}$$

where a and b are positive constants, $1<q<2$, $4<p<2^*=6$, $V(x),f(x),g(x)$ are continuous functions and satisfy suitable conditions.

Problem (1.1) can be written in the general form

$$\begin{gathered}-(a+b\int_{\mathbb{R}^N}|\nabla u|^2\,dx)\Delta u+V(x)u=h(x,u),\\ u\in H^1(\mathbb{R}^N),\end{gathered}\tag{1.2}$$

where $V:\mathbb{R}^N\to\mathbb{R}$ is a continuous potential, $a,b>0$ are constants. For the case of the nonlinearity h is asymptotically linear or superlinear, it has been studied extensively by many authors, see [12, 13, 17, 18, 19, 20, 24, 26, 31, 38, 39] and their references therein.

A special case of (1.2) is the well-known equation

$$\begin{gathered}-(a+b\int_{\Omega}|\nabla u|^2\,dx)\Delta u=h(x,u)\quad\text{in }\Omega,\\ u=0\quad\text{on }\partial\Omega,\end{gathered}\tag{1.3}$$

2010 *Mathematics Subject Classification.* 35A01, 35A15.

Key words and phrases. Kirchhoff type problem; Ekeland variational principle; Nehari manifold.

Submitted July 25, 2016. Published November 25, 2016.

where Ω is a bounded domain in $\mathbb{R}^N$. This problem (1.3) is related to the stationary analogue of the equation

$$\begin{gathered} u_{tt}-(a+b\int_\Omega |\nabla u|^2\,dx)\Delta u = h(x,u) \quad \text{in } \Omega, \\ u=0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.4}$$

proposed by Kirchhoff [21] in 1883 as an extension of the classical D'Alembert's wave equation for free vibration of elastic strings. Kirchhoff's model takes into account the changes in length of the string produced by transverse vibrations. In (1.4), u denotes the displacement, $h(x,u)$ the external force and b the initial tension while a is related to the intrinsic properties of the string (such as Young's modulus). Such problems are often viewed as nonlocal because of the presence of the integral term $\int_\Omega |\nabla u|^2\,dx$ which implies that problem (1.4) is no longer a pointwise identity. This phenomenon causes some mathematical difficulties which makes the study of such a class of problem particularly interesting. Besides, similar nonlocal problem also appears in other fields such as physical and biological systems, where u describes a process that depends on the average of itself, for example, the population density.

Here we are interested in the nonlinearity h made up of the combination of a sublinear term and a superlinear term. This case was considered by Willem [35] for the elliptic equation

$$\begin{gathered} -\Delta u = \lambda|u|^{q-2}u+\mu|u|^{p-2}u, \\ u\in H_0^1(\Omega), \end{gathered} \tag{1.5}$$

where $1<q<2<p<2^*(2^*=2N/(N-2)$ if $N\geq 3, 2^*=\infty$ if $N=1,2)$ and Ω is a bounded domain in $\mathbb{R}^N$. The author proved that for every $\mu>0, \lambda\in\mathbb{R}$, problem (1.5) has a sequence of high energy solutions, and for every $\lambda>0,\mu\in\mathbb{R}$, problem (1.5) has a sequence of negative energy solutions.

Ambrosetti, Brezis and Cerami [1] studied equation (1.5) when $\mu=1$. The authors used sub- and super-solutions to prove that when $\mu=1$ there exists $\Lambda>0$ such that (1.5) admits at least two positive solutions for $\lambda\in(0,\Lambda)$, one positive solution for $\lambda=\Lambda$ and no positive solution for $\lambda>\Lambda$. Note that if $\mu\neq 0$ in (1.5), then by scaling (1.5) becomes the situation of $\mu=1$.

Wu [36] also studied the concave-convex elliptic equation

$$\begin{gathered} -\Delta u+u = f_\lambda(x)u^{q-1}+g_\mu(x)u^{p-1} \quad \text{in } \mathbb{R}^N, \\ u\geq 0 \quad \text{in } \mathbb{R}^N, \\ u\in H^1(\mathbb{R}^N) \end{gathered} \tag{1.6}$$

where $1<q<2<p<2^*$ $(2^*=2N/(N-2)$ if $N\geq 3$, $2^*=\infty$ if $N=1,2)$,

$$f_\lambda(x)=\lambda f_+(x)+f_-(x)(f_\pm(x):=\pm\max\{0,\pm f(x)\}\neq 0)$$

is sign-changing, $g_\mu(x)=a(x)+\mu b(x)$ and the parameters $\lambda,\mu\geq 0$. When the functions $f_+(x), f_-(x), a(x), b(x)$ satisfy suitable conditions, he proved the multiplicity of positive solutions for the problem (1.6).

Recently, Chen, Kuo and Wu [9] considered the Kirchhoff type problem

$$\begin{gathered} -(a+b\int_\Omega |\nabla u|^2\,dx)\Delta u = \lambda f(x)|u|^{q-2}u+g(x)|u|^{p-2}u, \\ u\in H_0^1(\Omega), \end{gathered} \tag{1.7}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$ with $1 < q < 2 < p < 2^*(2^* = 2N/(N-2)$ if $N \geq 3, 2^* = \infty$ if $N = 1, 2)$, the parameters $a, b, \lambda > 0$ and the weight functions $f, g \in C(\overline{\Omega})$. Using Nehari manifold they proved the existence of solutions for (1.7) with $p > 4, p = 4$ and $p < 4$, respectively.

Motivated by these papers [9, 35, 36], we consider the Kirchhoff problem (1.1) with potential $V(x)$ and concave-convex nonlinearities on the whole space $\mathbb{R}^3$. To the best of our knowledge, there are few papers which deal with this type of Kirchhoff problem (1.1). The main difficulties lie in the unboundedness of the domain $\mathbb{R}^3$, the presence of the nonlocal term and the concave-convex nonlinearities.

Assume that $V(x), f(x), g(x)$ satisfy the following conditions:

- (A1) $V(x) \in C(\mathbb{R}^3, \mathbb{R})$, $V_0 := \inf_{\mathbb{R}^3} V(x) > 0$ and for any $M > 0$, there exists a constant $r_0 > 0$ such that $\text{meas}(\{x \in B_{r_0}(y) : V(x) \leq M\}) \to 0$ as $|y| \to +\infty$, where $B_{r_0}(y)$ denotes the ball centered at y with radius r_0, meas denotes the Lebesgue measure in $\mathbb{R}^3$.
- (A2) $f \in C(\mathbb{R}^3) \cap L^{q^*}(\mathbb{R}^3)$, where $q^* = p/(p-q)$.
- (A3) $g \in C(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ and $g(x) > 0$, for almost every $x \in \mathbb{R}^3$.

Let $\sigma := (p-2)(2-q)^{(2-q)/(p-2)}(\frac{S_p}{p-q})^{(p-q)/(p-2)}$ and $0 < \sigma^* := \frac{q}{p-2}\sigma < \sigma$, where S_p is the best Sobolev constant described in the following Lemma 2.2.

Theorem 1.1. *Under the assumptions* (A1)–(A3)*, if* $|f|_{q^*}|g|_\infty^{(2-q)/(p-2)} \in (0, \sigma)$*, the problem* (1.1) *has at least two positive solutions, one of which has negative energy. In particular, if* $|f|_{q^*}|g|_\infty^{(2-q)/(p-2)} \in (0, \sigma^*)$*, the solution corresponding to the negative energy is a positive ground state solution and the other one corresponds to positive energy.*

In problem (1.1), because of the unboundedness of the domain $\mathbb{R}^3$ there is no compactness, thus we bring in the hypothesis (A1) to recover the compactness. Moreover, the presence of the nonlocal term and the concave-convex nonlinearities prevents us from using the Nehari manifold method in a standard way as [12, 17, 18, 19, 20, 24, 26, 31, 38, 39]. Motivated by papers [6, 9, 36], we connect the Nehari manifold with the fibering map and split the Nehari manifold into three parts which are then considered separately. By putting suitable conditions on continuous functions $f(x)$, $g(x)$ and restricting $|f|_{q^*}|g|_\infty^{(2-q)/(p-2)}$ to a suitable range, we use Ekeland variational principle and the theory of Lagrange multipliers to obtain two positive solutions of the problem (1.1). In addition, from the condition (A2), we easily see that $f(x)$ is allowed to be sign-changing as [36].

Remark 1.2. The condition (A1) was first introduced by Bartsch and Wang in [5]. Note that it is weaker than both conditions

1. $V(x) \in C(\mathbb{R}^3, \mathbb{R}), \inf_{\mathbb{R}^3} V(x) > 0, V(x) \to +\infty$ as $|x| \to +\infty$ (See [19]).
2. $V(x) \in C(\mathbb{R}^3, \mathbb{R}), \inf_{\mathbb{R}^3} V(x) > 0$, for each $M > 0$, $\text{meas}(\{x \in \mathbb{R}^3 : V(x) \leq M\}) < \infty$ (See [10, 37, 40]).

These conditions are often used to recover compactness.

Remark 1.3. We can also consider the Kirchhoff problem

$$
\begin{gathered}
(a + b\int_{\mathbb{R}^N}(|\nabla u|^2 + V(x)u^2)\,dx)(-\Delta u + V(x)u) = f(x)|u|^{q-2}u + g(x)|u|^{p-2}u, \\
u \in H^1(\mathbb{R}^N),
\end{gathered} \tag{1.8}
$$

where the parameters $a, b > 0$ and $1 < q < 2 < p < 2^*(2^* = 2N/(N-2)$ if $N \geq 3, 2^* = \infty$ if $N = 1, 2)$. Under the assumptions of Theorem 1.1, restricting $|f|_{q^*}|g|_\infty^{(2-q)/(p-2)}$ to a suitable range as in Theorem 1.1, we can also obtain the existence of solutions for (1.8). It is worth noting that (1.8) is similar to (1.7) in the case of the whole space $\mathbb{R}^N$ and that there is compactness because of condition (A1). Hence we can also obtain the results in [9]. However, this type of Kirchhoff (1.8) is different from that of Kirchhoff (1.1).

This article is organized as follows: Section 2 is dedicated to our abstract framework and some preliminary results. Section 3 is concerned with the proof of Theorem 1.1. Throughout this paper, C and C_i are used in various places to denote distinct constants. $L^p(\mathbb{R}^N)$ is the usual Lebesgue space endowed with the standard norm $|u|_p = (\int_{\mathbb{R}^N} |u|^p dx)^{1/p}$ for $1 \leq p < \infty$ and $|u|_\infty = \sup_{x \in \mathbb{R}^N} |u(x)|$ for $p = \infty$. When it causes no confusion, we still denote by $\{u_n\}$ a subsequence of the original sequence $\{u_n\}$.

2. Preliminary results

In this section, we recall some preliminaries and establish the variational setting for our problem. Under the assumption (A1), define

$$E := \{u \in H^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} V(x)u^2\, dx < +\infty\},$$

with the associated norm

$$\|u\| = \Big(\int_{\mathbb{R}^3} (a|\nabla u|^2 + V(x)u^2)\, dx\Big)^{1/2},$$

where $H^1(\mathbb{R}^3)$ is the well known Sobolev space.

Then the energy functional corresponding to (1.1) is

$$\begin{aligned} I(u) = \frac{1}{2}\int_{\mathbb{R}^3} (a|\nabla u|^2 + V(x)u^2)\, dx + \frac{b}{4}\Big(\int_{\mathbb{R}^3} |\nabla u|^2\, dx\Big)^2 - \frac{1}{q}\int_{\mathbb{R}^3} f(x)|u|^q\, dx \\ - \frac{1}{p}\int_{\mathbb{R}^3} g(x)|u|^p\, dx, \quad u \in E. \end{aligned} \tag{2.1}$$

Lemma 2.1. *If* (A1)–(A3) *hold, then the functional* $I \in C^1(E, \mathbb{R})$ *and for any* $u, v \in E$,

$$\begin{aligned} \langle I'(u), v\rangle = \Big(a + b\int_{\mathbb{R}^3} |\nabla u|^2\, dx\Big)\int_{\mathbb{R}^3} \nabla u \nabla v\, dx + \int_{\mathbb{R}^3} V(x)uv\, dx \\ - \int_{\mathbb{R}^3} f(x)|u|^{q-2}uv\, dx - \int_{\mathbb{R}^3} g(x)|u|^{p-2}uv\, dx. \end{aligned} \tag{2.2}$$

The proof of the above lemma is a direct computation, and can be found in [35, 37]. As pointed out previously, assumption (A1) is used to recover compactness of embedding theorem, which is given in the next lemma.

Lemma 2.2 ([5, 40])**.** *Under assumption* (A1)*, the embedding* $E \hookrightarrow L^p(\mathbb{R}^3)$ *is continuous for* $p \in [2, 2^*]$ *and compact for* $p \in [2, 2^*)$*. Throughout this paper, we denote by* S_p *the best Sobolev constant for the embedding* $E \hookrightarrow L^p(\mathbb{R}^3)$ *which is given by*

$$S_p = \inf_{u \in E\setminus\{0\}} \frac{\|u\|^2}{(\int_{\mathbb{R}^3} |u|^p\, dx)^{2/p}} > 0.$$

In particular,
$$|u|_p \le S_p^{-1/2}\|u\|, \quad \forall u \in E\backslash\{0\}.$$

It is well-known that finding a weak solution of problem (1.1) is equivalent to finding a critical point of the corresponding functional I. In the following, we are devoted to finding the critical point of the corresponding functional I.

As usual, some energy functional such as I in (2.1) is not bounded from below on E but, as we will see, is bounded from below on an appropriate subset of E and a minimizer on this set (if it exists) may give rise to a solution of corresponding differential equation. A good exemplification for an appropriate subset of E is the so-called Nehari manifold
$$\mathcal{N} = \{u \in E : \langle I'(u), u\rangle = 0\},$$
where $\langle\cdot,\cdot\rangle$ denotes the usual duality between E and E^*. It is clear to see that $u \in \mathcal{N}$ if and only if
$$\|u\|^2 + b|\nabla u|_2^4 = \int_{\mathbb{R}^3} f(x)|u|^q\,dx + \int_{\mathbb{R}^3} g(x)|u|^p\,dx. \tag{2.3}$$
Obviously, $\mathcal{N}$ contains all solutions of (1.1). Below, we shall use the Nehari manifold methods to find critical points for functional I.

The Nehari manifold $\mathcal{N}$ is closely linked to the behavior of functions of the form $K_u : t \to I(tu)$ for $t > 0$. Such maps are known as fibering maps, which were introduced by Drábek and Pohozaev in [15]. For $u \in E$, let
$$\begin{aligned}
K_u(t) &= I(tu) = \frac{1}{2}t^2\|u\|^2 + \frac{b}{4}t^4|\nabla u|_2^4 - \frac{1}{q}t^q\int_{\mathbb{R}^3} f(x)|u|^q\,dx - \frac{1}{p}t^p\int_{\mathbb{R}^3} g(x)|u|^p\,dx;\\
K_u'(t) &= t\|u\|^2 + t^3 b|\nabla u|_2^4 - t^{q-1}\int_{\mathbb{R}^3} f(x)|u|^q\,dx - t^{p-1}\int_{\mathbb{R}^3} g(x)|u|^p\,dx;\\
K_u''(t) &= \|u\|^2 + 3t^2 b|\nabla u|_2^4 - (q-1)t^{q-2}\int_{\mathbb{R}^3} f(x)|u|^q\,dx\\
&\quad - (p-1)t^{p-2}\int_{\mathbb{R}^3} g(x)|u|^p\,dx.
\end{aligned}$$

Lemma 2.3. *Let $u \in E\backslash\{0\}$ and $t > 0$. Then $tu \in \mathcal{N}$ if and only if $K_u'(t) = 0$, that is, the critical points of $K_u(t)$ correspond to the points on the Nehari manifold. In particular, $u \in \mathcal{N}$ if and only if $K_u'(1) = 0$.*

Proof. The result is an immediate consequence of the fact that
$$K_u'(t) = \langle I'(tu), u\rangle = \frac{1}{t}\langle I'(tu), tu\rangle.$$
□

Thus, it is natural to split $\mathcal{N}$ into three parts corresponding to local minima, points of inflection and local maxima. Accordingly, we define
$$\begin{gathered}
\mathcal{N}^+ = \{u \in \mathcal{N} : K_u''(1) > 0\},\\
\mathcal{N}^0 = \{u \in \mathcal{N} : K_u''(1) = 0\},\\
\mathcal{N}^- = \{u \in \mathcal{N} : K_u''(1) < 0\}.
\end{gathered}$$
It is easy to see that
$$K_u''(1) = \|u\|^2 + 3b|\nabla u|_2^4 - (q-1)\int_{\mathbb{R}^3} f(x)|u|^q\,dx - (p-1)\int_{\mathbb{R}^3} g(x)|u|^p\,dx. \tag{2.4}$$

Define

$$\begin{aligned}\Psi(u) &= K_u'(1) = \langle I'(u), u\rangle \\ &= \|u\|^2 + b|\nabla u|_2^4 - \int_{\mathbb{R}^3} f(x)|u|^q\,dx - \int_{\mathbb{R}^3} g(x)|u|^p\,dx.\end{aligned}\tag{2.5}$$

Then for $u \in \mathcal{N}$,

$$\begin{aligned}\frac{d}{dt}\Psi(tu)|_{t=1} &= \langle \Psi'(u), u\rangle = \langle \Psi'(u), u\rangle - \langle I'(u), u\rangle = K_u''(1) \\ &= \|u\|^2 + 3b|\nabla u|_2^4 - (q-1)\int_{\mathbb{R}^3} f(x)|u|^q\,dx - (p-1)\int_{\mathbb{R}^3} g(x)|u|^p\,dx.\end{aligned}$$

For each $u \in \mathcal{N}$, $\Psi(u) = K_u'(1) = 0$. Thus, we have

$$\begin{aligned}K_u''(1) &= K_u''(1) - (q-1)\Psi(u) \\ &= (2-q)\|u\|^2 + (4-q)b|\nabla u|_2^4 - (p-q)\int_{\mathbb{R}^3} g(x)|u|^p\,dx\end{aligned}\tag{2.6}$$

and

$$\begin{aligned}K_u''(1) &= K_u''(1) - (p-1)\Psi(u) \\ &= (2-p)\|u\|^2 + (4-p)b|\nabla u|_2^4 + (p-q)\int_{\mathbb{R}^3} f(x)|u|^q\,dx.\end{aligned}\tag{2.7}$$

To ensure the Nehari manifold $\mathcal{N}$ to be a C^1-manifold, we need the following proposition. Let $\sigma := (p-2)(2-q)^{(2-q)/(p-2)}(\frac{S_p}{p-q})^{(p-q)/(p-2)}$.

Proposition 2.4. *If $|f|_{q^*}|g|_\infty^{(2-q)/(p-2)} \in (0,\sigma)$, then the set $\mathcal{N}^0 = \{0\}$.*

Proof. Suppose, on the contrary, there exists $u \in \mathcal{N}\backslash\{0\}$ such that $K_u''(1) = 0$. By Lemma 2.2,

$$\int_{\mathbb{R}^3} g(x)|u|^p\,dx \le |g|_\infty S_p^{-p/2}\|u\|^p.\tag{2.8}$$

Noting that $1 < q < 2$ and $4 < p < 6$, from (2.6) we have

$$(2-q)\|u\|^2 \le (p-q)|g|_\infty S_p^{-p/2}\|u\|^p,$$

and so

$$\|u\| \ge \Big(\frac{(2-q)S_p^{\frac{p}{2}}}{(p-q)|g|_\infty}\Big)^{\frac{1}{p-2}}.\tag{2.9}$$

Moreover, by Hölder inequality and Lemma 2.2, we have

$$\begin{aligned}\int_{\mathbb{R}^3} f(x)|u|^q\,dx &\le \Big(\int_{\mathbb{R}^3} |f(x)|^{q*}\,dx\Big)^{1/q*}\Big(\int_{\mathbb{R}^3} |u|^p\,dx\Big)^{q/p} \\ &= |f|_{q*}|u|_p^q \le |f|_{q*}S_p^{-q/2}\|u\|^q.\end{aligned}\tag{2.10}$$

From (2.7) we have

$$(p-2)\|u\|^2 \le (p-q)|f|_{q*}S_p^{-q/2}\|u\|^q,$$

which implies that

$$\|u\| \le \Big(\frac{(p-q)|f|_{q*}}{(p-2)S_p^{\frac{q}{2}}}\Big)^{\frac{1}{2-q}}.\tag{2.11}$$

Combining (2.9) and (2.11) we deduce that

$$|f|_{q^*}|g|_\infty^{\frac{2-q}{p-2}} \ge \Big(\frac{(2-q)S_p^{\frac{p}{2}}}{p-q}\Big)^{\frac{2-q}{p-2}}\frac{p-2}{p-q}S_p^{\frac{q}{2}} = (p-2)(2-q)^{\frac{2-q}{p-2}}\Big(\frac{S_p}{p-q}\Big)^{\frac{p-q}{p-2}},$$

which contradicts the assumptions. The proof is complete. $\square$

Let
$$h_b(t) := t^{2-q}\|u\|^2 + t^{4-q}b|\nabla u|_2^4 - t^{p-q}\int_{\mathbb{R}^3} g(x)|u|^p\,dx,$$
then we have
$$K_u'(t) = t^{q-1}\Big(h_b(t) - \int_{\mathbb{R}^3} f(x)|u|^q\,dx\Big). \tag{2.12}$$
Clearly, $h_b(0) = 0, h_b(t) > 0$ for t is small enough and $h_b(t) \to -\infty$ as $t \to \infty$. From $1 < q < 2,\ 4 < p < 2^* = 6$ and
$$\begin{aligned} h_b'(t) &= t^{p-q-1}\Big((2-q)t^{2-p}\|u\|^2 + (4-q)t^{4-p}b|\nabla u|_2^4 - (p-q)\int_{\mathbb{R}^3} g(x)|u|^p\,dx\Big) \\ &= 0, \end{aligned}$$
we can infer that there is a unique $t_{b,\max} > 0$ such that $h_b(t)$ achieves its maximum at $t_{b,\max}$, increasing for $t \in [0, t_{b,\max})$ and decreasing for $t \in (t_{b,\max}, \infty)$ with $\lim_{t\to\infty} h_b(t) = -\infty$.

Proposition 2.5. *Suppose that $|f|_{q^*}|g|_\infty^{(2-q)/(p-2)} \in (0, \sigma)$ and $u \in E\backslash\{0\}$. Then*

(i) if $\int_{\mathbb{R}^3} f(x)|u|^q\,dx \le 0$, then there is a unique $t^- > t_{b,\max}$ such that $t^-u \in \mathcal{N}^-$ and
$$I(t^-u) = \sup_{t\ge 0} I(tu);$$
(ii) if $\int_{\mathbb{R}^3} f(x)|u|^q\,dx > 0$, then there are unique t^+ and t^- with $0 < t^+ < t_{b,\max} < t^-$ such that $t^+u \in \mathcal{N}^+$, $t^-u \in \mathcal{N}^-$ and
$$I(t^+u) = \inf_{0\le t\le t_{b,\max}} I(tu), \quad I(t^-u) = \sup_{t\ge t_{b,\max}} I(tu).$$

Proof. Since $b > 0$, we have
$$h_b(t) > h_0(t) = t^{2-q}\|u\|^2 - t^{p-q}\int_{\mathbb{R}^3} g(x)|u|^p\,dx,$$
where $h_0(t) = h_b(t)|_{b=0}$. It is clear that $h_0(t)$ has a unique critical point at $t_{0,\max} = t_{0,\max}(u)$, where
$$t_{0,\max} = \Big(\frac{(2-q)\|u\|^2}{(p-q)\int_{\mathbb{R}^3} g(x)|u|^p\,dx}\Big)^{\frac{1}{p-2}}.$$
It follows that
$$\begin{aligned} h_0(t_{0,\max}) &= \|u\|^q\Big(\frac{\|u\|^p}{\int_{\mathbb{R}^3} g(x)|u|^p\,dx}\Big)^{\frac{2-q}{p-2}}\Big(\frac{2-q}{p-q}\Big)^{\frac{2-q}{p-2}}\frac{p-2}{p-q} \\ &\ge \|u\|^q\Big(\frac{\|u\|^p}{|g|_\infty S_p^{-p/2}\|u\|^p}\Big)^{\frac{2-q}{p-2}}\Big(\frac{2-q}{p-q}\Big)^{\frac{2-q}{p-2}}\frac{p-2}{p-q} \\ &= \|u\|^q\Big(\frac{(2-q)S_p^{\frac{p}{2}}}{|g|_\infty(p-q)}\Big)^{\frac{2-q}{p-2}}\frac{p-2}{p-q} > 0. \end{aligned} \tag{2.13}$$
Thus, $h_b(t_{b,\max}) > h_0(t_{0,\max}) > 0$.

From $|f|_{q^*}|g|_\infty^{(2-q)/(p-2)} \in (0, \sigma)$, (2.10) and (2.13) we also have
$$\begin{aligned} \int_{\mathbb{R}^3} f(x)|u|^q\,dx &< \|u\|^q\Big(\frac{(2-q)S_p^{\frac{p}{2}}}{|g|_\infty(p-q)}\Big)^{\frac{2-q}{p-2}}\frac{p-2}{p-q} \\ &\le h_0(t_{0,\max}) < h_b(t_{b,\max}). \end{aligned} \tag{2.14}$$

(i) If $\int_{\mathbb{R}^3} f(x)|u|^q\,dx \le 0$, noting that $h_b(0)=0$ and $h_b(t)\to-\infty$ as $t\to\infty$, by (2.12) there is a unique $t^- > t_{b,\max}$ such that $K_u'(t^-)=0$, that is $t^-u\in\mathcal{N}$. Moreover, for $tu\in\mathcal{N}$, $K_u'(t)=0$. By (2.12) we obtain that

$$K_u''(t) = t^{q-1}h_b'(t) < 0.$$

Thus, if $\int_{\mathbb{R}^3} f(x)|u|^q\,dx \le 0$, there is a unique $t^- > t_{b,\max}$ such that $t^-u\in\mathcal{N}^-$ and

$$I(t^-u) = \sup_{t\ge 0} I(tu).$$

(ii) If $\int_{\mathbb{R}^3} f(x)|u|^q\,dx > 0$, by (2.12) and (2.14) we know there are unique t^+ and t^- with $0<t^+<t_{b,\max}<t^-$ such that $K_{t^+u}'(1)=0$, $K_{t^-u}'(1)=0$, that is $t^+u, t^-u\in\mathcal{N}$.

From $K_u''(t)=t^{q-1}h_b'(t)$ and $h_b'(t^+)>0>h_b'(t^-)$, we have $t^+u\in\mathcal{N}^+$, $t^-u\in\mathcal{N}^-$ and

$$I(t^+u) = \inf_{0\le t\le t_{b,\max}} I(tu), \quad I(t^-u) = \sup_{t\ge t_{b,\max}} I(tu).$$

□

The forthcoming lemma obtains the minimizing sequence of the energy functional I on Nehari manifold $\mathcal{N}$.

Lemma 2.6. *The energy functional I is coercive and bounded from below on $\mathcal{N}$.*

Proof. For $u\in\mathcal{N}$, then, by Hölder inequality and Lemma 2.2,

$$\begin{aligned}
I(u) &= I(u) - \frac{1}{4}\langle I'(u),u\rangle \\
&= \frac{1}{4}\|u\|^2 - \Big(\frac{1}{q}-\frac{1}{4}\Big)\int_{\mathbb{R}^3} f(x)|u|^q\,dx + \Big(\frac{1}{4}-\frac{1}{p}\Big)\int_{\mathbb{R}^3} g(x)|u|^p\,dx \\
&\ge \frac{1}{4}\|u\|^2 - \Big(\frac{1}{q}-\frac{1}{4}\Big)|f|_{q^*}S_p^{-q/2}\|u\|^q.
\end{aligned}$$

This completes the proof. □

Lemma 2.7. *If $|f|_{q^*}|g|_\infty^{(2-q)/(p-2)}\in(0,\sigma)$, the set $\mathcal{N}^-$ is closed in E.*

Proof. Let $\{u_n\}\subset\mathcal{N}^-$ such that $u_n\to u$ in E. In the following we prove $u\in\mathcal{N}^-$. Indeed, by $\langle I'(u_n),u_n\rangle=0$ and

$$\langle I'(u_n),u_n\rangle - \langle I'(u),u\rangle = \langle I'(u_n)-I'(u),u\rangle + \langle I'(u_n),u_n-u\rangle \to 0, \quad \text{as } n\to\infty,$$

we have $\langle I'(u),u\rangle=0$. So $u\in\mathcal{N}$.

For any $u\in\mathcal{N}^-$, from (2.6) we have

$$(2-q)\|u\|^2 + (4-q)b|\nabla u|_2^4 < (p-q)\int_{\mathbb{R}^3} g(x)|u|^p dx.$$

Similar to the proof of (2.9), we have

$$\|u\| \ge \Big(\frac{(2-q)S_p^{\frac{p}{2}}}{(p-q)|g|_\infty}\Big)^{\frac{1}{p-2}}. \tag{2.15}$$

Hence $\mathcal{N}^-$ is bounded away from 0.

By (2.6), it follows that $K_{u_n}''(1)\to K_u''(1)$. From $K_{u_n}''(1)<0$, we have $K_u''(1)\le 0$. By Proposition 2.4, for $|f|_{q^*}|g|_\infty^{(2-q)/(p-2)}\in(0,\sigma), K_u''(1)<0$. Thus we deduce $u\in\mathcal{N}^-$. □

The following lemma yields a $(PS)_c$ sequence from the minimizing sequence of the energy functional I on Nehari manifold $\mathcal{N}$.

Lemma 2.8. *If $|f|_{q^*}|g|_\infty^{(2-q)/(p-2)} \in (0,\sigma)$, then for every $u \in \mathcal{N}^+$, there exist $\epsilon > 0$ and a differentiable function $\varphi^+ : B_\epsilon(0) \to \mathbb{R}_+ := (0,+\infty)$ such that*

$$\varphi^+(0) = 1, \quad \varphi^+(w)(u-w) \in \mathcal{N}^+, \quad \forall w \in B_\epsilon(0)$$

and

$$\langle (\varphi^+)'(0), w\rangle = L(u,w)/K_u''(1), \tag{2.16}$$

where

$$\begin{aligned} L(u,w) &= 2\langle u,w\rangle + 4b\int_{\mathbb{R}^3} |\nabla u|^2 \nabla u \nabla w\,dx - q\int_{\mathbb{R}^3} f(x)|u|^{q-2}uw\,dx \\ &\quad - p\int_{\mathbb{R}^3} g(x)|u|^{p-2}uw\,dx. \end{aligned}$$

Moreover, for any $C_1, C_2 > 0$, there exists $C > 0$ such that if $C_1 \le \|u\| \le C_2$,

$$|\langle (\varphi^+)'(0), w\rangle| \le C\|w\|.$$

Proof. We define $F : \mathbb{R} \times E \to \mathbb{R}$ by $F(t,w) = K'_{u-w}(t)$, it is easy to see F is differentiable. Since $F(1,0) = 0$ and

$$F_t(1,0) = K_u''(1) > 0,$$

we apply the implicit function theorem at point $(1,0)$ to obtain the existence of $\epsilon > 0$ and differentiable function $\varphi^+ : B_\epsilon(0) \to \mathbb{R}_+ := (0,+\infty)$ such that

$$\varphi^+(0) = 1, \quad F(\varphi^+(w), w) = 0, \quad \forall w \in B_\epsilon(0).$$

Thus,

$$\varphi^+(w)(u-w) \in \mathcal{N}, \quad \forall w \in B_\epsilon(0).$$

Next, we prove $\varphi^+(u-w) \in \mathcal{N}^+$, $\forall w \in B_\epsilon(0)$. Indeed, by $u \in \mathcal{N}^+$ and the set $\mathcal{N}^- \cup \mathcal{N}^0$ is closed, we know $\operatorname{dist}(u, \mathcal{N}^- \cup \mathcal{N}^0) > 0$. Since $\varphi^+(w)(u-w)$ is continuous with respect to w, when ϵ is small enough, we know for $w \in B_\epsilon(0)$

$$\|\varphi^+(w)(u-w) - u\| < \frac{1}{2}\operatorname{dist}(u, \mathcal{N}^- \cup \mathcal{N}^0),$$

so

$$\begin{aligned} \|\varphi^+(w)(u-w) - \mathcal{N}^- \cup \mathcal{N}^0\| &\ge \operatorname{dist}(u, \mathcal{N}^- \cup \mathcal{N}^0) - \operatorname{dist}(\varphi^+(w)(u-w), u) \\ &> \frac{1}{2}\operatorname{dist}(u, \mathcal{N}^- \cup \mathcal{N}^0) > 0. \end{aligned}$$

Thus, $\varphi^+(w)(u-w) \in \mathcal{N}^+$ for all $w \in B_\epsilon(0)$.

Also by the differentiability of the implicit function theorem, we have

$$\langle (\varphi^+)'(0), w\rangle = -\frac{\langle F_w(1,0), w\rangle}{F_t(1,0)}.$$

Note that $L(u,w) = -\langle F_w(1,0), w\rangle$ and $K_u''(1) = F_t(1,0)$. So we prove (2.16).

Next we prove that for any $C_1, C_2 > 0$, if $C_1 \le \|u\| \le C_2$, $u \in \mathcal{N}^+$, there exists $\delta > 0$ such that $K_u''(1) \ge \delta > 0$.

On the contrary. If there exists a sequence $\{u_n\} \in \mathcal{N}^+$, $C_1 \le \|u_n\| \le C_2$, such that for any δ_n sufficiently small, $K''_{u_n}(1) \le \delta_n$, $\delta_n \to 0$ as $n \to \infty$. From (2.6) we have

$$(2-q)\|u_n\|^2 + (4-q)b|\nabla u_n|_2^4 = (p-q)\int_{\mathbb{R}^3} g(x)|u_n|^p\,dx + O(\delta_n),$$

where $O(\delta_n) \to 0$ as $n \to \infty$.

Noting that $1 < q < 2$, $4 < p < 6$, $C_1 \le \|u_n\| \le C_2$ and (2.8), we have

$$(2-q)\|u_n\|^2 \le (p-q)|g|_\infty S_p^{-p/2}\|u_n\|^p + O(\delta_n),$$

and so

$$\|u_n\| \ge \Big(\frac{(2-q)S_p^{\frac{p}{2}}}{(p-q)|g|_\infty}\Big)^{\frac{1}{p-2}} + O(\delta_n). \tag{2.17}$$

From (2.7) we also have

$$(p-2)\|u_n\|^2 + (p-4)b|\nabla u_n|_2^4 = (p-q)\int_{\mathbb{R}^3} f(x)|u_n|^q\,dx + O(\delta_n).$$

In view of (2.10), we have

$$(p-2)\|u_n\|^2 \le (p-q)|f|_{q*} S_p^{-q/2}\|u_n\|^q + O(\delta_n),$$

which implies

$$\|u_n\| \le \Big(\frac{(p-q)|f|_{q*}}{(p-2)S_p^{\frac{q}{2}}}\Big)^{\frac{1}{2-q}} + O(\delta_n). \tag{2.18}$$

Combining (2.17) and (2.18) as $n \to \infty$, we deduce a contradiction.

Thus if $C_1 \le \|u\| \le C_2$, there exist $C > 0$ such that

$$|\langle (\varphi^+)'(0), w\rangle| \le C\|w\|.$$

This completes the proof. □

Similarly, we establish the following lemma.

Lemma 2.9. *Assume $|f|_{q^*}|g|_\infty^{(2-q)/(p-2)} \in (0,\sigma)$, then for every $u \in \mathcal{N}^-$, there exist $\epsilon > 0$ and a differentiable function $\varphi^- : B_\epsilon(0) \to \mathbb{R}_+ := (0,+\infty)$ such that*

$$\varphi^-(0) = 1, \quad \varphi^-(w)(u-w) \in \mathcal{N}^-, \quad \forall w \in B_\epsilon(0),$$
$$\langle (\varphi^-)'(0), w\rangle = L(u,w)/K_u''(1),$$

where $L(u,w)$ is defined in Lemma 2.8. Moreover, for any $C_1, C_2 > 0$, there exists $C > 0$ such that if $C_1 \le \|u\| \le C_2$,

$$|\langle (\varphi^-)'(0), w\rangle| \le C\|w\|.$$

The following lemma aims at obtaining the critical point of I on the whole space from the local minimizer for I on Nehari manifold .

Lemma 2.10. *Suppose that u is a local minimizer for I on $\mathcal{N}^+$ (or $\mathcal{N}^-$). Then $I'(u) = 0$.*

Proof. If $u \ne 0$, u is a local minimizer for I on $\mathcal{N}^+$ (or $\mathcal{N}^-$), then u is a nontrivial solution of the optimization problem

$$\text{minimize } I \text{ subject to } \Psi(u) = 0,$$

where $\Psi(u)$ is described in (2.5). Note that $\Psi'(u) \ne 0$, $\mathcal{N}^+$ (or $\mathcal{N}^-$) is a local differential manifold. So by the theory of Lagrange multipliers, there exists $\mu \in \mathbb{R}$ such that $I'(u) = \mu\Psi'(u)$. Thus

$$\langle I'(u), u\rangle = \mu\langle \Psi'(u), u\rangle.$$

Since $u \in \mathcal{N}^+$ (or $\mathcal{N}^-$), $\langle I'(u), u\rangle = 0$ and $\langle \Psi'(u), u\rangle = K_u''(1) \ne 0$. Hence, $\mu = 0$. Thus the proof is complete. □

3. Proof of Theorem 1.1

For proving Theorem 1.1, we first show that any Palais-Smale sequence of I has a strongly convergent subsequence in E.

Lemma 3.1. *Each Palais-Smale sequence $\{u_n\} \subset \mathcal{N}^+$ (or $\mathcal{N}^-$) for I on E has a strongly convergent subsequence.*

Proof. Assume that $\{u_n\} \subset \mathcal{N}^+$(or $\mathcal{N}^-$) such that $I(u_n) \to c$ and $I'(u_n) \to 0$ as $n \to \infty$. As in Lemma 2.6, we know the Palais-Smale sequence $\{u_n\} \subset \mathcal{N}^+$ (or $\mathcal{N}^-$) for I on E is bounded. And by Lemma 2.2, going if necessary to a subsequence, we have

$$
\begin{gathered}
u_n \rightharpoonup u \quad \text{in } E, \\
u_n \to u \quad \text{in } L^r(\mathbb{R}^3),\ r \in [2, 2^*).
\end{gathered}
$$

Note that

$$
\begin{aligned}
&(I'(u_n) - I'(u), u_n - u) \\
&= (I'(u_n), u_n - u) - (I'(u), u_n - u) \\
&= \Big(a + b\int_{\mathbb{R}^3} |\nabla u_n|^2\,dx\Big)\int_{\mathbb{R}^3} |\nabla(u_n - u)|^2\,dx + \int_{\mathbb{R}^3} V(x)|u_n - u|^2\,dx \\
&\quad - b\Big(\int_{\mathbb{R}^3} |\nabla u|^2\,dx - \int_{\mathbb{R}^3} |\nabla u_n|^2\,dx\Big)\int_{\mathbb{R}^3} \nabla u\nabla(u_n - u)\,dx \\
&\quad - \int_{\mathbb{R}^3} f(x)(|u_n|^{q-2}u_n - |u|^{q-2}u)(u_n - u)\,dx \\
&\quad - \int_{\mathbb{R}^3} g(x)(|u_n|^{p-2}u_n - |u|^{p-2}u)(u_n - u)\,dx \\
&\geq \|u_n - u\|^2 - b\Big(\int_{\mathbb{R}^3} |\nabla u|^2\,dx - \int_{\mathbb{R}^3} |\nabla u_n|^2\,dx\Big)\int_{\mathbb{R}^3} \nabla u\nabla(u_n - u)\,dx \\
&\quad - \int_{\mathbb{R}^3} f(x)(|u_n|^{q-2}u_n - |u|^{q-2}u)(u_n - u)\,dx \\
&\quad - \int_{\mathbb{R}^3} g(x)(|u_n|^{p-2}u_n - |u|^{p-2}u)(u_n - u)\,dx,
\end{aligned}
$$

then we can deduce that $\|u_n - u\| \to 0$ as $n \to \infty$. Indeed, from the boundedness of $\{u_n\}$ in E and Lemma 2.2, $\{u_n\}$ is bounded in $L^r(\mathbb{R}^3)$, $r \in [2, 6)$. By using twice Hölder inequality we obtain

$$
\begin{aligned}
&\Big|\int_{\mathbb{R}^3} f(x)(|u_n|^{q-2}u_n - |u|^{q-2}u)(u_n - u)\,dx\Big| \\
&\leq \Big(\int_{\mathbb{R}^3} |f|^{q^*}\,dx\Big)^{1/q^*}\Big(\int_{\mathbb{R}^3} ||u_n|^{q-2}u_n - |u|^{q-2}u|^{p/q}|u_n - u|^{p/q}\,dx\Big)^{q/p} \\
&\leq C|f|_{q^*}(|u_n|_p^{q-1} + |u|_p^{q-1})|u_n - u|_p \to 0, \quad \text{as } n \to \infty,
\end{aligned}
$$

where C is a positive constant. Similarly, we have

$$
\Big|\int_{\mathbb{R}^3} g(x)(|u_n|^{p-2}u_n - |u|^{p-2}u)(u_n - u)\,dx\Big| \to 0, \quad \text{as } n \to \infty\,.
$$

From

$$b\Big(\int_{\mathbb{R}^3}|\nabla u|^2\,dx-\int_{\mathbb{R}^3}|\nabla u_n|^2\,dx\Big)\int_{\mathbb{R}^3}\nabla u\nabla(u_n-u)\,dx\to 0,\quad \text{as } n\to\infty,$$

and

$$(I'(u_n)-I'(u),u_n-u)\to 0,\quad \text{as } n\to\infty,$$

we have $\|u_n-u\|\to 0$ as $n\to\infty$. This completes the proof. $\square$

Lemma 3.2. *If $|f|_{q^*}|g|_\infty^{(2-q)/(p-2)}\in(0,\sigma)$, then the minimization problem*

$$c_1=inf_{\mathcal{N}^+}I$$

is solved at a point $u_1\in\mathcal{N}^+$, moreover this value is a critical point of I.

Proof. First we prove the minimizing sequence $\{u_n\}\subset\mathcal{N}^+$ is a $(PS)_{c_1}$ sequence on E. Indeed, by Lemma 2.6 and Ekeland Variational Principle [35] on $\mathcal{N}^+\cup\mathcal{N}^0$, there exists a minimizing sequence $\{u_n\}\subset\mathcal{N}^+\cup N^0$ such that

$$\inf_{u\in\mathcal{N}^+\cup\mathcal{N}^0}I(u)\le I(u_n)<\inf_{u\in\mathcal{N}^+\cup\mathcal{N}^0}I(u)+\frac{1}{n},\tag{3.1}$$

$$I(u_n)-\frac{1}{n}\|v-u_n\|\le I(v),\quad \forall v\in\mathcal{N}^+\cup\mathcal{N}^0.\tag{3.2}$$

From Proposition 2.5, we know for each $u\in E\backslash\{0\}$, there is a unique t^+ such that $t^+u\in\mathcal{N}^+$, then $inf_{u\in\mathcal{N}^+}I\le I(t^+u)$. Now we prove that for each $u\in\mathcal{N}^+$, $I(u)<0$. Indeed, for each $u\in\mathcal{N}^+$, $K_u''(1)>0$. From (2.7), we have

$$(p-q)\int_{\mathbb{R}^3}f(x)|u|^q\,dx>(p-2)\|u\|^2+(p-4)b|\nabla u|_2^4.$$

Then for each $u\in\mathcal{N}^+$,

$$\begin{aligned}
I(u)&=I(u)-\frac{1}{p}\langle I'(u),u\rangle\\
&=\frac{p-2}{2p}\|u\|^2+\frac{p-4}{4p}b|\nabla u|_2^4-\frac{p-q}{pq}\int_{\mathbb{R}^3}f(x)|u|^q\,dx\\
&<\frac{p-2}{2p}\|u\|^2+\frac{p-4}{4p}b|\nabla u|_2^4-\frac{1}{pq}\big((p-2)\|u\|^2+(p-4)b|\nabla u|_2^4\big)\\
&=\frac{(p-2)(q-2)}{2pq}\|u\|^2+\frac{(p-4)(q-4)}{4pq}b|\nabla u|_2^4<0.
\end{aligned}\tag{3.3}$$

From the above, we know that $\inf_{u\in\mathcal{N}^+}I(u)<0$.

Since $I(0)=0$, we have $\inf_{u\in\mathcal{N}^+\cup\mathcal{N}^0}I(u)=\inf_{u\in\mathcal{N}^+}I(u)=c_1$. Thus we may assume $u_n\in\mathcal{N}^+$, $I(u_n)\to c_1<0$. By Lemma 2.8 , since $|f|_{q^*}|g|_\infty^{(2-q)/(p-2)}\in(0,\sigma)$, we can find $\epsilon_n>0$ and differentiable function $\varphi_n^+=\varphi_n^+(w)>0$ such that

$$\varphi_n^+(w)(u_n-w)\in\mathcal{N}^+,\quad \forall w\in B_{\epsilon_n}(0).$$

By the continuity of $\varphi_n^+(w)$ and $\varphi_n^+(0)=1$, without loss of generality, we can assume ϵ_n is sufficiently small such that $\frac{1}{2}\le\varphi_n^+(w)\le\frac{3}{2}$ for $\|w\|<\epsilon_n$. From $\varphi_n^+(w)(u_n-w)\in\mathcal{N}^+$ and (3.2), we have

$$I(\varphi_n^+(w)(u_n-w))\ge I(u_n)-\frac{1}{n}\|\varphi_n^+(w)(u_n-w)-u_n\|,$$

which implies

$$\langle I'(u_n),\varphi_n^+(w)(u_n-w)-u_n\rangle+o\big(\|\varphi_n^+(w)(u_n-w)-u_n\|\big)$$

$$\geq -\frac{1}{n}\|\varphi_n^+(w)(u_n - w) - u_n\|.$$

Consequently,

$$\begin{aligned}&\varphi_n^+(w)\langle I'(u_n), w\rangle + (1-\varphi_n^+(w))\langle I'(u_n), u_n\rangle\\&\leq \frac{1}{n}\|\left(\varphi_n^+(w)-1\right)u_n - \varphi_n^+(w)w\| + o\left(\|\varphi_n^+(w)(u_n-w)-u_n\|\right).\end{aligned}$$

By the choice of ϵ_n and $1/2 \leq \varphi_n^+(w) \leq 3/2$, we infer that there exists $C_3 > 0$ such that

$$|\langle I'(u_n), w\rangle| \leq \frac{1}{n}\|\langle(\varphi_n^+)'(0), w\rangle u_n\| + \frac{C_3}{n}\|w\| + o\left(|\langle(\varphi_n^+)'(0), w\rangle|(\|u_n\| + \|w\|)\right).$$

Next we prove for $\{u_n\} \subset \mathcal{N}^+$, $\inf_n \|u_n\| \geq C_1 > 0$, where C_1 is a constant. Indeed, if not, then $I(u_n)$ would converge to zero, which contradict with $I(u_n) \to c_1 < 0$. Moreover, by Lemma 2.6 we know that I is coercive on $\mathcal{N}^+$, $\{u_n\}$ is bounded in E. Thus, there exists $C_2 > 0$ such that $0 < C_1 \leq \|u_n\| \leq C_2$. From Lemma 2.8, $|\langle(\varphi_n^+)'(0), w\rangle| \leq C\|w\|$. So

$$|\langle I'(u_n), w\rangle| \leq \frac{C}{n}\|w\| + \frac{C}{n}\|w\| + o(\|w\|)$$

and

$$\begin{gathered}\|I'(u_n)\| = \sup_{w \in E\setminus\{0\}} \frac{|\langle I'(u_n), w\rangle|}{\|w\|} \leq \frac{C}{n} + o(1),\\ \|I'(u_n)\| \to 0, \quad \text{as } n \to \infty.\end{gathered} \tag{3.4}$$

Thus, $\{u_n\} \subset \mathcal{N}^+$ is $(PS)_{c_1}$ for I on E. From Lemma 3.1, there is a strongly convergent subsequence $\{u_n\}$, we still denote by $\{u_n\}$, $u_n \to u_1$ in E. From the above we know that there exist $C_1, C_2 > 0$ such that $0 < C_1 \leq \|u_n\| \leq C_2$, then $0 < C_1 \leq \|u_1\| \leq C_2$. Thus $u_1 \neq 0$.

Next we prove $u_1 \in \mathcal{N}^+$. Indeed, by (2.6), it follows that $K''_{u_n}(1) \to K''_{u_1}(1)$. From $K''_{u_n}(1) > 0$, we have $K''_{u_1}(1) \geq 0$. By Proposition 2.4, we know $K''_{u_1}(1) > 0$. Thus we deduce

$$u_1 \in \mathcal{N}^+, \quad I(u_1) = \lim_{n\to\infty} I(u_n) = \inf_{u \in \mathcal{N}^+} I(u).$$

We recall [16] that $\int_{\mathbb{R}^3} |\nabla|u||^2\,dx = \int_{\mathbb{R}^3} |\nabla u|^2\,dx$, therefore $I(u_1) = I(|u_1|)$ and $|u_1| \in \mathcal{N}^+$, then without loss of generality we may assume that u_1 is positive. Combining this with Lemma 2.10, we obtain the results. □

Lemma 3.3. *If $|f|_{q^*}|g|_\infty^{(2-q)/(p-2)} \in (0, \sigma)$, then the minimization problem*

$$c_2 = \inf_{\mathcal{N}^-} I$$

is solved at a point $u_2 \in \mathcal{N}^-$ which is a critical point for I.

Proof. From Lemma 2.7, $\mathcal{N}^-$ is closed in E. By Lemma 2.6, we know I is coercive on $\mathcal{N}^-$. So we use Ekeland Variational Principle [35] on $\mathcal{N}^-$ to obtain a minimizing sequence $\{u_n\} \subset \mathcal{N}^-$ such that

$$\begin{gathered}\inf_{u\in\mathcal{N}^-} I(u) \leq I(u_n) < \inf_{u\in\mathcal{N}^-} I(u) + \frac{1}{n},\\ I(u_n) - \frac{1}{n}\|v - u_n\| \leq I(v), \quad \forall v \in \mathcal{N}^-.\end{gathered}$$

In view of (2.15) and Lemma 2.6 we know the there exist $C_1, C_2 > 0$ such that

$$0 < C_1 \leq \|u_n\| \leq C_2.$$

Hence by Lemma 2.9, in the same way as Lemma 3.2, there exists a minimizing sequence $\{u_n\} \subset \mathcal{N}^-$ is the $(PS)_{c_2}$ sequence on E. From Lemmas 3.1, we know there is a strongly convergent subsequence $\{u_n\}$, we still denote by $\{u_n\}$, $u_n \to u_2$ in E. By Lemma 2.7 the set $\mathcal{N}^-$ is closed, we know $u_2 \in \mathcal{N}^-$. Thus, $I(u_2) = \lim_{n\to\infty} I(u_n) = \inf_{u\in\mathcal{N}^-} I(u)$. Since $I(u_2) = I(|u_2|)$ and $|u_2| \in \mathcal{N}^-$, then without loss of generality we may assume that u_2 is positive. Combining with Lemma 2.10, we prove the claim. ☐

Now we are in a position to give the proof of the main results.

Proof of theorem 1.1. From Lemmas 3.2 and 3.3, we know if $|f|_{q^*}|g|_\infty^{(2-q)/(p-2)} \in (0,\sigma)$, then problem (1.1) has at least two positive solutions u_1 and u_2. From the proof of Lemma 3.2 we know that if $|f|_{q^*}|g|_\infty^{(2-q)/(p-2)} \in (0,\sigma)$, then the positive solution of (1.1) u_1 belongs to $\mathcal{N}^+$ and $I(u_1) < 0$. If $0 < |f|_{q^*}|g|_\infty^{(2-q)/(p-2)} < \sigma^* := \frac{q}{p-2}\sigma < \sigma$, where σ is described in Proposition 2.4, then by (2.15) we can infer that

$$\begin{aligned}
I(u) &= I(u) - \frac{1}{4}\langle I'(u), u\rangle \\
&= \frac{1}{4}\|u\|^2 - (\frac{1}{q} - \frac{1}{4})\int_{\mathbb{R}^3} f(x)|u|^q\,dx + (\frac{1}{4} - \frac{1}{p})\int_{\mathbb{R}^3} g(x)|u|^p\,dx \\
&\geq \frac{1}{4}\|u\|^2 - (\frac{1}{q} - \frac{1}{4})|f|_{q_*} S_p^{-q/2}\|u\|^q \\
&= \|u\|^q\Big(\frac{1}{4}\|u\|^{2-q} - (\frac{1}{q} - \frac{1}{4})|f|_{q_*} S_p^{-q/2}\Big) \\
&\geq \Big(\frac{(2-q)S_p^{\frac{p}{2}}}{(p-q)|g|_\infty}\Big)^{\frac{q}{p-2}}\Big(\frac{1}{4}\Big(\frac{(2-q)S_p^{\frac{p}{2}}}{(p-q)|g|_\infty}\Big)^{\frac{2-q}{p-2}} - (\frac{1}{q} - \frac{1}{4})|f|_{q^*} S_p^{-q/2}\Big) \\
&\geq \Big(\frac{(2-q)S_p^{\frac{p}{2}}}{(p-q)|g|_\infty}\Big)^{\frac{q}{p-2}}\Big(\frac{1}{4}\Big(\frac{(2-q)S_p^{\frac{p}{2}}}{(p-q)|g|_\infty}\Big)^{\frac{2-q}{p-2}} - \frac{p-q}{4q}|f|_{q^*} S_p^{-q/2}\Big) > 0.
\end{aligned}$$

In fact, if $|f|_{q^*}|g|_\infty^{(2-q)/(p-2)} \in (0,\sigma^*)$, for any $u \in \mathcal{N}^-$, $I(u) > 0$, where $\sigma^* = q/(p-2)\sigma$. From Lemma 3.3 the positive solution of problem (1.1) $u_2 \in \mathcal{N}^-$, then for $|f|_{q^*}|g|_\infty^{(2-q)/(p-2)} \in (0,\sigma^*)$, $I(u_2) > 0$. From the above, we know that if $|f|_{q^*}|g|_\infty^{(2-q)/(p-2)} \in (0,\sigma^*)$, then $I(u_1) = \inf_{u\in\mathcal{N}} I(u)$, and u_1 is a positive ground state solution of (1.1). This completes the proof. ☐

Acknowledgments. This work is supported by the Innovation Project for college postgraduates in Jiangsu Province(KYLX15-0108), the Natural Science Foundation of China (Nos: 11371090, 11571140, 11601204, 11671077), the Natural Science Foundation of Jiangsu Province (Nos: BK 20140525, BK 20150478, BK 20160063) and the Natural Science Foundation of Outstanding Young Scholars of Jiangsu Province (Nos: BK 20160063).

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